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Osaka J. Math.
40 (2003), 313–326

PERFECT ISOMETRIES AND THE ISAACS CORRESPONDENCE

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(Received May 21, 2001)

1. Introduction

Suppose that S and G are finite groups such that S acts on G coprimely. Let B an S invariant p-block of G such that S centralizes some defect group D of B. In [10], Watanabe proved that whenever S is solvable, then there is a perfect isometry between B and the set of the Glauberman correspondents of the characters in B. Horimoto in [2] proved the case where S is nonsolvable.

Now, let G be a group of odd order. Let q be a prime and Q a Sylow q-subgroup of G. By $\mathrm{Irr}_{q'}(G)$, we denote the set of irreducible characters of G which have degree not divisible by q. When G is a solvable group of odd order, M. Isaacs constructed a natural one-to-one correspondence

$$^*\colon\ \mathrm{Irr}_{q'}(G) \to \mathrm{Irr}_{q'}(\mathbf{N}_G(Q))$$

which depends only on G and Q (see [3]).

In this paper, we show that there is also a perfect isometry between a block B where all irreducible characters of this have degree not divisible by q and the set of Isaacs correspondents of the characters in B. This complements the work by A. Watanabe and H. Horimoto.

Theorem A. *Suppose that G is a finite group of odd order and p and q are distinct prime numbers. Let B be a p-block of G such that every irreducible character of B has q'-degree. Let D be a defect group of B. Then there exists a unique p-block B^* of $\mathbf{N}_G(Q)$, for some $Q \in \mathrm{Syl}_q(G)$, with defect group D such that* $\mathrm{Irr}(B^*) = \{\chi^* \mid \chi \in \mathrm{Irr}(B)\}$. *Moreover, there exists a perfect isometry R such that $R(\chi) = \chi^*$ for* $\chi \in \mathrm{Irr}(B)$.

Some of the results of this paper were obtained while I was visiting Ohio University. I would like to thank the Mathematics Department for its hospitality. I would also like to thank G. Navarro for many helpful suggestions.

Research partially supported by DGICYT.

2. Preliminaries

In this section, we review the Isaacs correspondence and we prove some properties of this. We present the Isaacs correspondence for the prime p as it was defined in [3]. Let P be a Sylow p-subgroup of G.

Theorem 2.1 (Isaacs). *Suppose that G is a group of odd order. Suppose that $G = KH$, where K, $L \lhd G$, $K \cap H = L$ and K/L is abelian. Suppose that $H = L\mathbf{N}_G(P)$. Let $\theta \in \mathrm{Irr}(L)$ be P-invariant. If $\chi \in \mathrm{Irr}_{p'}(G)$ lies over θ, then*

$$\chi_H = \chi^{(H)} + 2\Delta + \beta\,,$$

where $\chi^{(H)}$ has p'-degree and lies over θ and no irreducible constituent of β lies over θ. Moreover the map $\chi \mapsto \chi^{(H)}$ is a bijection between $\mathrm{Irr}_{p'}(G \mid \theta)$ and $\mathrm{Irr}_{p'}(H \mid \theta)$.

Proof. This is Theorem 10.6 of [3]. □

Lemma 2.2. *Let G be a group of odd order. Suppose that $H = \mathbf{O}^{p'p}(G)'\,\mathbf{N}_G(P)$. Let $\mathbf{O}^{p'}(G) \subseteq J \subseteq G$. Let $\chi \in \mathrm{Irr}_{p'}(G)$ and let $\theta \in \mathrm{Irr}_{p'}(J)$. Then all irreducible constituents of χ_J have p'-degree and*

$$[\chi_J, \theta] = \left[(\chi^{(H)})_{J\cap H}, \theta^{(H\cap J)}\right].$$

In particular, if $\theta^G = \chi$, then

$$\left(\theta^{(H\cap J)}\right)^H = \chi^{(H)}.$$

Proof. Follows from Lemma 2.9 of [9]. □

Suppose that $G = G_0$ and write $G_{i+1} = \mathbf{O}^{p'p}(G_i)'\,\mathbf{N}_G(P)$. The Isaacs correspondence *: $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ is obtained by using Theorem 2.1 with respect to the chain

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{N}_G(P)\,.$$

First of all, we review some properties of the Isaacs correspondence.

Theorem 2.3. *Let G be a group of odd order. Suppose that H is a subgroup of G containing $\mathbf{O}^{p'p}(G)'\,\mathbf{N}_G(P)$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then $(\chi^{(H)})^* = \chi^*$.*

Proof. This is Theorem 2.3 of [9]. □

Theorem 2.4. *Let G be a group of odd order. Suppose that $P \subseteq J \subseteq G$ and let $\xi \in \mathrm{Irr}(J)$ such that $\xi^G = \chi \in \mathrm{Irr}(G)$. Let $\xi^* \in \mathrm{Irr}(\mathbf{N}_J(P))$ and $\chi^* \in \mathrm{Irr}(\mathbf{N}_G(P))$ be the*

Isaacs correspondents of ξ *and* χ*, respectively. Then* $(\xi^*)^{\mathbf{N}_G(P)} = \chi^*$.

Proof. This is Theorem A of [9]. □

A key tool for proving our main result is the following.

Theorem 2.5. *Let* G *be a group of odd order and let* $M \lhd G$*. Suppose that* $\chi \in \mathrm{Irr}_{p'}(G)$ *and* $\theta \in \mathrm{Irr}_{p'}(M)$ *is* P*-invariant. Then*
(a) $[\chi_M, \theta] \neq 0$ *if and only if* $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$ *for some* $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$.
(b) *If* $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$*, then*

$$I_G(\theta) \cap \mathbf{N}_G(P) = I_{\mathbf{N}_G(P)}\big((\varphi^*)_{\mathbf{N}_M(P)}\big) .$$

Moreover, if χ_θ *is the Clifford correspondent of* χ *over* θ *and* $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$*, then* χ_θ^* *is the Clifford correspondent of* χ^* *over* $(\varphi^*)_{\mathbf{N}_M(P)}$.

Notice that $(\varphi^*)_{\mathbf{N}_M(P)} \in \mathrm{Irr}(\mathbf{N}_M(P))$ since $\mathbf{N}_M(P)$ is a normal subgroup of $\mathbf{N}_{MP}(P)$ of p-index. As an inmediate consequence of this theorem, we have the following result.

Corollary 2.6. *Let* G *be a group of odd order and let* $P \subseteq M \lhd G$*. Suppose that* $\chi\, \mathrm{Irr}_{p'}(G)$ *and* $\theta \in \mathrm{Irr}_{p'}(M)$*. Then*
(a) $[\chi_M, \theta] \neq 0$ *if and only if* $[(\chi^*)_{\mathbf{N}_M(P)}, \theta^*] \neq 0$.
(b) *We have that*

$$I_G(\theta) \cap \mathbf{N}_G(P) = I_{\mathbf{N}_G(P)}(\theta^*) .$$

Moreover, if χ_θ *is the Clifford correspondent of* χ *over* θ*, then* χ_θ^* *is the Clifford correspondent of* χ^* *over* θ^*.

Proof. Follows from Theorem 2.5. □

First of all, we prove the Theorem 2.5 for the correpondence described in Theorem 2.1.

Proposition 2.7. *Let* G *be a group of odd order and let* $\mathbf{O}^{p'p}(G) \subseteq M \lhd G$*. Suppose that* $\chi \in \mathrm{Irr}_{p'}(G)$ *and* $\theta \in \mathrm{Irr}_{p'}(M)$ *is* P*-invariant. If* $H = \mathbf{O}^{p'p}(G)'\,\mathbf{N}_G(P)$*, then*
(a) $[\chi_M, \theta] \neq 0$ *if and only if* $[(\chi^{(H)})_{H \cap MP}, \varphi^{(H \cap MP)}] \neq 0$ *for some* $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$.
(b) *If* $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$*, then*

$$I_G(\theta) \cap H = I_H\big((\varphi^{(H \cap MP)})_{H \cap M}\big) .$$

Moreover, if χ_θ is the Clifford correspondent of χ over θ and $[(\chi^{(H)})_{H\cap MP}, \varphi^{(H\cap MP)}] \neq 0$, then $\chi_\theta^{(H\cap I_G(\theta))}$ is the Clifford correspondent of $\chi^{(H)}$ over $(\varphi^{(H\cap MP)})_{H\cap M}$.

Proof. Since $MP \lhd G$, we have that every irreducible constituent of χ_{MP} has p'-degree. And, by Lemma 2.2 we have that $[\chi_{MP}, \varphi] = [(\chi^{(H)})_{H\cap MP}, \varphi^{(H\cap MP)}]$. Hence, the part (a) follows.

Now, suppose that $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$. Let $\xi \in \mathrm{Irr}(\mathbf{O}^{p'p}(G)')$ be a P-invariant character which lies under θ and hence, under φ. By Theorem 2.1, we have that

$$\varphi_{H\cap MP} = \varphi^{(H\cap MP)} + 2\Delta + \beta$$

where $\varphi^{(H\cap MP)}$ has p'-degree and lies over ξ and no irreducible constituent of β lie over ξ (and hence over any of the H-conjugate of ξ). Write $\alpha = (\varphi^{(H\cap MP)})_{H\cap M}$. We have that

$$\theta_{H\cap M} = \varphi_{H\cap M} = \left(\varphi^{(H\cap MP)}\right)_{H\cap M} + 2\Delta_{H\cap M} + \beta_{H\cap M} = \alpha + 2\Delta_{H\cap M} + \beta_{H\cap M}$$

Let $h \in I_G(\theta) \cap H$. Since α and α^h are irreducible constituents of θ with odd multiplicity lying over ξ, it follows that $\alpha^h = \alpha$. And $h \in I_H(\alpha)$. Now, suppose that $h \in I_H(\alpha)$ Notice that if $h \in I_H(\varphi^{(H\cap MP)}) = I_G(\varphi) \cap H$ then $h \in I_G(\theta)$. Thus, we may assume that $h \notin I_H(\varphi^{(H\cap MP)})$. Hence, by Gallagher's lemma, we have that $(\varphi^{(H\cap PM)})^h = \varphi^{(H\cap PM)}\lambda$ for some linear character $\lambda \in \mathrm{Irr}(H \cap PM/H \cap M)$ (we can also see $\lambda \in \mathrm{Irr}(PM/M)$). By Lemma 2.2, we deduce that $\varphi^h = \varphi\lambda$. Then $\theta^h = (\varphi\lambda)_M = \theta$. Therefore, $h \in I_G(\theta) \cap H$ and the first part of (b) is proven.

Now, suppose that χ_θ be the Clifford correspondent of χ over θ and assume that $[(\chi^{(H)})_{H\cap MP}, \varphi^{(H\cap MP)}] \neq 0$. Write $I = I_G(\theta)$. By the first part of (b) we have that $\chi_\theta^{(H\cap I)} \in \mathrm{Irr}_{p'}(I_H(\alpha) \mid \alpha)$. By Lemma 2.2, it follows that $\chi^{(H)} = (\chi_\theta^{(H\cap I)})^H$. Therefore $\chi_\theta^{(H\cap I)}$ is the Clifford correspondent of $\chi^{(H)}$ over α, as desired. □

As consequence we have the following result which is a particular case of Theorem 2.5.

Proposition 2.8. *Let G be a group of odd order and let $\mathbf{O}^{p'p}(G) \subseteq M \lhd G$. Suppose that $\chi \in \mathrm{Irr}_{p'}(G)$ and $\theta \in \mathrm{Irr}_{p'}(M)$ is P-invariant. Then*
(a) *$[\chi_M, \theta] \neq 0$ if and only if $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$ for some $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$.*
(b) *If $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$, then*

$$I_G(\theta) \cap \mathbf{N}_G(P) = I_{\mathbf{N}_G(P)}((\varphi^*)_{\mathbf{N}_G(P)\cap M})\,.$$

Moreover, if χ_θ is the Clifford correspondent of χ over θ and $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$, then χ_θ^* is the Clifford correspondent of χ^* over $(\varphi^*)_{\mathbf{N}_G(P)\cap M}$.

Proof. We argue by induction on $|G|$. Let $H = \mathbf{O}^{p'p}(G)'\mathbf{N}_G(P)$. If $H = G$, then there is nothing to prove, and thus we suppose that $H < G$. Let $\chi^{(H)} \in \mathrm{Irr}_{p'}(H)$ the correspondent of χ by the correspondence of Theorem 2.1. We have that $\chi^* = (\chi^{(H)})^* \in \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ by Theorem 2.3. By Proposition 2.7 we have that $[\chi_M, \theta] \neq 0$ if and only if $[(\chi^{(H)})_{H\cap MP}, \varphi^{H\cap MP}] \neq 0$ for some $\varphi \in \mathrm{Irr}(MP \mid \theta)$. Since $H \cap MP \lhd H$, by induction, we have that $[(\chi^{(H)})_{H\cap MP}, \varphi^{(H\cap MP)}] \neq 0$ if and only if $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$. And (a) follows.

Now, let $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$. By Proposition 2.7, we know that

$$I_G(\theta) \cap H = I_H\left((\varphi^{(H\cap MP)})_{H\cap M}\right).$$

Since $H \cap M \lhd H$ by induction (applied to $(\varphi^{(H\cap MP)})_{H\cap M}$) we have that

$$I_H\left((\varphi^{(H\cap MP)})_{H\cap M}\right) \cap \mathbf{N}_G(P) = I_{\mathbf{N}_G(P)}\left((\varphi^*)_{\mathbf{N}_G(P)\cap M}\right).$$

Hence,

$$I_G(\theta) \cap \mathbf{N}_G(P) = I_{\mathbf{N}_G(P)}\left((\varphi^*)_{\mathbf{N}_G(P)\cap M}\right)$$

as desired.

Now, suppose that χ_θ be the Clifford correspondent of χ over θ and assume that $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$. Notice that $[(\chi^{(H)})_{H\cap MP}, \varphi^{(H\cap MP)}] \neq 0$. Write $I = I_G(\theta)$. By Proposition 2.7, we have that $\chi_\theta^{(H\cap I)}$ is the Clifford correspondent of $\chi^{(H)}$ over $(\varphi^{(H\cap MP)})_{H\cap M}$. And, by induction (applied to $(\varphi^*)_{\mathbf{N}_G(P)\cap M}$) we have that $\chi^*_\theta = (\chi_\theta^{(H\cap I)})^*$ is the Clifford correspondent of $\chi^* = (\chi^{(H)})^*$ over $(\varphi^*)_{\mathbf{N}_G(P)\cap M}$, as desired. □

Now, we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. We argue by induction on $|G|$. Write $N = \mathbf{N}_G(P)$. Let T be the inertia group of θ in G. We may assume that $K = \mathbf{O}^{p'p}(G) \nsubseteq M$. Otherwise $KP = \mathbf{O}^{p'}(G) \subseteq T$ and the result follows by Proposition 2.8.

Let KM/R be a chief factor of G such that $M \subseteq R < KM$. It follows that $K \nsubseteq R$, and hence $RN < G$. Now $K/(K \cap R)$ is an abelian p'-chief factor of G. Thus, $K' \subseteq K \cap R \subseteq R$. Hence, $K'N \subseteq RN$, and $\chi^{(RN)} \in \mathrm{Irr}_{p'}(RN)$ is defined. Since $N \subseteq RN$, it follows that $\chi^{(RN)}$ lies over all P-invariant irrdeucible characters of χ_M. Hence, $[\chi_M, \theta] \neq 0$ if and only if $[(\chi^{(RN)})_M, \theta] \neq 0$. By induction, we have that $[(\chi^{(RN)})_M, \theta] \neq 0$ if and only if $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$ for some $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$, and (a) follows.

Now, let $\varphi \in \mathrm{Irr}_{p'}(MP \mid \theta)$, by induction we have that

$$I_{RN}(\theta) \cap N = I_N((\varphi^*)_{N\cap M}).$$

Hence,

$$I_G(\theta) \cap N = I_N((\varphi^*)_{N\cap M})$$

as desired. Now, let χ_θ be the Clifford correspondent of χ over θ. Since $N \subseteq RN$, it follows that $\chi^{(RN)}$ lies over θ. Let $\delta \in \mathrm{Irr}(T \cap NR \mid \theta)$ be the Clifford correspondent of $\chi^{(RN)}$ over θ. We have that $\chi^{(RN)} = \delta^{NR}$. By Theorem 2.4, it follows that $\chi^* = (\chi^{(RN)})^* = (\delta^{NR})^* = (\delta^*)^N$.

Now, it follows that $\mathbf{O}^{p'p}(T)'(N \cap T) \subseteq T \cap RN$. Hence by Theorem 2.3, we have that $\chi_\theta^* = (\chi_\theta^{(RN\cap T)})^*$. We want to show that $\delta = \chi_\theta^{(RN\cap T)}$. For this, we prove that $[(\chi_\theta)_{T\cap NR}, \delta]$ is odd. We have that $[(\chi_\theta)_{T\cap NR}, \delta] = [\chi_{T\cap NR}, \delta] = [\chi_{NR}, \delta^{NR}]$ which is odd. And by induction it follows that χ_θ^* is the Clifford correspondent of χ^* over $(\varphi^*)_{N\cap M}$, where $[(\chi^*)_{\mathbf{N}_{MP}(P)}, \varphi^*] \neq 0$ □

3. Some perfect isometries

Let p be a prime, and let R be the ring of algebraic integers in $\mathbb{C}$. We let $U = \{\alpha/m \mid \alpha \in \mathrm{R}, m \in \mathbb{Z} - p\mathbb{Z}\}$ be the ring of p-local integers. We fix $(K, \mathcal{D}, F)$ a p-modular system, where K is algebraically closed and $U \subseteq \mathcal{D}$ (see [5]).

M. Broué introduced the notion of a perfect isometry in [1]. Suppose that G and H are finite groups, and let B and b a block of G and H, respectively. An isometry $\hat{}: \mathbb{Z}[\mathrm{Irr}(B)] \to \mathbb{Z}[\mathrm{Irr}(b)]$ is *perfect* if the following two conditions are satisfied:

(i) for all $g \in G$, $h \in H$, we have that

$$\frac{1}{|\mathbf{C}_G(g)|}\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\hat{\chi}(h) \quad \text{and} \quad \frac{1}{|\mathbf{C}_H(h)|}\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\hat{\chi}(h)$$

belong to $\mathcal{D}$;

(ii) if $\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\hat{\chi}(h) \neq 0$, then g is p-regular if and only if h is p-regular.

The following lemma is well-known and, together with weak block orthogonality, guarantees that the identity is a perfect isometry.

Lemma 3.1. *Suppose that B is a p-block of G. Then*

$$\frac{1}{|\mathbf{C}_G(g)|}\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\chi(h) \in U$$

for $g, h \in G$.

Proof. If $e_\chi \in \mathbb{C}G$ is the central idempotent associated to $\chi \in \mathrm{Irr}(G)$ and $f_B = \sum_{\chi\in\mathrm{Irr}(B)} e_\chi$, just compute the coefficient of $\hat{L}$ in $f_B\hat{K}$, where K is the conjugacy class of g, L is the conjugacy class of h^{-1} and $\hat{X}$ is the sum of the elements of $X \subseteq G$. Now apply that the coefficients of f_B lie in U (see the proof of Corollary 3.8 of [7]). □

It is not difficult to see that the composition of perfect isometries is a perfect isometry. (see, for instance, Lemma 1 of [10].)

Another example of perfect isometry is given by the Fong-Reynolds correspondence.

Lemma 3.2. *Let $N \triangleleft G$ and let b be a p-block of N. Let $T(b)$ be the stabilizer of b in G. Suppose that B^G is the Fong-Reynolds correspondent of $B \in \mathrm{Bl}(T(b) \mid b)$. Then the map $\mathrm{Irr}(B) \to \mathrm{Irr}(B^G)$ given by $\psi \mapsto \psi^G$ defines a perfect isometry for B and B^G.*

Proof. Let T be $T(b)$. Let $g \in G$ and $t \in T$. First we show that if

$$\sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t) \neq 0\,,$$

then g is p-regular if and only if t is p-regular. Notice that if no G-conjugate of g lies in T, then $\psi^G(g) = 0$. Hence we have that some G-conjugate of g lies in T. Let $g_1, \ldots, g_t$ be representatives for the classes of T contained in the G-conjugacy class of g. By using the formula of page 64 of [4], we have that

$$\begin{aligned}\sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t) &= \sum_{\psi \in \mathrm{Irr}(B)} \left(|\mathbf{C}_G(g)| \sum_{i=1}^{t} \frac{\psi(g_i)}{|\mathbf{C}_T(g_i)|} \right) \psi(t) \\ &= \sum_{i=1}^{t} \frac{|\mathbf{C}_G(g)|}{|\mathbf{C}_T(g_i)|} \left(\sum_{\psi \in \mathrm{Irr}(B)} \psi(g_i)\psi(t) \right).\end{aligned}$$

If this is nonzero, then g is p-regular if and only if t is p-regular by weak block orthogonality applied in T.

Now, we prove that

$$\frac{1}{|\mathbf{C}_G(g)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t) \quad \text{and} \quad \frac{1}{|\mathbf{C}_T(t)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t)$$

are elements of U. As above, we may assume that some G-conjugate of g lies in T. By Lemma 3.1, and using the same notation as before, we have that

$$\frac{1}{|\mathbf{C}_G(g)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t) = \sum_{i=1}^{t} \frac{1}{|\mathbf{C}_T(g_i)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi(g_i)\psi(t) \in U\,.$$

Also,

$$\frac{1}{|\mathbf{C}_T(t)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi^G(g)\psi(t) = \sum_{i=1}^{t} \frac{|\mathbf{C}_G(g)|}{|\mathbf{C}_T(g_i)|} \left(\frac{1}{|\mathbf{C}_T(t)|} \sum_{\psi \in \mathrm{Irr}(B)} \psi(g_i)\psi(t) \right) \in U$$

by using Lemma 3.1 and the fact that

$$\frac{|\mathbf{C}_G(g)|}{|\mathbf{C}_T(g_i)|} = \frac{|\mathbf{C}_G(g_i)|}{|\mathbf{C}_T(g_i)|}$$

is an integer. □

Our next goal is to find certain perfect isometries associated to normal p'-sections of groups.

If H is a subgroup of G and $\theta \in \mathrm{Irr}(H)$, we denote by $\mathrm{Irr}(G \mid \theta)$ the set of irreducible constituents of θ^G.

Theorem 3.3. *Suppose that K is a normal p'-subgroup of G. Let $H \subseteq G$ with $KH = G$ and write $L = K \cap H$. Suppose that $\theta \in \mathrm{Irr}(K)$ is G-invariant and such that $\theta_L \in \mathrm{Irr}(L)$. Let G^0 be the set of p-regular elements of G. If $\chi, \psi \in \mathrm{Irr}(G \mid \theta)$, then*

$$\sum_{x \in G^0} \chi(x)\psi(x^{-1}) = |K : L| \sum_{y \in H^0} \chi(y)\psi(y^{-1}).$$

Lemma 3.4. *Suppose that K is a normal subgroup of G. Let $H \subseteq G$ with $KH = G$ and write $L = K \cap H$. Suppose that $\theta \in \mathrm{Irr}(K)$ is G-invariant and such that $\theta_L \in \mathrm{Irr}(L)$. Let $h \in H$. Then if $\chi, \psi \in \mathrm{Irr}(G \mid \theta)$, we have*

$$\sum_{k \in K} \chi(kh)\psi((kh)^{-1}) = |K : L| \sum_{l \in L} \chi(lh)\psi((lh)^{-1}).$$

Proof. Consider the group $W = K\langle h \rangle$. Note that $V = W \cap H = L\langle h \rangle$. Since θ is W-invariant, there is some $\hat{\theta} \in \mathrm{Irr}(W)$ extending θ. Hence

$$\chi_W = \hat{\theta}\Delta_\chi,$$

where Δ_χ is a character of W/K by Gallagher's theorem (Corollary 6.17 of [4]). Also, by the same reason, we have that

$$\psi_W = \hat{\theta}\Delta_\psi,$$

where Δ_ψ is a character of W/K. Now,

$$\begin{aligned}\sum_{k \in K} \chi(kh)\overline{\psi(kh)} &= \sum_{k \in K} \hat{\theta}(kh)\Delta_\chi(h)\overline{\hat{\theta}(kh)\Delta_\psi(h)} \\ &= \Delta_\chi(h)\overline{\Delta_\psi(h)} \sum_{k \in K} \hat{\theta}(kh)\overline{\hat{\theta}(kh)} = |K|\Delta_\chi(h)\overline{\Delta_\psi(h)},\end{aligned}$$

by Lemma 8.14.c of [4]. Now, we have that

$$\chi_V = \hat{\theta}_V(\Delta_\chi)_V,$$

where $(\Delta_\chi)_V$ is a character of V/L, and

$$\psi_V = \hat{\theta}_V (\Delta_\psi)_V ,$$

where $(\Delta_\psi)_V$ is a character of V/L. Arguing as before and using that $\hat{\theta}_V \in \mathrm{Irr}(V)$, we get

$$\sum_{l \in L} \chi(lh)\overline{\psi(lh)} = |L| \Delta_\chi(h) \overline{\Delta_\psi(h)} .$$

This proves the lemma. □

Proof of Theorem 3.3. If $x \in G$, notice that x is p-regular iff all the elements in Kx are p-regular. This follows from the fact that x is p-regular iff $K\langle x\rangle$ is a p'-group. By the same argument (in H with L), we may write

$$H^0 = \bigcup_{t \in \mathcal{T}} Lt ,$$

as a disjoint union. We claim that

$$G^0 = \bigcup_{t \in \mathcal{T}} Kt ,$$

is also a disjoint union. If $x \in G^0$, then $x = kh$ for some $k \in K$ and $h \in H$. Since $K\langle x\rangle$ is a p'-group, it follows that $h \in H^0$. Hence, $h = lt$ for some $t \in \mathcal{T}$ and $l \in L$. Hence $x \in Kt$. Also, if $z \in Kt \cap Ks$ for $t, s \in \mathcal{T}$, then $ts^{-1} \in K \cap H = L$ and $Lt = Ls$. Hence $t = s$, as claimed. Now the result follows from Lemma 3.4. □

Corollary 3.5. *Suppose that K is a normal p'-subgroup of G. Let $H \subseteq G$ with $KH = G$ and write $L = K \cap H$. Suppose that $\theta \in \mathrm{Irr}(K)$ is G-invariant and such that $\theta_L \in \mathrm{Irr}(L)$. Let B be a block of G such that $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \theta)$. Then there is a unique block b of H such that $\mathrm{Irr}(b) = \{\chi_H \mid \chi \in \mathrm{Irr}(B)\}$. In this case, restriction defines a perfect isometry between B and b.*

Proof. By Lemma 10.5 of [3], we have that restriction defines a bijection $\mathrm{Irr}(G \mid \theta)$ onto $\mathrm{Irr}(H \mid \theta_H)$. By Theorem 3.3 above and Theorem 3.19 of [7], we have that $b = \{\chi_H \mid \chi \in \mathrm{Irr}(B)\}$ is a block of H. Now, Lemma 3.1 and weak block orthogonality guarantee that restriction is a perfect isometry between B and b. □

Our last result in this section, is to find a perfect isometry associated to certain odd fully ramified sections of a group.

A five-tuple $(G, K, L, \theta, \varphi)$ is a *character five* if K/L is a normal abelian section of G and φ is a G-invariant irreducible character of L fully ramified with respect to K/L; that is to say, $\varphi^K = e\theta$ with $e^2 = |K/L|$ for some $\theta \in \mathrm{Irr}(K)$.

If $(G, K, L, \theta, \varphi)$ is a character five, the *good* elements of G with respect to the character five are defined in Definition 3.1 of [3], and are relevant for our purposes here.

The following is one of the key tools when studying character theory of groups of odd order.

Theorem 3.6. *Let $(G, K, L, \theta, \varphi)$ be a character five. Assume that $|G : K|$ or $|K : L|$ is odd. Then there exists a character $\psi \in \mathrm{Char}(G/K)$ and $H \subseteq G$ be such that*:

(a) *$HK = G$ and $H \cap K = L$.*

(b) *The equation $\chi_H = \psi_H \hat{\chi}$, for $\chi \in \mathrm{Irr}(G \mid \theta)$ and $\hat{\chi} \in \mathrm{Irr}(H \mid \varphi)$ defines a bijection between these sets of characters.*

(c) *If $|G : L|$ is odd, then χ and $\hat{\chi}$ correspond above if and only if $[\chi_H, \hat{\chi}]$ is odd.*

(d) *Every element of H is good with respect to $(G, K, L, \theta, \varphi)$.*

(e) *$|\psi(g)|^2 = |\mathbf{C}_{K/L}(g)|$ for $g \in G$.*

(f) *If $\chi \in \mathrm{Irr}(G \mid \theta)$, then $\chi(g) = 0$ unless g lies in some G-conjugate of H.*

(g) *H^a is G-conjugate to H for all automorphism a of G fixing K, L, θ and φ.*

Proof. See Theorem 9.1 of [3]. Part (d), which is not explicitly stated in [3], can be found in Theorem 3.2 of [6]. □

We shall refer to such subgroups H in Theorem 3.6 as the *good complements* with respect to $(G, K, L, \theta, \varphi)$.

The next theorem is also in [2]. Here, we show another proof of this.

Theorem 3.7. *Assume the hypotheses and notation of* Theorem 3.6, *and suppose that K is a p'-group. Let B be a p-block of G with $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \theta)$. Then there is a unique p-block of H such that $\{\hat{\chi} \mid \chi \in \mathrm{Irr}(B)\} = \mathrm{Irr}(b)$ is a p-block of H. Also, the map $\chi \mapsto \hat{\chi}$ is an isometry between B and b.*

Proof. If χ, $\mu \in \mathrm{Irr}(G \mid \theta)$, first we claim that

$$\sum_{x \in G^0} \chi(x)\overline{\mu(x)} = |K : L| \sum_{x \in H^0} \chi(x)\overline{\mu(x)}.$$

Arguing as in the proof of Theorem 3.3, we may write

$$G^0 = \bigcup_{t \in \mathcal{T}} Kt\,,$$

as disjoint union, where

$$H^0 = \bigcup_{t \in \mathcal{T}} Lt\,,$$

is also a disjoint union. Now, since the elements of H are good, by Corollary 3.3 of [3], we have that

$$|\mathbf{C}_{K/L}(h)| \sum_{x \in Kh} \chi(x)\overline{\mu(x)} = |K : L| \sum_{x \in Lh} \chi(x)\overline{\mu(x)}.$$

By Theorem 3.6,

$$|K : L| \sum_{x \in Lh} \chi(x)\overline{\mu(x)} = |K : L||\psi(h)|^2 \sum_{x \in Lh} \hat{\chi}(x)\overline{\hat{\mu}(x)},$$

and our claim easily follows.

Now, by Theorem 3.19 of [7], we have that there is a unique p-block b of H such that $\{\hat{\chi} \mid \chi \in \mathrm{Irr}(B)\} = \mathrm{Irr}(b)$ is a p-block of H.

Now, if $g \in G$ and $h \in H$, we let

$$\alpha(g, h) = \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\hat{\chi}(h).$$

We wish to prove that

$$\frac{1}{|\mathbf{C}_G(g)|}\alpha(g, h) \quad \text{and} \quad \frac{1}{|\mathbf{C}_H(h)|}\alpha(g, h)$$

lie in U and that if $\alpha(g, h) \neq 0$, then g is p-singular if and only if h is p-singular. By Theorem 3.6 (f), we may assume that $g \in H$. Hence,

$$\alpha(g, h) = \psi(g) \sum_{\chi \in \mathrm{Irr}(B)} \hat{\chi}(g)\hat{\chi}(h).$$

From here, weak block orthogonality in H, and the fact that the character ψ is never zero, we deduce that whenever $\alpha(g, h) \neq 0$, then g is p-singular if and only if h is p-singular. Also, we have that

$$\frac{1}{|\mathbf{C}_H(h)|}\alpha(g, h) = \frac{\psi(g)}{|\mathbf{C}_H(h)|} \sum_{\chi \in \mathrm{Irr}(B)} \hat{\chi}(g)\hat{\chi}(h) \in U$$

by Lemma 3.1. Finally,

$$\begin{aligned}\frac{1}{|\mathbf{C}_G(g)|}\alpha(g, h) &= \frac{1}{|\mathbf{C}_G(g)|} \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\hat{\chi}(h) = \frac{1}{\psi(h)} \frac{1}{|\mathbf{C}_G(g)|} \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\chi(h) \\ &= \frac{\overline{\psi}(h)}{|\mathbf{C}_{K/L}(h)|} \frac{1}{|\mathbf{C}_G(g)|} \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\chi(h),\end{aligned}$$

by Theorem 3.6 (e). This element belongs to U by Lemma 3.1 and the fact that K/L is a p'-group. □

4. Proof of Theorem A

Lemma 4.1. *Let G be a solvable group. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $p \neq q$. If $P \subseteq \mathbf{N}_G(Q)$, then $\mathbf{O}^{q'q}(G) \subseteq \mathbf{O}_{p'}(G)$.*

Proof. It suffices to prove $Q \subseteq \mathbf{O}_{p'}(G)$. We have that $\mathbf{O}_{p'p}(G) \subseteq P\mathbf{O}_{p'}(G)$. Since $P \subseteq \mathbf{N}_G(Q)$, it follows that

$$\left[\frac{\mathbf{O}_{p'p}(G)}{\mathbf{O}_{p'}(G)}, \frac{Q\mathbf{O}_{p'}(G)}{\mathbf{O}_{p'}(G)}\right] = \left[\mathbf{O}_p\left(\frac{G}{\mathbf{O}_{p'}(G)}\right), \frac{Q\mathbf{O}_{p'}(G)}{\mathbf{O}_{p'}(G)}\right] = 1\,.$$

By Hall-Higman's lemma 1.2.3, it follows that $Q \subseteq \mathbf{O}_{p'}(G)$, as desired. □

In order to prove Theorem A, we need the following result.

Theorem 4.2. *Let p, q be primes, let G be a finite $\{p, q\}$-separable group, and let B be a p-block of G such that all of its ordinary irreducible characters have degree not divisible by q. Then a defect group of B normalizes some Sylow q-subgroup of G.*

Proof. This is Theorem A of [8]. □

We are ready to prove Theorem A.

Proof of Theorem A. We argue by induction on $|G|$. By Theorem 4.2, let Q be a Sylow q-subgroup G such that $D \subseteq \mathbf{N}_G(Q)$. Let N be a normal p'-subgroup of G. Let $\chi \in \mathrm{Irr}(B)$. We have that $\chi \in \mathrm{Irr}_{q'}(G)$. Now, let θ be an Q-invariant irreducible constituent of χ_N. Then $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \theta)$, because the block B covers the block $\{\theta\}$, as desired. Write $T = I_G(\theta)$, the stabilizer of θ in G. We claim that we may assume that θ is G-invariant. Otherwise, by the Fong-Reynolds correspondence (Theorem 9.14 of [7]), there exists a unique block b of T covering θ such that $\mathrm{Irr}(B) = \{\psi^G \mid \psi \in \mathrm{Irr}(b)\}$ and such that D is a defect group of b. We have that $b \subseteq \mathrm{Irr}_{q'}(T \mid \theta)$. Since $|T| < |G|$, by induction we have that there exists a unique block b^* of $T \cap \mathbf{N}_G(Q)$ with defect group D, with

$$\mathrm{Irr}(b^*) = \{\chi^* \mid \chi \in \mathrm{Irr}(b)\}$$

and such that the map $\psi \mapsto \psi^*$ is an isometry. Now, by Theorem 2.5, it follows that there is $\varphi \in \mathrm{Irr}(NQ \mid \theta)$ such that $\mathrm{Irr}(b^*) \subseteq \mathrm{Irr}(I_{\mathbf{N}_G(Q)}(\alpha^*) \mid \alpha^*)$ where $\alpha^* = \varphi^*_{\mathbf{N}_G(Q)\cap N}$. We also know that $(\chi^*)^{\mathbf{N}_G(Q)} = (\chi^G)^*$ for every $\chi \in \mathrm{Irr}(T \mid \theta)$ by Theorem 2.4. By the Fong-Reynolds correspondence, we conclude that $(b^*)^{\mathbf{N}_G(Q)} = \{(\chi^*)^{\mathbf{N}_G(Q)} \mid \chi \in \mathrm{Irr}(b)\} = B^*$ is a block of $\mathbf{N}_G(Q)$ with defect group D. Also, in this case, by using twice Lemma 3.2 and the fact that composition of perfect isometries is a perfect isometry, the proof of the theorem is complete.

Now, by the previous paragraph applied to $\mathbf{O}_{p'}(G)$ and Theorem 10.20 of [7], we have that $B = \mathrm{Irr}(G \mid \theta)$ and that D is a Sylow p-subgroup of G, where $\theta \in \mathrm{Irr}(\mathbf{O}_{p'}(G))$. Hence, we have that $D \subseteq \mathbf{N}_G(Q)$. By Lemma 4.1, it follows that $\mathbf{O}_{p'}(G)\mathbf{N}_G(Q) = G$. In particular, we have that $\mathbf{O}_{p'}(G) \cap \mathbf{N}_G(Q) = \mathbf{O}_{p'}(\mathbf{N}_G(Q))$. Let $\theta^* \in \mathrm{Irr}_{q'}(\mathbf{O}_{p'}(\mathbf{N}_G(Q)))$ be the Isaacs correspondent of θ. By Corollary 2.6, we have that α^* is $\mathbf{N}_G(Q)$-invariant. Hence, by Theorem 10.20 of [7] it follows that $\mathrm{Irr}(\mathbf{N}_G(Q) \mid \alpha^*)$ is a block of $\mathbf{N}_G(Q)$ with defect group D. By Corollary 2.6, we have that the Isaacs correspondence maps $\mathrm{Irr}(G \mid \theta)$ onto $\mathrm{Irr}(\mathbf{N}_G(Q) \mid \alpha^*)$.

Write $K = \mathbf{O}^{q'q}(G) \subseteq \mathbf{O}_{p'}(G)$. If K is trivial, then $G = \mathbf{N}_G(Q)$ and there is nothing to prove. Thus we suppose that K is not trivial. Let K/L be a chief factor of G. By coprime action, notice that $\mathbf{C}_{K/L}(Q) = 1$ (because K/L is abelian and $[K/L, Q] = K/L$). Hence, $L\mathbf{N}_G(Q)$ is a complement of K/L in G. Notice that $L\mathbf{N}_G(Q)$ is the unique complemet of K/L containing Q. Let $\xi \in \mathrm{Irr}(K)$ and $\epsilon \in \mathrm{Irr}(L)$ be Q-invariant characters such that $\mathrm{Irr}(B)$ covers $\{\xi\}$ and $\{\epsilon\}$. We already know that we may assume that ξ and ϵ are G-invariant. Hence, by the going down theorem (Theorem 6.18 of [4]), we will have that either $\xi_L = \epsilon$ or that ϵ is fully ramified with respect to K/L.

Suppose that $\xi_L = \epsilon$. Notice that if $\chi \in \mathrm{Irr}(G \mid \xi)$, then $(\chi_{L\mathbf{N}_G(Q)})^* = \chi^*$ by Theorem 2.1. In this case, the theorem follows by induction applied in LN and Corollary 3.5.

Suppose now that $\xi_L = e\epsilon$ with $e^2 = |K : L|$. By Theorem 3.6 (g) we may assume that there is a good complement H which contains Q and by uniqueness, we have that $L\mathbf{N}_G(Q) = H$ is a good complement, in the language of Theorem 3.6. In this case, the theorem follows from Theorem 3.7, Theorem 3.3 and the inductive hypothesis. □

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