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AN ELEMENTARY PROOF OF SMALL'S FORMULA FOR NULL CURVES IN PSL(2, *C*) AND AN ANALOGUE FOR LEGENDRIAN CURVES IN PSL(2, *C*)

Dedicated to Professor Katsuei Kenmotsu on his sixtieth birthday

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1. Introduction

Let M^2 be a Riemann surface, *which might not be simply connected*. A meromorphic map F from M^2 into $\mathrm{PSL}(2,\boldsymbol{C}) = \mathrm{SL}(2,\boldsymbol{C})/\{\pm\mathrm{id}\}$ is a map which is represented as

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sqrt{h}\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \qquad \bigl(AD - BC = 1\bigr), \tag{1.1}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and h are meromorphic functions on M^2. Though $\sqrt{h}$ is a multi-valued function on M^2, F is well-defined as a $\mathrm{PSL}(2,\boldsymbol{C})$-valued mapping.

A meromorphic map F as in (1.1) is called a *null curve* if the pull-back of the Killing form by F vanishes, which is equivalent to the condition that the derivative $F_z = \partial F/\partial z$ with respect to each complex coordinate z is a degenerate matrix everywhere. It is well-known that the projection of a null curve in $\mathrm{PSL}(2,\boldsymbol{C})$ into the hyperbolic 3-space $H^3 = \mathrm{PSL}(2,\boldsymbol{C})/\mathrm{PSU}(2)$ gives a constant mean curvature one surface (see [2, 10]). For a non-constant null curve F, we define two meromorphic functions

$$G := \frac{dA}{dC} = \frac{dB}{dD}, \qquad g := -\frac{dB}{dA} = -\frac{dD}{dC}. \tag{1.2}$$

(For a precise definition, see Definition 2.1 in Section 2). We call G the *hyperbolic Gauss map* of F and g the *secondary Gauss map*, respectively [12]. In 1993, Small [8] discovered the following expression

$$F = \begin{pmatrix} G\dfrac{da}{dG} - a & G\dfrac{db}{dG} - b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix}, \qquad \left(a := \sqrt{\frac{dG}{dg}},\ b := -ga\right) \tag{1.3}$$

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for null curves such that both G and g are non-constant. (*We shall give a simple proof of this formula in* Section 2. *Sa Earp and Toubiana* [3] *gave an alternative proof, which is quite different from ours. On the other hand, Lima and Roitman* [7] *explained this formula via the method of Bianchi* [1] *from the 1920's. Recently, Small* [9] *gave some remarks on this formula from the viewpoint of null curves in* $\boldsymbol{C}^4$.) In this expression, F is expressed by only the derivation of two Gauss maps. Accordingly, the formula is valid even if M^2 is not simply connected.

By the formula (1.3), it is shown that the set of non-constant null curves on M^2 with non-constant Gauss maps corresponds bijectively to the set of pairs (G, g) of meromorphic functions on M^2 such that $g \not\equiv a \star G$ (that is, g is not identically equal to $a \star G$) for any $a \in \mathrm{SL}(2, \boldsymbol{C})$. Here, for a matrix $a = (a_{ij}) \in \mathrm{SL}(2, \boldsymbol{C})$, we denote by $a \star G$ the Möbius transformation of G:

$$a \star G := \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}. \tag{1.4}$$

For this correspondence, see also [11].

On the other hand, according to Gálvez, Martínez and Milán ([4, 5]), a meromorphic map

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.5}$$

from M^2 into $\mathrm{PSL}(2, \boldsymbol{C})$ is called a *Legendrian curve* (or a *contact curve*) if the pull-back of the holomorphic contact form

$$D\,dA - B\,dC \tag{1.6}$$

on $\mathrm{PSL}(2, \boldsymbol{C})$ by E vanishes. For a Legendrian curve E, two meromorphic functions

$$G = \frac{A}{C}, \qquad G_* = \frac{B}{D} \tag{1.7}$$

are defined. In [4], G and G_* are called the *hyperbolic Gauss maps*. We define a meromorphic 1-form ω on M^2 as

$$\omega := \frac{dA}{B} = \frac{dC}{D}. \tag{1.8}$$

(For a precise definition, see Definition 3.1 and Lemma 3.2 in Section 3.) We shall call ω the *canonical form*.

As an analogue of the Bryant representation formula [2, 10] for constant mean curvature one surfaces in H^3, Gálvez, Martínez and Milán [4] showed that any simply connected flat surface in hyperbolic 3-space can be lifted to a Legendrian curve in

PSL(2, $\boldsymbol{C}$), where the complex structure of the surface is given so that the second fundamental form is hermitian. It is natural to expect that there is a Small-type formula for Legendrian curves in PSL(2, $\boldsymbol{C}$).

In this paper, we shall give a representation formula for Legendrian curves in terms of G and G_* (Theorem 3.3). Namely, for an arbitrary pair of non-constant meromorphic functions (G, G_*) such that $G \not\equiv G_*$ (G is not identically equal to G_*), the Legendrian curve E with hyperbolic Gauss maps G and G_* is written as

$$E = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix} \qquad \left(\xi = c \exp \int_{z_0}^{z} \frac{dG}{G - G_*} \right), \tag{1.9}$$

where $z_0 \in M^2$ is a base point and $c \in \boldsymbol{C} \setminus \{0\}$ is a constant. As a corollary of this formula, we shall give a Small-type representation formula for Legendrian curves (Corollary 3.4):

$$E = \begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix}, \qquad \left(C := i\sqrt{\frac{\omega}{dG}}, \quad A := GC \right). \tag{1.10}$$

It should be remarked that the formula (1.10) has appeared implicitly in [4, p. 423] by a different method.

In Section 4, we shall give new examples of flat surfaces with complete ends using these representation formulas. Though these examples might have singularities, they can be lifted as a Legendrian immersion into the unit cotangent bundle of H^3, and so we call them *flat* (*wave*) *fronts*. See [6] for a precise definition and global properties of flat fronts with complete ends.

Small's formula is an analogue of the classical representation formula for null curves in $\boldsymbol{C}^3$, which is closely related to the Weierstrass representation formula for minimal surfaces in $\boldsymbol{R}^3$. For the reader's convenience, we give a simple proof of the classical formula in the appendix.

2. A simple proof of Small's formula

In this section, we shall introduce a new proof of Small's formula (Theorem 2.4), which is an analogue of the classical representation formula for null curves in $\boldsymbol{C}^3$ (see the appendix). We fix a Riemann surface M^2, which is not necessarily simply connected.

Let

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

be a null curve in PSL(2, $\boldsymbol{C}$) defined on M^2.

DEFINITION 2.1. For a non-constant null curve F as in (2.1), we define

$$G := \begin{cases} \dfrac{dA}{dC} & \bigl(\text{if } (dA, dC) \not\equiv (0,0)\bigr), \\ \dfrac{dB}{dD} & \bigl(\text{if } (dB, dD) \not\equiv (0,0)\bigr), \end{cases} \qquad g := \begin{cases} -\dfrac{dB}{dA} & \bigl(\text{if } (dA, dB) \not\equiv (0,0)\bigr), \\ -\dfrac{dD}{dC} & \bigl(\text{if } (dC, dD) \not\equiv (0,0)\bigr). \end{cases}$$

Since F is null,

$$\frac{dA}{dC} = \frac{dB}{dD} \qquad \text{if } (dA, dC) \not\equiv (0,0) \text{ and } (dB, dD) \not\equiv (0,0),$$
$$-\frac{dB}{dA} = -\frac{dD}{dC} \qquad \text{if } (dA, dB) \not\equiv (0,0) \text{ and } (dC, dD) \not\equiv (0,0)$$

hold. We call G and g the *hyperbolic Gauss map* and the *secondary Gauss map* of F, respectively.

Lemma 2.2. *Let F be a meromorphic null curve as in* (2.1). *If either $dA \equiv dC \equiv 0$ or $dB \equiv dD \equiv 0$ holds, then the hyperbolic Gauss map G is constant. Similarly, if either $dA \equiv dB \equiv 0$ or $dC \equiv dD \equiv 0$ holds, then the secondary Gauss map g is constant.*

Proof. Assume $dA \equiv dC \equiv 0$. Since $AD - BC = 1$, we have

$$0 = d(AD - BC) = D\,dA + A\,dD - B\,dC - C\,dB = A\,dD - C\,dB.$$

Here, since $(A, C) \not\equiv (0,0)$, $dB/dD \in \boldsymbol{C} \cup \{\infty\}$ is constant. The other statements are proved in the same way. □

Lemma 2.3 ([11], [13]). *Let F be a non-constant null meromorphic curve such that the secondary Gauss map g is non-constant. Set*

$$F^{-1}\,dF = \alpha, \qquad \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}. \tag{2.2}$$

Then the secondary Gauss map g of F is represented as

$$g = \frac{\alpha_{11}}{\alpha_{21}} = \frac{\alpha_{12}}{\alpha_{22}}.$$

Proof. Let F be as in (2.1). If α_{11} and α_{21} vanish identically, so is α_{22}, because α is an $\mathfrak{sl}(2, \boldsymbol{C})$-valued 1-form. Then, since $dF = F\alpha$, we have $dA \equiv dB \equiv dD \equiv 0$, which implies g is constant. Hence $(\alpha_{11}, \alpha_{21}) \not\equiv (0,0)$. Similarly, $(\alpha_{12}, \alpha_{22}) \not\equiv (0,0)$. Here $\det\alpha = 0$ because F is null. Hence we have $\alpha_{11}/\alpha_{21} = \alpha_{12}/\alpha_{22}$.

Since $AD - BC = 1$, it holds that $D\,dA - B\,dC = -A\,dD + C\,dB$. Then, using the relations $dB = -g\,dA$ and $dD = -g\,dC$, we have

$$\frac{\alpha_{11}}{\alpha_{21}} = \frac{D\,dA - B\,dC}{-C\,dA + A\,dC} = \frac{-A\,dD + C\,dB}{-C\,dA + A\,dC} = \frac{g(-CdA + AdC)}{-CdA + AdC} = g.$$

This completes the proof. □

Theorem 2.4 (Small [8]). *For an arbitrary pair of non-constant meromorphic functions* (G, g) *on* M^2 *such that* $g \not\equiv a \star G$ *for any* $a \in \mathrm{PSL}(2, \boldsymbol{C})$, *a meromorphic map* F *given by* (1.3) *is a non-constant null curve in* $\mathrm{PSL}(2, \boldsymbol{C})$ *whose hyperbolic Gauss map and secondary Gauss map are* G *and* g *respectively.*

Conversely, any meromorphic null curve in $\mathrm{PSL}(2, \boldsymbol{C})$ *whose hyperbolic Gauss map* G *and secondary Gauss map* g *are both non-constant are represented in this way.*

An analogue of this formula for null curves in $\boldsymbol{C}^3$ is mentioned in Appendix 4.

Proof of Theorem 2.4. Let (G, g) be a pair as in the statement of the theorem and set as in (1.3). Then

$$\det F = -a\frac{db}{dG} + b\frac{da}{dG} = -a^2\frac{d}{dG}\left(\frac{b}{a}\right) = a^2\frac{dg}{dG} = 1,$$

and

$$\frac{dF}{dG} = \begin{pmatrix} \dfrac{dA}{dG} & \dfrac{dB}{dG} \\ \dfrac{dC}{dG} & \dfrac{dD}{dG} \end{pmatrix} = \begin{pmatrix} G\dfrac{d^2a}{dG^2} & G\dfrac{d^2b}{dG^2} \\ \dfrac{d^2a}{dG^2} & \dfrac{d^2b}{dG^2} \end{pmatrix}.$$

Hence $\operatorname{rank} dF \leq 1$, and F is a meromorphic null curve in $\mathrm{PSL}(2, \boldsymbol{C})$. The hyperbolic Gauss map of F is obtained as

$$\frac{dA}{dC} = \frac{dA/dG}{dC/dG} = G.$$

On the other hand, the secondary Gauss map is obtained by Lemma 2.3 as

$$\frac{\alpha_{11}}{\alpha_{21}} = \frac{D\dfrac{dA}{dG} - B\dfrac{dC}{dG}}{-C\dfrac{dA}{dG} + A\dfrac{dC}{dG}} = -\frac{GD - B}{GC - A} = -\frac{b}{a} = g.$$

Next, we prove that F is non-constant. Assume F is constant. Then by (1.3), $da/dG = p = \text{constant}$. Thus we have $a = \sqrt{dG/dg} = pG + q$, where p and q are complex numbers. Hence

$$\frac{dg}{dG} = \frac{1}{(pG+q)^2}.$$

Integrating this, we have that g is obtained as a Möbius transformation of G, a contradiction. Thus the first part of the theorem is proved.

Conversely, let F be a null curve as in (2.1). By Definition 2.1, we have

$$dA = G\,dC, \qquad dB = G\,dD. \tag{2.3}$$

We set

$$a := GC - A, \qquad b := GD - B.$$

By (2.3), we have $da = C\,dG$ and $db = D\,dG$. Since G is not constant, we have

$$C = \frac{da}{dG}, \qquad D = \frac{db}{dG}. \tag{2.4}$$

Then F can be expressed in terms of a and b as follows:

$$F = \begin{pmatrix} G\dfrac{da}{dG} - a & G\dfrac{db}{dG} - b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix}.$$

Since $\det F = 1$, we have

$$\det \begin{pmatrix} -a & -b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix} = 1. \tag{2.5}$$

Taking the derivative of this equation,

$$\det \begin{pmatrix} -a & -b \\ d\left(\dfrac{da}{dG}\right) & d\left(\dfrac{db}{dG}\right) \end{pmatrix} = 0 \tag{2.6}$$

holds. Here, since g is non-constant, $(dC, dD) \not\equiv 0$ by Lemma 2.2. Then by (2.4) and (2.6), it holds that

$$g = \frac{dD}{dC} = -\frac{d(db/dG)}{d(da/dG)} = -\frac{b}{a}.$$

This yields

$$b = -ga, \qquad db = -(da)g - a(dg). \tag{2.7}$$

Again by (2.5)

$$dG = \det \begin{pmatrix} -a & -b \\ da & db \end{pmatrix} = \det \begin{pmatrix} -a & ag \\ da & -(da)g - a(dg) \end{pmatrix} = a^2 dg.$$

By this and (2.7), we have $a = \sqrt{dG/dg}$ and $b = -ga$ which implies (1.3). □

By Theorem 2.4, we can prove the uniqueness of null curves with given hyperbolic Gauss map and secondary Gauss map. Hence we have

Corollary 2.5. *Let $\mathcal{N}(M^2)$ be the set of non-constant null curves in* PSL(2, $\boldsymbol{C}$) *defined on a Riemann surface M^2 with non-constant hyperbolic Gauss map and secondary Gauss map. Then $\mathcal{N}(M^2)$ corresponds bijectively to the set*

$$\left\{ (G, g) \;\middle|\; \begin{array}{l} G \text{ and } g \text{ are non-constant meromorphic functions on } M^2 \\ \text{such that } G \not\equiv a \star g \text{ for any } a \in \mathrm{SL}(2, \boldsymbol{C}). \end{array} \right\}.$$

It should be remarked that (G, g) satisfies the following important relation (see [11]):

$$S(g) - S(G) = 2Q, \tag{2.8}$$

where Q is the Hopf differential of F defined by $Q := (A\,dC - C\,dA)\,dg$ and S is the Schwarzian derivative defined by

$$S(G) = \left[\left(\frac{G''}{G'} \right)' - \frac{1}{2} \left(\frac{G''}{G'} \right)^2 \right] dz^2 \qquad \left(' = \frac{d}{dz} \right)$$

with respect to a local complex coordinate z on M^2. Though meromorphic 2-differentials $S(g)$ and $S(G)$ depend on complex coordinates, the difference $S(g) - S(G)$ does not depend on the choice of complex coordinates.

3. Legendrian curves in PSL(2, $\boldsymbol{C}$)

In this section, we shall give a representation formula for Legendrian curves in terms of two meromorphic functions G and G_*, which are called the *hyperbolic Gauss maps*. We fix a Riemann surface M^2, which might not be simply connected. Let E be a meromorphic Legendrian curve on M^2 as in (1.5). Since $AD - BC = 1$, we can define two meromorphic functions G and G_* as in (1.7). We call G and G_* the *hyperbolic Gauss maps* of E. (The geometric meaning of these hyperbolic Gauss maps is described in [4].)

DEFINITION 3.1. Let E be a meromorphic Legendrian curve as in (1.5). Then we can write

$$E^{-1}\,dE = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}, \tag{3.1}$$

where ω and θ are meromorphic 1-forms on M^2. We call ω the *canonical form* and θ the *dual canonical form* of E.

For a Legendrian curve E, we define another Legendrian curve $\hat{E}$ by

$$\hat{E} = E \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We call $\hat{E}$ the *dual* of E. The hyperbolic Gauss maps $\hat{G}$ and $\hat{G}_*$ of $\hat{E}$ satisfy $\hat{G} = G_*$ and $\hat{G}_* = G$, and the canonical form and the dual canonical form of $\hat{E}$ are θ and ω respectively. Roughly speaking, the duality exchanges the role of (G, ω) and (G_*, θ).

The following lemma holds.

Lemma 3.2. *For a non-constant meromorphic Legendrian curve E as in* (1.5), *the following identities hold*:

$$\omega = \begin{cases} \dfrac{dA}{B} & (\text{if } dA \not\equiv 0 \text{ or } B \not\equiv 0), \\[2mm] \dfrac{dC}{D} & (\text{if } dC \not\equiv 0 \text{ or } D \not\equiv 0), \end{cases} \tag{3.2}$$

$$\theta = \begin{cases} \dfrac{dB}{A} & (\text{if } dB \not\equiv 0 \text{ or } A \not\equiv 0), \\[2mm] \dfrac{dD}{C} & (\text{if } dD \not\equiv 0 \text{ or } C \not\equiv 0). \end{cases} \tag{3.3}$$

Here $dA \not\equiv 0$ (resp. $B \not\equiv 0$) means a 1-form dA (resp. a function B) is not identically 0. In particular, if all cases in (3.2) *and* (3.3) *are well-defined*,

$$\omega = \frac{dA}{B} = \frac{dC}{D} \qquad \text{and} \qquad \theta = \frac{dB}{A} = \frac{dD}{C}$$

hold.

Proof. Since E is Legendrian, $D\,dA - B\,dC = 0$ holds, and $\omega = A\,dC - C\,dA$ by (3.1). Hence we have

$$B\omega = AB\,dC - BC\,dA = AD\,dA - BC\,dA = (AD - BC)\,dA = dA$$

and

$$D\omega = AD\,dC - CD\,dA = (AD - BC)\,dC = dC,$$

which imply (3.2).

On the other hand, differentiating $AD - BC = 1$, we have

$$0 = d(AD - BC) = (D\,dA - B\,dC) + (A\,dD - C\,dB) = A\,dD - C\,dB.$$

Since $\theta = D\,dB - B\,dD$, we have then

$$A\theta = AD\,dB - AB\,dD = (AD - BC)\,dB = dB \qquad \text{and} \qquad C\theta = dD,$$

which imply (3.3). □

Theorem 3.3. *Let G and G_* be non-constant meromorphic functions on M^2 such that G is not identically equal to G_*. Assume that*
(i) *all poles of the 1-form $dG/(G - G_*)$ are of order 1, and*
(ii) *$\int_\gamma dG/(G - G_*) \in \pi i\boldsymbol{Z}$ holds for each loop γ on M^2.*
Set

$$\xi(z) := c \exp \int_{z_0}^{z} \frac{dG}{G - G_*}, \tag{3.4}$$

where $z_0 \in M^2$ is a base point and $c \in \boldsymbol{C} \setminus \{0\}$ is an arbitrary constant. Then

$$E := \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix} \tag{3.5}$$

is a non-constant meromorphic Legendrian curve in PSL(2, **C**) *whose hyperbolic Gauss maps are G and G_*. The canonical form ω of E is written as*

$$\omega = -\frac{dG}{\xi^2}. \tag{3.6}$$

Moreover, a point $p \in M^2$ is a pole of E if and only if $G(p) = G_*(p)$ holds.

Conversely, any meromorphic Legendrian curve in PSL(2, **C**) *with non-constant hyperbolic Gauss maps G and G_* is obtained in this way.*

Proof. By the assumptions (i) and (ii), ξ^2 is a meromorphic function on M^2. Hence E as in (3.5) is a meromorphic curve in PSL(2, **C**). One can easily see that $\det E = 1$ and $D\,dA - B\,dC = 0$, that is, E is a Legendrian map with hyperbolic Gauss maps G and G_*. The canonical form ω is obtained as (3.6) using

$$d\xi = \frac{\xi dG}{G - G_*}.$$

Since $G = A/C$ is non-constant, so is E.

Next, we fix a point $p \in M^2$. By a matrix multiplication $E \mapsto \widetilde{E} = aE$ ($a \in \mathrm{SL}(2, \boldsymbol{C})$), we have another Legendrian map $\widetilde{E}$ with hyperbolic Gauss maps $\widetilde{G} = a \star G$ and $\widetilde{G}_* = a \star G_*$, where $\star$ denotes the Möbius transformation (1.4). If necessary replacing E by $\widetilde{E}$, we may assume $G(p) \neq \infty$ and $G_*(p) \neq \infty$. Let z be a local complex coordinate on M^2 such that $z(p) = 0$.

Assume E is holomorphic at p. Then by (3.5), $CD = 1/(G - G_*)$ is holomorphic at p. Hence we have $G(p) \neq G_*(p)$. On the other hand, if $G(p) \neq G_*(p)$, ξ is holomorphic at p and $\xi(p) \neq 0$. Then by (3.5), E is holomorphic at p. Thus, we have shown that $\{p \in M^2 \mid G(p) \neq G_*(p)\}$ is the set of poles of E.

Finally, we shall prove the converse statement. Let E as in (1.5) be a meromorphic Legendrian curve. Then by (3.2), we have

$$dG = d\left(\frac{A}{C}\right) = \frac{C\,dA - A\,dC}{C^2} = -\frac{\omega}{C^2} = -\frac{dC}{C^2 D}. \tag{3.7}$$

On the other hand, we have

$$G - G_* = \frac{A}{C} - \frac{B}{D} = \frac{AD - BC}{CD} = \frac{1}{CD}. \tag{3.8}$$

By (3.7) and (3.8),

$$d \log C = -\frac{dG}{G - G_*} \tag{3.9}$$

holds. Since E is a meromorphic map into $\mathrm{PSL}(2, \boldsymbol{C})$, C is written as in the form $\sqrt{h}\,\hat{C}$, where h and $\hat{C}$ are meromorphic functions. Then if we set ξ as in (3.4), ξ^2 is a meromorphic function on M^2. Hence we have (i) and (ii) in the statement of the theorem. Integrating (3.9), we have $C = 1/\xi$ and $A = GC = G/\xi$. Moreover, since

$$1 = AD - BC = \left(\frac{G}{\xi}\right) D - G_* D \left(\frac{1}{\xi}\right) = \frac{D}{\xi}(G - G_*),$$

we have $D = \xi/(G - G_*)$ and $B = G_* D = G_* \xi/(G - G_*)$. Thus we obtain (3.5). $\square$

As a corollary of Theorem 3.3, we give a Small-type formula for Legendrian curves, which has appeared implicitly in [4] by a different method.

Corollary 3.4. *For an arbitrary pair* (G, ω) *of a non-constant meromorphic function and a non-zero meromorphic* 1*-form on* M^2*, a meromorphic map*

$$E = \begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix} \qquad \left(C := i\sqrt{\frac{\omega}{dG}},\ A := GC\right) \tag{3.10}$$

is a meromorphic Legendrian curve in PSL(2, $\boldsymbol{C}$) *whose hyperbolic Gauss map and canonical form are* G *and* ω*, respectively.*

Conversely, let E *be a meromorphic Legendrian curve in* PSL(2, $\boldsymbol{C}$) *defined on* M^2 *with the non-constant hyperbolic Gauss map* G *and the non-zero canonical form* ω*. Then* E *is written as in* (3.10)*.*

REMARK. There is a correponding simple formula (without integration) for Legendrian curves in $\boldsymbol{C}^3$ as follows: A meromorphic map $E: M^2 \to \boldsymbol{C}^3$ is called *Legendrian* if the pull-back of the holomorphic contact form $dx^1 - x^3\,dx^2$ vanishes, where (x^1, x^2, x^3) is the canonical coordinate system on $\boldsymbol{C}^3$. For a pair (f, g) of meromorphic functions on a Riemann surface M^2, $E := (f, g, df/dg)$ trivially gives a meromorphic Legendrian curve, which is an analogue of (3.10).

Proof of Corollary 3.4. If we set E by (3.10), we have $AD - BC = 1$ and $D\,dA - B\,dC = 0$. Hence E is a meromorphic Legendrian map.

Conversely, let E be a meromorphic Legendrian curve on M^2 with the non-constant hyperbolic Gauss map G and the non-zero canonical form ω. Then by (3.7), we have

$$C = i\sqrt{\frac{\omega}{dG}}, \qquad A = GC.$$

On the other hand, by Lemma 3.2, we have $B = dA/\omega$ and $D = dC/\omega$. Hence we have (3.10). □

We have the following corollary:

Corollary 3.5. *Let* $\mathcal{L}(M^2)$ *be the set of meromorphic Legendrian curves in* PSL(2, $\boldsymbol{C}$) *defined on a Riemann surface* M^2 *with non-constant hyperbolic Gauss maps and non-zero canonical forms. Then* $\mathcal{L}(M^2)$ *corresponds bijectively to the following set*:

$$\left\{ (G, \omega) \;\middle|\; \begin{array}{l} G \textit{ is a non-constant meromorphic function on } M^2, \\ \textit{and } \omega \textit{ is a non-zero meromorphic } 1\textit{-form on } M^2. \end{array} \right\}.$$

The symmetric product of the canonical form ω and the dual form θ

$$Q := \omega\theta \tag{3.11}$$

is called the *Hopf differential* of the Legendrian curve. By (3.7), we have

$$dG = -\frac{\omega}{C^2}.$$

Similarly, it holds that

$$dG_* = \frac{\theta}{D^2}.$$

Thus, by (3.8) we have

$$Q = -C^2 D^2\, dG\, dG_* = -\frac{dG\, dG_*}{(G - G_*)^2}.$$

As pointed out in [4], the following identities hold:

$$S(g) - S(G) = 2Q, \qquad S(g_*) - S(G_*) = 2Q,$$

where g (resp. g_*) is a meromorphic function defined on the universal cover of M^2 such that $dg = \omega$ (resp. $dg_* = \theta$).

4. Examples of flat surfaces in H^3

As an application of Corollary 3.4, we shall give new examples of flat surfaces in hyperbolic 3-space H^3. Though these examples might have singularities, all of them are obtained as projections of Legendrian immersions into the unit cotangent bundle $T_1^* H^3$. Usually, a projection of a Legendrian immersion is called a (wave) front. So we call them *flat fronts*. For details, see [6].

Hyperbolic 3-space H^3 has an expression

$$H^3 = \mathrm{PSL}(2, \boldsymbol{C})/\,\mathrm{PSU}(2) = \{aa^* \mid a \in \mathrm{PSL}(2, \boldsymbol{C})\} \qquad (a^* = {}^t\bar{a}).$$

As shown in [4], the projection

$$f := EE^* : M^2 \longrightarrow H^3$$

of a holomorphic Legendrian curve $E : M^2 \to \mathrm{PSL}(2, \boldsymbol{C})$ is a flat immersion if f induces positive definite metric on M^2. For a Legendrian curve E, we can write

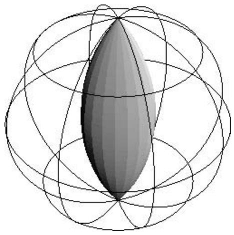
$$E^{-1}\, dE = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}. \tag{4.1}$$

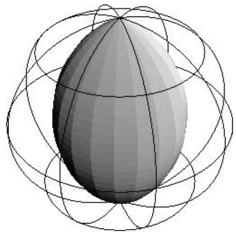
Then the first fundamental form ds^2 and the second fundamental form $d\sigma^2$ of f is written as

$$ds^2 = \omega\theta + \overline{\omega\theta} + |\omega|^2 + |\theta|^2 = (\omega + \bar{\theta})(\bar{\omega} + \theta), \tag{4.2}$$

$$d\sigma^2 = |\theta|^2 - |\omega|^2. \tag{4.3}$$

Common zeros of ω and θ correspond to branch points of the surface where the first fundamental form vanishes. At the point where $|\omega| = |\theta|$, ds^2 in (4.2) is written





$k = \sqrt{2}$ $k = 2$

Fig. 1. Surfaces equidistant from a geodesic (Example 4.1). The figures are shown in the Poincaré model of H^3.

as

$$ds^2 = \frac{\omega}{\theta}(\bar{\omega} + \theta)^2$$

which implies the metric degenerates at these points. Let ν be the unit normal vector field of f. For each $p \in M^2$, the asymptotic class of the geodesic with initial velocity $\nu(p)$ (resp. $-\nu(p)$) determines a point $G(p)$ (resp. $G_*(p)$) of the ideal boundary of H^3 which is identified with $\boldsymbol{C} \cup \{\infty\} = \boldsymbol{CP}^1$. Then G and G_* coincide with the hyperbolic Gauss maps of the lift E.

EXAMPLE 4.1 (Surfaces equidistant from a geodesic). Let $M^2 = \boldsymbol{C} \setminus \{0\}$ and

$$G = z, \qquad \omega = \frac{k}{2z}\,dz \qquad (k > 0).$$

Then by Corollary 3.4, the corresponding Legendrian curve E is written as

$$E = \frac{i}{\sqrt{2}} \begin{pmatrix} \sqrt{kz} & \sqrt{\dfrac{z}{k}} \\ \sqrt{\dfrac{k}{z}} & -\dfrac{1}{\sqrt{kz}} \end{pmatrix}.$$

Then the corresponding flat surface $f = EE^*$ is a surface equidistant from a geodesic in H^3 . The hyperbolic Gauss maps of f are given by $(G, G_*) = (z, -z)$ (see Fig. 1, and see also [4, p. 426]).

EXAMPLE 4.2 (Flat fronts of revolution). Let $M^2 = \boldsymbol{C} \setminus \{0\}$ and set

$$G = \sqrt{\frac{\mu - 1}{\mu + 1}}\, z \qquad \text{and} \qquad \omega = \frac{\sqrt{\mu^2 - 1}}{2}\, z^{\mu - 1}\, dz \qquad (\mu \in \boldsymbol{R}_+ \setminus \{1\}).$$

If $\mu \notin \boldsymbol{Z}$, ω is not well-defined on M^2, but defined on the universal cover $\widetilde{M}^2$ of M^2. If we consider G as a function on $\widetilde{M}^2$, the corresponding Legendrian curve $E \colon \widetilde{M}^2 \to$

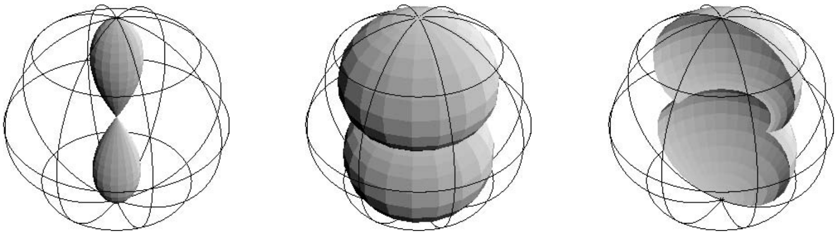


Fig. 2. Flat fronts of revolution (Example 4.2).

$\mathrm{PSL}(2, \boldsymbol{C})$ is given by

$$E = \frac{i}{\sqrt{2}} \begin{pmatrix} \sqrt{\mu-1}\, z^{(\mu+1)/2} & \sqrt{\mu+1}\, z^{-(\mu-1)/2} \\ \sqrt{\mu+1}\, z^{(\mu-1)/2} & \sqrt{\mu-1}\, z^{-(\mu+1)/2} \end{pmatrix}.$$

Let τ be the deck transformation of $\widetilde{M}^2$ corresponding to the loop on M^2 surrounding 0. Then

$$E \circ \tau = E \begin{pmatrix} -e^{\pi i \mu} & 0 \\ 0 & -e^{-\pi i \mu} \end{pmatrix}$$

holds. Hence the corresponding surface $f = EE^*$ is well-defined on M^2. The dual canonical form θ as in (4.1) is given by

$$\theta = -\frac{\sqrt{\mu^2-1}}{2} z^{-\mu-1}\, dz.$$

Then the metric induced by f degenerates on the set $\{|z| = 1\}$ when $\mu \neq 0$ (see Fig. 2). The hyperbolic Gauss maps of f are given by

$$(G, G_*) = \left(\sqrt{\frac{\mu-1}{\mu+1}}\, z, \sqrt{\frac{\mu+1}{\mu-1}}\, z \right).$$

EXAMPLE 4.3 (Flat fronts with dihedral symmetry). Let $n \geq 2$ be an integer. We set

$$M^2 := \boldsymbol{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\} \qquad \left(\zeta = \exp \frac{2\pi i}{n} \right).$$

and let $\pi \colon \widetilde{M}^2 \to M^2$ be the universal cover of M^2. Let

$$G_0(z) = z \quad \text{and} \quad \omega = k(z^n - 1)^{-2/n}\, dz \qquad (k > 0), \tag{4.4}$$

where z is the canonical coordinate on $\boldsymbol{C}$. Then $G := G_0 \circ \pi$ and ω are considered as a meromorphic function and a holomorphic 1-form on $\widetilde{M}^2$. Then by Corollary 3.4, there exists a holomorphic Legendrian curve $E : \widetilde{M}^2 \to \mathrm{PSL}(2, \boldsymbol{C})$. Let τ_j be a deck transformation of $\pi : \widetilde{M}^2 \to M^2$ corresponding a loop on M^2 around ζ^j $(j = 0, \dots, n-1)$. Then we have

$$G \circ \tau_j = G, \qquad \omega \circ \tau_j = \zeta^{-2}\omega.$$

Hence by (3.10), we have

$$E \circ \tau_j = E \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \qquad (j = 0, \dots, n-1).$$

This implies $f := EE^*$ is well-defined on M^2 itself. Thus, we have a one parameter family of flat surfaces in H^3, parametrized by k in (4.4). The parameter k corresponds to a parallel family of flat surfaces (see [4, page 426]). Moreover, by (4.2), one can see that each end ζ^j is complete. On the other hand, at the points where $|\omega| = |\theta|$, the immersion f has singularities. The automorphisms of M^2 as

$$z \longmapsto \zeta z\,, \qquad z \longmapsto \frac{1}{z}$$

do not change the first and second fundamental forms as in (4.2). This implies such surfaces have dihedral symmetry (see Fig. 3). The hyperbolic Gauss maps of f are given by

$$(G, G_*) = \left(z, z^{1-n}\right).$$

EXAMPLE 4.4 (A flat front with tetrahedral symmetry). Let

$$M^2 = \boldsymbol{C} \cup \{\infty\} \setminus \{1, \zeta, \zeta^2, \infty\} \qquad \left(\zeta = \exp \frac{2\pi i}{3}\right).$$

Set

$$G(z) = z \quad \text{and} \quad \omega = k(z^3 - 1)^{-1/2}\, dz \quad (k > 0).$$

Then, in the same way as in Example 4.3, we have a one parameter family of flat surfaces $f_k : M^2 \to H^3$ with four complete ends at $z = 1, \zeta, \zeta^2, \infty$. Such surfaces have the tetrahedral symmetry. The hyperbolic Gauss maps of f_k are given by

$$(G, G_*) = \left(z, \frac{4 - z^3}{3z^2}\right).$$

In Figs. 2 and 3, it seems that the surfaces admit singularities. It might be interesting problem to study singularities of flat fronts (see [6]).

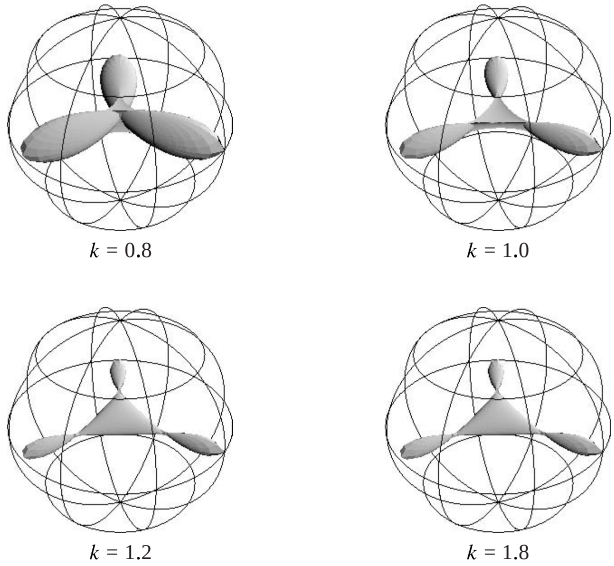


Fig. 3. Parallel family of flat fronts in Example 4.3 for $n = 3$.

Appendix A. Null curves in $\boldsymbol{C}^3$

Let M^2 be a Riemann surface, which is not necessarily simply connected. A meromorphic map $F = (F^1, F^2, F^3)\colon M^2 \to \boldsymbol{C}^3$ is said to be *null* if the $\boldsymbol{C}^3$-valued 1-form dF is null, that is,

$$\sum_{j=1}^{3} dF^j \cdot dF^j = 0. \tag{A.1}$$

It is well-known that a minimal surface in $\boldsymbol{R}^3$ is locally given by the projection of a null curve in $\boldsymbol{C}^3$ to $\boldsymbol{R}^3$.

For a null meromorphic map $F = (F^1, F^2, F^3)$, we put

$$\omega := d(F^1 - iF^2), \qquad g := \frac{dF^3}{\omega}. \tag{A.2}$$

Then we have

$$dF = \frac{1}{2}\bigl((1 - g^2)\omega, i(1 + g^2)\omega, 2g\omega\bigr) \tag{A.3}$$

by (A.1). Conversely, by integrating (A.3) for a given pair (g, ω), we obtain a null meromorphic map F. The integration of (A.3) is known as the *Weierstrass formula* and the pair (g, ω) is called the *Weierstrass data* of F.

On the other hand, let $F\colon M^2 \to \boldsymbol{C}^3$ be a meromorphic map defined by

$$F = \begin{pmatrix} 1 & g & (1-g^2)/2 \\ i & ig & i(1+g^2)/2 \\ 0 & -1 & g \end{pmatrix} \begin{pmatrix} h \\ h_1 \\ h_2 \end{pmatrix} \qquad \left(h_1 = \frac{dh}{dg},\ h_2 = \frac{dh_1}{dg} \right) \tag{A.4}$$

for a pair (g, h) of two meromorphic functions, then F is null. Conversely, any null meromorphic map $F\colon M^2 \to \boldsymbol{C}^3$ is represented by this formula (A.4). The Weierstrass formula (A.3) and the formula (A.4) are related by $(g, \omega) = (g, dh_2)$.

The remarkable feature of the formula (A.4) is that arbitrary null meromorphic maps can be represented in the integral-free form.

We introduce here a way to derive the formula (A.4).

Let $F\colon M^2 \to \boldsymbol{C}^3$ be a null curve and (g, ω) its Weierstrass data. We let

$$h_2 := F^1 - iF^2, \quad \psi := -F^1 - iF^2, \quad \varphi := F^3, \tag{A.5}$$

then their differentials satisfy

$$dh_2 = \omega, \tag{A.6}$$

$$d\varphi = g\omega, \tag{A.7}$$

$$d\psi = g^2\omega. \tag{A.8}$$

Now, we define a function h_1 by

$$\varphi = h_2 g - h_1. \tag{A.9}$$

Using (A.6) and (A.7), we compute that

$$g\omega = d\varphi = d(h_2 g - h_1) = g\omega + h_2 dg - dh_1,$$

hence

$$h_2 = \frac{dh_1}{dg}. \tag{A.10}$$

Moreover, we define a function h by

$$\psi = h_2 g^2 - 2h_1 g + 2h, \tag{A.11}$$

then

$$\begin{aligned} g^2\omega &= d(h_2 g^2 - 2h_1 g + 2h) \\ &= g^2\, dh_2 + 2h_2 g\, dg - 2g\, dh_1 - 2h_1\, dg + 2\, dh \end{aligned}$$

$$= g^2\omega + 2h_2 g\,dg - 2h_2 g\,dg - 2h_1\,dg + 2\,dh$$
$$= g^2\omega - 2h_1\,dg + 2\,dh,$$

by (A.6)–(A.8), hence

$$h_1 = \frac{dh}{dg}. \tag{A.12}$$

Substituting (A.9)–(A.12) into (A.5), we obtain the formula (A.4).

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