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Osaka J. Math.
40 (2003), 455–467

MANIFOLD WITH IDEAL BOUNDARIES OF DIFFERENT DIMENSIONS

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(Received October 26, 2001)

1. Introduction. Basic notions, definitions and formulation of the main result

Ideal boundaries of open manifolds describe their coarse geometry at infinity and prove to be very useful for the study of the geometry of a given manifold itself. In this note we construct on the Euclidean eight dimensional space $\mathbb{R}^8$ a Riemannian metric g of nonnegative Ricci curvature such that the open Riemannian manifold $(\mathbb{R}^8, g)$ has Gromov's ideal boundaries of different dimensions. Our construction is based on the methods developed in the paper of J. Cheeger and T.H. Colding [3], where are given examples of metrically non-equivalent tangent cones at infinity, but keeping their dimensions fixed. These examples are so-called double warped-products depending on some functions (called warping functions). See u and v below. The construction of our example consists of constructing these functions in such a way that on one hand the Ricci curvature of the obtained metric is positive (and we verify this using formulas for the Ricci curvature again from [3], see formulas (I) and (II) below), and on the other hand the claim of our main result holds.

Before presenting our main result (see Theorem A below) we recall the notions of an ideal boundary $M^n(\infty)$ of an open manifold M^n and its tangent cone at infinity $T_\infty M$.

Ideal boundaries. Everywhere below (M^n, g) denotes an open (complete non-compact and without boundary) connected n-dimensional Riemannian manifold with a Riemannian metric g. A pointed manifold is a manifold with one point fixed, we denote it by (M^n, p, g) and call the fixed point p a base point. A metric ball in the manifold (M^n, g) with a center at the point q and radius r is denoted by $B(q, r, g)$. Below we consider sequences of pointed manifolds (M_i^n, p_i, g_i). Correspondingly, $B(p_i, r_i, g_i)$ denotes the metric ball in (M_i^n, p_i, g_i) with a center at p_i and radius r_i. In the same way a metric space (X, d), where d denotes the distance function, is called pointed if some of its point a is fixed. We denote a pointed metric space by (X, a, d), or simply (X, a), and call the fixed points a a base point. In a Riemannian manifold (M, g) the

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1980 *Mathematics Subject Classification* : (1985 *Revision)* 53C20, 53C21.
supported by CNPq, UNICAMP-Brasil.

Riemannian metric g generates the distance function which we denote by $d(g)$, and we consider the Riemannian manifold also as a metric space $(M, d(g))$.

Gromov-Hausdorff distance. Given two metric spaces (in particular, two Riemannian manifolds (M_1, g_1) and (M_2, g_2) considered as metric spaces $(M_1, d(g_1))$ and $(M_2, d(g_2))$) Gromov in [6] introduced a way to measure their closeness: for two subsets A, $B \subset Z$ of a metric space (Z, d), the Hausdorff distance d_H between A and B is

$$d_H(A, B) := \inf\{\varepsilon \mid T_\varepsilon(A) \supset B \text{ and } T_\varepsilon(B) \supset A\},$$

where $T_\varepsilon(A)$ denotes the ε-neighborhood of A in Z. Now, for arbitrary metric spaces X, Y the Gromov-Hausdorff distance d_{GH} between X and Y is defined by

$$d_{GH}(X, Y) = \inf_Z \inf_{f,g} d_H(f(X), g(Y)),$$

where the infimum is taken over all possible Z and all possible isometric immersions $f : X \to Z$, $g : Y \to Z$. This distance defines a topology on the set of all metric spaces: the sequence of metric spaces X_n tends to a metric space X if $d_{GH}(X_n, X) \to 0$.

For pointed metric spaces the definition of a converging sequence is slightly different, see [5].

DEFINITION. We say that a sequence of pointed metric spaces $\{(X_n, a_n)\}$ converges to (X, a) in the sense of Gromov-Hausdorff if for every $r > 0$ the sequence of balls $B(a_n, r) \subset (X_n, a_n)$ converges to a ball $B(a, r) \subset (X, a)$ in the Gromov-Hausdorff topology.[1]

For manifolds and pointed manifolds we say that (M_i^n, p_i, g_i) converge in Gromov-Hausdorff topology to some metric space (X, a, d) if $(M_i^n, p_i, d(g_i))$ tends to this space.

The Gromov Precompactness Theorem (see [6]) claims that an arbitrary set of connected Riemannian n-manifolds of non-negative Ricci curvature is pre-compact in Gromov-Hausdorff topology if their diameters are uniformly bounded. For instance, any sequence of unit metric balls $B(q_i, 1, g_i)$ from different Riemannian manifolds (M_i^n, g_i) contains some subsequence $B(q_j, 1, g_j)$ converging to some metric space (B, g) in Gromov-Hausdorff topology: $(B, g) = \lim_{d_{GH}} B(q_j, 1, g_j)$ as $j \to \infty$ if all (M_i^n, g_i) are of non-negative Ricci curvature. Now, if in a given open manifold (M^n, g) of non-negative Ricci curvature we divide its metric tensor g by an arbitrary positive number r^2 the resulting metric $r^{-2}g$ will be also of non-negative Ricci curvature because the curvature of the metric $r^{-2}g$ equals the curvature of g multiplied by r^2.

[1]This does not imply that $d_{GH}(X_n, X) \to 0$.

Therefore, if $B(p, 1, g_i)$ denote the unit metric ball with the fixed center at the point p in a manifold M^n with a metric $g_i = r_i^{-2} g$ for an arbitrary sequence of positive numbers r_i, then the sequence $B(p, 1, g_i)$ is pre-compact in Gromov-Hausdorff topology and contains some subsequence $B(p, 1, g_j)$ converging to some metric space (B, d). For instance, if $r_i \to 0$ then the sequence $B(p, 1, g_i)$ converge to the unit ball in the tangent space $T_p M^n$ (which, of course, is isometric to the unit ball in the standard flat Euclidean space). The situation is much more interesting when $r_i \to \infty$. In this case the limit space might depend on the sequence $\{r_i\}$ and is called "the tangent cone at infinity". In [5] Gromov introduced this new concept which provides a nice tool for the study of the asymptotic behavior at infinity of open manifolds.

DEFINITION. Let (M^n, p, g) be a pointed open Riemannian n-manifold of nonnegative Ricci curvature $\mathrm{Ric}(M^n) \geq 0$. Its *Tangent Cone at Infinity*, or *Asymptotic Cone* $T_\infty(M^n, g)$ under a sequence of real numbers $r_i \to \infty$ is the Gromov-Hausdorff limit of the sequence of unit metric balls $B(p, 1, r_i^{-2} g)$ with a center p in pointed open manifolds $(M^n, p, r_i^{-2} g)$, if such a limit exists. The boundary of $T_\infty(M^n, g)$ is called an *Ideal Boundary* $M^n(\infty)$ of M^n. By d_∞ we denote the metric of $T_\infty(M^n, g)$.

By our definition

$$(T_\infty(M^n, g), d_\infty) = \lim_{d_{GH}} \big(B(p, 1, r_i^{-2} g), d(r_i^{-2} g)\big),$$

while the ideal boundary $M^n(\infty)$ is the Gromov-Hausdorff limit of unit spheres $S(p, 1, d_i)$ in (M^n, p, g_i) with a center p, where $g_i = r_i^{-2} g$ and $d_i = d(r_i^{-2} g)$.[2]

Non-uniqueness of $T_\infty(M, g)$. Main result. Gromov Precompactness Theorem implies the existence of a tangent cone at infinity $T_\infty(M, g)$ of a manifold M^n of non-negative Ricci curvature. If the sectional curvature $\mathrm{Sec}(M, g)$ of M^n is non-negative then, beyond the existence of $T_\infty(M, g)$, the uniqueness holds as well (see [2]), i.e., for arbitrary $r_i \to \infty$ our sequence $B(p, 1, g_i)$ of unit metric balls from the definition above converges to the same metric space. But generally $T_\infty(M, g)$ depends on the sequence $r_i \to \infty$. We denote this dependence by $T_\infty(M, g)\{r_i\}$, and the associated ideal boundary by $M^n(\infty)\{r_i\}$. The first example of an open manifold of nonnegative Ricci curvature having different cones at infinity is due to G. Perelman [10]. Until now in all such examples all cones at infinity had the same dimension.

[2]The reader may compare this definition with the well-known definitions of the ideal boundary of an Hadamard manifold V^n (complete simply connected manifold of non-positive curvature) usually called the sphere at infinity $V^n(\infty)$, see [2]. According to one of them $V^n(\infty)$ is the set of rays $l(t), 0 \leq t < \infty$ issuing from a given point p with a metric d_∞ (called Tits metric) defined as follows: $d_\infty(l_1, l_2) = \lim_{t\to\infty} d(l_1(t), l_2(t))/t$. Interesting to note that in [2] are given three equivalent definitions of the ideal boundary of an Hadamard manifold which in the case of open manifolds of non-negative sectional curvature may lead to different, non-homeomorphic ideal boundaries, see [8].

Here we construct in Euclidean space $\mathbb{R}^8$ a complete metric of nonnegative Ricci curvature having ideal boundaries and consequently tangent cones at infinity of different dimensions. Our main result is the following.

Theorem A. *There exists a metric g on the Euclidean 8-dimensional space $\mathbb{R}^8$ such that $(\mathbb{R}^8, g)$ is the manifold of nonnegative Ricci curvature having ideal boundaries of different dimensions. For some sequence $r_l \to \infty$ it holds $\mathbb{R}^8(\infty)\{r_l\} = S^3$, while for another $r_m \to \infty$ the ideal boundary $\mathbb{R}^8(\infty)\{r_m\}$ is a point.*

The metric g in Theorem A can be constructed of class $\mathcal{C}^k$ for arbitrary k, or even of the class $\mathcal{C}^\infty$. Our construction basically follows from [3] (essentially the Section §8 there). As in [3] our example is a so-called double warped-product depending on some functions u and v, called warping functions. Thus, the proving of our main result consists of finding u and v such that: first, the obtained metric has positive Ricci curvature (we verify this by using formulas I–II below from [3]), and secondly, we present two sequences r_l and r_m such that $\mathbb{R}^8(\infty)\{r_l\}$ is a three-dimensional sphere S^3 while $\mathbb{R}^8(\infty)\{r_m\}$ is a point.

The author would like to express his sincere gratitude to his supervisor Professor Valery Marenich for introducing me to this subject, for pointing out this problem, cooperation and patience in reading the numerous versions of this paper.[3] He is also deeply indebted to Professor D. Alekseevsky for his careful reading and valuable suggestions on this paper.

2. On relations between geometries of an open manifold and its tangent cone at infinity

Before proceeding with the construction of our example we would like to present some results showing that the knowledge of the geometric structure of tangent cones at infinity and ideal boundaries of a given open manifold allows us to inquire topological and geometric information about the manifold itself and better understanding of the structure of the class of manifolds of nonnegative Ricci curvature. For instance, in [9] the following was proved.

Theorem B (Marenich-Bessa). *Let (M^n, p, g) be a pointed, open, Riemannian n-manifold of non-negative Ricci curvature and quadratic curvature decay*

$$\mathrm{Sec}_{(M,g)}(q) \geq -k\,\mathrm{dist}^{-2}(p, q),$$

where $k > 0$ and $\mathrm{Sec}_{(M,g)}(q)$ denotes the sectional curvature of (M^n, g) at the point q. If $T_\infty(M, g)\{r_i\} = (\mathbb{R}^n, g_{\mathrm{can}})$ for some sequence $\{r_i\}$ then $(M, g) = (\mathbb{R}^n, g_{\mathrm{can}})$, where g_{can} is the flat metric on $\mathbb{R}^n$.

[3] which is a part of author's Ph.D. thesis

In [4] the same result was obtained without the assumption of a quadratic decay of the sectional curvature.

Denote by $\operatorname{diam}_M(p, r)$ the supremum over all the diameters of the boundary components of $M \backslash B(p, r)$.

Theorem C (Marenich-Bessa). *Let $(M^n g)$ be an open Riemannian n-manifold with non-negative Ricci curvature. If $T_\infty(M, g) = (\mathbb{R}^k, g_{\mathrm{can}})$, $k < n$, then either*
1. *$M = \overline{M} \times \mathbb{R}^k$ where $\overline{M}$ is compact, or*
2. *the no-line factor $\overline{M}$ of (M, g) has dimension $\dim(\overline{M}) \geq 3$ and is of the maximum diameter growth, i.e.,*

$$\lim_{r\to\infty} \frac{\operatorname{diam}_{\overline{M}}(p, r)}{r} = 2.$$

If the diameter growth is maximal and $T_\infty(\overline{M}, g_{\overline{M}})$ is a metric cone, then $T_\infty(\overline{M}, g_{\overline{M}}) = C(N) \times \mathbb{R}$, where $C(N)$ is a cone over compact length space N.

If we ask: is there any open manifold M^n having ideal boundaries of different dimensions, the first interesting fact to note is that in such example no ideal boundary could have the maximal possible dimension $(n - 1)$. If, for instance, in our case we have $n = 8$, and some M^8 has ideal boundaries of different dimensions, then neither of these dimensions equal 7. This is due to the following fact.

Theorem D (Marenich). *Let (M^n, g) be an open manifold of nonnegative Ricci curvature. If for some sequence $r_i \to \infty$ it holds $\dim T_\infty(M^n, g) = n$, then for any other sequence $r_j \to \infty$ it holds also $\dim T_\infty(M^n, g) = n$.*

In order to prove this, we use the following result from [3, Theorem 5.9] (adapted to our aim).

Lemma (Cheeger-Colding). *Let (Y, y) be the pointed Gromov-Hausdorff limit of a pointed sequence of Riemannian n-manifolds (M_i^n, p_i, g_i) of non-negative Ricci curvature. Denote by $B(p_i, r)$ the ball with radius r and center p_i in M_i. If for all i*

$$\operatorname{vol}(B(p_i, 1)) \geq v > 0,$$

then $\dim Y = n$, and for any $r > 0$

$$\lim_{i\to\infty} \operatorname{vol}(B(q_i, r)) = \mathcal{H}^n(B(y, r)),$$

where $B(y, r)$ is a ball in Y with a center y and radius r, and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

Proof of Theorem D. By definition $T_\infty(M,g)\{r_i\}$ is the Gromov-Hausdorff limit of unit metric balls $B(p,1,g_i)$ with fixed center p in open manifold M^n with rescaled metric $g_i = r_i^{-2}g$. Hence, for the volume of $B(p,1,g_i)$ we have $\mathrm{vol}(B(p,1,g_i)) = \mathrm{vol}(B(p,r_i,g))/r_i^n$. If the dimension of $T_\infty(M,g)\{r_i\}$ equals n then $\mathcal{H}^n(T_\infty(M,g)\{r_i\}) = \alpha > 0$, and according to the Lemma above we conclude

$$\frac{\mathrm{vol}(B(p,r_i,g))}{r_i^n} = \mathrm{vol}(B(p,1,g_i)) \to \mathcal{H}^n(T_\infty(M,g)\{r_i\}) = \alpha > 0$$

as $i \to \infty$. By the Bishop-Gromov Volume Comparison Theorem [6] the function $\mathrm{vol}(B(p,r,g))/r^n$ is monotonically decreasing in r. Therefore, for any other sequence $\{r_j\}$ it also holds that

$$\frac{\mathrm{vol}(B(p,r_j,g))}{r_j^n} \to \alpha > 0.$$

Due to $\mathrm{vol}(B(p,1,g_j)) = \mathrm{vol}(B(p,r_j,g))/r_j^n$ and Lemma above, it follows that $\mathcal{H}^n(T_\infty(M,g)\{r_j\}) = \alpha > 0$ (i.e., all cones at infinity have the same n-volume), or that the dimension of $T_\infty(M,g)\{r_j\}$ equals n.

Theorem D is proved. $\square$

Other interesting results on ideal boundaries, tangent cones at infinity and its relationship with the geometry of the manifold can be found in works of A. Kasue [7], W. Ballmann, M. Gromov and V. Schroeder [2], T. Shioya [11], etc.

3. Warp-products and their Ricci curvature

All manifolds which we are considering below are so-called double warp-products. We recall here their description taken from [3] as well as the formulas (I)–(II) for their Ricci curvature below.[4]

Let $\{\tilde{y}_i\}$ be a local orthonormal basis in a Riemannian manifold (M,g), and $g^M(r)$ the family of Riemannian metrics on M such that $\{\tilde{y}_i\}$ are orthogonal for all r from some interval $r \in I \subset R$. Then for some functions $u_i(r)$ the basis $\{y_i = \tilde{y}_i/u_i(r)\}$ is a local orthonormal basis in $(M, g^M(r))$. Let further the tangent bundle of M admits a decomposition $TM = E_1 \oplus \cdots \oplus E_d$, while the metric $g^M(r)$ can be written as $g^M(r) = u_1^2(r)k_1 + \cdots + u_d^2(r)k_d$ where $g^M(r)|_{E_i} = u_i^2(r)k_i$ and k_i annihilates E_j for $i \neq j$. Assume in addition that if a geodesic γ in (M,g) is tangent to a distribution E_i at some point $\gamma(0)$ then it stays tangent to it in all other points $\gamma(t)$. In [3] it was proved that under these conditions for the Ricci tensor of the metric $g = dr^2 + g^M(r)$

[4]Double warp-products are manifolds whose tangent bundle is represented as a sum of orthogonal distributions E_i. We vary their metric by multiplying it along E_i by some functions u_i. The well-known warp-products $(M_1,g_1)\times_u (M_2,g_2)$ are the particular case when these distributions are integrable, tangent to the factors M_1 and M_2.

on the product $I \times M$ the following is true.

$$
\text{(I)} \qquad \begin{cases} \operatorname{Ric}(\xi,\xi) = \displaystyle\sum_{i=1}^{n} -\frac{u_i''}{u_i}, \\ \operatorname{Ric}|_{(\{r\}\times M, g^M(r))} = \widetilde{\operatorname{Ric}}_{(M,g^M(r))} - \displaystyle\sum_i \left(\frac{u_i''}{u_i} + \frac{u_i'}{u_i} \sum_{i \neq j} \frac{u_j'}{u_j} \right) y_i^* \otimes y_i^*, \end{cases}
$$

where $\xi = \partial/\partial r$ is tangent to I in the product $I \times M$, $\{y_i^*\}$ denote the dual basis to $\{y_i\}$ and $\widetilde{\operatorname{Ric}}$ is the Ricci tensor of $(M, g^M(r))$. Particularly, consider a Riemannian submersion $X^l \hookrightarrow M^n \xrightarrow{\pi} W^{n-l}$, with totally geodesic fibres X^l; let g^M and g^W be the Riemannian metrics for M and W, respectively, and write $g^M = k_1 + k_2$, where $k_2 = \pi^*(g^W)$ and k_1 is the metric component responsible by the fibres (that is, if $TM = E_1 \oplus E_2$ with E_1 the vertical subbundle and E_2 the horizontal subbundle, then $g^M|_{E_1} = k_1|_{E_1}$, $k_1|_{E_2} \equiv 0$, and $g^M|_{E_2} = \pi^*(g^W)|_{E_2} = k_2|_{E_2}$, $k_2|_{E_1} \equiv 0$). A family of Riemannian metrics $g^M(r)$ on M is given by

$$
g^M(r) = u_1^2(r) k_1 + u_2^2(r) k_2.
$$

Then, the above equations (I) for the Ricci tensor take the form:

(II)

$$
\begin{cases} \operatorname{Ric}(\xi,\xi) = -l\dfrac{u_1''}{u_1} - (n-l)\dfrac{u_2''}{u_2}, \\ \operatorname{Ric}|_{(\{r\}\times M, g^M(r))} = \widetilde{\operatorname{Ric}}_{(M,g^M(r))} - \left(\dfrac{u_1''}{u_1} + (l-1)\left(\dfrac{u_1'}{u_1}\right)^2 + (n-l)\dfrac{u_1'}{u_1}\dfrac{u_2'}{u_2} \right) \displaystyle\sum_{i=1}^{l} y_i^* \otimes y_i^* \\ \qquad\qquad - \left(\dfrac{u_2''}{u_2} + (n-l-1)\left(\dfrac{u_2'}{u_2}\right)^2 + l\dfrac{u_1'}{u_1}\dfrac{u_2'}{u_2} \right) \displaystyle\sum_{i=l+1}^{n} y_i^* \otimes y_i^*, \end{cases}
$$

where $\{y_i, i = 1, \ldots, l\}$ are vertical local fields orthonormal with respect to $g^M(r)$, and $\{y_j, j = l+1, \ldots, n\}$ are orthonormal horizontal fields.

4. Construction of the Example

Now we proceed with the construction of our example in four steps:

1. Considering the space $\mathbb{R}^8$ as the union of a disc D centered at the origin and its complement N we first provide a smooth metric in N of nonnegative Ricci curvature.

2. We extend — of class $\mathcal{C}^k$ — the metric obtained in the first step from N to the disc D minus the origin (where the metric would stay singular), thus getting a metric of nonnegative Ricci curvature in $\mathbb{R}^8\backslash\{0\}$.

3. We show that it is possible to "desingularize" the obtained metric at the origin. This gives us a $\mathcal{C}^k$-metric on $\mathbb{R}^8$ of nonnegative Ricci curvature.

As the result of these three steps we will construct a family of smooth metrics depending on some parameters R, $\alpha \in (0,1)$, R_α, $R_\alpha(k)$ and c_0, c_1. We show that for all R sufficiently big (depending on all other parameters) and all c_0, c_1 sufficiently small our metric g on $\mathbb{R}^8$ has non-negative Ricci curvature.

4. Finally, we show that this metric has ideal boundaries of different dimensions.

STEP 1.

1.1. We consider $\mathbb{R}^8$ as the quotient space $[0,\infty)\times S^7/\{0\}\times S^7$ or as the product $(0,\infty)\times S^7$ with one point (origin) attached. For some $R > 0$ decompose $\mathbb{R}^8$ as the union of the disc $D = [0,R]\times S^7/\{0\}\times S^7$ of radius R and the space $N = [R,\infty)\times S^7$. We denote their common boundary, $D\cap N = \{R\}\times S^7$, by $S^7(R)$. We will endow these sets with doubly warped product metrics and $\mathcal{C}^k$-glue them afterwards.

Our metric in the space N is $\overline{g} = dr^2 + g^{S^7}(r)$, where $g^{S^7}(r)$ is of doubly warped product type, obtained from the Hopf fibration $(S^3, g^{S^3}) \to (S^7, g^{S^7}) \xrightarrow{\pi} (S^4, (1/4)g^{S^4})$. Here π is a Riemannian submersion with totally geodesic fibres, and g^{S^n} denotes the canonical metric of constant curvature 1. As in the previous section, we write $g^{S^7} = g_1 + g_2$ where $g_2 = \pi^*((1/4)g^{S^4})$ and g_1 is the metric component responsible by the fibres S^3. Define a new metric $g^{S^7}(r) = \overline{f}(r)^2 g_1 + \overline{h}(r)^2 g_2$ on S^7. Then, according to formulas I–II above N will have positive Ricci curvature if and only if the functions $\overline{f}$, $\overline{h}$ satisfy the following inequalities:

$$
\text{(III)} \qquad \begin{cases}
\operatorname{Ric}(\xi,\xi) = -3\dfrac{\overline{f}''}{\overline{f}} - 4\dfrac{\overline{h}''}{\overline{h}} > 0,\\[2ex]
\operatorname{Ric}\left(\dfrac{V}{\overline{f}}, \dfrac{V}{\overline{f}}\right) = \dfrac{2}{\overline{f}^2} - \dfrac{\overline{f}''}{\overline{f}} - 2\left(\dfrac{\overline{f}'}{\overline{f}}\right)^2 - 4\dfrac{\overline{f}'\overline{h}'}{\overline{f}\overline{h}} > 0,\\[2ex]
\operatorname{Ric}\left(\dfrac{W}{\overline{h}}, \dfrac{W}{\overline{h}}\right) = \dfrac{3}{\overline{h}^2} - \dfrac{\overline{h}''}{\overline{h}} - 3\left(\dfrac{\overline{h}'}{\overline{h}}\right)^2 - 3\dfrac{\overline{f}'\overline{h}'}{\overline{f}\overline{h}} > 0,\\[2ex]
\operatorname{Ric}\left(\xi, \dfrac{V}{\overline{f}}\right) = \operatorname{Ric}\left(\xi, \dfrac{W}{\overline{h}}\right) = \operatorname{Ric}\left(\dfrac{V}{\overline{f}}, \dfrac{W}{\overline{h}}\right) = 0,
\end{cases}
$$

where ξ is the unitary field tangent to $[R,\infty)$, V vertical unitary field with respect to g_1, and W horizontal unitary field with respect to g_2 corresponding the decomposition of the tangent bundle $TN = \mathbb{R}\xi \oplus E_1 \oplus E_2$.

Introduce new functions $u(r)$ and $v(r)$ such that $\overline{f} = c_0 r u(r)$ and $\overline{h} = c_1 r v(r)$ for some constants c_0, c_1 which we will chose below. The system (III) in terms of these new functions u, v is equivalent to:

(IV)
$$\begin{cases} \mathrm{Ric}(\xi,\xi) = -\left[\dfrac{3}{u}\left(\dfrac{2u'}{r}+u''\right)+\dfrac{4}{v}\left(\dfrac{2v'}{r}+v''\right)\right] > 0, \\ \mathrm{Ric}\left(\dfrac{V}{\overline{f}},\dfrac{V}{\overline{f}}\right) = \dfrac{2}{c_0^2r^2u^2}-\dfrac{6}{r^2}-\dfrac{u''}{u}-\dfrac{10u'}{ru}-2\left(\dfrac{u'}{u}\right)^2-\dfrac{4v'}{v}\left(\dfrac{u'}{u}+\dfrac{1}{r}\right) > 0, \\ \mathrm{Ric}\left(\dfrac{W}{\overline{h}},\dfrac{W}{\overline{h}}\right) = \dfrac{3}{c_1^2r^2v^2}-\dfrac{6}{r^2}-\dfrac{v''}{v}-\dfrac{11v'}{rv}-3\left(\dfrac{v'}{v}\right)^2-\dfrac{3u'}{u}\left(\dfrac{v'}{v}+\dfrac{1}{r}\right) > 0. \end{cases}$$

Take some $\alpha \in (0,1)$. We define $u = 1/\ln r + (\sin(\ln(\ln r)))^2$ and $v = r^{-\alpha}$. Then

(V)
$$\begin{aligned} \frac{u'}{u} &= \frac{1}{ru\ln r}\left(-\frac{1}{\ln r}+\sin 2x\right), \\ \frac{u''}{u} &= \frac{1}{r^2u\ln r}\left[-\sin 2x+\frac{1}{\ln r}\left(1+\frac{2}{\ln r}-\sin 2x+2\cos 2x\right)\right], \\ \frac{v'}{v} &= -\frac{\alpha}{r} \qquad \text{and} \qquad \frac{v''}{v} = \frac{\alpha(\alpha+1)}{r^2}, \end{aligned}$$

where $x = \ln(\ln r)$.

1.2. Substituting equalities (V) into (IV) above we conclude that

$$\mathrm{Ric}(\xi,\xi) = \frac{1}{r^2}\left\{\frac{3}{u\ln r}\left[-\sin 2x+\frac{z(r)}{\ln r}\right]+4\alpha(1-\alpha)\right\},$$

where $z(r) = 1 - 1/\ln r + \sin 2x - 2\cos 2x$. If $r > e$ then $z(r) \in (-3,4)$. Therefore, there exists some R_α sufficiently large such that for $r \geq R_\alpha$ we have

$$\left|\frac{\sin 2x}{u\ln r}\right| = \left|\frac{\sin 2x}{1+\ln r(\sin x)^2}\right| < \frac{4}{3}\alpha(1-\alpha)-\frac{3}{\ln r},$$

which implies $\mathrm{Ric}(\xi,\xi) > 0$.

1.3. Again, from (IV) and (V) we deduce for $\mathrm{Ric}(V/\overline{f}, V/\overline{f})$:

$$\mathrm{Ric}\left(\frac{V}{\overline{f}},\frac{V}{\overline{f}}\right) = \frac{1}{r^2}\left[\frac{2}{c_0^2u^2}-6+4\alpha+\phi(r)\right],$$

where

$$\phi(r) = -\frac{1}{u\ln r}\left[(9-4\alpha)\sin 2x+\frac{1}{\ln r}\left(-9+4\alpha+\frac{2}{\ln r}-\sin 2x+2\cos 2x\right)\right.$$

$$+\frac{2}{u\ln r}\left(-\frac{1}{\ln r}+\sin 2x\right)^2\Bigg].$$

We see that $|\phi(r)| < 21$ for any $\alpha \in (0,1)$ and $r > e$, and $1/u \in (1/2,\infty)$ for any $c_0 \in (0,1/8]$. This in turn implies $\mathrm{Ric}(V/\overline{f}, V/\overline{f}) > 0$.

1.4. Finally,

$$\mathrm{Ric}\left(\frac{W}{\overline{h}},\frac{W}{\overline{h}}\right)=\frac{1}{r^2}\left[\frac{3r^\alpha}{c_1^2}-6-4\alpha^2+10\alpha-\frac{3(1-\alpha)}{u\ln r}\left(-\frac{1}{\ln r}+\sin 2x\right)\right].$$

As direct computations show for $R \geq \max\{e, R_\alpha\}$ and $r > R$ for arbitrary $c_1^2 \in (0, R^\alpha/3)$ it holds that $\mathrm{Ric}(W/\overline{h}, W/\overline{h}) > 0$.

Summarizing our construction above we see that for arbitrary $\alpha \in (0,1)$ there exists R_α such that for all $R > R_\alpha$ and all c_0, c_1 sufficiently small our metric $\overline{g}$ on N has positive Ricci curvature.

Note also that R, c_0 can be chosen so that on a small open neighborhood, say V, of the boundary sphere $S^7(R)$ we have $\mathrm{Ric}\,|_V > \varepsilon R^{-2}$ for some $\varepsilon > 0$ sufficiently small.

STEP 2.

2.1 Next we endow $D = [0,R] \times S^7/\{0\} \times S^7$ with some metric $\tilde{g} = dr^2 + f_0(r)^2 g_1 + h_0(r)^2 g_2$, and then attach it to $(N, \overline{g})$ along their common boundary $S^7(R)$ so that the resulting manifold will be of class $\mathcal{C}^k$ outside the origin $\{0\}$.

Let the warping functions f_0, h_0 of the metric $\tilde{g}$ on D are given by $f_0(r) = c_0 r u_0(r)$, $h_0(r) = c_1 r v_0(r)$, where

$$u_0(r)=\sum_{i=0}^{k}\frac{u^{(i)}(R)}{i!}(r-R)^i \qquad \text{and} \qquad v_0(r)=\sum_{i=0}^{k}\frac{v^{(i)}(R)}{i!}(r-R)^i.$$

Without loss of generality we may assume that our sufficiently big R is such that $\ln(\ln R) = m\pi$ for some sufficiently big integer m. Under this assumption the values of our function $u(r)$ and its derivatives $u'(r), \ldots, u^{(k)}(r)$ when $r = R$ are:

$$u(R)=\frac{1}{\ln R}, u'(R)=-\frac{1}{R(\ln R)^2},\ldots,u^{(k)} \;=\frac{(-1)^k}{R^k(\ln R)^2}\left(a_1+\frac{a_2}{\ln R}+\cdots+\frac{a_k}{(\ln R)^{k-1}}\right).$$

Taking functions f_0 and h_0 as follows

$$
\text{(VI)}\qquad
\begin{cases}
f_0(r) = c_0 r\left[\dfrac{1}{\ln R} + \dfrac{1}{R(\ln R)^2}(R-r) + \cdots \right.\\
\qquad\qquad \left. + \dfrac{1}{k!R^k(\ln R)^2}\left(a_1 + \dfrac{a_2}{\ln R} + \cdots + \dfrac{a_k}{(\ln R)^{k-1}}\right)(R-r)^k\right],\\
h_0(r) = c_1 r\left[\dfrac{1}{R^\alpha} + \dfrac{\alpha}{R^{1+\alpha}}(R-r) + \cdots \right.\\
\qquad\qquad \left. + \dfrac{1}{k!R^{k+\alpha}}\alpha(\alpha+1)\cdots(\alpha+k-1)(R-r)^k\right],
\end{cases}
$$

we see that the k-jets of $\overline{g}$ and $\tilde{g}$ coincide along their common boundary $S^7(R)$. From this, it follows in particular that $\mathrm{Ric}_{\tilde{g}} = \mathrm{Ric}_{\overline{g}}$ in all points of the sphere $S^7(R)$.

Notice that if we consider a suitable $\mathcal{C}^\infty$ convex combination (see for instance the step 3 below) of the metrics $\overline{g}$ and $\tilde{g}$ in a neighborhood of the sphere $S^7(R)$, it is possible to show that the new metric obtained is of class $\mathcal{C}^\infty$ in $\mathbb{R}^8 \setminus \{0\}$, and also has positive Ricci curvature.

2.2 As in the first step above, direct computations (substituting (VI) into (IV)) show that the metric $\tilde{g}$ has positive Ricci curvature in D outside the origin. In addition $\mathrm{Ric}_{\tilde{g}}(x) \to \infty$ as $x \to 0$. Indeed, for arbitrary $r \in (0, R]$ it holds that

$$
\mathrm{Ric}(\xi,\xi) \geq \frac{1}{r}\left[\frac{1}{R}C(\alpha,k) - \frac{1}{R\ln R}\left(b_1 + \frac{b_2}{\ln R} + \cdots + \frac{b_k}{(\ln R)^{k-1}}\right)\right],
$$

where $C(\alpha, k)$ is a constant depending only on α and k, while b_i are constants depending only on k. Hence, $\mathrm{Ric}(\xi,\xi) > 0$ if only R is bigger than some $R_\alpha(k)$.

For other components of the Ricci tensor $\mathrm{Ric}(V/f_0, V/f_0)$ and $\mathrm{Ric}(W/h_0, W/h_0)$ similar computations show that for R big enough and c_0, c_1 small enough it holds that $\mathrm{Ric}(V/f_0, V/f_0)$ and $\mathrm{Ric}(W/h_0, W/h_0) > 0$, and that they are of order $\mathcal{O}(1/r^2)$ as $r \to 0$.

Again, summarizing our construction of the metric $\tilde{g}$ on D we see that arbitrary $\alpha \in (0,1)$ there exists $R_\alpha(k)$ such that for all $R > R_\alpha(k)$ and all c_0, c_1 sufficiently small our metric $\tilde{g}$ on D has positive Ricci curvature.

STEP 3. To smooth the conical singularity at the origin we deform $\tilde{g}$ in the following way. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be the function

$$
\phi(r) = \begin{cases}
0, & \text{if} \quad r \leq 0,\\
\dfrac{1}{1+\exp(1/r - 1/(1-r))}, & \text{if} \quad 0 < r < 1,\\
1, & \text{if} \quad r \geq 1.
\end{cases}
$$

For $0 \le r \le R$ define

$$\underline{g} = dr^2 + \underline{f}(r)^2 g_1 + \underline{h}(r)^2 g_2,$$

where $\underline{f}(r) = r\,[(1-\phi(r))l_1 + \phi(r)c_0u_0(r)]$, $\underline{h}(r) = r\,[(1-\phi(r))l_2 + \phi(r)c_1v_0(r)]$ and l_1, l_2 are some positive constants. Taking them sufficiently small ($l_1 \le 1/\sqrt{3}$, $l_2 \le 1/\sqrt{2}$ would be enough) and controlling the value of the constants c_0, c_1 after some lengthy, but straightforward calculations one sees that $(\mathbb{R}^8, g)$ with the metric given in the disc D by $\underline{g}$ and in the exterior domain N by $\overline{g}$ is smooth at the origin. Its has positive Ricci curvature in $\mathbb{R}^8 \setminus \{0\}$ which goes to zero exponentially as $r \to 0$.

STEP 4. Finally, we verify that our manifold $(\mathbb{R}^8, g)$ has ideal boundaries of different dimensions. By construction its double warp-product metric has the form $g = dr^2 + f(r)^2 g_1 + h(r)^2 g_2$ where $f(r) = \underline{f}(r)$, $h(r) = \underline{h}(r)$ for $0 \le r \le R$, and $f(r) = \overline{f}(r)$, $h(r) = \overline{h}(r)$ if $r \ge R$. If we consider the sequence $r_m = \exp(\exp m\pi)$ we notice that

$$\frac{f(r_m)}{r_m} = \frac{c_0}{\ln r_m} \to 0 \text{ and } \frac{h(r_m)}{r_m} = c_1 r_m^{-\alpha} \to 0 \text{ as } m \to \infty,$$

which means that the ideal boundary of $\mathbb{R}^8$ under this sequence $\mathbb{R}^8(\infty)\{r_m\}$ is a point: when r_m becomes arbitrarily large the warped sphere $(S^7, r_m^{-2} g^{S^7}_{r_m})$ becomes arbitrarily small, and in the limit it collapses to a point. On the other hand for the other sequence $r_l = \exp(\exp(\pi/2 + l\pi))$ it holds

$$\frac{f(r_l)}{r_l} = c_0\left(\frac{1}{\ln r_l} + 1\right) \to c_0 \text{ and } \frac{h(r_l)}{r_l} = c_1 r_l^{-\alpha} \to 0 \text{ as } l \to \infty.$$

That is, under this sequence the sphere $(S^7, r_l^{-2} g^{S^7}_{r_l})$ with our warped product metric collapses along the directions normal to the fibres of the Hopf fibration, i.e., the sphere S^7 collapses onto the fibre S^3. In other words, in the Gromov-Hausdorff distance $(S^7, r_l^{-2} g^{S^7}_{r_l})$ tends as $l \to \infty$ to the sphere $S^3(c_0)$ of radius c_0 or equivalently, the ideal boundary $\mathbb{R}^8(\infty)\{r_l\}$ equals $S^3(c_0)$.

Theorem A is proved.

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