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FINITENESS OF CONFORMAL BLOCKS OVER COMPACT RIEMANN SURFACES

TOSHIYUKI ABE[1] and KIYOKAZU NAGATOMO[2]

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Introduction

We study conformal blocks (the space of correlation functions) over compact Riemann surfaces associated to vertex operator algebras which are the sum of highest weight modules for the underlying Virasoro algebra. Under a fairly general condition, for instance, C_2-finiteness, we prove that conformal blocks are finite-dimensional. This, in particular, shows the finiteness of conformal blocks for many well-known conformal field theories including WZNW model and the minimal model.

In [1] we showed that conformal blocks over the *projective line* associated to a vertex operator algebra (VOA) V are finite-dimensional if modules for V satisfy some finiteness condition. In this paper we generalize these results to conformal blocks over any *compact Riemann surfaces*. More precisely we will prove that if V-modules of our concern as well as V are C_2-finite then corresponding conformal blocks are finite-dimensional. The main reason why we need C_2-finiteness of V in this case is caused by Weierstrass gaps, i.e., we are not able to find meromorphic differentials with poles of some exceptional orders.

Though in this paper the notion of conformal blocks are defined in a purely mathematical way, the definition goes back to the notion of correlation functions in conformal field theory (CFT) initiated by [4]. CFT's are supposed to have at least two properties, one of which is the finiteness of conformal blocks, and the other is the factorization property; the latter enables us to determine the dimension of conformal blocks by fusion rules (the space of 3-point correlation functions or its dimension). Like other objects in physics every CFT has its own symmetry group (Lie algebra): affine Lie algebras for WZNW model and the Virasoro algebra for the minimal model, for instance. We will study "general" CFT's, where "general" means that the symmetry is described by a VOA. Such CFT's were first proposed and studied by Zhu [21], however two main issues, i.e., finiteness of conformal blocks and the factorization theorem of these CFT's were left open.

We should point out two main differences between our general CFT's and the

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known CFT's mentioned above. Conformal blocks are the space of correlation functions of primary fields and the Virasoro fields. On the one hand, for instance, in WZNW model, the space of states is generated by currents and the Virasoro field of the theory; the currents are primary fields with conformal weight 1, and the Virasoro field is a quasi-primary field with conformal weight 2. The Virasoro field is obtained in terms of the currents, and so we only need information on meromorphic functions on a Riemann surface to study conformal blocks. However, in general, we have primary fields of higher conformal weight Δ, and we have to know the geometry of the line bundle $\kappa^{1-\Delta}$ where κ is the canonical line bundle. The minimal model is generated by a conformal weight 2 field (the Virasoro field) and the analysis of the line bundle κ^{-1} is necessary, though it is still not so complicated.

Part of ideas in [19] were generalized to VOA's in [21]: Zhu first generalized the notion of currents and the energy-momentum field to the notion of *global vertex operators* associated to any primary states and then gave a very general definition of conformal blocks; more precisely, conformal blocks are defined in almost the same way as in [19]. However this definition based on primary fields and the Virasoro field is not convenient because Fourier modes of primary fields and the Virasoro field do not form a Lie algebra. Zhu then introduced so-called *quasi-global vertex operators* which are defined by using quasi-primary states. The quasi-global vertex operators now form a Lie algebra under a fairly general assumption for V. The point is that we can characterize conformal blocks in terms of quasi-global vertex operators. This is one of the main results in [21].

In many examples of CFT's a key fact for the finiteness of conformal blocks is the finite dimensionality of the space of coinvariants; several examples are known such as WZNW model and the minimal model. The notion in VOA theory corresponding to the finiteness of "coinvariants" is the C_2-finiteness condition introduced in [20]. Using the notion of Frenkel-Zhu bimodules [12] the finiteness of fusion rules is proved in [17] for C_2-finite modules; more precisely, the weaker condition called B_1-finiteness is enough for the finiteness of fusion rules.

In this paper we prove the finiteness of conformal blocks over pointed compact Riemann surfaces associated to C_2-finite vertex operator algebras which are sum of highest weight modules of the Virasoro algebra and B_1-finite V-modules; we should mention that our notion of B_1-finiteness is different from Li's. The method used here basically follows [19], while we work in a fairly general set-up. The proof of finiteness of conformal blocks over compact Riemann surfaces is reduced to finding a nontrivial meromorphic section with poles of specified positions and orders. A main difference between the case of the projective line [1] and the case studied in the paper is that on a general Riemann surface we are not always able to find a meromorphic form which has poles at prescribed points and orders; this point will be elaborated in the paper.

The conformal blocks over pointed projective lines are also studied in [13]. The definition of conformal blocks looks different from the one of [21] and [1], but their method has a great influence on our work.

In Section 1 we review some basic fact about vertex operator algebras. In Section 2 we recall the notion of C_n-finiteness condition ($n \geq 2$) and B_1-finiteness condition for modules, and state several known results concerning these finiteness conditions. The notion of conformal blocks introduced in [21] is explained in Section 3; conformal blocks are built on a triple of a compact Riemann surface Σ, a finite set A of distinct points on Σ and V-modules $W^i\,(i \in A)$. We introduce a filtration on the Lie algebra $\mathcal{Q}(\widetilde{\Sigma})$ of all quasi-global vertex operators on a compact Riemann surface Σ. The Lie algebra $\mathcal{Q}(\widetilde{\Sigma})$ acts on the tensor product vector space $W_A = \bigotimes_{i\in A} W^i$. We define a filtration on W_A which makes W_A a filtered $\mathcal{Q}(\widetilde{\Sigma})$-module. These filtrations have their origins in [19], but of course they are appropriately generalized to fit our purposes. Section 4 is the core of this paper, where we prove that conformal blocks associated to a C_2-finite quasi-primary generated vertex operator algebra and B_1-finite V-modules are finite-dimensional; we prove the main theorem by showing the existence of a surjective map $\bigotimes_{i\in A} W^i/B_1(W^i) \to \mathrm{gr}_\bullet\, W_A/\mathcal{Q}(\widetilde{\Sigma})W_A$. In particular if all modules $W^i\,(i \in A)$ are C_2-finite then the corresponding conformal block is finite-dimensional. Finally in Section 5 we discuss several examples of C_2-finite vertex operator algebras whose irreducible modules are B_1-finite.

After we completed the work we learned Buhl's result [3] that any finitely generated weak module for a C_2-finite (which is called C_2 co-finite in [3]) vertex operator algebra is C_n-finite for all $n \geq 2$. Therefore our B_1-finiteness assumption for modules is not necessary.

1. Vertex operator algebras and their modules

Let V be a vertex operator algebra with the vacuum element $\mathbf{1}$ and the Virasoro element ω (see [11], [10], [18]), i.e., the vector space V is equipped with countably many bilinear operation $(a, b) \mapsto a(n)b\,(a,\, b \in V)$ for any integer n. For any $a \in V$, we denote the vertex operator associated to a by $Y(a, x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$ where $a(n) \in \mathrm{End}(V)$ is defined by $b \mapsto a(n)b$ for all $b \in V$. The operators $L_n := \omega(n+1)\,(n \in \mathbb{Z})$ form a representation of the Virasoro algebra on V, and the vector space V is $\mathbb{N}$-graded with the grading operator L_0, i.e., $V = \bigoplus_{n=0}^{\infty} V(n)$, $L_0 \mid V(n) = n$ id. The operator L_{-1} is assumed to satisfy $(\partial/\partial x)Y(a, x) = Y(L_{-1}a, x)$, i.e., $-na(n-1) = (L_{-1}a)(n)$ for all $a \in V$ and $n \in \mathbb{Z}$.

An element $a \in V$ satisfying $L_1 a = L_2 a = 0$ is called a *primary vector*, while an element satisfying only $L_1 a = 0$ is called a *quasi-primary vector*. Let $\mathcal{P}(V)$ and $\mathcal{Q}(V)$ be the set of all primary and quasi-primary vectors, respectively. We see that those two vector subspaces of V are graded, i.e., $\mathcal{P}(V) = \bigoplus_{n=0}^{\infty} \mathcal{P}(V) \cap V(n)$ and $\mathcal{Q}(V) = \bigoplus_{n=0}^{\infty} \mathcal{Q}(V) \cap V(n)$.

DEFINITION 1.1. A vertex operator algebra V satisfying $V = \sum_{k=0}^{\infty} L_{-1}^{k}\mathcal{Q}(V)$ is called *quasi-primary generated.*

It is known that V is quasi-primary generated if and only if $V_1 \subset \mathcal{Q}(V)$ ([6]). For V quasi-primary generated we see that $V = \bigoplus_{k=0}^{\infty} L_{-1}^{k}\mathcal{Q}(V)$ if and only if $L_{-1}V(0) = 0$.

DEFINITION 1.2. A *weak V-module* is a vector space W equipped with a linear map

$$\begin{aligned} Y_W\colon V &\to (\operatorname{End} W)[[x, x^{-1}]] \\ a &\mapsto Y_W(a, x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}, \quad (a(n) \in \operatorname{End} W) \end{aligned}$$

which satisfies the following conditions for all a, $b \in V$ and $w \in W$; $Y_W(a, x)w \in W((x))$, $Y_W(\mathbf{1}, x) = \mathrm{id}_W$, and for all integers p, q, $r \in \mathbb{Z}$,

$$\begin{aligned} &\sum_{i=0}^{\infty} \binom{p}{i}(a(r+i)b)(p+q-i)w \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \big(a(p+r-i)(b(q+i)w) - (-1)^r b(q+r-i)(a(p+i)w)\big). \end{aligned} \tag{1.1}$$

The identity (1.1) is equivalent to the set of the following two formulas for a, $b \in V$ and $w \in W$ (cf. [18, § 4.3]); one is called the *associativity formula*

$$\begin{aligned} &(a(-n)b)\ (-q)w \\ &= \sum_{i=0}^{\infty} \binom{-n}{i}(-1)^i(a(-n-i)b(-q+i)w - (-1)^n b(-n-q-i)a(i)w), \end{aligned} \tag{1.2}$$

and the other is called the *commutator formula*

$$[a(p), b(q)]w = \sum_{i=0}^{\infty} \binom{p}{i}(a(i)b)(p+q-i)w.$$

By the commutator formula we see that $L_n\,(n \in \mathbb{Z})$ form a representation on W of the Virasoro algebra. Since $L_{-1}a = (L_{-1}a)(-1)\mathbf{1} = a(-2)\mathbf{1}$ for any $a \in V$ the associativity formula for $(a(-2)\mathbf{1})(q)w$ shows that $(L_{-1}a)(q)w = -qa(q-1)w$ for all $a \in V$ and $w \in W$.

DEFINITION 1.3. A *V-module* W is a weak V-module on which L_0 acts semisimply, i.e., $W = \bigoplus_{\lambda\in\mathbb{C}} W(\lambda)$, $L_0 \mid W(\lambda) = \lambda\ \mathrm{id}$, and for fixed $\lambda \in \mathbb{C}$, $W(\lambda + n) = 0$

for all sufficiently large integers n. For $\lambda \in \mathbb{C}$, a nonzero vector w in $W(\lambda)$ is called *homogeneous vector of weight* λ, and its weight is denoted by $|w|$.

Whenever we write $|w|$ the element w is supposed to be homogeneous of weight $|w|$.

2. Finiteness conditions of vertex operator algebras

We recall the notion of C_n-finiteness (see [20] for $n = 2$, and [17] for general $n(\geq 2)$). We review the notion of B_1-finiteness in [1], and state several results; the most important is that for a C_2-finite vertex operator algebra V any B_1-finite weak V-module is C_n-finite for all $n \geq 2$, which was proved in [1].

DEFINITION 2.1 ([20], [17]). For any positive integer $n(\geq 2)$ we denote by $C_n(W)$ the subspace of W, which is linearly spanned by $a(-n)w$ for all $a \in V$ and $w \in W$. A weak V-module W is called *C_n-finite* ($n \geq 2$) if the vector space $W/C_n(W)$ is finite-dimensional.

Since $(L_{-1}a)(q)w = -qa(q-1)w$ for all $a \in V$ and $w \in W$ we see that $C_2(W) \supset C_3(W) \supset \cdots \supset C_n(W) \supset \cdots$, and that any C_n-finite module for some $n \geq 2$ is C_2-finite. We now let V be a C_2-finite vertex operator algebra:

Proposition 2.2 ([13, Proposition 8]). *Let $V = \bigoplus_{n=0}^{\infty} V(n)$ be a vertex operator algebra with $V(0) = \mathbb{C}\mathbf{1}$, and U be a graded subspace such that $V = U \oplus C_2(V)$. Then V is linearly spanned by the vectors*

$$\alpha_1(-n_1)\alpha_2(-n_2)\cdots\alpha_k(-n_k)\mathbf{1} \text{ for all } \alpha_i \in U \text{ and } n_1 > n_2 > \cdots > n_k > 0. \tag{2.1}$$

Let U be a graded subspace such that $V = U \oplus C_2(V)$. By Proposition 2.2 we see that the vectors (2.1) for $k \geq n$ belong to $C_n(V)$. Suppose that V is C_2-finite. Then U is finite-dimensional, and we have:

Proposition 2.3 ([13, Theorem 11]). *Let $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ be a C_2-finite vertex operator algebra. Then V is C_n-finite for all $n \geq 2$.*

Let W be a weak V-module. We denote by $B_1(W)$ the subspace of W spanned by $a(-1)w$ for all homogeneous $a \in V$ with positive weight, i.e., $|a| > 0$ and all $w \in W$; we note that $B_1(W) \supset C_2(W)$.

DEFINITION 2.4. A weak V-module W is called *B_1-finite* if the vector space $W/B_1(W)$ is finite-dimensional.

Note 2.5. (1) A B_1-finite weak module is called *quasirational* in [13].
(2) The notion of B_1-finiteness is slightly different from Li's one [17].

Note that C_n-finite module for $n \geq 2$ is B_1-finite. Conversely we get:

Theorem 2.6 ([1]). *Let $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ be a C_2-finite vertex operator algebra. Then any B_1-finite weak V-module is C_n-finite for all $n \geq 2$.*

3. Conformal blocks

We recall the definition of conformal blocks and review some properties of them. Most of material in this section except a filtration on quasi-global vertex operators are taken from [21].

Let Σ be a compact Riemann surface and κ be the canonical line bundle on Σ, and let us fix N distinct points $Q_1, Q_2, \ldots, Q_N$ on Σ. For any integer n we denote by $\Gamma(\Sigma; Q_1, Q_2, \ldots, Q_N; \kappa^n)$ the vector space of global meromorphic sections of κ^n with possible poles at $Q_1, Q_2, \ldots, Q_N$.

DEFINITION 3.1. Let Σ be a compact Riemann surface and $Q_1, Q_2, \ldots, Q_N$ be distinct points on Σ. Let z_i be local coordinates around the Q_i such that $z_i(Q_i) = 0$. A collection of datum $\overline{\Sigma} = (\Sigma; Q_1, Q_2, \ldots, Q_N; z_1, z_2, \ldots, z_N)$ is called an *N-pointed Riemann surface*. An N-pointed Riemann surface $\overline{\Sigma}$ with a set of V-modules W^i being attached to each point Q_i

$$\widetilde{\Sigma} = (\Sigma; Q_1, Q_2, \ldots, Q_N; z_1, z_2, \ldots, z_N; W^1, W^2, \ldots, W^N) \tag{3.1}$$

is called an *N-labeled Riemann surface*.

A covering of coordinate charts $\{(U_\alpha, z_\alpha)\}$ of Σ is called a *projective structure* on Σ if transition functions $z_\beta \circ z_\alpha^{-1}$ are Möbius transformations for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$; any Riemann surface has a projective structure. Let $\{(U_\alpha, z_\alpha)\}$ be a projective structure on Σ and $Q_1, Q_2, \ldots, Q_N$ be distinct points of Σ. For each Q_i we choose a local coordinate (U_α, z_α) such that $Q_i \in U_\alpha$, and define a new coordinate near Q_i by $z_i = z_\alpha - z_\alpha(Q_i)$. Then we obtain an N-pointed Riemann surface $\overline{\Sigma}$: such a $\overline{\Sigma}$ is called *projective*. The notion of projective N-labeled Riemann surface $\widetilde{\Sigma}$ is defined in the same way.

Let $\overline{\Sigma} = (\Sigma; Q_1, Q_2, \ldots, Q_N; z_1, z_2, \ldots, z_N)$ be an N-pointed Riemann surface. We will define a Lie algebra $\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$ associated to $\overline{\Sigma}$. Let V be a vertex operator algebra. We set $\widehat{V} = V \otimes \mathbb{C}((t))$ where $\mathbb{C}((t))$ is the ring of formal Laurent power series. It is well known that the commutative associative algebra $\mathbb{C}((t))$ with the derivation d/dt naturally becomes a vertex algebra by

$$Y(f(t), x)g(t) = \left(e^{x\frac{d}{dt}} f(t)\right)g(t).$$

The tensor product $\widehat{V} = V \otimes \mathbb{C}((t))$ has a structure of vertex algebra, which is given by

$$Y(a \otimes f(t), x)b \otimes g(t) = Y(a, x)b \otimes \left(e^{x\frac{d}{dt}} f(t)\right) g(t)$$

for all $a \otimes f(t)\,(a \in V,\ f(t) \in \mathbb{C}((t)))$. The translation operator is $D = L_{-1} \otimes \mathrm{id} + \mathrm{id} \otimes (d/dt)$. We set $\mathfrak{g}(V) = \widehat{V}/D\widehat{V}$. Then it is well known that the 0-th product on $\widehat{V}$ induces a Lie algebra structure on $\mathfrak{g}(V)$. The important point is that any weak V-module becomes a $\mathfrak{g}(V)$-module by

$$(a \otimes f(t))u = \operatorname*{Res}_{t=0} Y(a, t) f(t) u.$$

Let $A = \{1, 2, \dots, N\}$, and set $\mathfrak{g}(V)_A = \bigoplus_{i \in A} \mathfrak{g}(V)_{(i)}$ where $\mathfrak{g}(V)_{(i)} = \mathfrak{g}(V)$ is a copy of $\mathfrak{g}(V)$. We now define a linear map

$$j_{\widetilde{\Sigma}} \colon \bigoplus_{d=0}^{\infty} V(d) \otimes \Gamma(\Sigma\,; Q_1,\ Q_2, \dots, Q_N; \kappa^{1-d}) \longrightarrow \mathfrak{g}(V)_A$$

by sending $a \otimes f$ to $\sum_{i \in A} a \otimes f_i(t)$ for each $a \in V(d)$ and $f \in \Gamma(\Sigma\,; Q_1,\ Q_2, \dots, Q_N; \kappa^{1-d})$, where $\iota_{z_i} f(z_i) = \sum_{n \in \mathbb{Z}} c_n z_i^n$ is the Laurent series expansion of the meromorphic function $f(z_i)$ near Q_i given by $f = f(z_i)(dz_i)^{-|a|+1}$, and $f_i(t) = \sum_{i \in \mathbb{Z}} c_i t^i \in \mathbb{C}((t))$. We denote the image of $j_{\widetilde{\Sigma}}$ by $\mathfrak{g}(V)^{out}_{\widetilde{\Sigma}}$;

$$\mathfrak{g}(V)^{out}_{\widetilde{\Sigma}} = j_{\widetilde{\Sigma}} \left(\bigoplus_{d=0}^{\infty} V(d) \otimes \Gamma(\Sigma\,; Q_1,\ Q_2, \dots, Q_N; \kappa^{1-d}) \right).$$

Proposition 3.2 ([21]). *If $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ is a quasi-primary generated vertex operator algebra and $\widetilde{\Sigma}$ is projective, then $\mathfrak{g}(V)^{out}_{\widetilde{\Sigma}}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}(V)_A$.*

Let $\widetilde{\Sigma} = (\Sigma\,; Q_1,\ Q_2, \dots, Q_N; z_1,\ z_2, \dots, z_N; W^1,\ W^2, \dots, W^N)$ be an N-labeled Riemann surface. We set $W_A = \bigotimes_{i \in A} W^i$. We denote by ρ_{Q_i} the action of $\mathfrak{g}(V)_i$ on the i-th component of W_A, and set $\rho_{\widetilde{\Sigma}} = \bigoplus_{i \in A} \rho_{Q_i}$; the Lie algebra $\mathfrak{g}(V)_{\Sigma}$ acts on W_A, and so the Lie subalgebra $\mathfrak{g}(V)^{out}_{\widetilde{\Sigma}}$. In other words we have a homomorphism $\rho_{\widetilde{\Sigma}} \colon \mathfrak{g}(V)^{out}_{\widetilde{\Sigma}} \to \mathrm{End}(W_A)$.

We now set

$$\mathcal{Q}(\widetilde{\Sigma}) = \rho_{\widetilde{\Sigma}} \left(j_{\widetilde{\Sigma}} \left(\bigoplus_{d=0}^{\infty} \mathcal{Q}(V)(d) \otimes \Gamma(\Sigma\,; Q_1,\ Q_2, \dots, Q_N; \kappa^{1-d}) \right) \right) \subset \mathrm{End}(W_A).$$

An element in $\mathcal{Q}(\widetilde{\Sigma})$ is called a *quasi-global vertex operator*. If V is quasi-primary generated and $\widetilde{\Sigma}$ is projective, then $\mathcal{Q}(\widetilde{\Sigma}) = \rho_{\widetilde{\Sigma}}\left(\mathfrak{g}(V)^{out}_{\widetilde{\Sigma}}\right)$ is a Lie algebra by Proposition 3.2. For any $a \otimes f$ we often denote $\rho_{\widetilde{\Sigma}}(j_{\widetilde{\Sigma}}(a \otimes f))$ by $a(f, \widetilde{\Sigma})$ for simplicity.

The vector space W_A is a module for the Lie algebra $\mathcal{Q}(\widetilde{\Sigma})$. We denote the space of coinvariants $W_A/\mathcal{Q}(\widetilde{\Sigma})W_A$ by $\mathcal{QV}(\widetilde{\Sigma})$.

If a is a primary vector, the quasi-global vertex operator $a(f, \widetilde{\Sigma})$ is called a *global vertex operator* because it satisfies the transformation law as if it were $(1 - |a|)$-differentials under coordinate changes. Let $\mathcal{G}(\widetilde{\Sigma})$ be the vector subspace of $\mathrm{End}(W_A)$ spanned by all global vertex operators and quasi-global vertex operators $\omega(f, \widetilde{\Sigma})$ for all meromorphic vector fields $f \in \Gamma(\Sigma\,; Q_1, Q_2, \dots, Q_N; \kappa^{-1})$. Then the *space of covacua* is defined to be $\mathcal{V}(\widetilde{\Sigma}) = W_A/\mathcal{G}(\widetilde{\Sigma})W_A$. A main ingredient of this paper, the *space of vacua* or the *conformal block* associated to the N-labeled Riemann surface $\widetilde{\Sigma}$, is defined to be $\mathcal{V}^\dagger(\widetilde{\Sigma}) = \mathrm{Hom}_{\mathbb{C}}(W_A/\mathcal{G}(\widetilde{\Sigma})W_A, \mathbb{C})$.

This definition of the space of covacua or the conformal block is not convenient because in general $\mathcal{G}(\widetilde{\Sigma})$ is not a Lie subalgebra. However, due to the following theorem of Zhu, it suffices for us to consider the Lie algebra $\mathcal{Q}(\widetilde{\Sigma})$.

Theorem 3.3 ([21, Theorem 5.2]). *Let $V = \bigoplus_{n=0}^{\infty} V(n)$ be a vertex operator algebra with $V(0) = \mathbb{C}\mathbf{1}$, and $\widetilde{\Sigma}$ a projective N-labeled Riemann surface. Suppose that V is a sum of highest weight modules for the Virasoro algebra. Then $\eta \in (W_A)^*$ belongs to $\mathcal{V}^\dagger(\widetilde{\Sigma})$ if and only if $\eta(\mathcal{Q}(\widetilde{\Sigma})W_A) = 0$, i.e., there is a natural isomorphism as vector spaces*

$$\mathcal{V}^\dagger(\widetilde{\Sigma}) \cong \mathcal{QV}(\widetilde{\Sigma})^*,$$

where $\mathcal{V}^\dagger(\widetilde{\Sigma})$ is identified with the subset $\{\eta \in (W_A)^* \mid \eta(\mathcal{G}(\widetilde{\Sigma})W_A) = 0\}$ of $(W_A)^*$.

In Section 4 the filtration on $\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$ being introduced here plays a very important role. Let us start with an N-pointed Riemann surface $\overline{\Sigma} = (\Sigma\,; Q_1, Q_2, \dots, Q_N; z_1, z_2, \dots, z_N)$. For a given meromorphic differential f on Σ whose poles are located at Q_1, $Q_2, \dots, Q_N$ with order $a_1, a_2, \dots, a_N$, we define the order of f by

$$\mathrm{ord}\, f = \max\{a_1, a_2, \dots, a_N\}.$$

The filtration $\mathcal{F}_p\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$ $(p \in \mathbb{N})$ on $\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$ is defined by

$$\mathcal{F}_p\mathfrak{g}(V)^{out}_{\overline{\Sigma}} = \mathrm{span}_{\mathbb{C}}\{j_{\overline{\Sigma}}(a \otimes f) \mid |a| - 1 + \mathrm{ord}\, f \le p\}. \tag{3.2}$$

Then the Lie algebra $\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$ becomes a filtered Lie algebra.

We next define a filtration on $\mathfrak{g}(V)^{out}_{\overline{\Sigma}}$-module W_A. We first recall that any V-module W is a direct sum of V-modules of the form $\bigoplus_{d=0}^{\infty} W(\lambda_i + d)$ $(i \in I)$ with lowest weight λ_i such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for some index set I. We set $W_d = \bigoplus_{i\in I} W(\lambda_i + d)$ so that $W = \bigoplus_{d=0}^{\infty} W_d$. The filtration $\mathcal{F}_pW_A$ $(p \in \mathbb{N})$ on W_A is defined by

$$\mathcal{F}_pW_A = \bigoplus_{0\le d\le p} W_{A,d}, \quad W_{A,d} = \sum_{d_1+\cdots+d_N=d} W^1_{d_1} \otimes \cdots \otimes W^N_{d_N}.$$

The $\mathfrak{g}(V)^{out}_{\Sigma}$-module W_A becomes a filtered $\mathfrak{g}(V)^{out}_{\Sigma}$-module by this filtration.

Let $\mathcal{F}_p \mathcal{Q}\mathcal{V}(\widetilde{\Sigma})\,(p \in \mathbb{N})$ be the induced filtration on $\mathcal{Q}\mathcal{V}(\widetilde{\Sigma})$, i.e.,

$$\mathcal{F}_p \mathcal{Q}\mathcal{V}(\widetilde{\Sigma}) := s(\mathcal{F}_p W_A) = (\mathcal{F}_p W_A + \mathcal{Q}(\widetilde{\Sigma}) W_A)/\mathcal{Q}(\widetilde{\Sigma}) W_A,$$

where s is the natural projection $s\colon W_A \to \mathcal{Q}\mathcal{V}(\widetilde{\Sigma})$. We have the canonical surjection

$$\pi\colon W_A = \bigoplus_{p=0}^{\infty} W_{A,p} \longrightarrow \operatorname{gr}_{\bullet} \mathcal{Q}\mathcal{V}(\widetilde{\Sigma}) := \bigoplus_{p=0}^{\infty} \operatorname{gr}_p \mathcal{Q}\mathcal{V}(\widetilde{\Sigma}),$$

defined by $\pi(w) = s(w) + \mathcal{F}_{p-1}\mathcal{V}(\widetilde{\Sigma}) \in \operatorname{gr}_p \mathcal{V}(\widetilde{\Sigma})$ for $w \in W_{A,p}$, where $\operatorname{gr}_p \mathcal{Q}\mathcal{V}(\widetilde{\Sigma}) := \mathcal{F}_p \mathcal{Q}\mathcal{V}(\widetilde{\Sigma})/\mathcal{F}_{p-1}\mathcal{Q}\mathcal{V}(\widetilde{\Sigma})$.

4. Finiteness of conformal blocks

We will prove the finiteness of conformal blocks over projective N-labeled Riemann surfaces $\widetilde{\Sigma}\colon \widetilde{\Sigma} = (\Sigma\,; Q_1, \ldots, Q_N\,; z_1, \ldots, z_N\,; W^1, \ldots, W^N)$. For a proof we basically follow the argument in [1]; however, we need to remedy difficulties arising from the lack of global meromorphic sections with lower order poles of κ^{1-n} for positive integer n, i.e., Weierstrass gaps, which do not appear in the case of the projective line.

In [1] we explicitly constructed global meromorphic sections of $\kappa^{1-n}\,(n \geq 1)$ for the canonical bundle κ of the projective line with poles of desired orders at a prescribed point, and are holomorphic elsewhere. By Riemann-Roch theorem for a compact Riemann surface Σ, there exists a global meromorphic section which has a pole at $Q \in \Sigma$, and is holomorphic on $\Sigma \setminus \{Q\}$; however in general the order is large so that we are not able to find such a meromorphic section with lower order poles at Q.

Lemma 4.1. *Let Σ be a compact Riemann surface of genus g. We fix a point $Q \in \Sigma$ and a positive integer $n \in \mathbb{Z}_{>0}$.*
(1) *There exists a nontrivial global meromorphic section f of κ^{1-n} which has a pole at Q and is holomorphic on $\Sigma \setminus \{Q\}$.*
(2) *Let ν be the order of the pole at Q of the global meromorphic section f in* (1). *Set $M = \nu + 2g$. Then for any $m \geq M$, there exists a global meromorphic section of κ^{1-n} which has a pole of order m at Q and is holomorphic on $\Sigma \setminus \{Q\}$.*

Proof. The assertion (1) is found in [7, Theorem 29.16, page 225] or it is directly proved by using Riemann-Roch theorem. By Weierstrass gap theorem ([2, page 202]), for any $i \in \mathbb{N}$ there exists a meromorphic function h on Σ such that h has a pole of order $i + 2g$ at Q and is holomorphic of $\Sigma \setminus \{Q\}$. Then hf has a pole at Q of order $2g + \nu + i$. ☐

Let U be a subspace of V, which is linearly spanned by finitely many homogeneous quasi-primary vectors. Let r_U be the maximum of the weights of homogeneous vectors of U. Using Lemma 4.1 we can find a positive integer M_U such that for any $n \leq r_U$, $m \geq M_U$ and $i \in A$, there exists a global meromorphic section over κ^{1-n} which has a pole of order m at Q_i and is holomorphic on $\Sigma \setminus \{Q_i\}$.

We denote by $C_m(U, W)\,(m \geq 2)$ the subspace of W which is linearly spanned by $a(-n)w$ for all $a \in U$, $w \in W$ and $n \geq m$.

Lemma 4.2. *Let $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ be a quasi-primary generated vertex operator algebra. Let U be a subspace of V spanned by finitely many quasi-primary vectors. Then for any $i \in A$ the set $W^1 \otimes \cdots \otimes C_{M_U}(U, W^i) \otimes \cdots \otimes W^N$ is in the kernel of the surjective linear map $\pi\colon W_A \to \mathrm{gr}_\bullet\, \mathcal{QV}(\widetilde{\Sigma})$. In particular, the map π induces a surjective linear map*

$$\pi\colon W^1/C_{M_U}(U, W^1) \otimes \cdots \otimes W^N/C_{M_U}(U, W^N) \longrightarrow \mathrm{gr}_\bullet\, \mathcal{QV}(\widetilde{\Sigma}).$$

Proof. It suffices to show that for a homogeneous $a \in U$

$$\pi(w_1 \otimes \cdots \otimes a(-m)w_i \otimes \cdots \otimes w_N) = 0 \text{ for all } w_i \in W^i_{d_i} \text{ and } m \geq M_U.$$

For any $m \geq M_U$, there exists $f \in \Gamma(\Sigma\,; Q_i\,; \kappa^{1-|a|})$ whose Laurent series expansion is $\iota_{z_i} f = z_i^{-m} + \sum_{l>-m} c_l z_i^l$ at Q_i. Then we see that

$$\begin{aligned} &a(f, \widetilde{\Sigma})(w_1 \otimes \cdots \otimes w_N) \\ &\quad = w_1 \otimes \cdots \otimes a(-m)w_i \otimes \cdots \otimes w_N + \sum_{l>-m} c_l w_1 \otimes \cdots \otimes a(l)w_i \otimes \cdots \otimes w_N \\ &\qquad + \sum_{\substack{j\in A\\ j\neq i}} w_1 \otimes \cdots \otimes (\mathrm{Res}_{z_j} Y(a, z_j)\iota_{z_j} f)w_j \otimes \cdots \otimes w_N \\ &\quad = w_1 \otimes \cdots \otimes a(-m)w_i \otimes \cdots \otimes w_N + \text{lower weight (degree) terms} \in \mathcal{Q}(\widetilde{\Sigma})W_A, \end{aligned}$$

where we have used the fact that f is holomorphic at $Q_j\,(j \neq i)$. This implies that

$$w_1 \otimes \cdots \otimes a(-m)w_i \otimes \cdots \otimes w_N \in \mathcal{F}_{\sum d_i+|a|+m-2} W_A + \mathcal{Q}(\widetilde{\Sigma})W_A.$$

Since $w_1 \otimes \cdots \otimes a(-m)w_i \otimes \cdots \otimes w_N \in W_{A,\sum d_i+|a|+m-1}$, this element belongs to the kernel of the map π. $\square$

Let U be a graded subspace of V such that $V = U \oplus C_2(V)$. Recall that V is linearly spanned by vectors $\alpha_1(-n_1)\cdots\alpha_r(-n_r)\mathbf{1}\,(\alpha_i \in U)$ with $n_1 > \cdots > n_r > 0$. For any positive integers m and q, we set

$$C_{m,q}(W) = \{\, (\alpha_1(-n_1)\cdots\alpha_r(-n_r)\mathbf{1})(-p)w \mid m \geq n_1 > \cdots > n_r > 0,\ \alpha_i \in U,\ p \geq q \,\}.$$

Lemma 4.3. *Let m, q be positive integers. Then $C_q(W) \subset C_{m,q}(W) + C_m(U, W)$.*

Proof. By Proposition 2.2 it suffices to show that for any $\alpha_i \in U$ and $n_1 > \cdots > n_r > 0$

$$(\alpha_1(-n_1)\cdots\alpha_r(-n_r)\mathbf{1})(-p)w \in C_{m,q}(W) + C_m(U, W) \tag{4.1}$$

for all $p \geq q$; we can assume that $n_1 > m$ by definition of $C_{m,q}(W)$.

We now see that

$$(\alpha(-n)\mathbf{1})(-p)w = (-1)^{p-1}\binom{-n}{p-1}\alpha(-n-p+1)w \in C_m(U, W) \text{ for all } \alpha \in U.$$

This proves the case $r = 1$. Suppose that (4.1) holds for any $r < r_0$ for some $r_0 \geq 2$. We set $\beta = \alpha_2(-n_2)\cdots\alpha_{r_0}(-n_{r_0})\mathbf{1}$, and use the associativity formula (1.2) to get

$$\begin{aligned}&(\alpha_1(-n_1)\beta)(-p)w\\ &= \sum_{i=0}^{\infty}\binom{-n_1}{i}(-1)^i\left(\alpha_1(-n_1-i)\beta(-p+i)w - (-1)^{n_1}\beta(-n_1-p-i)\alpha_1(i)w\right).\end{aligned}$$

Then the first and the second terms of the right hand side belong to $C_m(U, V)$ and $C_{m,q}(W) + C_m(U, W)$, respectively, where we have used inductive hypothesis to the second term. ☐

Lemma 4.4. *Let m be a positive integer. For any $a \in V(|a| \geq 1)$ and $w \in W$ we have $a(-q)w \in C_m(U, W)$ for all q such that $q \geq m|a|$.*

Proof. If $|a| = 1$ then $a \in U$ because $V(1) \cap C_2(V) = \{0\}$. Hence $a(-q)w \in C_m(U, W)$ for any $q \geq m(= |a|m)$. Suppose that $|a| > 1$. If $a \in U$ then $a(-q)w \in C_m(U, W)$ for any $q \geq |a|m(> m)$. We can now assume that $a \in C_2(V)$. Suppose that $(0 \neq)\, a = b'(-2)c$ for some b' and c. Then $a = (L_{-1}b')(-1)c = b(-1)c$ and $1 \leq |b| \leq |a|$, $|c| < |a|$.

By the associativity formula we have

$$a(-q)w = (b(-1)c)(-q)w = \sum_{i=0}^{\infty}\left(b(-1-i)c(-q+i)w + c(-1-q-i)b(i)w\right)$$

for any $q \in \mathbb{Z}$. Since $q \geq |a|m > |c|m$, using inductive hypothesis to the element c, we see that $c(-1-q-i)b(i)w \in C_m(U, W)$ for $i \geq 0$. We will show that $b(-1-i) \times c(-q+i)\,w \in C_m(U, W)$ for any $q \geq |a|m$ and $i \geq 0$. If $i \geq |b|m$ then $b(-1-i) \times c(-q+i)w \in C_m(U, W)$ for any $q \in \mathbb{Z}$ by inductive hypothesis. Otherwise, i.e., $i <$

$|b|m$, recall the commutator formula

$$b(-1-i)c(-q+i)w = c(-q+i)b(-1-i)w + \sum_{j=0}^{\infty} \binom{-1-i}{j} (b(j)c)(-1-q-j)w.$$

Now since $q - i \geq |a|m - |b|m + 1 = (|a| - |b|)m + 1 = |c|m + 1 > |c|m$, using inductive hypothesis we see that the first term $c(-q+i)b(-1-i)w$ belongs to $C_m(U, W)$. Finally since $|a| > |b(j)c|$ for any $j \geq 0$, inductive hypothesis shows that $(b(j)c)(-1-q-j) \times w \in C_m(U, W)$. ☐

Proposition 4.5. *Let* $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ *be a* C_2*-finite vertex operator algebra, and* U *be a finite dimensional graded subspace of* V *such that* $V = U \oplus C_2(V)$. *Let* m *be a positive integer. Then there exists a positive integer* k *such that* $C_k(W) \subset C_m(U, W)$.

Proof. By Lemma 4.3 we know that $C_k(W) \subset C_{m,k}(W) + C_m(U, W)$ for any positive integer k. Then it suffices to show that there exists a positive integer k such that $C_{m,k}(W) \subset C_m(U, W)$. Let s_U be the maximum of the weights of homogeneous elements in U. For any $\alpha_i \in U\,(1 \leq i \leq r)$ and $m \geq n_1 > \cdots > n_r > 0$ we see that

$$\begin{aligned} |\alpha_1(-n_1) \cdots \alpha_r(-n_r)\mathbf{1}| &= \sum_{i=1}^{r} (|\alpha_i| - 1) + \sum_{i=1}^{r} n_i \\ &\leq r(s_U - 1) + \sum_{i=1}^{r} (m - i + 1) = -\frac{1}{2}r^2 + r\left(s_U + m - \frac{1}{2}\right), \end{aligned}$$

which is bounded from above by some positive integer $k_0 \geq \left(s_U + m - 1/2\right)^2/2$. We note that the constant k_0 depends only on m and U. Setting $k = k_0 m$ we get $C_{m,k}(W) \subset C_m(U, W)$ by Lemma 4.4 because any element in $C_{m,k}(W)$ is a linear combination of $a(-p)w$ for $|a| \leq k_0$, $w \in W$ and $p \geq k\,(\geq m|a|)$. ☐

Proposition 4.6. *Let* $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ *be a* C_2*-finite vertex operator algebra and* W *a weak* V*-module. If the module* W *is* B_1*-finite then* $W/C_m(U, W)$ *is finite-dimensional for any* $m > 0$.

Proof. By Proposition 4.5 there exists a positive integer k such that $C_k(W) \subset C_m(U, W)$. Since V is C_2-finite and W is B_1-finite, W is C_k-finite by Theorem 2.6. Thus $W/C_k(W)$ is finite-dimensional, and so is $W/C_m(U, W)$. ☐

Theorem 4.7. *Let* $V = \bigoplus_{n=0}^{\infty} V(n)$, $V(0) = \mathbb{C}\mathbf{1}$ *be a quasi-primary generated,* C_2*-finite vertex operator algebra such that* V *is a sum of highest weight modules for the Virasoro algebra, and let* $\widetilde{\Sigma} = (\Sigma\,; Q_1, \ldots, Q_N; z_1, \ldots, z_N; W^1, \ldots, W^N)$ *be a* *projective N-labeled Riemann surface. If all V-modules W^i ($i \in A$) are B_1-finite, then the conformal block $\mathcal{V}^\dagger(\widetilde{\Sigma})$ is finite-dimensional.*

Proof. Let U be a finite-dimensional graded subspace of V such that $V = U \oplus C_2(V)$. Since V is quasi-primary generated any vectors from U are linear combinations of $L^i_{-1}a$ for some $i \in \mathbb{N}$ and quasi-primary vectors a. Then we can further assume that any elements of U are quasi-primary because $L_{-1}a = a(-2)\mathbf{1} \in C_2(V)$ for any $a \in V$.

By Theorem 3.3 it suffices to prove that $\mathcal{QV}(\widetilde{\Sigma})$ is finite-dimensional. We set $M = M_U > 0$. The constant M_U is defined in the paragraph just before Lemma 4.2; recall that in order to define M_U we assume that U is linearly spanned by quasi-primary vectors. By Proposition 4.6 $W^i/C_M(U, W^i)$ is finite-dimensional for any $i \in A$. Thus Lemma 4.2 shows that $\mathrm{gr}_\bullet\ \mathcal{QV}(\widetilde{\Sigma})$ is finite-dimensional and so is $\mathcal{QV}(\widetilde{\Sigma})$. ☐

5. Examples

We present several examples of C_2-finite vertex operator algebras; affine vertex operator algebras (with positive integral level k), Virasoro vertex operator algebras (with minimal central charge $c_{p,q}$) and lattice vertex operator algebras. We will prove that all irreducible modules for these vertex operator algebras are B_1-finite. Then we see that those modules are all C_2-finite by Theorem 2.6. In fact C_2-finiteness for irreducible modules for affine and Virasoro vertex operator algebra is well known (cf. [5] for affine case, and [9] for Virasoro case). The C_2-finiteness for irreducible modules for lattice vertex operator algebras seems to be known, though we are not able to find any published material so far.

In order to prove the B_1-finiteness in the Virasoro case we follow the same argument being used in the proof of C_2-finiteness, however, we will see that verifying B_1-finiteness is much easier than C_2-finiteness.

EXAMPLE 5.1 (Affine vertex operator algebras). Let $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be an affine Lie algebra where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. We denote by $\{\Lambda_0, \dots, \Lambda_n\}$ the set of fundamental weights for $\hat{\mathfrak{g}}$, and by P^k_+ the set of all level k dominant integral weights. We denote the irreducible highest weight module of $\hat{\mathfrak{g}}$ with highest weight Λ by $L(\Lambda)$. It is known that if $k \neq -h^\vee, 0$ where $h^\vee$ is the dual Coxeter number of $\hat{\mathfrak{g}}$, then $L_k = L(k\Lambda_0)$ is a vertex operator algebra. Moreover, if k is a positive integer any irreducible L_k-module is realized as an irreducible $\hat{\mathfrak{g}}$-module $L(\Lambda)$ for some $\Lambda \in P^k_+$ (see [12]). The C_2-finiteness of L_k is known ([20], [5]).

We now prove the B_1-finiteness of irreducible L_k-modules. Since $L(\Lambda)$ is linearly spanned by vectors $a_1(-n_1) \cdots a_r(-n_r)v$ with $n_i > 0$, $a_i \in L_k(1) (\cong \mathfrak{g})$ and $v \in V_{\overline{\Lambda}}$, where $V_{\overline{\Lambda}}$ is the irreducible highest weight module for $\mathfrak{g}$ with the highest weight $\overline{\Lambda}$ and highest weight vector $v_{\overline{\Lambda}}$, where $\overline{\Lambda}$ is the classical part of Λ. We now see that $L(\Lambda) = V_{\overline{\Lambda}} + B_1(L(\Lambda))$, and that $L(\Lambda)$ is B_1-finite because $V_{\overline{\Lambda}}$ is finite-dimensional.

EXAMPLE 5.2 (Lattice vertex operator algebras). Let L be an even lattice of finite rank with a positive definite symmetric $\mathbb{Z}$-bilinear form $\langle\cdot\mid\cdot\rangle$. We set $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L$ and $\hat{\mathfrak{h}}=\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\mathfrak{h}\oplus\mathbb{C}K$; the latter is the affinization of the abelian Lie algebra $\mathfrak{h}$. Let L° be the dual lattice of L, and $\mathbb{C}[L^{\circ}]=\bigoplus_{\beta\in L^{\circ}}\mathbb{C}e_{\beta}$ be the twisted group algebra of L° with some cocycle which represents a central extension of L°. For any subset M of L° we write $\mathbb{C}[M]=\bigoplus_{\beta\in M}\mathbb{C}e_{\beta}$, and set $V_M=U(\hat{\mathfrak{h}}^{-})\otimes\mathbb{C}[M]$ where $\hat{\mathfrak{h}}^{-}=t^{-1}\mathbb{C}[t^{-1}]\otimes_{\mathbb{C}}\mathfrak{h}$ is a Lie subalgebra of $\hat{\mathfrak{h}}$. We note that the Lie algebra $\hat{\mathfrak{h}}$ canonically acts on V_M. Then it is known that V_L is a vertex operator algebra, and that $V_{\lambda+L}$ for $\lambda\in L^{\circ}$ give all irreducible V_L-modules (see [11]). The vertex operator associated to $e_{\alpha}\,(\alpha\in L)$ is

$$Y(e_{\alpha},x)=\exp\left(\sum_{n=1}^{\infty}\frac{\alpha(-n)}{n}x^{n}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\alpha(n)}{n}x^{-n}\right)e_{\alpha}x^{\alpha(0)},\quad \alpha(n)=t^{n}\otimes\alpha$$

where e_{α} acts on $\mathbb{C}[L^{\circ}]$ by the left multiplication, and the action of $x^{\alpha(0)}$ on $V_{L^{\circ}}$ is defined by $x^{\alpha(0)}(u\otimes e_{\mu})=x^{\langle\alpha|\mu\rangle}(u\otimes e_{\mu})$ for all $\mu\in L^{\circ}$ and $u\in U(\hat{\mathfrak{h}}^{-})$.

We now prove that for any $\lambda\in L^{\circ}$ the irreducible V_L-module $V_{\lambda+L}$ is B_1-finite. We set $\Gamma_{\lambda}=\{\,\beta\in L\mid\langle\beta-\alpha\mid\alpha+\lambda\rangle<0$ for any $\alpha\in L$ such that $\alpha\neq\beta,-\lambda\,\}$. The following lemma is due to H. Shimakura.

Lemma 5.3. *Let $\lambda\in L^{\circ}$. Then Γ_{λ} is a finite set.*

Proof. Let ϕ be the translation map on L° defined by $\phi(\gamma)=\gamma+\lambda$. We see that $\Gamma_{\lambda}=\{\,\beta\in L\mid\langle\delta\mid\beta+\lambda-\delta\rangle<0$ for any $\delta\in L$ such that $\delta\neq 0,\lambda+\beta\,\}$, and that

$$\phi(\Gamma_{\lambda})=\{\,\gamma\in\lambda+L\mid\langle\delta\mid\gamma-\delta\rangle<0 \text{ for any } \delta\in L \text{ such that } \delta\neq 0,\gamma\}.$$

Let $\alpha_1,\dots,\alpha_l$ be a basis of L, and let $\Lambda_1,\dots,\Lambda_l$ be the basis of L° such that $\langle\Lambda_i\mid\alpha_j\rangle=\delta_{ij}$. Let $\gamma\in\phi(\Gamma_{\lambda})$ and $\gamma\neq\pm\alpha_i\,(1\leq i\leq l)$. By the definition of $\phi(\Gamma_{\lambda})$ we have $\langle\gamma-(\pm\alpha_i)\mid\pm\alpha_i\rangle<0$ $(i=1,\dots,l)$, and we see that $\gamma\in\{\,\sum_{i=1}^{l}m_i\Lambda_i\mid m_i\in\mathbb{Z}$ and $|m_i|\leq\langle\alpha_i\mid\alpha_i\rangle$ for any $i\,\}$. Therefore, $\phi(\Gamma_{\lambda})$ is a finite set, and so is Γ_{λ}. □

We see that $V_{\lambda+L}=\sum_{\alpha\in L}\mathbb{C}e_{\lambda+\alpha}+B_1(V_{\lambda+L})$, and for any α, $\beta\in L$ we have $e_{\beta-\alpha}(-\langle\beta-\alpha\mid\lambda+\alpha\rangle-1)e_{\lambda+\alpha}=\pm e_{\lambda+\beta}$. Hence we find that $V_{\lambda+L}=\sum_{\alpha\in\Gamma_{\lambda}}\mathbb{C}e_{\lambda+\alpha}+B_1(V_{\lambda+L})$. Then Lemma 5.3 shows that $V_{\lambda+L}$ is B_1-finite.

EXAMPLE 5.4 (The Virasoro vertex operator algebras). Let $\mathcal{V}ir=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n\oplus\mathbb{C}C$ be the Virasoro algebra. Let $M(c,h)$ be the Verma module for the Virasoro algebra with a highest weight $h\in\mathbb{C}$ and central charge $c\in\mathbb{C}$. We denote by $v_{h,c}$ the highest weight vector, i.e., $v_{h,c}$ satisfies $L_nv_{h,c}=\delta_{n,0}hv_{h,c}\,(n\geq 0)$ and $Cv_{h,c}=cv_{h,c}$. The Verma module $M(c,h)$ is a rank one free $U(\mathcal{V}ir^{-})$-module with the generator $v_{h,c}$

where $\mathcal{V}ir^{-} = \bigoplus_{n \in \mathbb{Z}_{>0}} \mathbb{C}L_{-n}$. Let $L(c, h)$ be the irreducible quotient of $M(c, h)$. Then it is known that $L(c, 0)$ is a vertex operator algebra.

Let p, q be coprime positive integers. We set $c_{p,q} = 1 - 6(p - q)^2/pq$. For any integers r and s such that $1 \leq r < q$, $1 \leq s < p$ we denote $h_{p,q;r,s} = ((rp - sq)^2 - (p - q)^2)/4pq$. Then any irreducible $L(c_{p,q}, 0)$-module is isomorphic to $L(c_{p,q}, h_{p,q;r,s})$. We prove:

Proposition 5.5. *$L(c_{p,q}, 0)$ is C_2-finite and any irreducible $L(c_{p,q}, 0)$-module $L(c_{p,q}, h_{p,q;r,s})$ is B_1-finite.*

In order to prove the proposition we recall several properties of singular vectors v in $M(c_{p,q}, h_{p,q;r,s})$, i.e., $L_n v = v$ for $n > 0$. It is known that for any positive integers r and s satisfying $1 \leq r < q$ and $1 \leq s < p$ there exists a unique singular vector $u_{r,s} \in M(c_{p,q}, h_{p,q;r,s})$ such that $L_0 u_{r,s} = (h_{p,q;r,s} + rs)u_{r,s}$ up to scalar multiples. The explicit form of the singular vector is not known, but we have a partial formula which expresses this singular vector as explained below.

Let us fix central charge $c = c_{p,q}$ and highest weight $h = h_{p,q;r,s}$. We set $\mathcal{V}ir^{\leq -3} = \bigoplus_{n \geq 3} \mathbb{C}L_{-n}$. The set $\mathcal{V}ir^{\leq -3}$ is a Lie subalgebra of $\mathcal{V}ir$. We define a linear isomorphism $\phi\colon \mathbb{C}[x, y] \to M(c, h)/\mathcal{V}ir^{\leq -3}M(c, h)$ by

$$x^i y^j \longmapsto L_{-2}^j L_{-1}^i v_{h_{r,s},c} + \mathcal{V}ir^{\leq -3}M(c, h).$$

Let $f\colon M(c, h) \to M(c, h)/\mathcal{V}ir^{\leq -3}M(c, h)$ be the canonical projection. We define $\pi = \phi^{-1} \circ f$. The following proposition is proved in [8]:

Proposition 5.6. *Let $c = c_{p,q}$ and $h = h_{p,q;r,s}$. Let $u_{r,s} \in M(c, h)$ be the singular vector such that $L_0 u_{r,s} = (h + rs)u_{r,s}$. Then $\pi(u_{r,s}) = \alpha F_{r,s}(x, y; p/q)$ for some nonzero constant α, where $F_{r,s}(x, y; t)$ is a polynomial of $\mathbb{C}[x, y, t, t^{-1}]$ given by*

$$F_{r,s}(x, y; t)^2 = \prod_{k=0}^{r-1} \prod_{l=0}^{s-1} (x^2 - \{(r - 2k - 1)t^{1/2} - (s - 2l - 1)t^{-1/2}\}^2 y).$$

We now can prove Proposition 5.5.

Proof of Proposition 2.2. The canonical projection $\psi\colon M(c, h) \to L(c, h)$ maps the subspace $\mathcal{V}ir^{\leq -3}M(c, h)$ into $C_2(L(c, h))$. Any singular vectors in $M(c, h)$ are in the kernel of this map. We note that $h_{p,q;r,s} = h_{p,q;q-r,p-s}$, in particular, there exists a singular vector $u_{q-r,p-s}$ such that $L_0 u_{q-r,p-s} = (h + (p - s)(q - r))u_{q-r,p-s}$. We see that the composition of ϕ and ψ induces a surjective linear map

$$\psi\colon \mathbb{C}[x, y]/(F_{r,s}(x, y; p/q), F_{q-r,p-s}(x, y; p/q)) \longrightarrow L(c, h)/C_2(L(c, h)). \tag{5.1}$$

First we find that $F_{1,1}(x, y; p/q) = x$ and $F_{q-1,p-1}(x, y; p/q) \equiv \alpha' y^{(p-1)(q-1)/2}$

mod (x) for some nonzero $\alpha' \in \mathbb{C}$. Thus $\mathbb{C}[x, y]/(F_{1,1}(x, y; p/q), F_{q-1,p-1}(x, y; p/q))$ is finite-dimensional. So $L(c, 0)$ is C_2-finite.

Finally we prove that any irreducible module $L(c, h)$ is B_1-finite. We find that the surjective map (5.1) induces a surjective map

$$\mathbb{C}[x, y]/(y, F_{r,s}(x, y; p/q), F_{q-r,p-s}(x, y; p/q)) \to L(c, h)/B_1(L(c, h)).$$

Since

$$\mathbb{C}[x, y]/(y, F_{r,s}(x, y; p/q), F_{q-r,p-s}(x, y; p/q)) \cong \mathbb{C}[x]/(x^{rs}, x^{(q-r)(p-s)})$$

is finite-dimensional we see that $L(c, h)$ is B_1-finite for any $1 \leq r < q$ and $1 \leq s < p$. □

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Toshiyuki Abe
Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka, Osaka 560-0043, Japan

Current address:
Graduate School of Mathematical Sciences
University of Tokyo
Komaba 3-8-1, Tokyo, 153-8914, Japan
e-mail: abe@ms.u-tokyo.ac.jp

Kiyokazu Nagatomo
Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka, Osaka 560-0043, Japan
e-mail: nagatomo@math.sci.osaka-u.ac.jp