Laurençot, P.
Osaka J. Math.
**40** (2003), 271–278

# DECAY OF MASS FOR A SEMILINEAR PARABOLIC SYSTEM: THE CRITICAL CASE

PHILIPPE LAURENÇOT

(Received September 10, 2001)

## 1. Introduction and main result

In the recent paper [1], Amour & Raoux have studied the large-time behaviour of the $L^1$-norm of nonnegative and integrable solutions $(u, v)$ to

$$\begin{cases} u_t - \Delta u + |\nabla v|^q = 0 \\ v_t - \Delta v + |\nabla u|^p = 0 \end{cases} \quad \text{in } (0,+\infty)\times\mathbb{R}^N, \tag{1}$$

with initial data

$$u(0) = u_0, \quad v(0) = v_0 \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $p$ and $q$ are real numbers satisfying $1 \leq p \leq q$ and $N$ is a positive integer. Assuming that $u_0$ and $v_0$ are nonnegative functions in $L^1(\mathbb{R}^N)$ with

$$\int |x| \; u_0(x) \; dx < \infty$$

and that $(u, v)$ is a solution to (1)–(2) with $u \geq 0$ and $v \geq 0$, they show that

$$\lim_{t\to+\infty} \|u(t) + v(t)\|_{L^1} > 0 \quad \text{if } q > q_p, \tag{3}$$

while

$$\lim_{t\to+\infty} \|u(t) + v(t)\|_{L^1} = 0 \quad \text{if } q < q_p, \tag{4}$$

where

$$q_p = \frac{1}{N+1} + \frac{1}{p}\;\frac{N+2}{N+1}, \tag{5}$$

the critical case $q = q_p$ being left opened [1]. It is the purpose of this note to fill this gap and prove that (4) also holds true if $q = q_p$. More precisely, we assume that

$$1 \leq p \leq q \leq q_p, \tag{6}$$

and observe that (6) implies that

$$1 \leq p \leq \frac{N+2}{N+1}. \tag{7}$$

We next assume that

$$\begin{cases} u_0 \text{ and } v_0 \text{ are nonnegative functions in } L^1(\mathbb{R}^N) \text{ with} \\ \qquad\qquad \int |x|^{((N+2)-p(N+1))/p} \ u_0(x) \ dx < \infty. \end{cases} \tag{8}$$

Our result then reads as follows.

**Theorem 1.** *Assume that* $p$, $q$, $u_0$ *and* $v_0$ *fulfil the conditions* (6) *and* (8), *and let* $(u, v)$ *be a nonnegative solution to* (1)–(2), *that is,* $u$ *and* $v$ *are nonnegative functions satisfying*

$$u \in \mathcal{C}\big([0,+\infty); L^1(\mathbb{R}^N)\big) \quad \textit{with} \quad \nabla u \in L^p\left((0,+\infty)\times\mathbb{R}^N\right),$$
$$v \in \mathcal{C}\big([0,+\infty); L^1(\mathbb{R}^N)\big) \quad \textit{with} \quad \nabla v \in L^q\left((0,+\infty)\times\mathbb{R}^N\right),$$

*and* $u$ *and* $v$ *are mild solutions to the first and the second equation of* (1), *respectively, with* $(u, v)(0) = (u_0, v_0)$. *Then*

$$\lim_{t\to+\infty} t^{((N+2)-p(N+1))/(2p)} \ \|u(t)\|_{L^1} = \lim_{t\to+\infty} \|v(t)\|_{L^1} = 0,$$

*and thus*

$$\lim_{t\to+\infty} \|u(t)+v(t)\|_{L^1} = 0.$$

Let us stress here that only the case $q = q_p$ is new in Theorem 1. However our proof works for the whole range of parameters $(p, q)$ given by (6) and differs from the one used in [1] to handle the case $q < q_p$. We will thus give it in the general case described by (6). As in [1], the first step towards the proof of Theorem 1 is the following properties enjoyed by $(u, v)$ which follow at once from (1) and the nonnegativity of $u$ and $v$:

$$t \mapsto \|u(t)\|_{L^1} \text{ and } t \mapsto \|v(t)\|_{L^1} \text{ are nonincreasing functions on } [0,+\infty), \tag{9}$$

$$\int_0^\infty \int \left(|\nabla u(t,x)|^p + |\nabla v(t,x)|^q\right) \ dxdt < \infty, \tag{10}$$

and

$$u(t,x) \leq \left(e^{t\Delta}u_0\right)(x) \quad \text{and} \quad v(t,x) \leq \left(e^{t\Delta}v_0\right)(x) \tag{11}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^N$, where $\left(e^{t\Delta}\right)_{t\geq 0}$ denotes the linear heat semigroup in $\mathbb{R}^N$. The second step, which is the main contribution of this work, is to deduce that

$$\lim_{t\to+\infty} t^{((N+2)-p(N+1))/(2p)} \;\|u(t)\|_{L^1} = 0 \tag{12}$$

by a careful use of (1) and (10). Notice that, for $p < (N+2)/(N+1)$, (12) improves [1, Lemma 2] where the weaker bound

$$\sup_{t\geq 0}\; t^{((N+2)-p(N+1))/(2p)-\varepsilon} \;\|u(t)\|_{L^1} < \infty$$

is proved for each $\varepsilon > 0$. The estimate (12) is actually the cornerstone of the proof of Theorem 1. Combining (10) and (12) then leads us to the expected result.

REMARK 2. Since $0 \leq (N+2)-p(N+1) \leq p$ by (7) (with equality only if $p = 1$) the additional integrability property in (8) on $u_0$ is weaker than the one required in [1].

Let us finally mention that we do not consider here the question of the existence of nonnegative solutions to (1)–(2) and refer to [2] for results in that direction. Moreover, the techniques developed in [5, 6] could possibly give further results.

## 2. Proof of Theorem 1

From now on, we fix $p$, $q$, $u_0$ and $v_0$ fulfilling the conditions (6) and (8) and let $(u, v)$ be a nonnegative solution to (1)–(2). We recall that (1)–(2) and the nonnegativity of $u$ and $v$ yield that

$$\int_0^\infty \int \left(|\nabla u(t,x)|^p + |\nabla v(t,x)|^q\right)\; dxdt \leq \|u_0\|_{L^1} + \|v_0\|_{L^1} \tag{1}$$

after integration of (1) over $(0, +\infty) \times \mathbb{R}^N$. We then put

$$\omega(t) = \left(\int_{t/2}^\infty \int |\nabla u(s,x)|^p \; dxds\right)^{1/p} \tag{2}$$

for $t \geq 0$ and notice that $\omega \in \mathcal{C}([0, +\infty))$ is a nonincreasing function which satisfies

$$\lim_{t\to+\infty} \omega(t) = 0, \tag{3}$$

thanks to (1). Observe that we may assume that $\omega(t) > 0$ for each $t \geq 0$. Indeed, if $\omega(t_0) = 0$ for some $t_0 \geq 0$, we realize that $\nabla u(t) \equiv 0$ for $t \geq t_0$, whence $u(t) \equiv 0$ for $t \geq t_0$ by the integrability of $u(t)$. By (1), this also implies that $\nabla v(t) \equiv 0$ for $t \geq t_0$ and thus $v(t) \equiv 0$ for $t \geq t_0$. Theorem 1 is then obvious in that case.

We now state some preliminary estimates which will be used throughout the paper. We first recall a Morrey-type inequality established in [4, Eq. (2.1)].

**Lemma 3** ([4]). *If $w \in W^{1,1}(\mathbb{R}^N)$ and $R > 0$, there holds*

$$\|w\|_{L^1} \le 2R \int_{\{|x|\le 3R\}} |\nabla w(x)|\ dx + 2 \int_{\{|x|>R\}} |w(x)|\ dx. \tag{4}$$

Next, since both $u$ and $v$ are subsolutions to the linear heat equation, a control of $u(t)$ and $v(t)$ for large values of $x$ and $t$ is available and is a consequence of [4, Lemma 2.1].

**Lemma 4** ([4]). *If $r \in \mathcal{C}([0,+\infty))$ is a nonnegative function such that*

$$\lim_{t\to+\infty} r(t)\ t^{-1/2} = +\infty, \tag{5}$$

*then*

$$\lim_{t\to+\infty} \int_{\{|x|\ge r(t)\}} \big(u(t,x) + v(t,x)\big)\ dx = 0. \tag{6}$$

We finally adapt a technique from the proof of [3, Proposition 14] to obtain another estimate on $u(t)$ for large values of $x$. We fix a function $\varrho \in \mathcal{C}^\infty(\mathbb{R}^N)$ satisfying $0 \le \varrho \le 1$ with

$$\varrho(x) = 0 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varrho(x) = 1 \quad \text{if} \quad |x| \ge 2.$$

For $R > 0$ and $x \in \mathbb{R}^N$ we put $\varrho_R(x) = \varrho(x/R)$. In the following, we denote by $C$ any positive constant depending only on $N$, $p$, $q$, $u_0$, $v_0$ and $\varrho$.

**Lemma 5.** *For $t \ge 0$ and $R > 0$ we have*

$$\int_{\{|x|\ge 2R\}} u(t,x)\ dx \le \int_{\{|x|\ge R\}} u_0(x)\ dx + C\ R^{(p(N-1)-N)/p}\ t^{(p-1)/p}. \tag{7}$$

*Proof.* We multiply the first equation of (1) by $\varrho_R$ and integrate over $(0,t)\times\mathbb{R}^N$ to obtain

$$\int \varrho_R(x)\ u(t,x)\ dx \le \int \varrho_R(x)\ u_0(x)\ dx + \frac{1}{R} \int_0^t \int \nabla\varrho\left(\frac{x}{R}\right).\nabla u(s,x)\ dxds\,.$$

Owing to the properties of $\varrho$, we infer from the Hölder inequality that

$$\int_{\{|x|\ge 2R\}} u(t,x)\ dx \le \int_{\{|x|\ge R\}} u_0(x)\ dx + C\ R^{N(1-1/p)-1} \int_0^t \|\nabla u(s)\|_{L^p}\ ds$$

$$\leq \int_{\{|x|\geq R\}} u_0(x)\ dx$$
$$+\ C\ R^{(p(N-1)-N)/p}\ t^{(p-1)/p}\ \left(\int_0^t \|\nabla u(s)\|_{L^p}^p\ ds\right)^{1/p},$$

from which (7) follows, thanks to (1). □

REMARK 6. Observe that, if we take $R = r(t)/2$ in (7) with $r$ as in Lemma 4, (7) yields a stronger decay estimate than (6) if $p < (N+2)/(N+1)$.

We next prove the temporal decay estimate for $\|u(t)\|_{L^1}$ claimed in the Introduction.

**Proposition 7.** *There exists $\sigma \in \mathcal{C}([0,+\infty))$, positive and nonincreasing, such that*

$$t^\alpha\ \|u(t)\|_{L^1} \leq \sigma(t)\ \text{ for }\ t > 0 \quad \text{with} \quad \lim_{t\to+\infty}\ \sigma(t) = 0\,, \tag{8}$$

*where* $\alpha := ((N+2) - p(N+1))/(2p) \geq 0$.

Proof. Consider $t > 0$, $R > 0$ and $s \in [t/2, t]$. On the one hand, we infer from (8) that

$$\int_{\{|x|\geq R\}} u_0(x)\ dx \leq R^{-2\alpha}\ \int_{\{|x|\geq R\}} |x|^{2\alpha}\ u_0(x)\ dx.$$

Inserting this estimate in (7) yields

$$\int_{\{|x|\geq 2R\}} u(s,x)\ dx \leq C\ R^{-2\alpha}\ \left(I_R + \left(s\ R^{-2}\right)^{(p-1)/p}\right),$$

where

$$I_R := \int_{\{|x|\geq R\}} |x|^{2\alpha}\ u_0(x)\ dx.$$

After integrating this inequality with respect to $s$ over $(t/2, t)$, we obtain

$$\int_{t/2}^t \int_{\{|x|\geq 2R\}} u(s,x)\ dxds \leq C\ R^{-2\alpha}\ t\ \left(I_R + \left(t\ R^{-2}\right)^{(p-1)/p}\right). \tag{9}$$

On the other hand, it follows from the Hölder inequality and (2) that

$$\int_{t/2}^t \int_{\{|x|\leq 6R\}} |\nabla u(s,x)|\ dxds$$

$$\leq C \; \left(t \; R^N\right)^{(p-1)/p} \; \left(\int_{t/2}^t \|\nabla u(s)\|_{L^p}^p \; ds\right)^{1/p}$$

$$\leq C \; \left(t \; R^N\right)^{(p-1)/p} \; \omega(t)\,. \tag{10}$$

Combining (9), (10) and Lemma 3, we end up with

$$\int_{t/2}^t \|u(s)\|_{L^1} \; ds \leq C \; R^{-2\alpha} \; t \; \left(I_R + \left(t \; R^{-2}\right)^{(p-1)/p} + \omega(t) \; \left(t \; R^{-2}\right)^{-1/p}\right).$$

We take $R = R(t) := t^{1/2} \; \omega(t)^{-1/2}$ in the previous inequality to conclude that

$$\frac{2}{t} \; \int_{t/2}^t \|u(s)\|_{L^1} \; ds \leq C \; \left(\frac{\omega(t)}{t}\right)^{\alpha} \; \left(I_{R(t)} + \omega(t)^{(p-1)/p}\right).$$

Owing to (9), the left-hand side of the above inequality is bounded from below by $\|u(t)\|_{L^1}$ and we finally obtain that

$$\|u(t)\|_{L^1} \leq t^{-\alpha} \; \sigma(t)$$

where

$$\sigma(t) := C \; \omega(t)^{\alpha} \; \left(I_{R(t)} + \omega(t)^{(p-1)/p}\right) \,.$$

Now the monotonicity of $\omega$ and (3) warrant that $R(t)$ increases to $+\infty$ as $t \longrightarrow +\infty$ which implies that $t \mapsto I_{R(t)}$ is a nonincreasing function which converges to zero as $t \longrightarrow +\infty$ by (8). Using once more the monotonicity of $\omega$, (3) and (7) as well, it is straightforward to conclude that $\sigma$ is a nonincreasing function which converges to zero as $t \longrightarrow +\infty$, whence (8). □

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. Consider $t > 0$, $R > 0$ and $s \in (t/2, t)$. By Lemma 3 and the Hölder inequality, we have

$$\|v(s)\|_{L^1} \; \leq \; 2R \; \int_{\{|x|\leq 3R\}} |\nabla v(s,x)| \; dx + 2 \; \int_{\{|x|>R\}} |v(s,x)| \; dx$$

$$\|v(s)\|_{L^1} \; \leq \; C \; R^{(q(N+1)-N)/q} \; \|\nabla v(s)\|_{L^q} + 2 \; \int_{\{|x|>R\}} |v(s,x)| \; dx\,.$$

We integrate the previous inequality with respect to $s$ over $(t/2, t)$ and use again the Hölder inequality to obtain

$$\int_{t/2}^t \|v(s)\|_{L^1} \; ds \; \leq \; C \; R^{(q(N+1)-N)/q} \; t^{(q-1)/q} \; \left(\int_{t/2}^t \|\nabla v(s)\|_{L^q}^q \; ds\right)^{1/q}$$

$$+ 2 \int_{t/2}^t \int_{\{|x|>R\}} |v(s,x)| \ dxds .$$

On the one hand, the monotonicity (9) of $s \mapsto \|v(s)\|_{L^1}$ entails that

$$\|v(t)\|_{L^1} \leq \frac{2}{t} \int_{t/2}^t \|v(s)\|_{L^1} \ ds .$$

On the other hand, integrating the first equation of (1) over $(t/2, t) \times \mathbb{R}^N$ yields

$$\int_{t/2}^t \|\nabla v(s)\|_{L^q}^q \ ds \leq \left\| u\left(\frac{t}{2}\right)\right\|_{L^1} .$$

Combining the previous three inequalities and (8), we end up with

$$\begin{aligned} \|v(t)\|_{L^1} \ \leq \ & C \ R^{(q(N+1)-N)/q} \ t^{-(1+\alpha)/q} \ \sigma(t)^{1/q} \\ & + \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R\}} |v(s,x)| \ dxds , \end{aligned} \tag{11}$$

where $\alpha$ and $\sigma$ are defined in Proposition 7.

Finally, since $q \geq 1$, let $\delta$ be a positive real number such that

$$0 < \delta < \frac{1}{q(N+1) - N}$$

and take $R = R(t) := t^{1/2} \ \sigma(t)^{-\delta}$ in (11). Owing to the monotonicity of $\sigma$, $s \mapsto R(s)$ is an nondecreasing function and we deduce from (11) that

$$\begin{aligned} \|v(t)\|_{L^1} \ \leq \ & C \ t^{(N+1)(q-q_p)/(2q)} \ \sigma(t)^{(1-\delta(q(N+1)-N))/q} \\ & + \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R(s)\}} |v(s,x)| \ dxds . \end{aligned} \tag{12}$$

Now, by (6) and Proposition 7, we have

$$\lim_{t\to+\infty} \ t^{(N+1)(q-q_p)/(2q)} \ \sigma(t)^{(1-\delta(q(N+1)-N))/q} = 0.$$

Consequently, since $R(s) \ s^{-1/2} \to +\infty$ as $s \to +\infty$, we have

$$\lim_{t\to+\infty} \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R(s)\}} |v(s,x)| \ dxds = 0$$

by Lemma 4. We may then let $t \to +\infty$ in (12) and conclude that

$$\lim_{t\to+\infty} \ \|v(t)\|_{L^1} = 0.$$

Theorem 1 follows at once from this last assertion and Proposition 7. □

ACKNOWLEDGEMENTS. I thank Saïd Benachour for helpful remarks on this paper.

## References

[1] L. Amour and T. Raoux: *$L^1$ decay properties for a semilinear parabolic system*, Israel J. Math. **123** (2001), 157–177.
[2] L. Amour and T. Raoux: *The Cauchy problem for a coupled semilinear parabolic system*, preprint (2000).
[3] S. Benachour, Ph. Laurençot, D. Schmitt and Ph. Souplet: *Extinction and non-extinction for viscous Hamilton-Jacobi equations in $\mathbb{R}^N$*, Asymptot. Anal. **31** (2002), 229–246.
[4] M. Ben-Artzi and H. Koch: *Decay of mass for a semilinear parabolic equation*, Comm. Partial Differential Equations, **24** (1999), 869–881.
[5] M. Ben-Artzi, Ph. Souplet and F.B. Weissler: *The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces*, J. Math. Pures Appl. **81** (2002), 343–378.
[6] N. Boudiba: *Existence globale pour des systèmes de réaction-diffusion avec contrôle de masse*, Thèse de l'Université de Rennes I, 1999.

Mathématiques pour l'Industrie et la Physique
CNRS UMR 5640
Université Paul Sabatier – Toulouse 3
118 route de Narbonne
F–31062 Toulouse cedex 4, France
e-mail: laurenco@mip.ups-tlse.fr