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ON A HYBRID MEAN VALUE OF CERTAIN HARDY SUMS AND RAMANUJAN SUM

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1. Introduction

Let c be a natural number and d an integer prime to c. The classical Dedekind sum

$$S(d, c) = \sum_{j=1}^{c} \left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right), \tag{1}$$

with

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer,} \end{cases}$$

describes the behaviour of the logarithm of eta-function (cf. [7]) under modular transformations. B.C. Berndt [3] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 z), \qquad \operatorname{Im} z > 0.$$

Put $Vz = (az+b)(cz+d)$ with $a, b, c, d \in Z$, $c > 0$, and $ad - bc = 1$. Then

$$\log\theta(Vz) = \log\theta(z) + \frac{1}{2}\log(cz+d) - \frac{1}{4}\pi i + \frac{1}{4}\pi i S_1(d, c), \tag{2}$$

where

$$S_1(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

The sums $S_1(d, c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums (cf., e.g., Lemma 2). Some arithmetical

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properties of $S_1(d, c)$ can be found in R. Sitaramachandraro [10]. In [13], the author studied the $2m$-th power mean of $S_1(d, c)$, and proved that the asymptotic formula

$$\sum_{h=1}^{p-1} |S_1(h, p)|^{2m} = p^{2m} \frac{\zeta^2(2m)\left(1 - 1/4^m\right)}{\zeta(4m)\left(1 + 1/4^m\right)} + O\left(p^{2m-1} \exp\left(\frac{6 \ln p}{\ln \ln p}\right)\right)$$

holds for all odd prime p and positive integer m, where $\zeta(s)$ is the Riemann zeta-function and $\exp(y) = e^y$. In this paper, we shall study the distribution problem of the hybrid mean value involving $S_1(d, c)$ and Ramanujan's sum

$$R_c(d) = \sum_{b=1}^{c}{}' e\left(\frac{db}{c}\right),$$

where $e(y) = e^{2\pi i y}$, $\sum_b'$ denotes the summation over all b such that $(b, c) = 1$. In fact, we use the estimates for character sums and the mean value theorem of Dirichlet L-functions to obtain an interesting hybrid mean value formula involving $S_1(d, c)$ and $R_c(d)$. That is, we shall prove the following:

Theorem. *Let $c \geq 3$ be an odd number. Then we have the asymptotic formula*

$$\sum_{h=1}^{c}{}' R_c(2h+1) S_1(2h, c) = \phi^2(c) \prod_{p|c}\left(1 + \frac{1}{p}\right) \prod_{p\|c}\left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O\left(c^{1+\epsilon}\right),$$

where ϵ be any fixed positive number, $\phi(d)$ be the Euler function and $\prod_{p\|c}$ denotes the product over prime divisor p of c such that $p \mid c$ and $p^2 \nmid c$.

From this Theorem we may immediately deduce the following:

Corollary. *If $c \geq 3$ be a square-full odd number, then we have*

$$\sum_{h=1}^{c}{}' R_c(2h+1) S_1(2h, c) = \phi^2(c) \prod_{p|c}\left(1 + \frac{1}{p}\right) + O\left(c^{1+\epsilon}\right).$$

2. Some Lemmas

To complete the proof of Theorem, we need the following Lemmas.

Lemma 1. *Let integer $q \geq 3$ and $(h, q) = 1$. Then we have the identity*

$$S(h, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) \left|L(1, \chi)\right|^2,$$

where χ denotes a Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof (see reference [11]). □

Lemma 2. *Let integer $q \geq 2$ and $(h,q) = 1$. Then we have the identity*

$$S_1(h,q) = -8S(h+q, 2q) + 4S(h,q).$$

Proof. This formula is an immediate consequence of (5.9) and (5.10) in [10]. □

Lemma 3. *Let χ be a character modulo q, generated by the primitive character χ_m modulo m. Then we have the identity*

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right)\mu\left(\frac{q}{m}\right)\tau(\chi_m),$$

where $\mu(n)$ be the Möbius function.

Proof (see Lemma 1.3 of reference [2]). □

Lemma 4. *Let q and r be integers with $q \geq 2$ and $(r,q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of primitive characters modulo q.

Proof. From the properties of characters we know that for any character χ modulo q, there exists one and only one $d \mid q$ and primitive character χ_d^* modulo d such that $\chi = \chi_d^*\chi_q^0$, where χ_q^0 denotes the principal character modulo q. So we have

$$\sum_{\chi \bmod q} \chi(r) = \sum_{d|q} \sideset{}{^*}\sum_{\chi \bmod d} \chi(r)\chi_q^0(r) = \sum_{d|q} \sideset{}{^*}\sum_{\chi \bmod d} \chi(r).$$

Combining this formula and Möbius inversion, and noting the identity

$$\sum_{\chi \bmod q} \chi(r) = \begin{cases} \phi(q), & \text{if } r \equiv 1 \pmod q; \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d|q} \mu(d) \sum_{\chi \bmod (q/d)} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right) \phi(d).$$

Taking $r = 1$, we immediately get

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right).$$

This proves Lemma 4. □

Lemma 5. *Let $q > 1$ be an odd number and $(h, q) = 1$. Then we have*

$$S_1(h,q) = \begin{cases} -\dfrac{16}{\pi^2 q} \displaystyle\sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi\chi_2^0)|^2, & \text{if } 2 \mid h\ ; \\ 0, & \text{if } 2 \nmid h\ , \end{cases}$$

where χ_d^0 denotes the principal character $\bmod d$.

Proof. Note that $\sum_{d|2q} f(d) = \sum_{d|q} f(d) + \sum_{d|q} f(2d)$. So from Lemma 1 and Lemma 2 we get

$$\begin{aligned} S_1(h,q) &= -8S(h+q, 2q) + 4S(h,q) \\ &= -\frac{4}{\pi^2 q} \sum_{d|2q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h+q) |L(1,\chi)|^2 \\ &\quad + \frac{4}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1,\chi)|^2 \\ &= -\frac{4}{\pi^2 q} \sum_{d|q} \frac{(2d)^2}{\phi(2d)} \sum_{\substack{\chi \bmod 2d \\ \chi(-1)=-1}} \chi(h+q) |L(1,\chi)|^2. \end{aligned} \tag{3}$$

It is clear that 2 has only one principal character χ_2^0, so for any odd number q, from (3) we obtain

$$S_1(h,q) = -\frac{16}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h+q) \chi_2^0(h+q) |L(1, \chi\chi_2^0)|^2$$

$$= \begin{cases} -\dfrac{16}{\pi^2 q} \displaystyle\sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi\chi_2^0)|^2, & \text{if } 2 \mid h \ ; \\ 0, & \text{if } 2 \nmid h \ . \end{cases}$$

This completes the proof of Lemma 5. □

Lemma 6. *Let q be any integer with $q \geq 3$. Then for any integer $k \mid q$ with $k \geq 2$, we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} |L(1, \chi\chi_q^0)|^2 = \frac{\pi^2}{12} J(k) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O\left(q^\epsilon\right),$$

where χ_q^0 be the principal character.

Proof. First for any parameter $q \leq N \leq q^2$ and any non-principal character χ modulo k, applying Abel's identity (see reference [1]) we have

$$L(1, \chi\chi_q^0) = \sum_{n=1}^{\infty} \frac{\chi(n)\chi_q^0(n)}{n} = \sum_{1 \leq n \leq N} \frac{\chi(n)\chi_q^0(n)}{n} + \int_N^\infty \frac{A(y, \chi\chi_q^0)}{y^2} dy, \tag{4}$$

where $A(y, \chi\chi_q^0) = \sum_{N<n\leq y} \chi(n)\chi_q^0(n)$. Applying Pólya-Vinogradov inequality we have

$$|A(y, \chi\chi_q^0)| = \left| \sum_{N<n\leq y} \chi(n)\chi_q^0(n) \right| = \left| \sum_{d|q} \mu(d)\chi(d) \sum_{N/d<n\leq y/d} \chi(n) \right|$$
$$\ll \left(\sum_{d|q} |\mu(d)| \right) \sqrt{k} \ \ln k \ll 2^{\omega(q)} \sqrt{k} \ \ln k \ll k^{1/2} q^\epsilon,$$

where $\omega(q)$ denotes the number of all different prime divisors of q. From this estimate and (4) we get

$$L(1, \chi\chi_q^0) = \sum_{1\leq n\leq N} \frac{\chi(n)\chi_q^0(n)}{n} + O\left(\frac{k^{1/2} q^\epsilon}{N} \right)$$

and

$$\sideset{}{^*}\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \left| L(1, \chi\chi_q^0) \right|^2 = \sideset{}{^*}\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \left| \sum_{1\leq n\leq N} \frac{\chi(n)\chi_q^0(n)}{n} \right|^2$$

$$+ O\left(\frac{k^{1/2}q^{\epsilon}}{N}\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\left|\sum_{1\le n\le N}\frac{\chi(n)\chi_q^0(n)}{n}\right|\right) + O\left(\frac{k^2q^{\epsilon}}{N^2}\right)$$

(5)
$$= \sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\left|\sum_{1\le n\le N}\frac{\chi(n)\chi_q^0(n)}{n}\right|^2 + O\left(\frac{k^{3/2}q^{\epsilon}}{N}\right).$$

Note that for $(a,k)=1$, from Lemma 4 we have

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\chi(a) &= \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}(1-\chi(-1))\,\chi(a)\\ &= \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}\chi(a) - \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod k}\chi(-a)\\ &= \frac{1}{2}\sum_{u\mid(k,a-1)}\mu\left(\frac{k}{u}\right)\phi(u) - \frac{1}{2}\sum_{u\mid(k,a+1)}\mu\left(\frac{k}{u}\right)\phi(u).\end{aligned}$$

So that we have

$$\begin{aligned}&\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\left|\sum_{1\le n\le N}\frac{\chi(n)\chi_q^0(n)}{n}\right|^2\\ &= \sum_{1\le m\le N}\sum_{1\le n\le N}\frac{\chi_q^0(mn)}{mn}\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\chi(m)\overline{\chi}(n)\\ &= \frac{1}{2}\sum_{1\le m\le N}\sum_{1\le n\le N}\frac{\chi_q^0(mn)}{mn}\sum_{u\mid(k,m-n)}\mu\left(\frac{k}{u}\right)\phi(u)\\ &\quad - \frac{1}{2}\sum_{1\le m\le N}\sum_{1\le n\le N}\frac{\chi_q^0(mn)}{mn}\sum_{u\mid(k,m+n)}\mu\left(\frac{k}{u}\right)\phi(u)\\ &= \frac{1}{2}J(k)\sum_{\substack{1\le n\le N\\ (n,q)=1}}\frac{1}{n^2} + O\left(\sum_{u\mid k}\phi(u)\sum_{\substack{1\le n\le N\\ (n,q)=1}}\sum_{1\le h\le N/u}\frac{1}{n(hu+n)}\right)\\ &\quad + O\left(\sum_{u\mid k}\phi(u)\sum_{1\le n<u}\frac{1}{n(u-n)}\right) + O\left(\sum_{u\mid k}\sum_{\substack{1\le n\le N\\ (n,q)=1}}\sum_{1+(n/u)\le h\le N/u}\frac{\phi(u)}{n(hu-n)}\right)\end{aligned}$$

$$= \frac{1}{2} J(k)\zeta(2) \prod_{p|q}\left(1-\frac{1}{p^2}\right) + O\left(\sum_{u|k}\frac{\phi(u)}{u}\ln^2 N\right)$$

$$= \frac{\pi^2}{12} J(k) \prod_{p|q}\left(1-\frac{1}{p^2}\right) + O\left(q^{\epsilon}\right), \tag{6}$$

where $\zeta(n)$ be the Riemann zeta-function and $\zeta(2) = \pi^2/6$.

Taking $N = q^2$, combining (4), (5) and (6) we may immediately obtain the asymptotic formula

$$\sideset{}{^*}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} |L(1, \chi\chi_q^0)|^2 = \frac{\pi^2}{12} J(k) \prod_{p|q}\left(1-\frac{1}{p^2}\right) + O\left(q^{\epsilon}\right).$$

This completes the proof of Lemma 6. □

3. Proof of the Theorem

In this section, we complete the proof of Theorem. First for any primitive character χ_m modulo m, from the properties of Gauss sums we have

$$\tau(\chi_m)\tau(\overline{\chi_m}) = -m \text{ if } \chi(-1) = -1$$

and for any odd number c and $d \mid c$ with any character χ modulo d, note that

$$\sideset{}{'}\sum_{h=1}^{c} \chi(2h)R_c(2h+1) = \sideset{}{'}\sum_{b=1}^{c}\sum_{h=1}^{c} \chi(2h)\chi_c^0(2h)e\left(\frac{(2h+1)b}{c}\right)$$

$$= \sideset{}{'}\sum_{b=1}^{c} \overline{\chi}(b)\chi_c^0(b)e\left(\frac{b}{c}\right) \sideset{}{'}\sum_{h=1}^{c} \chi(h)\chi_c^0(h)e\left(\frac{h}{c}\right) = \tau(\overline{\chi}\chi_c^0)\tau(\chi\chi_c^0). \tag{7}$$

From (7), Lemma 3 and Lemma 5 we have

$$\sideset{}{'}\sum_{h=1}^{c} S_1(2h, c)R_c(2h+1)$$

$$= -\frac{16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \left(\sideset{}{'}\sum_{h=1}^{c} \chi(2h)R_c(2h+1)\right) |L(1, \chi\chi_2^0)|^2$$

$$= -\frac{16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \tau(\overline{\chi}\chi_c^0)\tau(\chi\chi_c^0)|L(1, \chi\chi_2^0)|^2$$

$$= -\frac{16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)}\sum_{m|d}\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \tau(\overline{\chi}\chi_c^0)\tau(\chi\chi_c^0)|L(1,\chi\chi_d^0\chi_2^0)|^2$$

$$= \frac{-16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)}\sum_{m|d}\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \left|\chi\left(\frac{c}{m}\right)\right|^2\mu^2\left(\frac{c}{m}\right)\tau(\overline{\chi})\tau(\chi)|L(1,\chi\chi_{2d}^0)|^2$$

$$= \frac{16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)}\sum_{m|d}\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} m\left|\chi\left(\frac{c}{m}\right)\right|^2\mu^2\left(\frac{c}{m}\right)|L(1,\chi\chi_{2d}^0)|^2. \tag{8}$$

Let $c = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Now for any $m \mid c$ and χ modulo m, it is clear that $\chi(c/m) = 0$, if $(c/m, m) > 1$. $\chi(c/m)\mu(c/m) \neq 0$ if and only if $m = ut$, where $t \mid v$. From these properties, (8) and Lemma 6 we have

$$\sideset{}{'}\sum_{h=1}^{c} R_c(2h+1)S_1(2h,q)$$

$$= \frac{16}{\pi^2 c}\sum_{d|c}\frac{d^2}{\phi(d)}\sum_{m|d}\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} m\left|\chi\left(\frac{c}{m}\right)\right|^2\mu^2\left(\frac{c}{m}\right)|L(1,\chi\chi_{2d}^0)|^2$$

$$= \frac{16}{\pi^2 c}\sum_{d|v}\frac{u^2d^2}{\phi(ud)}\sum_{m|d}\sideset{}{^*}\sum_{\substack{\chi \bmod um\\ \chi(-1)=-1}} mu|L(1,\chi\chi_{2ud}^0)|^2$$

$$= \frac{16u^2}{\pi^2 v\phi(u)}\sum_{d|v}\frac{d^2}{\phi(d)}\sum_{m|d} m \sideset{}{^*}\sum_{\substack{\chi \bmod um\\ \chi(-1)=-1}} |L(1,\chi\chi_{2ud}^0)|^2$$

$$= \frac{16u^2}{\pi^2 v\phi(u)}\sum_{d|v}\frac{d^2}{\phi(d)}\sum_{m|d} m\left(\frac{\pi^2}{12}J(um)\prod_{p|2ud}\left(1-\frac{1}{p^2}\right) + O\left((ud)^{\epsilon}\right)\right)$$

$$= \frac{16u^2J(u)}{12v\phi(u)}\sum_{d|v}\frac{d^2}{\phi(d)}\sum_{m|d} mJ(m)\prod_{p|2ud}\left(1-\frac{1}{p^2}\right) + O\left(c^{1+\epsilon}\right)$$

$$= \frac{u^2\phi^2(u)}{u\phi(u)}\frac{1}{v}\prod_{p|u}\left(1-\frac{1}{p^2}\right)\sum_{d|v}\frac{d^2}{\phi(d)}\prod_{p|d}\left(1-\frac{1}{p^2}\right)\sum_{m|d} mJ(m) + O\left(c^{1+\epsilon}\right)$$

$$= \phi^2(u)\prod_{p|u}\left(1+\frac{1}{p}\right)\cdot\frac{1}{v}\sum_{d|v} d\prod_{p|d}\left(1+\frac{1}{p}\right)(1+p(p-2)) + O\left(c^{1+\epsilon}\right)$$

$$= \phi^2(u)\prod_{p|u}\left(1+\frac{1}{p}\right)\cdot\frac{1}{v}\prod_{p|v}\left(1+(p+1)(p-1)^2\right) + O\left(c^{1+\epsilon}\right)$$

$$= \phi^2(uv) \prod_{p|vu} \left(1 + \frac{1}{p}\right) \cdot \prod_{p|v} \left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O\left(c^{1+\epsilon}\right)$$
$$= \phi^2(c) \prod_{p|c} \left(1 + \frac{1}{p}\right) \prod_{p\|c} \left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O\left(c^{1+\epsilon}\right),$$

where we have used the identity $J(u) = \phi^2(u)/u$, if u be a square-full number. This completes the proof of Theorem.

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