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SURFACES OF REVOLUTION WITH PERIODIC MEAN CURVATURE

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1. Introduction

The surfaces of revolution with constant mean curvature in R^3 are classified by Delaunay [2] in 1841. They are locally plane, catenoid, sphere, circular cylinder, unduloid, and nodoid up to isometries of R^3. On these 10 years, new and interesting examples of non-zero constant mean curvature surfaces are discovered. In the global study of complete surfaces with constant mean curvature, unduloids and nodoids play important role as the models of ends of such surfaces (see [5], [6]). The work by Delaunay is now revived after 150 years of his discovery.

The purpose of this paper is to study surfaces of revolution with periodic mean curvature in order to extend the theory of constant mean curvature surfaces. In general such a surface is not periodic, because the catenoid gives the counter-example. First we show the criterion for a periodic function to be the mean curvature of a periodic surface of revolution and second describe a method how to construct these periodic surfaces of revolution whose mean curvatures are periodic functions satisfying the criterion.

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The figures of this paper are all programmed by the author using the symbolic manipulation program Mathematica. I also thank my graduate student Shinya Hirakawa for his help to simplify the programing.

2. Criterion of the periodicity

Let $C = (x(s), y(s))$, $s \in I$, be a smooth plane curve parametrized by arc length on the plane $z = 0$ of R^3. We assume that the domain of the definition I is an open interval including zero and $y(s) > 0$, $s \in I$. A surface of revolution S on I is defined

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by

$$S = \left\{(x(s), y(s)\cos\theta, y(s)\sin\theta) \in R^3 \mid s \in I,\ 0 \le \theta < 2\pi\right\},$$

where C is called the profile curve of S. Since the mean curvature H of S at (s, θ) is independent of the coordinate θ, we write $H = H(s)$. The coordinate functions of the profile curve satisfy the system of differential equations:

$$\begin{aligned} 2H(s)y(s) - x'(s) - x''(s)y(s)y'(s) + x'(s)y(s)y''(s) &= 0, \\ x'(s)^2 + y'(s)^2 &= 1, \quad s \in I. \end{aligned} \tag{1}$$

In the formulas above, the mean curvature is computed for the "inward" unit normal vector of the surface. We put

$$\eta(u) = 2\int_0^u H(s)\,ds, \quad s \in I, \tag{2}$$

$$F(s) = \int_0^s \sin\eta(u)\,du, \quad G(s) = \int_0^s \cos\eta(u)\,du\ . \tag{3}$$

Then, the profile curve C is expressed as

$$y(s) = \{(F(s) - c_1)^2 + (G(s) - c_2)^2\}^{1/2} \tag{4}$$

$$x(s) = \int_0^s \frac{(G(s) - c_2)F'(s) - (F(s) - c_1)G'(s)}{\{(F(s) - c_1)^2 + (G(s) - c_2)^2\}^{1/2}}\,ds, \quad s \in I, \tag{5}$$

where c_1 and c_2 are some constants [3].

Conversely given a continuous function $H(s)$ defined on I and any real numbers c_1 and c_2 with $c_1^2 + c_2^2 > 0$, we define $y(s)$ on I by (4), which is continuous and non-negative on I. Let I_0 be the maximal subinterval of I on which $y(s)$ is positive. We define $x(s)$, $s \in I_0$, by (5). Then $y(s)$ and $x(s)$ satisfy the system (1) on I_0. Hence the curve $(x(s), y(s))$, $s \in I_0$, generates a surface of revolution on I_0 with the mean curvature $H(s)$. Let us consider an $s_1 \in I$ with $y(s_1) = 0$. We claim that there exist the derivatives of $y(s)$ and $x(s)$ at $s = s_1$ such that $y'_-(s_1) = -1$, $y'_+(s_1) = 1$, and $x'(s_1) = 0$.

Proof of the claim. $y(s_1) = 0$ is equivalent to $F(s_1) = c_1$ and $G(s_1) = c_2$. We see that

$$\lim_{s\to s_1} \frac{G(s) - c_2}{F(s) - c_1} = \lim_{s\to s_1} \frac{G'(s)}{F'(s)} = \frac{\cos\eta(s_1)}{\sin\eta(s_1)}. \tag{6}$$

Since we know that

$$y'(s) = \frac{(F(s) - c_1)F'(s) + (G(s) - c_2)G'(s)}{\{(F(s) - c_1)^2 + (G(s) - c_2)^2\}^{1/2}}\ ,\ s \in I_0,$$

(6) implies $y'_-(s_1) = -1$ and $y'_+(s_1) = 1$. The derivative of $x(s)$ at $s = s_1$ is similarly computed as $x'(s_1) = 0$. □

Hence, in the case where we have a continuous function $H = H(s)$ on I, the curve $(x(s), y(s))$ defined by (4) and (5) is continuous on I, satisfies (1) on I_0, and at the point of $y(s) = 0$, it reaches orthogonally at the x-axis. I consists of I_0 and some discrete set (possibly empty) on which $y(s) = 0$. Therefore, the formulas (4) and (5) define two parameters family of continuous surfaces of revolution on I such that each element of the family is smooth on the dense subset I_0 of I and the mean curvature at $s \in I_0$ is $H(s)$.

A surface of revolution S is called *periodic* if the coordinate function $y(s)$ of the profile curve is periodic, that is, there is a positive number L such that $y(s+L) = y(s)$ for all $s \in R$. L is said to be the period of S.

Any closed curve on the upper half plane of R^2 generates a periodic surface of revolution, and hence it has the periodic mean curvature. Since the converse does not hold as catenoid shows it, the following question naturally raise: Which periodic function is the mean curvature of some periodic surface of revolution? The answer is

Theorem 1. *Let $H(s)$ be a continuous periodic function on R with period L. Then, the function $H(s)$ is the mean curvature of a periodic surface of revolution S with period L if and only if it satisfies the condition*:

$$\frac{\int_0^L \cos\eta(u)\,du}{\sin\eta(L)} = \frac{\int_0^L \sin\eta(u)\,du}{1-\cos\eta(L)} \quad . \tag{7}$$

Convention. A denominator in (7) *vanishes if and only if the corresponding numerator also does.*

Proof. Let $C = (x(s), y(s))$, $s \in R$, be the profile curve of a periodic surface of revolution S with period L, where s denotes arc length parameter of C. Since we have $y(s + L) = y(s)$, $s \in R$, the system (1) implies

$$x'(s + L) = x'(s), \quad s \in R. \tag{8}$$

We may assume that $y(0)$ is the minimum. From $y'(0) = 0$ and the differentiation of (4), we have $c_2 = 0$. By differentiating the formula (4) with $c_2 = 0$ at $s = L$, we have

$$(F(L) - c_1)\sin\eta(L) + G(L)\cos\eta(L) = 0. \tag{9}$$

By (5) and (8) we have

$$G(s + L)\sin\eta(s + L) - (F(s + L) - c_1)\cos\eta(s + L)$$

$$= G(s)\sin\eta(s) - (F(s) - c_1)\cos\eta(s)\ , \quad s \in R.$$

Inserting $s = 0$ in the formula above, we get

$$G(L)\sin\eta(L) - F(L)\cos\eta(L) = c_1(1 - \cos\eta(L)). \tag{10}$$

If $\sin\eta(L) \neq 0$, then we have $1 - \cos\eta(L) \neq 0$, and (9) and (10) imply

$$\frac{G(L)}{\sin\eta(L)} = \frac{F(L)}{1 - \cos\eta(L)}\ ,$$

which is not zero and the same formula as (7). If $\sin\eta(L) = 1 - \cos\eta(L) = 0$, then we have $G(L) = F(L) = 0$. If $\sin\eta(L) = 1 + \cos\eta(L) = 0$, then we have $G(L) = 0$ and $F(L) = 2c_1 \neq 0$. These and the convention prove (7).

Conversely assume that a continuous function $H(s)$, $s \in R$, is periodic with period L and satisfies the condition (7). The functions $F(s)$ and $G(s)$ are defined by (3) using the function $H(s)$ given above. Since we have $\eta(u + L) = \eta(L) + \eta(u)$, we see that

$$\begin{cases} F(s + L) = F(L) + \sin\eta(L)\cdot G(s) + \cos\eta(L)\cdot F(s) \\ G(s + L) = G(L) + \cos\eta(L)\cdot G(s) - \sin\eta(L)\cdot F(s). \end{cases} \tag{11}$$

Firstly assume that $\sin\eta(L) \neq 0$. Define the functions $y(s)$ and $x(s)$ by (4) and (5) with $c_1 = c$ and $c_2 = 0$, where the constant c takes the common values of the ratios in the condition (7). Then direct computation using (11) shows that $y(s + L) = y(s)$ and $y(0) = |c|$. We can also prove that $x'(s + L) = x'(s)$, $s \in R$.

Secondly we study a periodic function $H(s)$ which satisfies $\sin\eta(L) = 1 - \cos\eta(L) = 0$ and (7). We have $F(L) = G(L) = 0$ by the convention. For any positive number $c > 0$, define the functions $y(s)$ and $x(s)$ by (4) and (5) with $c_1 = c$ and $c_2 = 0$. Then we see $y(s + L) = y(s)$ and $y(0) = c$. We also prove $x'(s + L) = x'(s)$.

Finally let us study a periodic function $H(s)$ which satisfies $\sin\eta(L) = 1 + \cos\eta(L) = 0$ and (7). We have $G(L) = 0$ and $F(L) \neq 0$ by the convention. Define the functions $y(s)$ and $x(s)$ by (4) and (5) with $c_1 = F(L)/2$ and $c_2 = 0$. Then, we see that $y(s + L) = y(s)$ and $y(0) = |F(L)|/2$. We can also show $x'(s + L) = x'(s)$ proving Theorem 1. □

Let H be any positive number. The constant function defined by $H(s) = H$ satisfies the condition (7) with period π/H, in which all denominators and numerators of (7) vanish. Hence, for any positive number c, there is the profile curve $(x(s), y(s))$ with $x(0) = 0$ and $y(0) = c$ such that its mean curvature of the resultant surface of revolution is H. This defines the one-parameter family of surfaces of revolution with constant mean curvature including unduloids and nodoids. This is the case of Delaunay [2].

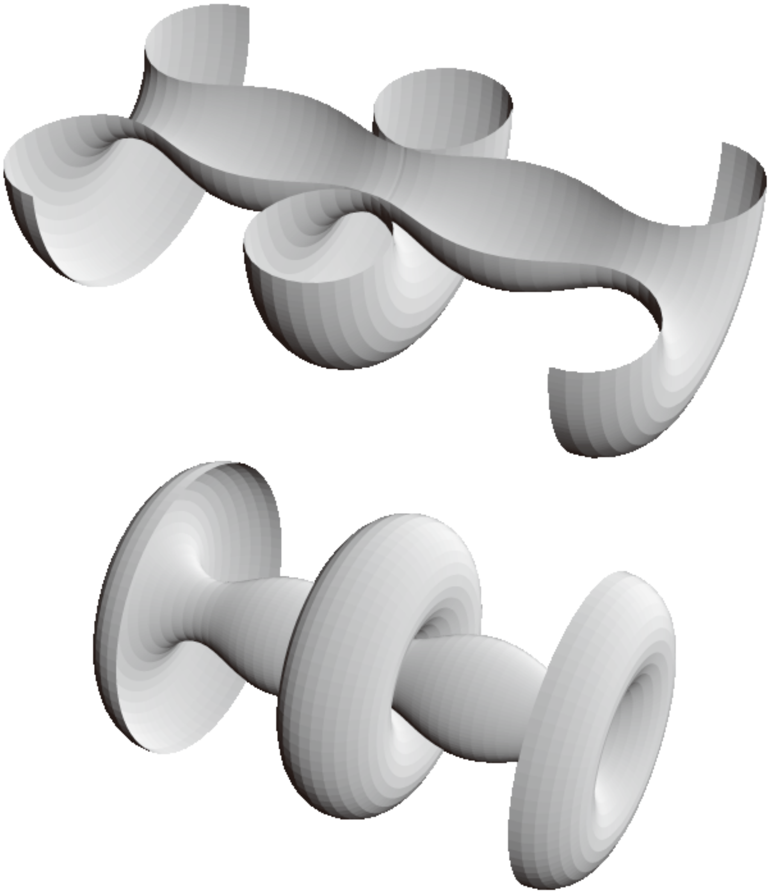

Fig. 1. A periodic surface of revolution whose mean curvature is $1.2025\cos s$.

Cosine function does not satisfy the condition of Theorem 1. Indeed, we have $\sin\eta(2\pi) = 1 - \cos\eta(2\pi) = 0$, but

$$\int_0^{2\pi} \cos\eta(u)\,du = 1.40675, \qquad \int_0^{2\pi} \sin\eta(u)\,du = 0,$$

where the computations above are made by computer. Hence, *any surface of revolution whose mean curvature is* $\cos s$ *can not be periodic*. But, there is a real number τ such that the function $\tau\cos s$ satisfies (7) with $L = 2\pi$. See Fig. 1.

3. Construction of the periodic surfaces

The condition (7) is an integral equation for unknown function $H = H(s)$. In this section, we will show a geometric method to find all solutions of the integral equation (7) by considering the curve $(G(s), F(s))$. We first handle the special case of (7), that is, all terms in (7) vanish at the same time. The general solutions of such a case are given by closed plane curves.

Theorem 2. *Let Γ be a C^2 closed plane curve parametrized by arc length s. For any positive number c, there is a periodic profile curve C with period L and through $(0, c)$ such that the surface of revolution S generated by C has the mean curvature $k(s)/2$, where L and $k(s)$ denote the length and the curvature of Γ, respectively.*

Proof. Put $\Gamma = (G(s), F(s))$. We may assume that $G(0) = F(0) = 0$, and $G'(0) = 1$, $F'(0) = 0$ by some isometries of R^3. The fundamental theorem of smooth curves theory tells us that, on $s \in [0, L]$,

$$G(s) = \int_0^s \cos\left(\int_0^u k(t)\,dt\right) du, \quad F(s) = \int_0^s \sin\left(\int_0^u k(t)\,dt\right) du. \tag{12}$$

By the assumption, we have $G(L) = F(L) = 0$. Since the total curvature of the closed curve is an integral multiple of 2π, we have $G'(L) = 1, F'(L) = 0$. These trivially imply the condition (7) putting $H(s) = k(s)/2$. Theorem 2 is proved by Theorem 1. □

Conversely, it is easily observed that the solutions of the case of $\sin\eta(L) = 1 - \cos\eta(L) = 0$ in (7) are exhausted by the curvatures of these smooth closed plane curves.

We say that a surface of revolution described in Theorem 2 is associated with the plane curve Γ. The relation $k(s) = 2H(s)$ between curvature $k(s)$ of Γ and mean curvature $H(s)$ of S associated with Γ is originally remarked by H. Reckziegel (see Remark 1 in p. 149 of [3]). Recently, Aiyama [1] extended this idea to generalized surfaces of revolution in complex Euclidean 2-space.

Applying Theorem 2 to a circle with radius r centered at $(0, r) \in R^2$, we have a surface of revolution with constant mean curvature $1/2r$. This is the case of Delaunay [2]. Next, we apply Theorem 2 to an ellipse and get interesting periodic surfaces of revolution, which give the explicit examples of the special f-surfaces studied by R. Sa Earp and E. Toubiana [7]. See Fig. 2.

Now we are in a position to study the general case of Theorem 1. First of all, we give many solutions of the integral equation (7). Put $P_0 = (0, 0)$, and let $P_1 = (r\cos\theta, r\sin\theta)$, $r \neq 0$, be any point of $R^2 - \{(0, 0)\}$. Let us consider a C^2 curve segment $\Gamma_0 = (G(s), F(s))$, $0 \leq s \leq L$, parametrized by arc length which starts from P_0 and terminates at P_1 such that the both tangent vectors at the end points are $(G'(0), F'(0)) = (1, 0)$, and $(G'(L), F'(L)) = (\cos 2\theta, \sin 2\theta)$. Moreover, we assume that the curvatures of the curve Γ_0 at the both end points have the same values. We extend Γ_0 by some isometries of R^2 so as the resultant curve Γ is C^2 and complete. The cur-

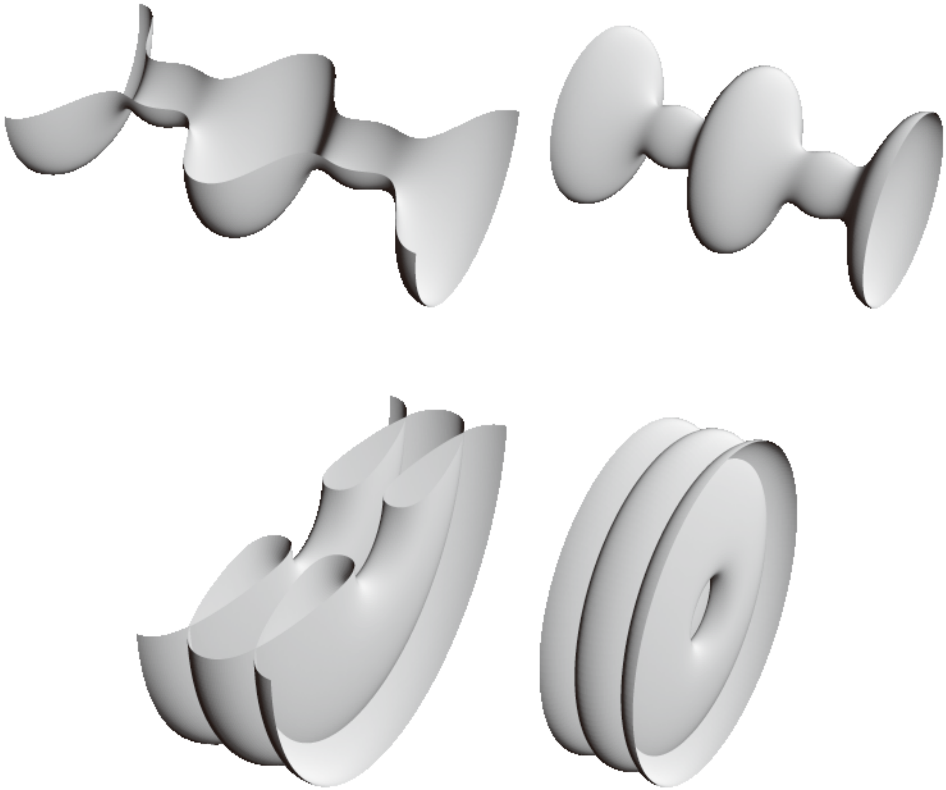

Fig. 2. The surfaces of revolution associated with an ellipse.

vature of Γ, $k(s)$, is a continuous and periodic function on R with period L and the same as the curvature of Γ_0 if it is restricted to $[0, L]$. Put $H(s) = k(s)/2$. Then the coordinate functions of the curve Γ satisfy the condition (7), hence we have a periodic surface of revolution S with the mean curvature $k(s)/2$ by Theorem 1.

Conversely, we prove that the non-trivial solutions of (7) are exhausted by the curvatures of these curves constructed above. Putting Σ the set of such curves Γ_0 defined above, we have

Theorem 3. *Let C be a complete profile curve parametrized by arc length s such that the resultant surface of revolution S has periodic mean curvature $H(s)$ with period L. Assume that the condition* (7) *is non-trivially satisfied by the $H(s)$. Then, for the curve $\Gamma = (G(s), F(s))$, there is a fundamental curve segment $\Gamma_0 \in \Sigma$ such that Γ is decomposed as $\Gamma = \bigcup_{-\infty<n<\infty} T_n\Gamma_0$, where T_n is an isometry of R^3, and n runs through integers.*

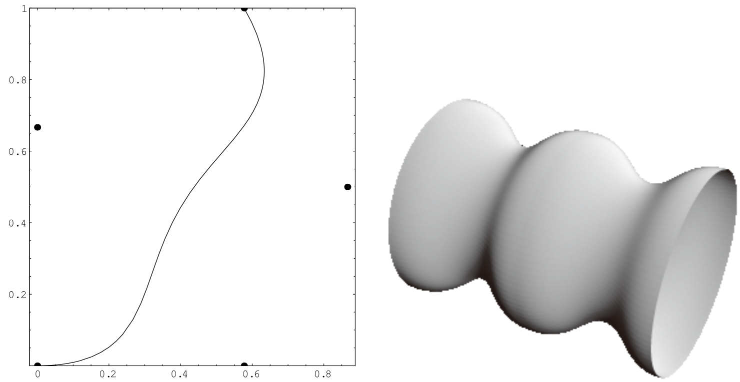

Fig. 3. A Bézier curve with 5 control points and the associated surface of revolution.

Proof. $\Gamma = (G(s), F(s))$ is defined on R by (3). Put $\Gamma_0 = \{(G(s), F(s)) \mid 0 \le s \le L\}$. The condition (7) implies $\Gamma_0 \in \Sigma$. By (11), we have

$$G(s+nL) + iF(s+nL) = (1 + e^{ih(L)} + \cdots + e^{(n-1)ih(L)}(G(L) + iF(L)) \\ + e^{nih(L)}(G(s) + iF(s)).$$

Hence, the curve segment $(G(s+nL), F(s+nL))$, $0 \le s \le L$, is transformed from Γ_0 by the isometry T_n which is defined by the formula above. This proves Theorem 3. □

The problem is now reduced how to find a curve Γ_0 from the given data of terminal points. Bézier curves supply such Γ_0's: Let $\mathbf{B}(t)$, $0 \le t \le 1$, be a Bézier curve with the control points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ such that $\mathbf{b}_0 = (0, 0)$, $\mathbf{b}_1 = (\cot\theta, 0), \ldots, \mathbf{b}_{n-1} = (\sin 2\theta, -\cos 2\theta)$, $\mathbf{b}_n = (\cot\theta, 1)$, $(\pi/4 < \theta < \pi/2)$, and $\mathbf{b}_{l-1}$ and $\mathbf{b}_{n-l+1}$, $l = 2, 3, \ldots$, are symmetric with respect to the bisector of the lines $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_{n-1}\mathbf{b}_n$, and $\mathbf{b}_{n/2}$ is on the bisector if n is even. The $\mathbf{B}(t)$ has the same curvature at the both end points and satisfies the condition (7). Hence we get a periodic surface of revolution from the Bézier curve $\mathbf{B}(t)$. We say such a surface is associated with the Bézier curve. See Fig. 3.

4. Remarks

We conclude this paper by giving some remarks and open problems. The periodic surfaces made by Theorems 2 and 3 have different properties. In fact, the surface associated with a plane closed curve has a one parameter family of these periodic surfaces with the same mean curvature function by changing the starting point $(0, c)$ of the curve. This resembles the H-deformation from unduloid to nodoid of constant mean curvature surfaces.

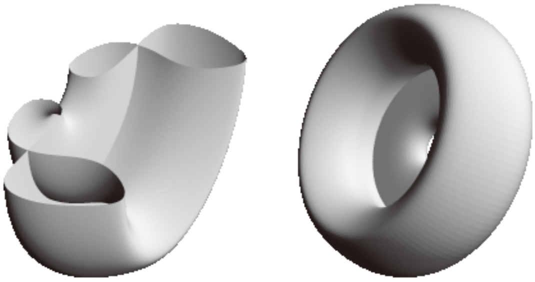

Fig. 4. A torus associated with the figure-eight curve.

On the other hand, the periodic surfaces associated with the curve segments in Theorem 3 are isolated, that is, we have the special value c such that the resultant surface of revolution starting at $(0, c)$ is periodic, where the constant c is given by the common ratio in (7) and for any other $c'(\neq c)$, the profile curve starting at $(0, c')$ is not periodic.

By (5), the periodic profile curve is closed if and only if the functions $G(s)$ and $F(s)$, moreover, satisfy

$$\int_0^L \frac{G(t)F'(t) - (F(t) - c)G'(t)}{\{(F(t) - c)^2 + G(t)^2\}^{1/2}}\, dt = 0. \tag{13}$$

Constant functions do not satisfy (13) when $c > 0$. Hence, any complete surface of revolution with constant mean curvature can not be compact except round sphere.

The function defined by

$$H(s) = \frac{1}{2}\left(-1 + \frac{\cos s}{2 - \cos s}\right), \quad -\infty < s < \infty, \tag{14}$$

non-trivially satisfies (7) with the common ratio $c = 1$ and (13), because this is the mean curvature of a round torus in R^3. Any surface of revolution whose mean curvature is (14) is not periodic if the starting point of the curve is not $(0, 1)$.

There is a closed surface of revolution associated with the figure-eight curve. See Fig. 4.

We remark that Theorem 1 answers to the question in p. 49 of the book [4].

Finally let us put some open problems in order to develop the theory of surfaces with periodic mean curvature in near future.

1. Find a method to solve the integral equation (13).
2. Extend these results of this paper to higher dimensional case.
3. Study isometric immersions from R^2 into R^3 with doubly periodic mean curvatures.

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