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CLOSABILITY OF POSITIVE SYMMETRIC BILINEAR FORMS WITH APPLICATIONS TO CLASSICAL AND STABLE FORMS ON FINITE AND INFINITE DIMENSIONAL STATE SPACES

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1. General framework and basic definitions

In the literature concerning Dirichlet forms and its applications, closability plays a crucial role. In fact, closedness is one of the defining properties of a Dirichlet form. According to this, a number of closability criterions are known in particular cases. An important question is under which conditions closability is kept after changing the reference measure.

M. Fukushima, K. Sato, and S. Taniguchi [5] treated this problem for a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ which is defined on a locally compact separable metric state space. Under technical conditions on some core $\mathcal{C} \subseteq D(\mathcal{E})$, they presented a complete solution if the Dirichlet form is either irreducible or transient. An earlier paper dealing with this subject is M. Röckner and N. Wielens [13]. Related results on Lusinean separable metric spaces were published in I. Shigekawa and S. Taniguchi [16].

The aim of this paper is to give general analytical conditions in order to keep closability when turning to a new reference measure. One particular purpose is to present an extension of an assertions in [5] (namely, Corollary 4.2) within a purely measure theoretic framework, i.e., the state space $(E, \mathcal{B})$ is just a measurable space. In particular, the set $\mathcal{C}$ is defined exclusively in terms of the initial form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$. The main results are Theorems 2.3, 2.4, and 2.5.

We proceed to give some basic definitions.

DEFINITION 1.1. Let $(H, \|.\|_H)$ be a separable Hilbert space and let $\mathcal{F}$ be a dense subset of H.
(i) A positive symmetric bilinear form (p.s.b.f.) $\mathcal{E}$ defined on $\mathcal{F}$ is said to be *closed* if $\mathcal{F}$, equipped with the $(\mathcal{E}_1)^{1/2}$-norm $\|f\|_{\mathcal{E}_1} := (\|f\|_H^2 + \mathcal{E}(f, f))^{1/2}$, is a Hilbert space.
(ii) Let $\mathcal{C}$ be a subspace of H. We say that a p.s.b.f. $(\mathcal{E}, \mathcal{C})$ is *pre-closable* on H if, for all sequences $u_n \in \mathcal{C}$, $n \in \mathbb{N}$, which are $\mathcal{E}$-Cauchy (i.e., $\mathcal{E}(u_n - u_m, u_n -$

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$u_m) \xrightarrow[m,n\to\infty]{} 0$) and satisfy $u_n \xrightarrow[n\to\infty]{} 0$ in H, we have $\mathcal{E}(u_n, u_n) \xrightarrow[n\to\infty]{} 0$. If $(\mathcal{E}, \mathcal{C})$ is pre-closable on H and, furthermore, $\mathcal{C}$ is dense in H, we say that $(\mathcal{E}, \mathcal{C})$ is *closable* on H.

If a p.s.b.f. $(\mathcal{E}, \mathcal{C})$ is closable on H then there exists a p.s.b.f. $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on H which is closed and extends $(\mathcal{E}, \mathcal{C})$ in the following sense: $\hat{\mathcal{F}} \supseteq \mathcal{C}$ and $\hat{\mathcal{E}}(u,u) = \mathcal{E}(u,u)$, $u \in \mathcal{C}$.

DEFINITION 1.1 (continuation). (iii) The smallest closed extension of $(\mathcal{E}, \mathcal{C})$ on H, i.e., that closed extension $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ on H satisfying $\overline{\mathcal{F}} \subseteq \hat{\mathcal{F}}$, for all closed extensions $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{C})$ on H, is called the *closure* of $(\mathcal{E}, \mathcal{C})$.

We concentrate on Hilbert spaces of the form $L^2(E, \mu)$, where E endowed with a σ-algebra $\mathcal{B}$ is a measurable space and μ is a σ-finite measure on $(E, \mathcal{B})$. As usual, we call a closed p.s.b.f. $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$ a (*symmetric*) *Dirichlet form* if $u \in D(\mathcal{E})$ implies $u \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u \wedge 1, u \wedge 1) \le \mathcal{E}(u, u)$. Moreover, we call a nonpositive definite self-adjoint operator A on $L^2(E, \mu)$ a *Dirichlet operator* whenever $f \in D(A)$ yields $\int Af\,(f-1)^+\,d\mu \le 0$. If $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form then the associated. Then the associated Dirichlet operator is the unique (nonpositive definite self-adjoint) operator satisfying $D(A) = \{f \in D(\mathcal{E}) : \text{there exists } g \in L^2(E, \mu) \text{ such that } \mathcal{E}(f, h) = \int g \cdot h\,d\mu \text{ for all } h \in D(\mathcal{E})\}$ and $-\int Af \cdot h\,d\mu = \mathcal{E}(f, h)$, $f \in D(A)$, $h \in D(\mathcal{E})$. For more details about the interplay between these two notions see, for example, N. Bouleau and F. Hirsch [3], M. Fukushima, Y. Oshima, and M. Takeda [4], and Z.M. Ma and M. Röckner [10].

Our presentation starts with an introduction to fractional powers of Dirichlet operators and Dirichlet forms (Section 1). Here, the aim is to summarize all basic facts from functional analysis which we need in order to discuss the applications of our closability criterions. In particular in Subsection 1.2, we provide a detailed comparision between three different representations of fractional powers of the Laplacian in finite dimension: We consider the representation in form of an (integro-) differential operator, the representation via spectral resolution, and the representation via multiplication in the Fourier image.

The starting point of Section 2 is a p.s.b.f. $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$ and the associated nonpositive definite self-adjoint operator A. The purpose of this section is to obtain a closability criterion for $(\mathcal{E}, \mathcal{C}) = (\mathcal{E}^M, \mathcal{C}^M)$ on some $L^2(E, M)$, where $\mathcal{C}^M$ is a certain set of M-classes. Each of those M-classes has a version that can be interpreted as a μ-class belonging to some subspace $\mathcal{C}^\mu$ of $D(\mathcal{E})$. Furthermore, $\mathcal{E}^M$ corresponds to $\mathcal{E}$ when turning from μ- to M-classes. We emphasize that the set $\mathcal{C}$ is constructed by means of the spectral resolution of A; i.e., $\mathcal{C}$ is not a core in the sense of [4] and [10]. In the case $M = \tau\mu$ with $\tau \in L^1(E, \mu) \cap L^\infty(E, \mu)$, under additional assumptions, we characterize a property of the M-negligible sets which is equivalent to the closability of $(\mathcal{E},\mathcal{C})$ on $L^2(E,M)$ (cf. Propositions 2.1, 2.2, and Theorem 2.3). These structural results are one central point of our investigations. However, for measures $M=\tau\mu$ with $\tau\in L^1(E,\mu)\cap L^\infty(E,\mu)$ and $\tau>0$ μ-a.e., we formulate simple conditions on A guaranteeing closability (Theorem 2.4). For example, if the following *spectral condition*

(SC) 0 is an isolated point in the spectrum of A and $\operatorname{Ker} A$ consists of the constant functions

is satisfied, then we have closability. But also in case that (SC) is not satisfied, we obtain verifiable closability conditions. Finally, the following should be mentioned. Under the conditions guaranteeing closability of $(\mathcal{E},\mathcal{C})$ on $L^2(E,M)$ in Theorems 2.3, 2.4, the closure of $(\mathcal{E},\mathcal{C})$ on $L^2(E,M)$ is a Dirichlet form whenever the p.s.b.f. $(\mathcal{E},D(\mathcal{E}))$ on $L^2(E,\mu)$ is a Dirichlet form (Theorem 2.5).

In Section 3, we discuss applications of the criterions obtained. In particular, we consider diffusion type forms and their fractional powers corresponding to second quantization (Subsection 3.1). In this example, we have the spectral condition (SC). However, we also investigate classical and stable Dirichlet forms on $\mathbb{R}^d$. Here, (SC) is not satisfied. In this case, Theorem 2.3 provides an explicit description of a property of the M-negligible sets in terms of Riesz potentials in order to have closability (see Subsection 3.2).

Finally, we refer to the fact that we use standard notations. However, note that Λ^d denotes the d-dimensional Lebesgue measure; but in integrals we also write $\int\ \cdot\ dx$.

1.1. Fractional powers of Dirichlet operators and Dirichlet forms. We start with the presentation of some classical results concerning fractional powers of Dirichlet operators due to V. Balakrishnan, T. Kato, and K. Yosida; see [18, IX. 11].

Let A be a Dirichlet operator on $L^2(E,\mu)$, i.e., A is a densely defined nonpositive definite self-adjoint operator in $L^2(E,\mu)$ with $(Af,(f-1)^+)_{L^2(E,\mu)}\le 0$, $f\in D(A)$. Furthermore, let $(P_t)_{t\ge 0}$ be the associated symmetric strongly continuous sub-Markov contraction semigroup in $L^2(E,\mu)$; sub-Markov means that $0\le f\le 1$ implies $0\le P_tf\le 1$, $t\ge 0$, $f\in L^2(E,\mu)$. Let $0<\alpha<1$. We introduce

$$f_{t,\alpha}(s):=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{zs-tz^\alpha}\,dz\,,\quad t>0,\ s\ge 0,\ \sigma>0,$$

where the branch of z^α is so taken that $Re(z^\alpha)>0$ for $Re(z)>0$. Note that $f_{t,\alpha}$ is independent of σ. We define

$$P_0^{(\alpha)}f:=f\ \text{ and }\ P_t^{(\alpha)}f:=\int_0^\infty f_{t,\alpha}(s)\,P_sf\,ds\,,\quad t>0,\ f\in L^2(E,\mu).$$

$\left(P_t^{(\alpha)}\right)_{t\ge 0}$ forms a symmetric strongly continuous semigroup in $L^2(E,\mu)$. As $f_{t,\alpha}(s)\ge 0$ for all $s,\,t\ge 0$ and $\int_0^\infty f_{t,\alpha}(s)\,ds=1$, $t>0$ (cf. [18, IX. 11, Propositions 2 and 3]),

$\left(P_t^{(\alpha)}\right)_{t\geq 0}$ is, moreover, contractive and sub-Markov. The corresponding 1-resolvent operator can be represented by

$$R_1^{(\alpha)} f = \frac{\sin \alpha\pi}{\pi} \int_0^\infty (r - A)^{-1} f \; \frac{r^\alpha \; dr}{1 - 2r^\alpha \cos \alpha\pi + r^{2\alpha}} , \quad f \in L^2(E, \mu).$$

Hence, the generator $A^{(\alpha)}$ of $\left(P_t^{(\alpha)}\right)_{t\geq 0}$ is given by

$$D\left(A^{(\alpha)}\right) = R_1^{(\alpha)}(L^2(E, \mu)) \tag{1.1}$$

and

$$A^{(\alpha)} f = -\frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1}(P_t f - f)\, dt , \quad f \in D(A) \left(\subseteq D\left(A^{(\alpha)}\right)\right) ,$$

(see [18, IX. 11, Theorem 2]) where Γ denotes the Gamma-function. Furthermore, we set $A^{(1)} := A$ and $\left(P_t^{(1)}\right)_{t\geq 0} := (P_t)_{t\geq 0}$. By the properties of $\left(P_t^{(\alpha)}\right)_{t\geq 0}$ mentioned above, the operator $A^{(\alpha)}$ is also a Dirichlet operator. Introducing $\left(E_\lambda^{(\alpha)}\right)_{\lambda\geq 0}$ as the (right continuous) resolution of the identity with respect to $-A^{(\alpha)}$, we have

$$D\left(-A^{(\alpha)}\right) = \left\{ f \in L^2(E, \mu) : \int_{[0,\infty)} \lambda^2\, d\|E_\lambda^{(\alpha)} f\|^2_{L^2(E,\mu)} < \infty \right\}$$

and

$$-A^{(\alpha)} = \int_{[0,\infty)} \lambda\, dE_\lambda^{(\alpha)} , \quad 0 < \alpha \leq 1. \tag{1.2}$$

We proceed to give the relations between the spectral resolutions of A and $A^{(\alpha)}$. We define

$$D\left((-A)^\alpha\right) := \left\{ f \in L^2(E, \mu) : \int_{[0,\infty)} \lambda^{2\alpha}\, d\|E_\lambda^{(1)} f\|^2_{L^2(E,\mu)} < \infty \right\}$$

and

$$(-A)^\alpha := \int_{[0,\infty)} \lambda^\alpha\, dE_\lambda^{(1)} , \quad 0 < \alpha \leq 1.$$

Proposition 1.2. *We have*

$$E_\lambda^{(\alpha)} = E_{\lambda^{1/\alpha}}^{(1)} , \;\; \lambda \geq 0 , \quad \textit{and} \quad -A^{(\alpha)} = (-A)^\alpha \tag{1.3}$$

which means, in particular, $D\left(-A^{(\alpha)}\right) = D((-A)^\alpha)$, *cf.* (1.1). *Furthermore,* $(-A)^\alpha$ *is a Dirichlet operator,* $0 < \alpha \leq 1$.

For the readers convenience, we recall the proof of these basic facts:

Proof. The case $\alpha = 1$ is trivial. Let $0 < \alpha < 1$. For $t > 0$ and $f \in L^2(E, \mu)$, we have

$$\begin{aligned}\int_{[0,\infty)} e^{-\lambda t}\, dE_\lambda^{(\alpha)} f &= P_t^{(\alpha)} f \\ &= \int_{[0,\infty)} P_s^{(1)} f\, f_{t,\alpha}(s)\, ds \\ &= \int_{s=0}^{\infty} \int_{\lambda\in[0,\infty)} e^{-\lambda s}\, dE_\lambda^{(1)} f\, f_{t,\alpha}(s)\, ds \\ &= \int_{\lambda\in[0,\infty)} \int_{s=0}^{\infty} e^{-\lambda s}\, f_{t,\alpha}(s)\, ds\, dE_\lambda^{(1)} f \\ &= \int_{[0,\infty)} e^{-\lambda^\alpha t}\, dE_\lambda^{(1)} f\,,\end{aligned}$$

where the last equality follows from [18, IX. 11, Proposition 1]. Hence,

$$\int_{[0,\infty)} e^{-\lambda t}\, dE_\lambda^{(\alpha)} f = \int_{[0,\infty)} e^{-\lambda t}\, dE_{\lambda^{1/\alpha}}^{(1)} f$$

which implies (1.3). The last assertion of the proposition follows from the fact that $-A^{(\alpha)}$ is, as mentioned above, a Dirichlet operator. □

The associated Dirichlet form can be expressed by

$$D\left(\mathcal{E}^{(\alpha)}\right) = D\left(\left(-A^{(\alpha)}\right)^{1/2}\right) := \left\{ f \in L^2(E,\mu) : \int_{[0,\infty)} \lambda\, d\|E_\lambda^{(\alpha)} f\|^2_{L^2(E,\mu)} < \infty \right\}$$

and

$$\begin{aligned}&\mathcal{E}^{(\alpha)}(f,g) \\ &:= \int \left(-A^{(\alpha)}\right)^{1/2} f \left(-A^{(\alpha)}\right)^{1/2} g\, d\mu\,, \quad f,\ g \in D\left(\mathcal{E}^{(\alpha)}\right),\ 0 < \alpha \le 1.\end{aligned} \tag{1.4}$$

From Proposition 1.2 and (1.4), we obtain immediately:

Proposition 1.3. *Let $0 < \alpha \le 1$. We have $D\left(\mathcal{E}^{(\alpha)}\right) = D\left((-A)^{\alpha/2}\right)$ and*

$$\begin{aligned}\mathcal{E}^{(\alpha)}(f,g) &= \mathcal{E}^{\alpha}(f,g) := \left((-A)^{\alpha/2} f, (-A)^{\alpha/2} g\right)_{L^2(E,\mu)} \\ &\left(= \int_{[0,\infty)} \lambda^\alpha\, d\left(E_\lambda^{(1)} f, E_\lambda^{(1)} g\right)_{L^2(E,\mu)} \right), \quad f, g \in D\left(\mathcal{E}^{(\alpha)}\right).\end{aligned}$$

Let $D(\mathcal{E}^\alpha) := D\left((-A)^{\alpha/2}\right)$. Then $(\mathcal{E}^\alpha, D(\mathcal{E}^\alpha))$ is a Dirichlet form.

To be consistent, we denote $D(\mathcal{E}) := D(\mathcal{E}^1)$ and $\mathcal{E} := \mathcal{E}^1$.

1.2. Fractional powers of the Laplacian. In this subsection, we introduce three different representations of the Laplacian on $\mathbb{R}^d$ and its fractional powers. In particular we discuss the interplay between the representation in form of an (integro-) differential operator, the representation via spectral resolution, and the representation via multiplication in the Fourier image. Basic facts are taken from M. Reed and B. Simon [11], E.H. Lieb and M. Loss [8], and S.G. Samko, A.A. Kilbas, and O.I. Marichev [14]. Further presentations of related topics are, for example, in N. Jacob [6], A.V. Skorokhod [15], and E.M. Stein and G. Weiss [17].

However, the applications we are interested in require extensions of these references. They are formulated and proved below.

Let $d \in \mathbb{N}$. Define

$$D(A) \equiv D(\Delta)$$

$$:= \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} f \in L^2(\mathbb{R}^d, \Lambda^d) \text{ in the sense of distributions} \right\} \tag{1.5}$$

and

$$Af \equiv \Delta f := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} f \,, \quad f \in D(A) \equiv D(\Delta). \tag{1.6}$$

According to [11, Theorem IX.27], A is selfadjoint. With

$$\mathcal{F}\varphi(\hat{x}) = \int e^{i\langle x,\hat{x}\rangle_d} \varphi(x)\, dx\,, \quad \hat{x} \in \mathbb{R}^d,$$

and

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^d} \int e^{-i\langle \hat{x},x\rangle_d} \varphi(\hat{x})\, d\hat{x}\,, \quad x \in \mathbb{R}^d,$$

we have

$$D(A) \equiv D(\Delta) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\hat{x}|^2 \mathcal{F} f \in L^2(\mathbb{R}^d, \Lambda^d) \right\} \tag{1.7}$$

and

$$-Af \equiv -\Delta f = \mathcal{F}^{-1}\left(|\hat{x}|^2 \mathcal{F} f \right)\,, \quad f \in D(A) \equiv D(\Delta), \tag{1.8}$$

cf. the same reference. In virtue of (1.8), $A \equiv \Delta$ is nonpositive. The operator $A \equiv \Delta$ is the generator of the semigroup $(P_t)_{t\geq 0}$ in $L^2(\mathbb{R}^d, \Lambda^d)$ given by

$$P_t f = \int f(y) \frac{1}{(4\pi t)^{d/2}} exp\left(-\frac{1}{4t} |\cdot - y|^2 \right) dy\,, \quad f \in L^2(\mathbb{R}^d, \Lambda^d).$$

Since $(P_t)_{t\geq 0}$ is a symmetric strongly continuous sub-Markov contraction semigroup, the operator $A \equiv \Delta$ is a Dirichlet operator. Let

$$\chi_{a,b}(x) := \begin{cases} 1 & a \leq |x| < b \\ 0 & \text{otherwise} \end{cases}, \quad x \in \mathbb{R}^d,\ 0 \leq a < b \leq \infty.$$

In order to compare representation (1.7) and (1.8) with

$$D(A) \equiv D(\Delta) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^2 \, d\|E_\lambda^{(1)} f\|^2 < \infty \right\}, \tag{1.9}$$

and

$$-Af \equiv -\Delta f = \int_{[0,\infty)} \lambda \, dE_\lambda^{(1)} f, \quad f \in D(A) \equiv D(\Delta), \tag{1.10}$$

let us state the following:

Proposition 1.4. *We have*

$$E_\lambda^{(1)} f = \mathcal{F}^{-1}(\chi_{0,\lambda^{1/2}} \mathcal{F} f), \quad f \in L^2(\mathbb{R}^d, \Lambda^d),\ \lambda \geq 0. \tag{1.11}$$

Proof. Formula (1.11) is an immediate consequence of the spectral resolution of the operator $D(M) := \{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\cdot|^2 f \in L^2(\mathbb{R}^d, \Lambda^d)\}$, $Mf(x) := |x|^2 f(x)$, $x \in \mathbb{R}^d$, $f \in D(M)$, and the fact that $(2\pi)^{d/2}\mathcal{F}^{-1}$ can be be regarded as a unitary operator $L^2(\mathbb{R}^d, \Lambda^d) \to L^2(\mathbb{R}^d, \Lambda^d)$. Consult also, for example, L.A. Ljusternik and W.I. Sobolev [9, VII, §9]. □

Subsequently, we are interested in fractional powers of $-A \equiv -\Delta$. In order to be compatible with the results of Subsection 1.1, we mainly concentrate on the case $\alpha \leq 1$.

Proposition 1.5. *Let* $0 < \alpha \leq 1$. *We have*

$$D((-A)^\alpha) \equiv D((-\Delta)^\alpha) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\hat{x}|^{2\alpha} \mathcal{F} f \in L^2(\mathbb{R}^d, \Lambda^d) \right\} \tag{1.12}$$

and

$$(-A)^\alpha f \equiv (-\Delta)^\alpha f = \mathcal{F}^{-1}\left(|\hat{x}|^{2\alpha} \mathcal{F} f \right), \quad f \in D((-A)^\alpha) \equiv D((-\Delta)^\alpha). \tag{1.13}$$

Proof. In virtue of (1.9), (1.10), and (1.11), it holds that

$$D((-\Delta)^\alpha) \equiv \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^{2\alpha} \, d\|E_\lambda^{(1)} f\|^2_{L^2(\mathbb{R}^d, \Lambda^d)} < \infty \right\}$$

$$= \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^{2\alpha}\, d\|\mathcal{F}^{-1}(\chi_{0,\lambda^{1/2}}\mathcal{F} f)\|^2_{L^2(\mathbb{R}^d,\Lambda^d)} < \infty \right\}$$
$$= \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^{2}\, d\|\mathcal{F}^{-1}(\chi_{0,\lambda^{1/2\alpha}}\mathcal{F} f)\|^2_{L^2(\mathbb{R}^d,\Lambda^d)} < \infty \right\}$$

and

$$(-\Delta)^\alpha f = \int_{[0,\infty)} \lambda^\alpha \, d\mathcal{F}^{-1}(\chi_{0,\lambda^{1/2}}\mathcal{F} f)$$
$$= \int_{[0,\infty)} \lambda \, d\mathcal{F}^{-1}(\chi_{0,\lambda^{1/2\alpha}}\mathcal{F} f), \quad f \in D((-\Delta)^\alpha).$$

Recalling that $(2\pi)^{d/2}\mathcal{F}^{-1}$ is a unitary operator $L^2(\mathbb{R}^d, \Lambda^d) \to L^2(\mathbb{R}^d, \Lambda^d)$, from the spectral resolution of the operator $D(M^{(\alpha)}) := \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\cdot|^{2\alpha} f \in L^2(\mathbb{R}^d, \Lambda^d) \right\}$, $M^{(\alpha)} f(x) := |x|^{2\alpha} f(x)$, $x \in \mathbb{R}^d$, $f \in D(M^{(\alpha)})$, relations (1.12) and (1.13) can be obtained. $\square$

For $0 < \alpha \leq 1$, define

$$D(J^{(\alpha)}) := \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^{-\alpha}\, d\|E^{(1)}_\lambda f\|^2_{L^2(\mathbb{R}^d,\Lambda^d)} < \infty \right\}$$

and

$$J^{(\alpha)} f := \int_{[0,\infty)} \lambda^{-\alpha/2}\, dE^{(1)}_\lambda f, \quad f \in D(J^{(\alpha)}).$$

As in Proposition 1.5, one can verify

$$D(J^{(\alpha)}) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\hat{x}|^{-\alpha}\mathcal{F} f \in L^2(\mathbb{R}^d, \Lambda^d) \right\} \tag{1.14}$$

and

$$J^{(\alpha)} f = \mathcal{F}^{-1}\left(|\hat{x}|^{-\alpha}\mathcal{F} f \right), \quad f \in D(J^{(\alpha)}). \tag{1.15}$$

Furthermore, for $0 < \alpha \leq 1$, define $\gamma_d(\alpha) := 2^\alpha \pi^{d/2}\Gamma(\alpha/2)/\Gamma((d-\alpha)/2)$, $D(I^{(\alpha)}) := L^2(\mathbb{R}^d, \Lambda^d)$, and

$$I^{(\alpha)} f := \frac{1}{\gamma_d(\alpha)} \int \frac{f(y)\, dy}{|\cdot - y|^{d-\alpha}}, \quad f \in D(I^{(\alpha)}).$$

In virtue of a theorem due to S.L. Sobolev (see, for example, [14], Theorem 25.2), $I^{(\alpha)}$ is a continuous operator $D(I^{(\alpha)}) = L^2(\mathbb{R}^d, \Lambda^d) \longrightarrow L^p(\mathbb{R}^d, \Lambda^d)$ whenever $\alpha < d/2$ and $p = 2d/(d - 2\alpha)$.

Proposition 1.6. *Let* $0 < \alpha \le 1$ *and* $\alpha < d/2$. *Then* $I^{(\alpha)} f = J^{(\alpha)} f$, $f \in D(J^{(\alpha)})$.

Proof. 1° For $\varphi \in C_0^\infty(\mathbb{R}^d)$, we know $\mathcal{F}\varphi \in L^2(\mathbb{R}^d, \Lambda^d) \cap L^\infty(\mathbb{R}^d, \Lambda^d)$. Recalling $\alpha < d/2$ this yields $|\hat{x}|^{-\alpha}\mathcal{F}\varphi \in L^2(\mathbb{R}^d, \Lambda^d)$. On account of (1.14), it holds that $\varphi \in D(J^{(\alpha)})$. It follows from [8], Theorem 5.9, that

$$I^{(\alpha)}\varphi = J^{(\alpha)}\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^d); \tag{1.16}$$

when turning to the reference above, take the special definition of the Fourier transform therein into account.

2° Let $f \in D(J^{(\alpha)})$ and $\varphi_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, be a sequence with $\varphi_n \xrightarrow[n\to\infty]{} f$ in $L^2(\mathbb{R}^d, \Lambda^d)$. Since $I^{(\alpha)}\colon L^2(\mathbb{R}^d, \Lambda^d) \to L^p(\mathbb{R}^d, \Lambda^d)$, continuously, there exists a subsequence n_k, $k \in \mathbb{N}$, such that

$$I^{(\alpha)}\varphi_{n_k} \xrightarrow[k\to\infty]{} I^{(\alpha)} f \quad \Lambda^d\text{-a.e.} \tag{1.17}$$

On the other hand, (1.15) implies

$$J^{(\alpha)}\varphi_n = \mathcal{F}^{-1}\left(\chi_{0,1}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_n\right)\right) + \mathcal{F}^{-1}\left(\chi_{1,\infty}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_n\right)\right), \quad n \in \mathbb{N}. \tag{1.18}$$

As $\varphi_n \xrightarrow[n\to\infty]{} f$ in $L^2(\mathbb{R}^d, \Lambda^d)$, we have $\chi_{0,1}\mathcal{F}\varphi_n \xrightarrow[n\to\infty]{} \chi_{0,1}\mathcal{F}f$ in $L^2(\mathbb{R}^d, \Lambda^d)$. The Schwarz inequality implies

$$\begin{aligned} \chi_{0,1}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_n\right) &= \left(\chi_{0,1}|\hat{x}|^{-\alpha}\right)\left(\chi_{0,1}\mathcal{F}\varphi_n\right) \\ &\xrightarrow[n\to\infty]{} \left(\chi_{0,1}|\hat{x}|^{-\alpha}\right)\left(\chi_{0,1}\mathcal{F}f\right) \\ &= \chi_{0,1}\left(|\hat{x}|^{-\alpha}\mathcal{F}f\right) \qquad \text{in } L^1(\mathbb{R}^d, \Lambda^d); \end{aligned}$$

note that, according to $\alpha < d/2$, we have $\chi_{0,1}|\hat{x}|^{-\alpha} \in L^2(\mathbb{R}^d, \Lambda^d)$. Furthermore, the Hausdorff-Young theorem yields

$$\mathcal{F}^{-1}\left(\chi_{0,1}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_n\right)\right) \xrightarrow[n\to\infty]{} \mathcal{F}^{-1}\left(\chi_{0,1}\left(|\hat{x}|^{-\alpha}\mathcal{F}f\right)\right) \quad \text{in } L^\infty(\mathbb{R}^d, \Lambda^d). \tag{1.19}$$

Finally, $\varphi_n \xrightarrow[n\to\infty]{} f$ in $L^2(\mathbb{R}^d, \Lambda^d)$ implies

$$\mathcal{F}^{-1}\left(\chi_{1,\infty}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_n\right)\right) \xrightarrow[n\to\infty]{} \mathcal{F}^{-1}\left(\chi_{1,\infty}\left(|\hat{x}|^{-\alpha}\mathcal{F}f\right)\right) \quad \text{in } L^2(\mathbb{R}^d, \Lambda^d). \tag{1.20}$$

It follows from (1.18)–(1.20) that there is a subsequence n_{k_l}, $l \in \mathbb{N}$, such that

$$\begin{aligned} J^{(\alpha)}\varphi_{n_{k_l}} &= \mathcal{F}^{-1}\left(|\hat{x}|^{-\alpha}\mathcal{F}\varphi_{n_{k_l}}\right) \\ &\xrightarrow[l\to\infty]{} \mathcal{F}^{-1}\left(|\hat{x}|^{-\alpha}\mathcal{F}f\right) \end{aligned}$$

$$= \quad J^{(\alpha)} f \quad \Lambda^d\text{-a.e.} \tag{1.21}$$

Relations (1.16), (1.17), and (1.21) show that $I^{(\alpha)} f = J^{(\alpha)} f$, $f \in D(J^{(\alpha)})$. □

Proposition 1.7 ([14, Theorem 26.3]). *Let $0 < \alpha < \min(1, d/2)$. Define $c_d(\alpha) := \int (1 - e^{it_1}) |t|^{-d-\alpha}\, dt$ where $t = (t_1, \ldots, t_d)$. Then, for $f \in D(I^{(\alpha)}) = L^2(\mathbb{R}^d, \Lambda^d)$ and $\varphi := I^{(\alpha)} f$, the limit*

$$D^{(\alpha)} \varphi := \frac{1}{c_d(\alpha)} \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi(\cdot) - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy \tag{1.22}$$

exists in $L^2(\mathbb{R}^d, \Lambda^d)$ and we have

$$D^{(\alpha)} I^{(\alpha)} f = D^{(\alpha)} \varphi = f\,.$$

For $0 < \alpha < 1$, define

$$D_\alpha := \left\{ \varphi \in L^2(\mathbb{R}^d, \Lambda^d) : \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy \text{ exists in } L^2(\mathbb{R}^d, \Lambda^d) \right\}. \tag{1.23}$$

Proposition 1.8. *Let $0 < \alpha < 1$. Then we have*

$$D_\alpha = D\left((-\Delta)^{\alpha/2} \right).$$

Furthermore, for $\varphi \in D_\alpha$, we have

$$(-\Delta)^{\alpha/2} \varphi = \frac{1}{c_d(\alpha)} \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy \quad \text{in } L^2(\mathbb{R}^d, \Lambda^d).$$

Proof. 1° Let $\varepsilon > 0$ and $\varphi \in L^2(\mathbb{R}^d, \Lambda^d)$. Since $\chi_{\varepsilon,\infty} |y|^{-(d+\alpha)} \in L^1(\mathbb{R}^d, \Lambda^d)$, it holds that $\mathcal{F} \int_{|y|>\varepsilon} |y|^{-(d+\alpha)} \varphi(\cdot - y)\, dy = \mathcal{F}\left(\chi_{\varepsilon,\infty} |y|^{-(d+\alpha)} \right) \cdot \mathcal{F}\varphi$, cf. [8, Theorem 5.8]. Therefore, we obtain

$$\begin{aligned}
\mathcal{F} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy &= \int_{|y|>\varepsilon} |y|^{-(d+\alpha)}\, dy \cdot \mathcal{F}\varphi - \mathcal{F}\left(\chi_{\varepsilon,\infty} |y|^{-(d+\alpha)} \right) \cdot \mathcal{F}\varphi \\
&= \int_{|y|>\varepsilon} \left(1 - e^{i\langle \hat{x}, y \rangle_d} \right) |y|^{-(d+\alpha)}\, dy \cdot \mathcal{F}\varphi \\
&= |\hat{x}|^\alpha \int_{|t|>\varepsilon|\hat{x}|} \left(1 - e^{i\langle \hat{x}/|\hat{x}|, t \rangle_d} \right) |t|^{-(d+\alpha)}\, dt \cdot \mathcal{F}\varphi \\
&= \int_{|t|>\varepsilon|\hat{x}|} \left(1 - e^{it_1} \right) |t|^{-(d+\alpha)}\, dt \cdot |\hat{x}|^\alpha \mathcal{F}\varphi\,. \qquad (1.24)
\end{aligned}$$

2° Define

$$c_d(\alpha;\delta) := \int_{|t|>\delta} \left(1 - e^{it_1}\right) |t|^{-(d+\alpha)}\, dt\,, \quad \delta > 0.$$

We have $|1 - e^{it_1}| = 2|\sin(t_1/2)|$ and, thus,

$$|c_d(\alpha;\delta)| \le 2 \int \left|\sin \frac{t_1}{2}\right| |t|^{-(d+\alpha)}\, dt < \infty\,. \tag{1.25}$$

Furthermore, by dominated convergence,

$$c_d(\alpha;\delta) \xrightarrow[\delta\to 0]{} c_d(\alpha)\,. \tag{1.26}$$

Therefore, (1.24) yields

$$\mathcal{F} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy \xrightarrow[\varepsilon\to 0]{} c_d(\alpha)\, |\hat{x}|^\alpha \mathcal{F}\varphi \quad \Lambda^d\text{-a.e.} \tag{1.27}$$

This implies that $\lim_{\varepsilon\downarrow 0} \mathcal{F} \int_{|y|>\varepsilon} \{\varphi - \varphi(\cdot - y)\}/|y|^{d+\alpha}\, dy$ exists in $L^2(\mathbb{R}^d, \Lambda^d)$ if and only if

$$|\hat{x}|^\alpha \mathcal{F}\varphi \in L^2(\mathbb{R}^d, \Lambda^d) \tag{1.28}$$

and

$$c_d(\alpha;\varepsilon|\hat{x}|)\, |\hat{x}|^\alpha \mathcal{F}\varphi \xrightarrow[\varepsilon\to 0]{} c_d(\alpha)\, |\hat{x}|^\alpha \mathcal{F}\varphi \quad \text{in } L^2(\mathbb{R}^d, \Lambda^d)\,, \tag{1.29}$$

cf. (1.24). Taking (1.25) and (1.26) into consideration, it follows from dominated convergence that (1.29) is a consequence of (1.28). Thus,

$$\lim_{\varepsilon\downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy \quad \text{exists in } L^2(\mathbb{R}^d, \Lambda^d)$$

if and only if we have (1.28). In this case, (1.27) implies

$$\frac{1}{c_d(\alpha)} \lim_{\varepsilon\downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}}\, dy = \mathcal{F}^{-1} |\hat{x}|^\alpha \mathcal{F}\varphi \quad \text{in } L^2(\mathbb{R}^d, \Lambda^d).$$

Now, the assertion of the proposition is a consequence of (1.12), (1.13), and (1.23). □

We summarize the efforts of Proposition 1.5 and Proposition 1.8 in the subsequent Theorem.

Theorem 1.9. (a) *Let* $0 < \alpha \le 1$ *and let* $(\Delta, D(\Delta))$ *be given by* (1.5), (1.6). *Furthermore, let* $\left(E_\lambda^{(1)}\right)_{\lambda \ge 0}$ *be the resolution of the identity with respect to* $-\Delta$, *cf.* (1.9) *and* (1.10). *Then, for the operator* $(-\Delta)^\alpha$ *defined by*

$$D((-\Delta)^\alpha) := \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_{[0,\infty)} \lambda^{2\alpha} \, d\|E_\lambda^{(1)} f\|^2_{L^2(\mathbb{R}^d,\Lambda^d)} < \infty \right\}$$

and

$$(-\Delta)^\alpha f := \int_{[0,\infty)} \lambda^\alpha \, dE_\lambda^{(1)} f \,, \quad f \in D((-\Delta)^\alpha),$$

we have

$$D((-\Delta)^\alpha) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : |\hat{x}|^{2\alpha} \mathcal{F} f \in L^2(\mathbb{R}^d, \Lambda^d) \right\}$$

and

$$(-\Delta)^\alpha f = \mathcal{F}^{-1} \left(|\hat{x}|^{2\alpha} \mathcal{F} f \right) \,, \quad f \in D((-\Delta)^\alpha).$$

(b) *Let* $0 < \alpha < 1$. *Then we have*

$$D\left((-\Delta)^{\alpha/2}\right) = \left\{ \varphi \in L^2(\mathbb{R}^d, \Lambda^d) : \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}} \, dy \ \textit{exists in} \ L^2(\mathbb{R}^d, \Lambda^d) \right\}$$

and

$$(-\Delta)^{\alpha/2} \varphi = \frac{1}{c_d(\alpha)} \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\varphi - \varphi(\cdot - y)}{|y|^{d+\alpha}} \, dy \quad \textit{in} \ L^2(\mathbb{R}^d, \Lambda^d) \,, \quad \varphi \in D\left((-\Delta)^{\alpha/2}\right).$$

2. General closability results

In this section, we ask for closability after changing the reference measure. More precisely, we start with a closed p.s.b.f. $(\mathcal{E}, D(\mathcal{E}))$ on some $L^2(E, \mu)$, keep the form $\mathcal{E}$ unchanged, and present criterions for closability on some $L^2(E, M)$. We emphasize that the underlying state space is a measurable space $(E, \mathcal{B})$ not necessarily endowed with a topological structure. This would suggest that the results below cannot be derived by using probabilistic methods. Moreover, we ask for a Dirichlet form on $L^2(E, M)$ whenever $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(E, \mu)$.
The main results are, on the one hand, a structural theorem (Theorem 2.3) which describes a property of the M-negligible sets in order to have closability and, on the other hand, a simple practicable criterion (Theorem 2.4).

2.1. Definitions and notations. Let $(E, \mathcal{B})$ be a measurable space. Furthermore, let $L^p(E, \nu)$, $1 \le p \le \infty$, denote the usual real L^p-spaces with respect to a σ-finite positive measure ν on $(E, \mathcal{B})$.

Suppose we are given a nontrivial σ-finite positive measure μ on $(E, \mathcal{B})$ and a nonpositive definite self-adjoint operator A in $L^2(E, \mu)$. Let $(E_\lambda)_{\lambda \geq 0}$ denote the (right continuous) resolution of the identity with respect to $-A$, i.e.,

$$-Af = \int_{[0,\infty)} \lambda \, dE_\lambda f \,, \quad f \in D(A) \,.$$

Note that, for all admissible functions φ and all a, $b \in (0, \infty)$ with $a < b$, we define $\int_{[a,b)} \varphi(\lambda) \, dE_\lambda f$ by $\varphi(a)(E_a f - E_{a-0} f) + \int_{(a,b)} \varphi(\lambda) \, dE_\lambda f$ and that $\int_{[0,b)} \varphi(\lambda) \, dE_\lambda f :=$ $\varphi(0) E_0 f \; + \int_{(0,b)} \varphi(\lambda) \, dE_\lambda f$.
We set $\mathbf{1}(x) := 1$, $x \in E$, and we introduce the *spectral condition*:

(SC) The point $\lambda = 0$ belongs to the spectrum of A. Furthermore, there exists $C < 0$ such that no $\lambda \in (C, 0)$ belongs to the spectrum of A. Finally, $\operatorname{Ker} A = \{c \cdot \mathbf{1} : c \in \mathbb{R}\}$.

REMARK. (1) We mention that if (SC) holds then μ is finite.
If (SC) is satisfied then we define

$$\gamma' := \inf \{C < 0 : \text{no } \lambda \in (C, 0) \text{ belongs to the spectrum of } A\} \,.$$

We set

$$\gamma := \begin{cases} -\gamma' & \text{if (SC) is satisfied} \\ 0 & \text{if (SC) is not satisfied} \end{cases} \,.$$

Throughout the paper, we suppose the validity of the following condition:

(C) If (SC) is not satisfied then $\operatorname{Ker} A = \{0\}$ and $\mu(E) = \infty$.

In other words, we suppose that either (SC) (which includes $\operatorname{Ker} A = \{c \cdot \mathbf{1} : c \in \mathbb{R}\}$ and $\mu(E) < \infty$), or $\operatorname{Ker} A = \{0\}$ and $\mu(E) = \infty$. However, this restriction is irrelevant for the applications we are interested in, see Section 3 below.

We define

$$D(J) := \left\{ f \in L^2(E, \mu) : \int_{[\gamma,\infty)} \lambda^{-1} \, d\|E_\lambda f\|^2_{L^2(E,\mu)} < \infty \right\}$$

and

$$Jf := \int_{[\gamma,\infty)} \lambda^{-1/2} dE_\lambda f \,, \quad f \in D(J).$$

Furthermore, we introduce

$$D\left((-A)^{1/2}\right) = \left\{ f \in L^2(E, \mu) : \int_{[0,\infty)} \lambda \, d\|E_\lambda f\|^2_{L^2(E,\mu)} < \infty \right\}$$

and

$$(-A)^{1/2}f = \int_{[0,\infty)} \lambda^{1/2}\, dE_\lambda f\,, \quad f \in D\left((-A)^{1/2}\right).$$

Finally, the closed p.s.b.f. $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,\mu)$ associated with A is given by

$$\mathcal{E}(f,g) = \int (-A)^{1/2} f\, (-A)^{1/2} g\, d\mu\,, \quad f,\ g \in D(\mathcal{E}) := D\left((-A)^{1/2}\right).$$

REMARKS. (2) Both, the definition of γ and condition (C) imply that, for all $f \in L^2(E,\mu)$, we have

$$f - E_0 f = \int_{[\gamma,\infty)} dE_\lambda f\,. \tag{2.1}$$

(3) Define

$$\mathcal{C}^\mu := \left\{ E_0 f + \int_{[a,b)} dE_\lambda f : 0 < a < b < \infty,\ f \in L^2(E,\mu) \right\}.$$

Under condition (C), the set $\mathcal{C}^\mu$ is dense in $L^2(E,\mu)$. Furthermore, since $\mathcal{C}^\mu \subseteq D(J)$, the set $D(J)$ is dense in $L^2(E,\mu)$. In particular, condition (SC) implies that $D(J) = L^2(E,\mu)$.

Let M be a nontrivial σ-finite positive measure on $(E,\mathcal{B})$. Let M_a denote its absolutely continuous part with respect to μ. Furthermore, let M_s denote the singular part of M w.r.t. μ. Throughout the paper, we suppose that M_a is nontrivial.

Let $\Gamma_s^\mu \in \mathcal{B}$ be a set, satisfying $\mu(\Gamma_s^\mu) = 0$ and $M_s(E \setminus \Gamma_s^\mu) = 0$. Furthermore, let μ_s denote the singular part of μ with respect to M. Let $\Gamma_s^M \in \mathcal{B}$ be a set with $M(\Gamma_s^M) = 0$ and $\mu_s(E \setminus \Gamma_s^M) = 0$. For every μ-class $f \in L^2(E,\mu)$, fix a version

$$f_\mu \in f \text{ with } f_\mu = 0 \text{ on } \Gamma_s^\mu$$

and let

$$f^M \text{ denote the } M\text{-class which satisfies } f_\mu \in f^M.$$

The mapping $L^2(E,\mu) \ni f \to f^M$ is independent of the choice of f_μ and, hence, linear. Moreover, define $S := \{\varphi \in L^2(E,\mu) : \varphi = 0 \text{ on } E \setminus \Gamma_s^M\}$. We observe that

$$f^M = g^M \quad \text{if and only if} \quad f - g \in S \cap \mathcal{C}^\mu\,, \quad f,\ g \in \mathcal{C}^\mu. \tag{2.2}$$

Furthermore, we define

$$\mathcal{C} := \{f^M : f \in \mathcal{C}^\mu\}\,.$$

Finally, we introduce the condition

(W) $S \cap \mathcal{C}^\mu = \{0\}$.

By (C), condition (W) is equivalent to $(-A)^{1/2}\varphi = 0$ for all $\varphi \in S \cap \mathcal{C}^\mu$. Thus, (W) is equivalent to

$$\mathcal{E}(u + \varphi, u + \varphi) = \mathcal{E}(u, u) \quad \text{for all } \varphi \in S \cap \mathcal{C}^\mu \text{ and all } u \in \mathcal{C}^\mu. \tag{2.3}$$

Since, under (W), for every μ-class $f \in \mathcal{C}^\mu$, there is no further μ-class $g \in \mathcal{C}^\mu$ with $g \neq f$ such that $f^M = g^M (\in \mathcal{C})$, see (2.2), we can identify

$$\mathcal{C} \cong \mathcal{C}^\mu .$$

The identification $\mathcal{C} \cong \mathcal{C}^\mu$ justifies the notations $\mathcal{E}(u^M, u^M)$ instead of $\mathcal{E}(u, u)$ and $(-A)^{1/2}u^M$ instead of $(-A)^{1/2}u$, $u \in \mathcal{C}^\mu$. In order to avoid confusion while reading the following text, we suggest to replace any $F \in \mathcal{C}$ by the common M- and μ-version $f_\mu \in f \cap f^M$ where $f \in \mathcal{C}^\mu$ such that $f^M = F$.
Relations (2.2) and (2.3) imply that (W) is necessary and sufficient for well-definiteness of $(\mathcal{E}, \mathcal{C})$ on $L^2(E, M)$ whenever $\mathcal{C} \subseteq L^2(E, M)$.

REMARKS. (4) The following relations are important for the proofs of the subsequent results: By the definition of $\mathcal{C}^\mu$ and relation (2.1), we observe that under condition (C),

$$\begin{aligned} \mathcal{C}^\mu &\subseteq D((-A)^{1/2}) \cap D(J)\,, \\ (-A)^{1/2}(\mathcal{C}^\mu) \oplus \operatorname{sign}\gamma \cdot \mathbf{1}\} &= \mathcal{C}^\mu\,, \\ J(\mathcal{C}^\mu) \oplus \{\operatorname{sign}\gamma \cdot \mathbf{1}\} &= \mathcal{C}^\mu\,, \\ J(-A)^{1/2} f &= f - E_0 f\,, \quad f \in \mathcal{C}^\mu. \end{aligned}$$

(5) Let (C) be satisfied. By definition, we have $\mathcal{C}^\mu \subseteq D(\mathcal{E})$. Moreover, $\mathcal{C}^\mu$ is dense in $D(\mathcal{E})$ with respect to $(\mathcal{E}_1)^{1/2}$-norm: Assume that there exists $\varphi \in D(\mathcal{E})$ with

$$\begin{aligned} 0 = (\varphi, f)_{\mathcal{E}_1} &= \int_{[0,\infty)} (1 + \lambda)\ d(E_\lambda \varphi, E_\lambda f)_{L^2(E,\mu)} \\ &= \left(\varphi, \int_{[0,\infty)} (1 + \lambda)\ dE_\lambda f\right)_{L^2(E,\mu)}, \quad f \in \mathcal{C}^\mu. \end{aligned}$$

Here, on account of $f = E_0 f + \int_{[a,b)} dE_\lambda f$ with $0 < a < b < \infty$, we have $\int_{[0,\infty)}(1 + \lambda)\, dE_\lambda f \in L^2(E, \mu)$. However, $\{\int_{[0,\infty)}(1 + \lambda)\, dE_\lambda f : f \in \mathcal{C}^\mu\} = \mathcal{C}^\mu$ is dense in $L^2(E, \mu)$, cf. Remark (3). Hence, $\varphi = 0$ in $L^2(E, \mu)$. Therefore, $\|\varphi\|_{\mathcal{E}_1} = 0$.
(6) Condition (W) is a condition on M. For example, (W) is satisfied if μ is absolutely continuous with respect to M.

(7) The following example demonstrates that condition (W) does not imply that μ is absolutely continuous with respect to M:

Let $E := [0,1)$, let $p\colon E \to [0,\infty)$ be a strictly increasing continuous function, and let $m\colon E \to [0,\infty)$ be a nondecreasing function satisfying

$$0 = m(0) = \lim_{y\to 0} m(y) < m(x) < \lim_{y\to 1} m(y)\,, \quad x \in E, \tag{2.4}$$

and

$$\int_{[0,1)} m(x)\,dp(x) = \infty\,, \quad \int_{[0,1)} p(x)^2\,dm(x) < \infty\,. \tag{2.5}$$

Accordingly, we have the limit circle case, 0 is a regular boundary, and 1 is an entrance boundary. Let μ be the measure generated by m. Furthermore, let ϑ^* denote the set of all real valued μ-classes $f \in L^2(E,\mu)$ such that there exist a μ-class $g \in L^2(E,\mu)$ and $a \in \mathbb{R}$ with

$$f(x) = a + \int_0^x \int_0^y g(s)\,\mu(ds)\,dp(y)\,, \quad x \in E.$$

We set $D_m D_p f := g$ and

$$D(A) := \left\{ f \in \vartheta^* : \int_{[0,1)} D_m D_p f\,d\mu = 0 \right\}\,.$$

Finally, let $A := \mathbf{D}_m\mathbf{D}_p$ be the restriction of $D_m D_p$ to the set $D(A)$.

According to U. Küchler [7, Proposition 1], A is a nonpositive definite self-adjoint operator in $L^2(E,\mu)$. It follows from the definitions of p as well as μ and from (2.4), (2.5) that $\mu(E) < \infty$, i.e., $\mathbf{1} \in L^2(E,\mu)$. By the definition of $D(A)$, the point $\lambda = 0$ belongs to the spectrum of A and we have $\operatorname{Ker} A = \{c \cdot \mathbf{1} : c \in \mathbb{R}\}$. In virtue of [7, Theorem 1], $\gamma > 0$. Hence, we have (C).

We specify the choice of μ: Let μ be absolutely continuous with respect to Λ^1 on $[0,1/2) \cup (1/2,1)$ and let $\mu(\{1/2\}) = 1$. Furthermore, let M be the restriction of the one-dimensional Lebesgue measure Λ^1 to $[0,1)$. We observe that μ is not absolutely continuous with respect to M.

We have $\Gamma_s^M = \{1/2\}$ and $S = \{c \cdot \chi_{\{1/2\}} : c \in \mathbb{R}\}$ where $\chi_{\{1/2\}}(x) = 0$, $x \in E \setminus \{1/2\}$, and $\chi_{\{1/2\}}(1/2) = 1$. Moreover, since every $f \in D(A)$ has a continuous version and $\mathcal{C}^\mu \subseteq D(A)$, it holds that $S \cap \mathcal{C}^\mu = \{0\}$, i.e., condition (W) is satisfied.

2.2. Formulation of the results. We consider the bilinear form

$$\mathcal{E}(u,v) = \int (-A)^{1/2}u\,(-A)^{1/2}v\,d\mu\,, \quad u,\ v \in \mathcal{C} \cong \mathcal{C}^\mu$$

and ask for closability on $L^2(E,M)$ whenever (W) is satisfied. Recall that condition (W) implies that the form $(\mathcal{E},\mathcal{C})$ is well-defined on $L^2(E,M)$ whenever $\mathcal{C} \subseteq$

$L^2(E, M)$.
Before stating the results, let us introduce some conditions. Recall that $D(J)$ is dense in $L^2(E, \mu)$.

(a) If $\gamma = 0$ then there exist p with $2 \le p \le \infty$ and a continuous operator $J' \colon L^2(E, \mu) \longrightarrow L^p(E, \mu)$ satisfying $J' f = Jf,\ f \in D(J)$.
(b) If $\gamma = 0$ then we have $\operatorname{Ker} J' = \{0\}$.

If $\gamma > 0$ then we set $p = 2$. If we have $\gamma = 0$ and (a) is satisfied then simplify the notation as follows: Write Jf instead of $J' f,\ f \in L^2(E, \mu)$.

REMARK. (8) An example of the validity of (a) and (b) in case of $\gamma = 0$ is discussed in Subsection 3.2 below.

(α) $\mathcal{C} \subseteq L^2(E, M)$.
(β) $L^p(E, \mu) \subseteq L^2(E, M_a)$, continuously (in the sense of

$$\left(\int f^2 \, dM_a\right)^{1/2} \le c \left(\int |f|^p \, d\mu\right)^{1/p}$$

for some $c > 0$ and all $f \in L^p(E, \mu)$).

Finally, we formulate a condition on the M_a-negligible sets. We mention that this condition makes sense only in case that (a) is satisfied.

(CL) If $\gamma = 0$ then there is no $\psi \in L^2(E, \mu)$ with $\psi \ne 0$ and $J\psi = 0$ M_a-a.e. If $\gamma > 0$ then there are no $c \in \mathbb{R}$ and no $\psi \in L^2(E, \mu)$ with $\psi \notin \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$ and $J\psi = c \cdot \mathbf{1}$ M_a-a.e.

Below, condition (CL) will play the role of the *closability condition*.

REMARK. (9) Suppose the validity of condition (C). If condition (CL) is satisfied then we have (W). This can be verified as follows: Suppose that (W) does not hold. Then there exist $\varphi \in S \cap \mathcal{C}^\mu$ and $0 < a < b < \infty$ such that $\varphi = E_0\varphi + \int_{[a,b)} dE_\lambda \varphi \notin \{d \cdot \operatorname{sign}\gamma \cdot \mathbf{1} : d \in \mathbb{R}\}$, recall that M_a is nontrivial. For $\psi := \int_{[a,b)} \lambda^{1/2} dE_\lambda \varphi$, we have $\psi \notin \{d \cdot \operatorname{sign}\gamma \cdot \mathbf{1} : d \in \mathbb{R}\}$. The definition of S and $J\psi + E_0\varphi = \varphi (\in S)$ imply $J\psi + E_0\varphi = 0$ M_a-a.e. Hence, (CL) does not hold.

Subsequently, we state the results of this section. We start with two technical propositions which are the frame for two of the main assertions, namely Theorems 2.3 and 2.4.

Proposition 2.1. *Suppose* (C), (a), *and* (α). *If condition (CL) is satisfied then* $(\mathcal{E}, \mathcal{C})$ *is pre-closable on* $L^2(E, M)$. *If, in addition,* $\mathcal{C} \subseteq L^2(E, M)$, *densely, then* $(\mathcal{E}, \mathcal{C})$ *is closable on* $L^2(E, M)$.

Proposition 2.2. *Suppose* (C), (W), (a), (α), *and* (β). *If* $(\mathcal{E}, \mathcal{C})$ *is pre-closable on* $L^2(E, M)$ *then* (CL) *is satisfied.*

We turn to the main purpose of this section. By means of condition (CL), we describe a property of the M-negligible sets in order to have closability whenever M is absolutely continuous with respect to μ and $\tau = dM/d\mu \in L^1(E, \mu) \cap L^\infty(E, \mu)$. Immediate consequences of Propositions 2.1 and 2.2 are:

Theorem 2.3. *Suppose* (C) *and* (a). *Let* $M = \tau\mu$ *with* $\tau \in L^1(E, \mu) \cap L^\infty(E, \mu)$. *If condition* (CL) *is satisfied then* $(\mathcal{E}, \mathcal{C})$ *is closable on* $L^2(E, M)$. *Conversely, if condition* (W) *is satisfied and* $(\mathcal{E}, \mathcal{C})$ *is closable on* $L^2(E, M)$ *then we have* (CL).

Theorem 2.4. *Suppose* (C), (a), *and* (b). *Let* $M = \tau\mu$ *with* $\tau \in L^1(E, \mu) \cap L^\infty(E, \mu)$ *and* $\tau > 0$ μ-a.e. *Then we have* (CL); *hence,* $(\mathcal{E}, \mathcal{C})$ *is closable on* $L^2(E, M)$.

We are now interested in Dirichlet forms on $L^2(E, M)$. Let $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ be the closure of $(\mathcal{E}, \mathcal{C}^\mu)$ on $L^2(E, \mu)$. In virtue of Remark (5), we have $(\mathcal{E}^\mu, \mathcal{F}^\mu) = (\mathcal{E}, D(\mathcal{E}))$. If $(\mathcal{E}, \mathcal{C})$ is closable on $L^2(E, M)$ then let $(\mathcal{E}^M, \mathcal{F}^M)$ denote the closure of $(\mathcal{E}, \mathcal{C})$ on $L^2(E, M)$.

Theorem 2.5. *Suppose* (C) *and* (a). *Let* $M = \tau\mu$ *with* $\tau \in L^1(E, \mu) \cap L^\infty(E, \mu)$. *Suppose* (CL), *or* (b) *and* $\tau > 0$ μ-a.e. *If* $(\mathcal{E}^\mu, \mathcal{F}^\mu) = (\mathcal{E}, D(\mathcal{E}))$ *is a Dirichlet form on* $L^2(E, \mu)$ *then* $(\mathcal{E}^M, \mathcal{F}^M)$ *is a Dirichlet form on* $L^2(E, M)$.

2.3. Coincidence of closures. This subsection is devoted to a general observation. Among other things, we demonstrate that the closure constructed in Theorem 2.3 coincides, for example, with those of M. Fukushima, K. Sato, and S. Taniguchi [5] or I. Shigekawa and S. Taniguchi [16] provided that all conditions in the related criterions are satisfied.
To this end, let (E, μ), $(\mathcal{E}, D(\mathcal{E}))$, and $(A, D(A))$ be as above. Choose two sets of μ-classes $\mathcal{C}^{1,\mu}$ and $\mathcal{C}^{2,\mu}$ dense in $(D(\mathcal{E}), (\mathcal{E}_1)^{1/2})$. As in Subsection 2.2, introduce two sets of M-classes, $\mathcal{C}^1$ and $\mathcal{C}^2$, such that we can identify $\mathcal{C}^1 \cong \mathcal{C}^{1,\mu}$ and $\mathcal{C}^2 \cong \mathcal{C}^{2,\mu}$ whenever

(W′) $S \cap \mathcal{C}^{1,\mu} = S \cap \mathcal{C}^{2,\mu} = \{0\}$;

note that the definition of the set S in Subsection 2.2 is independent of the choice of $\mathcal{C}^\mu$ there. For the well-definiteness of $(\mathcal{E}, \mathcal{C}^1)$ and $(\mathcal{E}, \mathcal{C}^2)$ on $L^2(E, M)$ whenever $\mathcal{C} \subseteq L^2(E, M)$, let us suppose the validity of condition (W′) in this subsection.

Proposition 2.6. *Let* $M = \tau\mu$ *with* $\tau \in L^\infty(E, \mu)$.
(a) *If one of the forms* $(\mathcal{E}, \mathcal{C}^1)$, $(\mathcal{E}, \mathcal{C}^2)$ *is closable on* $L^2(E, M)$ *then so is the other*

one.
(b) *Suppose that one of the forms* $(\mathcal{E},\mathcal{C}^1)$, $(\mathcal{E},\mathcal{C}^2)$ *is closable on* $L^2(E,M)$. *Then the corresponding closures* $(\mathcal{E}^1,\mathcal{F}^1)$ *and* $(\mathcal{E}^2,\mathcal{F}^2)$ *coincide.*

REMARKS. (10) Part (b) of the previous proposition follows from part (a) when choosing $\mathcal{C}^2$ to be maximal, i.e., $\mathcal{C}^2 = (D(\mathcal{E})\setminus S)\cup\{0\}$ and showing that $\mathcal{F}^2\subseteq\mathcal{F}^1$. Note that $(D(\mathcal{E})\setminus S)\cup\{0\}$ is dense in $D(\mathcal{E})$ with respect to $(\mathcal{E}_1)^{1/2}$-norm.
(11) A closablity criterion as in [5, Theorem 4.1], [16, Theorem 8.4], or Theorem 2.3 of the present paper consists of (a set of) conditions

(S) on the space $(E,\mathcal{B},\mu)$,
(F) on the initial closed form $(\mathcal{E},D(\mathcal{E}))$ on $L^2(E,\mu)$,
(M) restricting the class of measures M on $(E,\mathcal{B},\mu)$,
(W) guaranteeing well-definiteness in $L^2(E,M)$ of a subset of μ-classes $\mathcal{C}\cong\mathcal{C}^\mu$ dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1^{1/2}$-norm,

and a sufficient or even necessary and sufficient condition

(CL) guaranteeing closability of $(\mathcal{E},\mathcal{C})$ on $L^2(E,M)$ whenever (S), (F), (M), (W) are satisfied.

Let $(\mathcal{E},D(\mathcal{E}))$ be a non-negative closed form on some $L^2(E,\mu)$ and $\mathcal{C}$ be a subset of μ-classes dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1^{1/2}$-norm, satisfying the conditions (S_1), (F_1), (M_1), (W_1) and (S_2), (F_2), (M_2), (W_2) of two such closablity criterions. Suppose that these conditions imply $M=\tau\mu$ with $\tau\in L^\infty(E,\mu)$. If the related closability conditions (CL_1) and (CL_2) are both necessary and sufficient ones then they are equivalent. If (CL_1) is a necessary and sufficient condition and (CL_2) is a sufficient condition then (CL_2) implies (CL_1).

2.4. Proofs.

Proof of Proposition 2.1. Let $u_n\in\mathcal{C}^\mu$, $n\in\mathbb{N}$, be a sequence with

$$u_n^M \xrightarrow[n\to\infty]{} 0 \text{ in } L^2(E,M) \tag{2.6}$$

and

$$(-A)^{1/2}u_n \xrightarrow[n\to\infty]{} f \text{ in } L^2(E,\mu) \tag{2.7}$$

for some $f\in L^2(E,\mu)$. Recall $\mathcal{C}\cong\mathcal{C}^\mu\subseteq D((-A)^{1/2})$ and that, by (α), $u_n^M\in L^2(E,M)$. We show that $f=0$ in $L^2(E,\mu)$.
For the sake of clarity, we treat both cases, $\gamma=0$ and $\gamma>0$, separately.
1° Let $\gamma=0$. It follows from (2.7) that

$$u_n \xrightarrow[n\to\infty]{} Jf \text{ in } L^p(E,\mu)\,,$$

recall condition (a). Selecting a subsequence if necessary, we may assume that $u_n \xrightarrow[n\to\infty]{} Jf$ μ-a.e. On the other hand, selecting again a subsequence if necessary, relation (2.6) implies that $u_n^M \xrightarrow[n\to\infty]{} 0$ M_a-a.e. From these relations, we obtain $Jf = 0$ M_a-a.e. Finally, condition (CL) implies $f = 0$ in $L^2(E,\mu)$.
2° Let $\gamma > 0$. In this case, it follows from (2.7) that

$$u_n - E_0 u_n = J\left((-A)^{1/2} u_n\right) \xrightarrow[n\to\infty]{} Jf \text{ in } L^2(E,\mu)\,, \tag{2.8}$$

cf. Remark (4). We observe that $E_0 u_n = c_n \cdot \mathbf{1}$ for some $c_n \in \mathbb{R}$, $n \in \mathbb{N}$. Furthermore, we note that, on account of (2.6) and (2.8), all accumulation points of the sequence c_n, $n \in \mathbb{N}$, are finite. Let $-c \in \mathbb{R}$ be an accumulation point of the sequence c_n, $n \in \mathbb{N}$. Selecting a subsequence if necessary, from (2.8) we get

$$u_n \xrightarrow[n\to\infty]{} Jf - c\cdot\mathbf{1} \text{ in } L^2(E,\mu)\,.$$

Selecting again a subsequence if necessary, we verify

$$u_n \xrightarrow[n\to\infty]{} Jf - c\cdot\mathbf{1}\ \mu\text{-a.e.} \tag{2.9}$$

As in part 1° of the proof relation (2.6) implies that $u_n \xrightarrow[n\to\infty]{} 0$ M_a-a.e. (selecting a subsequence if necessary). According to (2.9), we have $Jf = c\cdot\mathbf{1}$ M_a-a.e. Furthermore, (2.7) implies $f \notin \{d\cdot\mathbf{1} : d \in \mathbb{R}\backslash\{0\}\}$. Finally, condition (CL) yields $f = 0$ in $L^2(E,\mu)$. □

Proof of Proposition 2.2. We suppose that (CL) does not hold and construct a sequence $u_n \in \mathcal{C}^\mu$, $n \in \mathbb{N}$, such that $u_n^M \xrightarrow[n\to\infty]{} 0$ in $L^2(E,M)$ and $(-A)^{1/2}u_n \xrightarrow[n\to\infty]{} f$ in $L^2(E,\mu)$ for some nontrivial $f \in L^2(E,\mu)$. In particular, we suppose that there exist $c \in \mathbb{R}$ and a function $\psi \in L^2(E,\mu)$ with

$$\psi \notin \{d\cdot \operatorname{sign}\gamma\cdot\mathbf{1} : d \in \mathbb{R}\} \text{ and } J\psi = c\cdot\operatorname{sign}\gamma\cdot\mathbf{1}\ M_a\text{-a.e.} \tag{2.10}$$

We set

$$u_n := \int_{[a_n,b_n)} \lambda^{-1/2} dE_\lambda\psi - c\cdot\operatorname{sign}\gamma\cdot\mathbf{1} \quad (\in \mathcal{C}^\mu),\ n \in \mathbb{N}, \tag{2.11}$$

where $0 < a_n < b_n < \infty$ and $a_n \xrightarrow[n\to\infty]{} 0$, $b_n \xrightarrow[n\to\infty]{} \infty$. According to (2.11), we have

$$\begin{aligned} (-A)^{1/2}u_n &= \int_{[a_n,b_n)} dE_\lambda\psi \xrightarrow[n\to\infty]{} \int_{[0,\infty)} dE_\lambda\psi - E_0\psi \\ &= \psi - E_0\psi \neq 0 \quad \text{in } L^2(E,\mu), \end{aligned} \tag{2.12}$$

where $\psi - E_0\psi \neq 0$ is a consequence of convention (C) and the first part of (2.10). Moreover, in virtue of condition (a), we have

$$\begin{aligned} u_n &= J(-A)^{1/2}u_n - c \cdot \operatorname{sign}\gamma \cdot \mathbf{1} \\ &\in L^p(E, \mu). \end{aligned}$$

By (2.12) and $J(E_0\psi) = 0$, it holds that

$$u_n \xrightarrow[n\to\infty]{} J\psi - c \cdot \operatorname{sign}\gamma \cdot \mathbf{1} \quad \text{in } L^p(E, \mu). \tag{2.13}$$

Taking into consideration that we have assumed (α) and (β), by the second part of (2.10) and by (2.13) we verify $u_n^M \xrightarrow[n\to\infty]{} 0$ in $L^2(E, M_a)$. Therefore in the sense of the identification $\mathcal{C} \cong \mathcal{C}^\mu$ (cf. Subsection 2.1)

$$u_n^M \xrightarrow[n\to\infty]{} 0 \quad \text{in } L^2(E, M).$$

Together with (2.12) we conclude that $(\mathcal{E}, \mathcal{C})$ is not pre-closable whenever (CL) does not hold. □

Proof of Theorem 2.3. 1° According to $M = \tau\mu$ and $\tau \in L^\infty(E, \mu)$, we have $\mathcal{C} \cong \mathcal{C}^\mu \subseteq L^2(E, \mu) \subseteq L^2(E, M)$, i.e., we have ($\alpha$). We show that $\mathcal{C}$ is dense in $L^2(E, M)$.
Let $T_1 \subseteq T_2 \subseteq \cdots$ ($T_n \in \mathcal{B}(E)$, $n \in \mathbb{N}$) be an increasing sequence with $E = \bigcup_{n=1}^\infty T_n$ and $\mu(T_n) < \infty$, $n \in \mathbb{N}$. Furthermore, let $\chi_{T_n}(x) = 1$, $x \in T_n$, and $\chi_{T_n}(x) = 0$, $x \in E \setminus T_n$, $n \in \mathbb{N}$.
Now, fix $\tilde{\Phi} \in L^2(E, M)$ and a function $\Phi \in \tilde{\Phi}$. We observe that $\varphi_n := ((\Phi \cdot \chi_{T_n} \wedge n) \vee (-n)) \in L^2(E, \mu) \cap L^2(E, M)$, $n \in \mathbb{N}$, and that $\varphi_n \xrightarrow[n\to\infty]{} \Phi$ in $L^2(E, M)$. (Simultaneously, we regard φ_n, $n \in \mathbb{N}$, as a μ- and as an M-class.) Let $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|\varphi_{n_0} - \Phi\|_{L^2(E,M)} < \varepsilon/2$. Furthermore, fix $\psi \in \mathcal{C}^\mu$ with $\|\psi - \varphi_{n_0}\|_{L^2(E,\mu)} < \varepsilon/2\|\tau\|^{-1/2}_{L^\infty(E,\mu)}$. Furthermore, keep in mind that $\mathcal{C}^\mu$ is dense in $L^2(E, \mu)$, cf. Remark (3). Then

$$\|\psi^M - \varphi_{n_0}\|_{L^2(E,M)} \leq \|\tau\|^{1/2}_{L^\infty(E,\mu)} \|\psi - \varphi_{n_0}\|_{L^2(E,\mu)} < \frac{\varepsilon}{2}.$$

Hence, $\|\psi^M - \Phi\|_{L^2(E,M)} < \varepsilon$. Therefore, $\mathcal{C}$ is dense in $L^2(E, M)$.
2° In virtue of $M = \tau\mu$ and $\tau \in L^\infty(E, \mu)$, we have

$$L^p(E, \mu) \subseteq L^p(E, M), \quad \text{continuously}. \tag{2.14}$$

Furthermore, by $2 \leq p \leq \infty$ and $M(E) < \infty$ it holds that $L^p(E, M) \subseteq L^2(E, M)$, continuously. With (2.14), we obtain (β). Now, Theorem 2.3 follows from Propositions 2.1 and 2.2. □

Proof of Theorem 2.4. 1° Let $\psi \in L^2(E,\mu)$ with $J\psi = c \cdot \operatorname{sign}\gamma \cdot \mathbf{1}$ M-a.e. for some $c \in \mathbb{R}$. On account of $\tau > 0$ μ-a.e., we have $J\psi = c \cdot \operatorname{sign}\gamma \cdot \mathbf{1}$ μ-a.e. By the definition of J, it holds that $c = 0$ if $\gamma > 0$. Hence, we have $J\psi = 0$ μ-a.e. in both cases $\gamma = 0$ and $\gamma > 0$.
2° Let $\gamma > 0$. Then (C) and $J\psi = 0$ μ-a.e. yield $\psi \in \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$. Together with 1° we observe that $J\psi = c \cdot \mathbf{1}$ M-a.e. for some $c \in \mathbb{R}$ implies $\psi \in \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$. Thus, we have (CL).
3° Let $\gamma = 0$. Then $\operatorname{Ker} J = \{0\}$ (see (b)) and relation $J\psi = 0$ μ-a.e. imply $\psi = 0$. In virtue of 1°, $J\psi = 0$ M-a.e. implies $\psi = 0$, i.e., we have (CL).
4° The assertion follows from 2°, 3°, and Theorem 2.3. □

Proof of Theorem 2.5. 1° Let $u \in \mathcal{F}^\mu$. By definition, $\mathcal{C}^\mu$ is dense in $\mathcal{F}^\mu$ with respect to $(\mathcal{E}_1^\mu)^{1/2}$-norm. Hence, we can choose a sequence $u_n \in \mathcal{C}^\mu$, $n \in \mathbb{N}$, with $u_n \xrightarrow[n\to\infty]{} u$ in $(\mathcal{E}_1^\mu)^{1/2}$-norm. In particular,

$$u_n \xrightarrow[n\to\infty]{} u \quad \text{in } L^2(E,\mu) \tag{2.15}$$

and

$$\mathcal{E}^\mu(u_m - u_n, u_m - u_n) \xrightarrow[m,n\to\infty]{} 0\,. \tag{2.16}$$

According to $M = \tau\mu$, $\tau \in L^\infty(E,\mu)$, and (2.15), we have

$$u_n^M \xrightarrow[n\to\infty]{} u^M \quad \text{in } L^2(E,M). \tag{2.17}$$

It follows from (2.16) that

$$\mathcal{E}^M\left(u_m^M - u_n^M, u_m^M - u_n^M\right) \xrightarrow[m,n\to\infty]{} 0\,. \tag{2.18}$$

Now, the closedness of $(\mathcal{E}^M, \mathcal{F}^M)$ on $L^2(E,M)$, $u_n \in \mathcal{C}^\mu \cong \mathcal{C} \subseteq \mathcal{F}^M$, $n \in \mathbb{N}$, and relations (2.17) as well as (2.18) yield $u_n^M \xrightarrow[n\to\infty]{} u^M$ in $(\mathcal{E}_1^M)^{1/2}$-norm and $u^M \in \mathcal{F}^M$. Finally,

$$\mathcal{E}^\mu(u,u) = \lim_{n\to\infty} \mathcal{E}^\mu(u_n,u_n) = \lim_{n\to\infty} \mathcal{E}^M(u_n^M, u_n^M) = \mathcal{E}^M(u^M,u^M)\,.$$

2° As $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is a Dirichlet form, $u \in \mathcal{C}^\mu \cong \mathcal{C}$ implies $u^+ \wedge 1 \in \mathcal{F}^\mu$. By the results of step 1°, we can establish $(u^M)^+ \wedge 1 = (u^+ \wedge 1)^M \in \mathcal{F}^M$ as well as

$$\mathcal{E}^M((u^M)^+ \wedge 1, (u^M)^+ \wedge 1) = \mathcal{E}^\mu(u^+ \wedge 1, u^+ \wedge 1) \le \mathcal{E}^\mu(u,u) = \mathcal{E}^M(u^M,u^M)\,, \quad u \in \mathcal{C}^\mu.$$

Since $(\mathcal{E}^M, \mathcal{F}^M)$ is the closure of $(\mathcal{E}, \mathcal{C})$ on $L^2(E,M)$, the set $\mathcal{C}$ is dense in $\mathcal{F}^M$ with respect to $(\mathcal{E}_1^M)^{1/2}$-norm. Now, the assertion above is a consequence of [10], Chapter 1, Proposition 4.10. □

Proof of Proposition 2.6. First, let us mention that under $M = \tau\mu$ with $\tau \in L^\infty(E,\mu)$, we have $\mathcal{C}^1$, $\mathcal{C}^2 \subseteq L^2(E,\mu) \subseteq L^2(E,M)$. Without loss of generality, we suppose that $\|\tau\|_{L^\infty(E,\mu)} = 1$.
Let us suppose that $(\mathcal{E},\mathcal{C}^1)$ is closable on $L^2(E,M)$. It is sufficient to show that, for $\mathcal{C}^2 = (D(\mathcal{E})\setminus S)\cup\{0\}$, the form $(\mathcal{E},\mathcal{C}^2)$ is also closable on $L^2(E,M)$ and that $\mathcal{F}^2 \subseteq \mathcal{F}^1$. For this, let $g \in L^2(E,M)$ and $u_n^{(2)} \in \mathcal{C}^2$, $n \in \mathbb{N}$, be a sequence with

$$u_n^{(2)} \xrightarrow[n\to\infty]{} g \quad \text{in } L^2(E,M) \tag{2.19}$$

and $\mathcal{E}(u_n^{(2)} - u_m^{(2)}, u_n^{(2)} - u_m^{(2)}) \xrightarrow[m,n\to\infty]{} 0$ which implies

$$(-A)^{1/2}u_n^{(2)} \xrightarrow[n\to\infty]{} f \quad \text{in } L^2(E,\mu) \tag{2.20}$$

for some $f \in L^2(E,\mu)$. As $u_n^{(2)} \in D(\mathcal{E})$, $n \in \mathbb{N}$, and $\mathcal{C}^1$ is dense in $D(\mathcal{E})$ with respect to $(\mathcal{E}_1)^{1/2}$-norm, there is a sequence $u_n^{(1)} \in \mathcal{C}^1$, $n \in \mathbb{N}$, with

$$\mathcal{E}(u_n^{(1)} - u_n^{(2)}, u_n^{(1)} - u_n^{(2)}) + \|u_n^{(1)} - u_n^{(2)}\|^2_{L^2(E,\mu)} = \mathcal{E}_1(u_n^{(1)} - u_n^{(2)}, u_n^{(1)} - u_n^{(2)}) < \frac{1}{n}\,.$$

Next, $M = \tau\mu$ and $\|\tau\|_{L^\infty(E,\mu)} = 1$ imply

$$\mathcal{E}(u_n^{(1)} - u_n^{(2)}, u_n^{(1)} - u_n^{(2)}) + \|u_n^{(1)} - u_n^{(2)}\|^2_{L^2(E,M)} < \frac{1}{n}\,, \quad n \in \mathbb{N}. \tag{2.21}$$

From (2.19), (2.20), and (2.21), we may conclude that

$$u_n^{(1)} \xrightarrow[n\to\infty]{} g \quad \text{in } L^2(E,M) \tag{2.22}$$

and

$$(-A)^{1/2}u_n^{(1)} \xrightarrow[n\to\infty]{} f \quad \text{in } L^2(E,\mu)\,. \tag{2.23}$$

Taking into consideration that $(\mathcal{E},\mathcal{C}^1)$ is closable on $L^2(E,M)$, for $g = 0$, the last two relations lead to $f = 0$. Now from (2.19) and (2.20), it follows that $(\mathcal{E},\mathcal{C}^2) = (\mathcal{E},(D(\mathcal{E})\setminus S)\cup\{0\})$ is closable on $L^2(E,M)$, as well. Furthermore, choosing $g \in \mathcal{F}^2$ arbitrarily, relations (2.22) and (2.23) show that the sequence $u_n^{(1)}$, $n \in \mathbb{N}$, converges in $(\mathcal{F}^1, \mathcal{E}_1^1)$ and that its limit is $g \in \mathcal{F}^1$, i.e., $\mathcal{F}^2 \subseteq \mathcal{F}^1$. $\square$

3. Applications

In this section, we give examples of infinite and finite dimensional forms which are closable on some $L^2(E,M)$. Here, closability will be shown by using Theorems 2.3 and 2.4. In particular, we consider diffusion type forms and their fractional powers corresponding to second quantization (Subsection 3.1) and classical and stable forms on $\mathbb{R}^d$ (Subsection 3.2).

The state space E of the forms discussed in 3.1 is, in general, a locally convex Hausdorff topological vector space which is, moreover, Souslinean. This example is, therefore, not covered by the results in [5], [16]. In the example in Subsection 3.2, the measure μ is not absolutely continuous with respect to M. Consequently, it does not meet the conditions of Corollary 4.2 of [5] or Theorem 2.4 of the present paper. However, closability on $L^2(E, M)$ can be verified by means of Theorem 2.3.

Furthermore, we would like to draw attention to the fact that the forms $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$ introduced in Subsection 3.1 satisfy the spectral condition (SC), i.e., conditions (a) and (b) are trivial. In 3.2, we have $\operatorname{Ker} A = \{0\}$ and $\mu(E) = \infty$ which means that we have to verify conditions (a) and (b).

3.1. Diffusion type forms corresponding to second quantization. Let us summarize the background facts taken from [2] and [19]. Let E be a locally convex Hausdorff topological vector space. Suppose, furthermore, that E is Souslinean. Let E' denote its topological dual and let $\mathcal{B}(E)$ denote the Borel σ-algebra on E. In this example, let μ be a mean zero Gaussian measure on $(E, \mathcal{B}(E))$, i.e., each $l \in E'$ has a mean zero distribution in $\mathbb{R}$ under μ and assume $\operatorname{supp} \mu = E$. We introduce H_1 as the real Hilbert space obtained by completing E' with respect to the norm associated with the inner product

$$\langle k_1, k_2 \rangle_{H_1} := \int_E {}_{E'}\langle k_1, z \rangle_E \, {}_{E'}\langle k_2, z \rangle_E \, \mu(dz) , \quad k_1, \ k_2 \in E'.$$

For $h \in H_1$ and a sequence $k_n \in E'$, $n \in \mathbb{N}$ with $k_n \xrightarrow[n\to\infty]{} h$ in H_1, we introduce $X_h \in L^2(E, \mu)$ by

$$X_h := \lim_{n\to\infty} {}_{E'}\langle k_n, \cdot \rangle_E \quad \text{in } L^2(E, \mu) .$$

Let L be a self-adjoint operator on H_1 such that

$$-L \geq c \operatorname{Id}_{H_1} \tag{3.1}$$

for some $c > 0$, where Id_{H_1} denotes the identity on H_1. According to the chaos decomposition

$$L^2(E, \mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

with $\mathcal{H}_0 = \mathbb{R}$ and $\bigoplus_{j=0}^n \mathcal{H}_j$, $n \in \mathbb{N}$, being the closed linear span of $\{\mathbf{1}\} \cup \{\prod_{j=1}^m X_{h_j} : h_1, \ldots, h_n \in H_1, \ m \leq n\}$ in $L^2(E, \mu)$, we can define a family $(T_t)_{t\geq 0}$ of operators on

$L^2(E, \mu)$ by

$$T_t \mathbf{1} = \mathbf{1} \quad \text{and} \quad T_t \left(: \prod_{j=1}^{n} X_{h_j} :_n \right) = : \prod_{j=1}^{n} X_{e^{tL} h_j} :_n \tag{3.2}$$

$n \in \mathbb{N}$, $h_1, \ldots, h_n \in H_1$ and linearity; here, $: . :_n$ stands for orthogonal projection onto $\mathcal{H}_n$.

By virtue of [1, Remark 3.1], and [2, Subsection 7.1], $(T_t)_{t \geq 0}$ forms a symmetric non-negative strongly continuous contraction semigroup on $L^2(E, \mu)$. Let A denote its generator and let $(\mathcal{E}, D(\mathcal{E}))$ be the corresponding p.s.b.f. This form can be represented by

$$\mathcal{E}(u, v) = \int \sqrt{-A}\, u \cdot \sqrt{-A}\, v \, d\mu\,, \quad u,\ v \in D(\mathcal{E}) := D(\sqrt{-A}\,)\,.$$

In addition, introduce the set

$$\begin{aligned} \mathcal{F}C_b^\infty(E) := \{u \colon E \to \mathbb{R} : u(z) = f(l_1(z), \ldots, l_m(z)), \\ z \in E,\ l_1, \ldots, l_m \in E',\ f \in C_b^\infty(\mathbb{R}^m),\ m \in \mathbb{N}\}\,, \end{aligned}$$

and the space $H := D(\sqrt{-L}\,)$ with the inner product $\langle h_1, h_2 \rangle_H := \langle \sqrt{-L}\, h_1, \sqrt{-L}\, h_2 \rangle_{H_1}$. Suppose $H \subseteq E$ densely and continuously. Now, we can restate a theorem by S. Albeverio and M. Röckner:

Theorem 3.1 ([2, Theorem 7.4]). *The form $(\mathcal{E}, D(\mathcal{E}))$ is the closure of*

$$\mathcal{E}(u, v) := \int \langle \nabla u, \nabla v \rangle_H \, d\mu\,, \quad u,\ v \in \mathcal{F}C_b^\infty(E),$$

where $\nabla u(z)$ is the unique element in H representing the continuous linear map $h \to (\partial u / \partial h)(z)$, $h \in H$. In particular, $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form and $(T_t)_{t \geq 0}$ is sub-Markov.

REMARK. (1) The term "diffusion type" in the headline of this Subsection corresponds to the choice of H.

Let us turn to the verification of the spectral condition (SC).

Proposition 3.2. *We have* (SC).

Proof. The main step is done by T. Zhang [19], Relation (2.11): With the convention $\langle u \rangle := \int u \, d\mu$, we have

$$\int (-Au) u \, d\mu \geq c \int (u - \langle u \rangle)^2 \, d\mu\,, \quad u \in D(A), \tag{3.3}$$

where c is the constant appearing in (3.1). Furthermore, by (3.2), (3.1), $\lambda = 0$ is an eigenvalue of $-A$ and $\operatorname{Ker} A = \{c \cdot \mathbf{1} : c \in \mathbb{R}\}$. Now, (SC) follows from (3.3). $\square$

Subsequently, we apply Theorems 2.3, 2.4, and 2.5 to a bilinear form associated with the nonpositive definite self-adjoint operator $-(-A)^\alpha$ in $L^2(E, \mu)$, $0 < \alpha \le 1$.

REMARKS. (2) Let us consider the sets $\mathcal{C}^\mu$ and $\mathcal{C}$ constucted on the basis of the resolution of the identity with respect to $-(-A)^\alpha$. It follows from the definitions of these sets and from Proposition 1.2, that $\mathcal{C}^\mu$ and $\mathcal{C}$ are independent of $0 < \alpha \le 1$. Thus, condition (W) is independent of $0 < \alpha \le 1$.
(3) Furthermore, we mention that under (SC), condition (CL) is also independent of $0 < \alpha \le 1$. To show this, we denote the resolution of the identity with respect to $-(-A^\alpha)$ by $\left(E_\lambda^{(\alpha)}\right)_{\lambda \ge 0}$, $0 < \alpha \le 1$. Moreover, in order to indicate that the operator J constucted on the basis of $\left(E_\lambda^{(\alpha)}\right)_{\lambda \ge 0}$ depends on $0 < \alpha \le 1$, we write J^α instead of J.

In fact, if there are $c \in \mathbb{R}$ and a function $\psi^{(1)} \in L^2(E, \mu)$ with $\psi^{(1)} \notin \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$ and $J\psi^{(1)} = c \cdot \mathbf{1}$ M_a-a.e. then $\psi^{(\alpha)} := \int_{[\gamma,\infty)} \lambda^{-(1-\alpha)/2}\, dE_\lambda^{(1)} \psi^{(1)}$ satisfies $\psi^{(\alpha)} \in L^2(E, \mu)$, $\psi^{(\alpha)} \notin \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$, and $J^\alpha \psi^{(\alpha)} = c \cdot \mathbf{1}$ M_a-a.e.; cf. Proposition 1.2. On the other hand, if there are $c \in \mathbb{R}$ and a function $\psi^{(\alpha)} \in L^2(E, \mu)$ with $\psi^{(\alpha)} \notin \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$ and $J^\alpha \psi^{(\alpha)} = c \cdot \mathbf{1}$ M_a-a.e. then $\psi^{(1)} := \int_{[\gamma,\infty)} \lambda^{(1-\alpha)/2}\, dE_\lambda^{(1)} \psi^{(\alpha)}$ satisfies $\psi^{(1)} \in L^2(E, \mu)$, $\psi^{(1)} \notin \{d \cdot \mathbf{1} : d \in \mathbb{R}\}$, and $J\psi^{(1)} = c \cdot \mathbf{1}$ M_a-a.e., $0 < \alpha \le 1$.

Defining

$$\mathcal{E}^\alpha(u, v) = \left((-A)^{\alpha/2} u, (-A)^{\alpha/2} v\right)_{L^2(E,\mu)}, \quad u,\ v \in \mathcal{C},\ 0 < \alpha \le 1,$$

the following two assertions are now a direct consequence of Theorem 2.3 and Theorem 2.4.

Theorem 3.3. *Let $M = \tau\mu$ with $\tau \in L^\infty(E, \mu)$ and $0 < \alpha \le 1$. If condition* (CL) *is satisfied then $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(E, M)$. Conversely, if condition* (W) *is satisfied and $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(E, M)$ then we have* (CL).

Theorem 3.4. *Let $M = \tau\mu$ with $\tau \in L^\infty(E, \mu)$ and $\tau > 0$ μ-a.e. Then, for $0 < \alpha \le 1$, $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(E, M)$.*

As the p.s.b.f. $(\mathcal{E}, D(\mathcal{E}))$ associated with the operator A is a Dirichlet form on $L^2(E, \mu)$, Proposition 1.3 and Theorem 2.5 imply:

Theorem 3.5. *Let $M = \tau\mu$ with $\tau \in L^\infty(E, \mu)$ and $0 < \alpha \le 1$. Suppose* (CL) *or $\tau > 0$ μ-a.e. Then the closure $(\mathcal{E}^{\alpha,M}, \mathcal{F}^{\alpha,M})$ of $(\mathcal{E}^\alpha, \mathcal{C})$ on $L^2(E, M)$ is a Dirichlet* *form.*

3.2. Discussion of the finite dimensional case. Let $d \in \mathbb{N}$ and let $A \equiv \Delta$ be the Laplacian on $\mathbb{R}^d$ with domain

$$D(A) \equiv D(\Delta) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} f \in L^2(\mathbb{R}^d, \Lambda^d) \text{ in the sense of distributions} \right\},$$

see (1.5). Here, $A \equiv \Delta$ is a nonpositive definite self-adjoint operator, where 0 is an accumulation point of the spectrum. Thus, condition (SC) is not satisfied. Consequently, we have $\gamma = 0$. Furthermore, we observe that $\operatorname{Ker} A = \{0\}$. Therefore, with $\mu := \Lambda^d$, condition (C) is satisfied. As a consequence, condition (C) is also satisfied for the operator $(-A)^\alpha \equiv (-\Delta)^\alpha$ defined on

$$D((-A)^\alpha) = \left\{ f \in L^2(\mathbb{R}^d, \Lambda^d) : \int_0^\infty \lambda^{2\alpha}\, d\|E_\lambda^{(1)} f\|^2_{L^2(\mathbb{R}^d, \Lambda^d)} < \infty \right\},$$

where $0 < \alpha \leq 1$ and $\left(E_\lambda^{(1)}\right)_{\lambda \geq 0}$ is the resolution of the identity with respect to $-A \equiv -\Delta$, cf. Section 1. For alternative representations of $(-A)^\alpha \equiv (-\Delta)^\alpha$, we refer to Theorem 1.9. Corresponding to the exponent $\alpha \in (0, 1]$, we denote the form $(\mathcal{E}, \mathcal{C})$ appearing in Theorems 2.3–2.5 by $(\mathcal{E}^\alpha, \mathcal{C})$ and the operators J and J' in conditions (a), (b) by J^α and J'^α, respectively. Note that $\mathcal{C}$ is independent of $0 < \alpha \leq 1$, recall the definitions of the sets $\mathcal{C}^\mu$ as well as $\mathcal{C}$ and Proposition 1.2.

In order to apply Theorems 2.3–2.5 to $(\mathcal{E}^\alpha, \mathcal{C})$, we have to check conditions (a) and (b) which are not trivial in this situation.

Proposition 3.6. *Let $0 < \alpha \leq 1$ and $\alpha < d/2$. Then, with $J'^\alpha := I^\alpha$, conditions* (C), (a), *and* (b) *are satisfied. Furthermore, the p.s.b.f. associated with $(-A)^\alpha \equiv (-\Delta)^{\alpha/2}$ is a Dirichlet form.*

Proof. 1° We consider the Riesz α-potential,

$$I^\alpha \psi = \frac{1}{\gamma_d(\alpha)} \int \frac{\psi(y)\, dy}{|\cdot - y|^{d-\alpha}}, \tag{3.4}$$

where $\gamma_d(\alpha) = 2^\alpha \pi^{d/2} \Gamma(\alpha/2)/\Gamma((d-\alpha)/2)$ and Γ denotes the Gamma-function. Furthermore, we recall that $I^\alpha \psi$ is a continuous mapping $L^2(\mathbb{R}^d, \Lambda^d) \longrightarrow L^p(\mathbb{R}^d, \Lambda^d)$ whenever $\alpha < d/2$ and $p = 2d/(d-2\alpha)$, see, for example, [14], Theorem 25.2. In this case, we have $J^\alpha f = I^\alpha f$ for all $f \in D(J^\alpha)$, cf. Proposition 1.6. Hence, with $J'^\alpha := I^\alpha$, condition (a) is satisfied. As a consequence of Proposition 1.7, we have $\operatorname{Ker} I^\alpha = \{0\}$. This implies condition (b).

2° The second assertion of the proposition is a consequence of Section 1, especially of Proposition 1.3. □

The closability result of Theorem 2.4 is certainly not surprising since, in finite dimension, we have the fundamental probabilistic solution to a similar closability problem by M. Fukushima, K. Sato, and S. Taniguchi [5]. However, Theorem 2.3 and Remark (9) of Section 2 assert that in case of $M = \tau\Lambda^d$ with $\tau \in L^1(\mathbb{R}^d, \Lambda^d) \cap L^\infty(\mathbb{R}^d, \Lambda^d)$, the form $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(\mathbb{R}^d, M)$ and condition (W) is satisfied if and only if there is no $\psi \in L^2(\mathbb{R}^d, \Lambda^d)$, $\psi \neq 0$ such that $I^\alpha\psi = 0$ M-a.e. In this sense, the explicit representation (3.4) of I^α provides an analytical characterization of the M-negligible sets in order to state closability of $(\mathcal{E}^\alpha, \mathcal{C})$ on $L^2(\mathbb{R}^d, M)$.

In order to demonstrate that by means of Theorem 2.3, we are really able to treat more complicated situations than by means of Theorem 2.4, we concentrate on the case $d = 1$. We show that there are sets $L \in \mathcal{B}(\mathbb{R})$ with $\Lambda^1(L) > 0$ and measures $M = \tau\Lambda^1$ with

$\tau \in L^1(E, \mu) \cap L^\infty(E, \mu)$,

$\tau > 0$ Λ^1-a.e. on $\mathbb{R} \setminus L$, and

$\tau = 0$ Λ^1-a.e. on L

such that $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(\mathbb{R}, M)$. We introduce such a set $L \subseteq [0, 1]$ with $\Lambda^1(L) = 1/2$ which is, moreover, closed and has no inner point with respect to the usual topology in $\mathbb{R}$. Choose $\beta \in (0, 1/2)$ and let $b := 2^{-\beta}/(1 - 2^{-\beta})$. Let $L_1 \equiv L_1^1 := [0, 1]$. We proceed by iteration. For $n \in \mathbb{N}$, construct

$$L_{n+1} = \bigcup_{k=1}^{2^n} L_{n+1}^k ,$$

where $L_{n+1}^1, \dots, L_{n+1}^{2^n}$ are disjoint closed intervals of equal length, as follows: Split L_n^k into three intervals

$$L_n^k = L_{n+1}^{2k-1} \cup (a_{n+1}^k, b_{n+1}^k) \cup L_{n+1}^{2k}$$

such that the right-hand side boundary point of L_{n+1}^{2k-1} is a_{n+1}^k, the left-hand side boundary point of L_{n+1}^{2k} is b_{n+1}^k, and

$$b_{n+1}^k - a_{n+1}^k = b^{-1}2^{-n(1+\beta)}, \quad k \in \{1, \dots, 2^{n-1}\}. \tag{3.5}$$

Set

$$L := \bigcap_{n\in\mathbb{N}} L_n .$$

The Lebesgue measure of $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}}(a_{n+1}^k, b_{n+1}^k)$ is

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{n-1} b^{-1} 2^{-n(1+\beta)} &= b^{-1} 2^{-(1+\beta)} \sum_{n=0}^{\infty} 2^{-n\beta} \\ &= \frac{1}{2}\,.\end{aligned}$$

Therefore,

$$\Lambda^1(L) = \Lambda^1\left([0,1] \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}}(a_{n+1}^k, b_{n+1}^k)\right) = \frac{1}{2}\,.$$

Lemma 3.7. *Let $g \in L^2(\mathbb{R}, \Lambda^1)$ with $g \neq 0$ in $L^2(\mathbb{R}, \Lambda^1)$ and $g = 0$ Λ^1-a.e. on $\mathbb{R}\setminus L$. Furthermore, suppose $g \leq 1$ Λ^1-a.e. on $\mathbb{R}$ and $g = 1$ Λ^1-a.e. on some $G \in \mathcal{B}(\mathbb{R})$ with $G \subseteq L$ and $\Lambda^1(G) > 0$. If $0 < \beta \leq \alpha < 1/2$ then $g \notin D\left((-\Delta)^{\alpha/2}\right)$.*

Proof. 1° Let $H := G \setminus \left\{a_{n+1}^k, b_{n+1}^k : n \in \mathbb{N},\ k \in \{1, \dots, 2^{n-1}\}\right\}$. We observe that, for all $x \in H$ with $g(x) = 1$, the function $\varphi_x : (0, \infty) \longrightarrow \mathbb{R}$ given by

$$\varphi_x(\varepsilon) := \int_{|y|>\varepsilon} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\, dy\,, \quad \varepsilon > 0,$$

is decreasing in $\varepsilon \in (0, \infty)$. Thus, Theorem 1.9 (b) and $g = 1$ Λ^1-a.e. on H as well as $\Lambda^1(H) > 0$ imply that it is sufficient to show that, for all $x \in H$ with $g(x) = 1$, there is a sequence $\varepsilon_n \equiv \varepsilon_n(x) > 0$, $n \in \mathbb{N}$, with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and

$$\lim_{n\to\infty} \int_{|y|>\varepsilon_n} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\, dy = \infty\,. \tag{3.6}$$

2° For $x \in L$, $n \in \mathbb{N}$, and $k \in \{1, \dots, 2^{n-1}\}$, define

$$c_{n+1}^k \equiv c_{n+1}^k(x) := \begin{cases} a_{n+1}^k & \text{if } x \leq a_{n+1}^k \\ b_{n+1}^k & \text{otherwise} \end{cases}$$

and

$$d_{n+1}^k \equiv d_{n+1}^k(x) := \begin{cases} b_{n+1}^k & \text{if } x \leq a_{n+1}^k \\ a_{n+1}^k & \text{otherwise} \end{cases}\,.$$

Keeping in mind the construction of L, for all $x \in L$, we have

$$\min_{k\in\{1,\dots,2^{n-1}\}} \left|x - c_{n+1}^k\right| < \frac{1}{2^n}\,, \quad n \in \mathbb{N}.$$

Let $l \equiv l(n)$ be a number such that $\left|x - c_{n+1}^{l}\right| = \min_{k\in\{1,\dots,2^{n-1}\}} \left|x - c_{n+1}^{k}\right|$, $n \in \mathbb{N}$. Then it holds that

$$\left|x - c_{n+1}^{l(n)}\right| < \frac{1}{2^n}, \quad n \in \mathbb{N}. \tag{3.7}$$

Finally, for $x \in L$, define

$$\varepsilon_n \equiv \varepsilon_n(x) = \min_{j\in\{1,\dots,n\}} \left|x - c_{j+1}^{l(j)}\right|, \quad n \in \mathbb{N}.$$

Obviously, $(\varepsilon_n)_{n\in\mathbb{N}} \equiv (\varepsilon_n(x))_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$.

3° We have $H \subseteq L$. By the properties of g, for all $x \in H$ with $g(x) = 1$, it holds that

$$\begin{aligned}\int_{|y|>\varepsilon_n} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\,dy &\geq \int_{\{y\in\mathbb{R}:|y|>\varepsilon_n,\ x-y\notin L\}} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\,dy \\ &= \int_{\{y\in\mathbb{R}:|y|>\varepsilon_n,\ x-y\notin L\}} \frac{1}{|y|^{1+\alpha}}\,dy\,.\end{aligned} \tag{3.8}$$

In virtue of the definition of $(\varepsilon_n)_{n\in\mathbb{N}} \equiv (\varepsilon_n(x))_{n\in\mathbb{N}}$ and relations (3.5), (3.7), for all $x \in H$ with $g(x) = 1$, we obtain from (3.8)

$$\begin{aligned}\int_{|y|>\varepsilon_n} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\,dy &\geq \sum_{j=1}^{n} \left(b_{j+1}^{l(j)} - a_{j+1}^{l(j)}\right)\left(d_{j+1}^{l(j)}\right)^{-(1+\alpha)} \\ &\geq \sum_{j=1}^{n} b^{-1}2^{-j(1+\beta)}\left(2^{-j} + b^{-1}2^{-j(1+\beta)}\right)^{-(1+\alpha)}.\end{aligned}$$

Let $j_0 \in \mathbb{N}$ be a number that guarantees $b^{-1}2^{-j\beta} < 1$, $j \geq j_0$. Then, for all $x \in H$ with $g(x) = 1$ and $n > j_0$, we get

$$\begin{aligned}\int_{|y|>\varepsilon_n} \frac{g(x) - g(x-y)}{|y|^{1+\alpha}}\,dy &\geq \sum_{j=j_0}^{n} b^{-1}2^{-j(1+\beta)}2^{(j-1)(1+\alpha)} \\ &= b^{-1}2^{-(1+\alpha)}\sum_{j=j_0}^{n} 2^{j(\alpha-\beta)}.\end{aligned}$$

Now, from $\alpha \geq \beta$, the validity of relation (3.6) can be concluded. □

Theorem 3.8. *Let $0 < \beta \leq \alpha < 1/2$. Furthermore, let $\tau \in L^1(\mathbb{R}, \Lambda^1) \cap L^\infty(\mathbb{R}, \Lambda^1)$. Suppose $\tau = 0$ Λ^1-a.e. on L and $\tau > 0$ Λ^1-a.e. on $\mathbb{R} \setminus L$. Define $M := \tau\Lambda^1$. Then $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(\mathbb{R}, M)$.*

Proof. 1° Assume that $(\mathcal{E}^\alpha, \mathcal{C})$ is not closable on $L^2(\mathbb{R}, M)$. Since conditions (C) and (a) are satisfied (by Theorem 3.5), under this assumption, condition (CL) is not satisfied (by Theorem 2.3). Thus, there exists $\psi \in L^2(\mathbb{R}, \Lambda^1)$ with $\psi \neq 0$ and

$$I^\alpha \psi = 0 \quad \Lambda^1\text{-a.e.} \quad \text{on} \quad \mathbb{R} \setminus L\,, \tag{3.9}$$

cf. Proposition 3.6. Recalling $\operatorname{Ker} I^\alpha = \{0\}$, without loss of generality, we may suppose that $I^\alpha \psi \geq 1$ on some $G \in \mathcal{B}(\mathbb{R})$ with $G \subseteq L$ and $\Lambda^1(G) > 0$. As mentioned in Subsection 1.2, for $p = 2/(1-2\alpha) > 2$, we have $I^\alpha \psi \in L^p(\mathbb{R}, \Lambda^1)$. Because of (3.9), this yields $I^\alpha \psi \in L^2(\mathbb{R}, \Lambda^1)$. Finally, from Propositions 1.7 and 1.8, we get $I^\alpha \psi \in D\left((-\Delta)^{\alpha/2}\right)$.

2° Proposition 1.3 implies $I^\alpha \psi \in D(\mathcal{E}^\alpha)$. Therefore,

$$g := I^\alpha \psi \wedge 1 \in D(\mathcal{E}^\alpha) = D\left((-\Delta)^{\alpha/2}\right)\,. \tag{3.10}$$

Since g satisfies the hypotheses of Lemma 3.7, under the assumption above, we get a contradiction. Thus, $(\mathcal{E}^\alpha, \mathcal{C})$ is closable on $L^2(\mathbb{R}, M)$. □

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