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ON A FRAMEWORK OF SCATTERING FOR DISSIPATIVE SYSTEMS

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1. Introduction

In this paper we study the existence of scattering solutions for some dissipative systems which contain elastic wave with dissipative boundary conditions in a half space of $\mathbf{R}^3$ (cf. Dermenjian-Guillot [1]). First we give a framework based on the idea of Simon [18] and apply it to elastic wave mentioned above. In applying the abstract framework, we shall use the Mellin transformation (cf. Perry [14]) as a key tool.

Let $\mathcal{H}$ be separable Hilbert space with inner $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The norm is denoted by $\| \cdot \|_{\mathcal{H}}$. Let $\{V(t)\}_{t \geqq 0}$ and $\{U_0(t)\}_{t \in \mathbf{R}}$ be a contraction semi- group in $\mathcal{H}$ and a unitary group in $\mathcal{H}_0$, respectively. We denote the generator of $V(t)$ and $U_0(t)$ by A and A_0, respectively ($V(t) = e^{-itA}$ and $U_0(t) = e^{-itA_0}$). We make the following assumptions on A and A_0.

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ or $[0, \infty)$.

(A2) $(A - i)^{-1} - (A_0 - i)^{-1}$ defined as a form is extended to a compact operator K in $\mathcal{H}$.

(A3) There exist non-zero projection operators in $\mathcal{H}$, P_+ and P_-, such that $P_+ + P_- = I_d$ and

$$\int_0^\infty \|KU_0(t)\psi(A_0)P_+\| \, dt < \infty, \tag{A3.1}$$

$$\int_0^\infty \|K^*U_0(t)\psi(A_0)P_+\| \, dt < \infty, \tag{A3.2}$$

$$\int_0^\infty \|K^*U_0(-t)\psi(A_0)P_-\| \, dt < \infty, \tag{A3.3}$$

$$\text{w} - \lim_{t \to +\infty} U_0(-t)\psi(A_0)P_- f_t = 0, \tag{A3.4}$$

for each $\psi \in C_0^\infty(\mathbf{R} \backslash 0)$ and $\{f_t\}_{t \in \mathbf{R}}$ satisfying $\sup_{t \in \mathbf{R}} \|f_t\|_{\mathcal{H}} < \infty$, where $\| \cdot \|$ is the operator norm of bounded operators in $\mathcal{H}$.

(A3.1), (A3.3) and (A3.4) will imply the existence of the wave operator. It will follow from (A3.2) that the wave operator is not zero as an operator in $\mathcal{H}$. The framework of [18] is due to Enss's method [2]. In order to cheak the applicability of the framework of [18] to dissipative systems (see also Stefanov-Georgiev [20] or [15]), we have to show the following type limit:

$$\lim_{n\to\infty}\int_0^\infty \|\tilde{K}(t)V(t_n)f\|_{\mathcal{H}} = 0 \tag{1.1}$$

for some one parameter compact operators, $\{\tilde{K}(t)\}_{t\in[0,\infty)}$, in $\mathcal{H}$, where t_n is as in (F2) below. This follows from Lebesgue's theorem and

$$\int_0^\infty \|\tilde{K}(t)\|\,dt < \infty. \tag{1.2}$$

(A3.1)$\sim$(A3.3) mean (1.2) (for details, see §2).

Let $\mathcal{H}_b$ be the space generated by the eigenvectors of A with real eigenvalues. We use the following facts (see Simon [18] and Petkov [15]):

(F1) $\{(A-i)^{-2}Af \in \mathcal{H} : f \in D(A)\cap\mathcal{H}_b^\perp\}$ is dense in $\mathcal{H}_b^\perp$.
(F2) There exists a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty$$

and

$$\mathrm{w}-\lim_{n\to\infty} V(t_n)f = 0, \quad \text{for any} \quad f \in \mathcal{H}_b^\perp.$$

In this abstract framework, we shall show

Theorem 1. *Assume that* (A1) $\sim$ (A3). *Then for any* $f \in \mathcal{H}_b^\perp$*, the wave operator*

$$Wf = \lim_{t\to\infty} U_0(-t)V(t)f$$

exists. Moreover W *is not zero as an operator from* $\mathcal{H}_b^\perp$ *to* $\mathcal{H}$*.*

As a corollary of Theorem 1, we can find scattering solutions of $d(V(t)f)/dt = -iAV(t)f,\ f \in D(A)$.

Corollary 2. *Assume that* (A1) $\sim$ (A3). *Then there exist non-trivial initial data* $f \in \mathcal{H}$ *and* $f_+ \in \mathcal{H}$ *such that for any* $k = 0, 1, 2, \ldots,$ *and* $\zeta_0 \in \mathbf{C}$ *satisfying* $\Re\zeta_0 > 0$

$$\lim_{t\to\infty} \|V(t)(A-\zeta_0)^{-k}f - U_0(t)(A_0-\zeta_0)^{-k}f_+\|_{\mathcal{H}} = 0.$$

We can also obtain the standard result concerning real eigenvalues of A as follows.

Theorem 3. *Any non-zero eigenvalues of A has finite multiplicity. Moreover the possible finite accumulation point of the real eigenvalues of A is zero.*

In §2 we shall give the proof of Theorem 1, Corollary 2 and Theorem 3.

In §3 we shall apply our framework to (1.3) as below, which describes elastic wave with dissipative boundary conditions in a half space of $\mathbf{R}^3$. It seems that there is no work concerning dissipative elastic wave in a half space (cf. Theorem 3.1).

Let $x = (x_1, x_2, x_3) = (y, x_3) \in \mathbf{R}^2 \times \mathbf{R}_+$, $\rho_0 > 0$, $\mu_0 > 0$ and $\lambda_0 \in \mathbf{R}$ satisfying $3\lambda_0 + 2\mu_0 > 0$. We use $O_{3\times 3}$ and $I_{3\times 3}$ as zero and unit matrix of 3×3 type, respectively.

We set

$$\varepsilon_{hj}(u(x)) = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_j} + \frac{\partial u_j}{\partial x_h}\right)$$

and

$$\sigma_{hj}(u(x)) = \lambda_0(\nabla_x \cdot u)\delta_{hj} + 2\mu_0\varepsilon_{hj}(u)$$

where $h,\ j = 1,\ 2,\ 3$, $u(x) = {}^t(u_1(x), u_2(x), u_3(x)) \in \mathbf{C}^3$ and $\nabla_x = (\partial/\partial_1, \partial/\partial_2, \partial/\partial_3)$.

We define operators $\tilde{L}_0$ as

$$(\tilde{L}_0 u)_h = -\sum_{j=1}^{3} \frac{1}{\rho_0} \frac{\partial \sigma_{hj}(u(x))}{\partial x_j} \quad (h = 1,\ 2,\ 3).$$

We consider two elastic wave equations as follows:

$$\text{(1.3)} \qquad \begin{cases} \partial_t^2 u(x,t) + \tilde{L}_0 u(x,t) = 0, & (x,t) \in \mathbf{R}_+^3 \times [0,\infty), \\ {}^t(\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u)) \mid_{x_3=0} = B(y)\partial_t u \mid_{x_3=0} \end{cases}$$

and

$$\text{(1.4)} \qquad \begin{cases} \partial_t^2 u(x,t) + \tilde{L}_0 u(x,t) = 0, & (x,t) \in \mathbf{R}_+^3 \times \mathbf{R}, \\ \sigma_{i3}(u) \mid_{x_3=0} = 0 \ (i = 1,\ 2,\ 3). \end{cases}$$

To state assumptions for $\lambda(x)$, $\mu(x)$ and $B(y)$ we introduce a function space $B^k(\Omega)$ as follows:

$$B^k(\Omega) = \left\{ u \in C^k(\Omega);\ \sum_{|\alpha| \leqq k} \|\partial^\alpha u\|_{L^\infty(\Omega)} < \infty \right\},$$

where $\Omega \subset \mathbf{R}^n$.

Assume that

(1.5) $B(y)$ belongs to $B^1(\mathbf{R}^2, \mathbf{M}_{3\times 3})$ and satisfies

$$O_{3\times 3} \leqq B(y) \leqq \varphi(|y|)I_{3\times 3},$$

where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbf{R}_+)$, and $\mathbf{M}_{3\times 3}$ is the totality of 3×3 matrix.

The following operator L_0 in $\mathcal{G} = L^2(\mathbf{R}^3_+, \mathbf{C}^3; \rho_0\, dx)$:

$$L_0 u = \tilde{L}_0 u$$

with

$$D(L_0) = \{u \in H^1(\mathbf{R}^3_+, \mathbf{C}^3); \tilde{L}_0 u \in \mathcal{G}, \sigma_{h3}(u)\ |_{x_3=0} = 0(h = 1,\ 2,\ 3)\}$$

is a non-negative and self-adjoint. In the Appendix we state basic results on L_0

REMARK 1.1. We can also deal with the perturbation of λ_0, μ_0 and ρ_0 as follows: $\lambda(x)$ and $\mu(x)$ belong to $B^1(\mathbf{R}^3_+)$. $\rho(x)$ belongs to $L^\infty(\mathbf{R}^3_+)$. Moreover these functions satisfy

$$0 < m \leqq 3\lambda(x) + 2\mu(x),\ \ \mu(x),\ \ \rho(x) \leqq M$$

for some M, $m > 0$ and

$$|\lambda(x) - \lambda_0|, \quad |\mu(x) - \mu_0|, \quad |\rho(x) - \rho_0| \leqq \varphi(|x|).$$

The projections $P_\pm$ in (A3) are considered by using the generalized Fourier transformation for L_0 and the negative (or positive) spectral projections of generators of dilation of $\mathbf{R}^3_+$ and $\mathbf{R}^2$. To make a check on (A3.1)$\sim$(A3.3) we use the Mellin transformation of $\mathbf{R}^3_+$ (resp. $\mathbf{R}^2$) in the range space of the generalized Fourier transformation of free waves (resp. Rayleigh wave). Perry [14] is the first to apply the Mellin transformation to show asymptotic completeness for Schrödinger equation. Kadowaki [6] has also used the Mellin transformation with the generalized eigenfunction expansion theorem of Wilcox [22] or Weder [21]. Recently, Soga [19] has considered the scattering theory of Lax-Phillips (cf. Lax-Phillips [11]) for (1.4). However (1.3) has not been dealt with.

[18], [20] and [15] have directly showed (1.1) by using pseudo-differential operator and the relation

$$\mathfrak{F}(u * v) = \mathfrak{F}u \times \mathfrak{F}v, \tag{1.6}$$

where

$$(u * v)(x) = \int_{\mathbf{R}^n} u(x - y)v(y)\, dy$$

and $\mathfrak{F}$ is the Fourier transformation. If we directly apply the framework of [18] to dissipative elastic wave in a half space, we have to require (1.6) to hold for the generalized Fourier transformation for elastic wave in a half space. But it is not clear. Therefore it seems difficult to directly apply the framework of [18] to elastic wave with dissipative boundary conditions in a half space. We can also deal with dissipative elastic wave equation in stratified (two layered) media (cf. Shimizu [17]) by our framework. As other applications (cf. §3) of our framework, we consider Schödinger equation with complex valued potential (cf. [18]) and acoustic wave equation with dissipative term in inhomogeneous media (cf. Mochizuki [13]). These applications are essentially the same results as in [18] and [13]. Although we do not pick up the exterior domain problem (for instance [11] and [20]), we can also treat them with some modifications.

Kadowaki [7] has considered the existence of scattering solution for acoustic wave equation with dissipative term in special stratified media. This equation is not included in our example. The reason is that, since the generalized eigenfunction of acoustic wave operator has singular points due to the thresholds (see [22] or [21]), the key estimate (cf. Lemma 3.2) of the neighborhood of each threshold (for detail, see [6, Lemma 2.1]) is hard to prove. In [7], Mochizuki's work [13] was used instead. His framework is based on resolvent estimates and Kato's smooth perturbation theory (cf. Kato [9]). In Kadowaki [8], we extended Mochizuki's framework and apply that to dissipative wave equations in stratified media which was not treated by [7].

Finally we give brief comments concerning energy decay solutions:

$$\lim_{t\to\infty} V(t)f = 0.$$

In the proof of $W \not\equiv 0$, we have to note the existence of energy decay solutions, for details see §2 (cf. Georgiev [3], Majda [12], [15] and [20]).

2. Proof of Theorem 1, Corollary 2 and Theorem 3

In this section we deal with the case $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ only. In the same way, another case can be dealt with. We set $F(\lambda) = (\lambda - i)^{-2}\lambda$ and $W(t) = U_0(-t)V(t)$. In this section C is used as positive constants.

Below we shall give the proof of Theorem 1. First we prove the existence of W by referring to Enss [2], Simon [18], Kuroda [10], Perry [14], Isozaki-Kitada [4], Stefanov-Georgiev [20] and Petkov [15]. But we sometimes omit to note the above references.

Proof of the existence of W. For any $f \in \mathcal{H}_b^{\perp} \cap D(A)$ and $t,\ s > t_n$, note (F1) and

$$\begin{aligned}&\|(W(t) - W(s))F(A)^2 f\|_{\mathcal{H}}\\ &\quad \leqq \|(W(t) - W(t_n))F(A)^2 f\|_{\mathcal{H}} + \|(W(s) - W(t_n))F(A)^2 f\|_{\mathcal{H}}.\end{aligned}$$

Thus the existence of W follows from

$$\lim_{n\to\infty}\overline{\lim_{t\to\infty}}\,\|(W(t)-W(t_n))F(A)^2 f\|_{\mathcal{H}}=0 \tag{2.1}$$

(cf. [4])

We estimate $\|(W(t)-W(t_n))F(A)^2 f\|_{\mathcal{H}}$ as follows (cf. [18]):

$$\begin{aligned}&\|(W(t)-W(t_n))F(A)^2 f\|_{\mathcal{H}}\\&\quad=\|U_0(-t)(V(t-t_n)-U_0(t-t_n))F(A)^2V(t_n)f\|_{\mathcal{H}}\\&\quad\leqq\sum_{j=1}^{5}\|T_j\|_{\mathcal{H}},\end{aligned}$$

where

$$\begin{aligned}T_1&=(V(t-t_n)-U_0(t-t_n))(F(A)^2-F(A_0)^2)V(t_n)f,\\T_2&=(V(t-t_n)-U_0(t-t_n))(I_d-\psi_M(A_0))F(A_0)^2V(t_n)f,\\T_3&=(V(t-t_n)-U_0(t-t_n))(\psi_M F)(A_0)P_+F(A_0)V(t_n)f,\\T_4&=(V(t-t_n)-U_0(t-t_n))(\psi_M F)(A_0)P_-F(A_0)(I_d-\psi_M(A_0))V(t_n)f,\\T_5&=(V(t-t_n)-U_0(t-t_n))(\psi_M F)(A_0)P_-(\psi_M F)(A_0)V(t_n)f\end{aligned}$$

and $\psi_M(\lambda)\in C_0^\infty(\mathbf{R})$ satisfies $0\leqq\psi_M(\lambda)\leqq 1$, $\psi_M(\lambda)=0(|\lambda|<1/2M, |\lambda|>2M)$ and $\psi_M(\lambda)=1(1/M<|\lambda|<M)$.

First, we note that for any ε, there exists $M>0$ such that

$$\|T_j\|_{\mathcal{H}}\leqq C\|(1-\psi_M)F\|_{L^\infty(\mathbf{R})}<\varepsilon\quad(j=2,\ 4)$$

Therefore once the limits

$$\lim_{n\to\infty}\overline{\lim_{t\to\infty}}\,\|T_j\|_{\mathcal{H}}=0,\quad(j=1,\ 3,\ 5) \tag{2.2}$$

are proved, we obtain (2.1). Below we shall show (2.2). For $j=1$, note $F(\lambda)=i(\lambda-i)^{-2}+(\lambda-i)^{-1}$. Therefore (A2) implies that $F(A)^2-F(A_0)^2$ is also a compact operator in $\mathcal{H}$. Using (F2) we have

$$\|T_1\|_{\mathcal{H}}\leqq C\|(F(A)^2-F(A_0)^2)V(t_n)f\|_{\mathcal{H}}\to 0\quad(n\to\infty)$$

For $j=3$, we decompose T_3 as follows:

$$T_3=T_{31}+T_{32}+T_{33},$$

where

$$T_{31}=V(t-t_n)(F(A_0)-F(A))(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$

$$T_{32} = (F(A) - F(A_0))U_0(t - t_n)(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$
$$T_{33} = F(A)(V(t - t_n) - U_0(t - t_n))\psi_M(A_0)P_+F(A_0)V(t_n)f.$$

The argument as in the proof of T_1 implies

$$\lim_{n\to\infty} \overline{\lim_{t\to\infty}} \|T_{31}\|_{\mathcal{H}} = 0.$$

Since we have by (A1)

$$\mathrm{w} - \lim_{t\to\infty} U_0(t - t_n)f = 0,$$

(A2) implies

$$\lim_{t\to\infty} \|T_{32}\|_{\mathcal{H}} = 0.$$

To estimate T_{33}, we use Cook-Kuroda method. Note that

$$\begin{aligned}\langle T_{33}, g\rangle_{\mathcal{H}} =& \langle \psi_M(A_0)P_+F(A_0)f_n, V^*(t - t_n)\overline{F}(A^*)g\rangle_{\mathcal{H}} \\ &- \langle U_0(t - t_n)\psi_M(A_0)P_+F(A_0)f_n, \overline{F}(A^*)g\rangle_{\mathcal{H}},\end{aligned}$$

where $g \in \mathcal{H}$ and $f_n = V(t_n)f$.

Then we have by (A2)

$$\begin{aligned}&\langle T_{33}, g\rangle_{\mathcal{H}} \\ &= i\int_0^{t-t_n} (\langle A_0U_0(s)\psi_M(A_0)P_+F(A_0)f_n, V^*(t - t_n - s)\overline{F}(A^*)g\rangle_{\mathcal{H}} \\ &\quad - \langle U_0(s)\psi_M(A_0)P_+F(A_0)f_n, A^*V^*(t - t_n - s)\overline{F}(A^*)g\rangle_{\mathcal{H}})\,ds \\ &= -i\int_0^{t-t_n} \langle V(t - t_n - s)A(A - i)^{-1}KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n, g\rangle_{\mathcal{H}}\,ds,\end{aligned}$$

where $\tilde{\psi}_M(\lambda) = (\lambda - i)\psi_M(\lambda)$.

Therefore we obtain

$$\begin{aligned}\|T_{33}\|_{\mathcal{H}} &\leqq C\int_0^\infty \|KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n\|_{\mathcal{H}}\,ds \\ &\leqq C\|f\|_{\mathcal{H}}\int_0^\infty \|KU_0(s)\tilde{\psi}_M(A_0)P_+\|\,ds.\end{aligned}$$

For each $s \geqq 0$ we have by (F2) and (A2) ,

$$\lim_{n\to\infty} \|KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n\|_{\mathcal{H}} = 0.$$

Therefore (A3.1) and Lebesgue's theorem imply

$$\lim_{n\to\infty}\overline{\lim_{t\to\infty}}\|T_{33}\|_{\mathcal{H}} = 0.$$

Now we obtain

$$\lim_{n\to\infty}\overline{\lim_{t\to\infty}}\|T_{3}\|_{\mathcal{H}} = 0.$$

We estimate T_5 as follows:

$$\begin{aligned}\|T_5\|^2_{\mathcal{H}_0} &\leqq C\|P_-(F\psi_M)(A_0)V(t_n)f\|^2_{\mathcal{H}}\\ &= C\sum_{j=1}^{3} T_{5j},\end{aligned}$$

where

$$\begin{aligned}T_{51} &= \langle \psi_M(A_0)P_-h_n, (F(A_0)-F(A))V(t_n)f\rangle_{\mathcal{H}}\\ T_{52} &= \langle \psi_M(A_0)P_-h_n, (V(t_n)-U_0(t_n))F(A)f\rangle_{\mathcal{H}}\\ T_{53} &= \langle U_0(-t_n)\psi_M(A_0)P_-h_n, F(A)f\rangle_{\mathcal{H}}\end{aligned}$$

and $h_n = (F\psi_M)(A_0)V(t_n)f$.

Note that

$$|T_{51}| \leqq C\|(F(A_0)-F(A))V(t_n)f\|_{\mathcal{H}}.$$

Thus (A2) and (F2) imply

$$\lim_{n\to\infty} T_{51} = 0.$$

(A3.4) implies

$$\lim_{n\to\infty} T_{53} = 0.$$

To estimate T_{52}, again we use Cook-Kuroda method. Note that

$$\begin{aligned}T_{52} =& i\int_0^{t_n}(\langle U_0(-s)\psi_M(A_0)P_-h_n, AV(t_n-s)F(A)f\rangle_{\mathcal{H}}\\ &\qquad - \langle A_0U_0(-s)\psi_M(A_0)P_-h_n, V(t_n-s)F(A)f\rangle_{\mathcal{H}})\,ds\\ =& i\int_0^{t_n}\langle V^*(t_n-s)A^*(A^*+i)^{-1}K^*U_0(-s)\overline{\tilde{\psi}}_M(A_0)P_-h_n, f\rangle_{\mathcal{H}}\,ds.\end{aligned}$$

Thus we have

$$|T_{52}| \leqq C\|f\|_{\mathcal{H}}\int_0^{\infty}\|K^*U_0(-s)\overline{\tilde{\psi}}_M(A_0)P_-h_n\|_{\mathcal{H}}\,ds.$$

Using (A2), (F2) and (A3.2) we have by Lebesgue's theorem

$$\lim_{n\to\infty} T_{52} = 0.$$

Now we obtain

$$\lim_{n\to\infty} \overline{\lim_{t\to\infty}} \|T_5\|_{\mathcal{H}} = 0.$$

Therefore the proof of the existence of W is completed. □

To show $W \not\equiv 0$, we introduce a subspace of $\mathcal{H}$, D, as follows:

$$D = \{f \in \mathcal{H} : \lim_{t\to\infty} V(t)f = 0\}.$$

Since

$$Af = \lambda f, \lambda \in \mathbf{R}, f \in \mathcal{H} \Longrightarrow A^* f = \lambda f$$

(see Petkov [15, Lemma 1.1.5]), we can easily show

$$D \subset \mathcal{H}_b^{\perp}.$$

We prepare

Proposition 2.1. *Assume that*

$$\mathcal{H}_b^{\perp} \ominus D = \{0\}.$$

Then one has

$$\text{w} - \lim_{t\to\infty} U_0(-t)V(t)f = 0 \tag{2.3}$$

for any $f \in \mathcal{H}$.

Proof. Now we can decompose any $f \in \mathcal{H}$ as $f = f_1 + f_2$, where $f_1 \in D = \mathcal{H}_b^{\perp}$ and $f_2 \in \mathcal{H}_b$. Then it is clear that

$$\text{w} - \lim_{t\to\infty} U_0(-t)V(t)f_1 = 0. \tag{2.4}$$

Note the definition of $\mathcal{H}_b$. Then for any $\varepsilon > 0$, there exist $N \in \mathbf{N}$, $c_j \in \mathbf{C}$, $\lambda_j \in \mathbf{R}$ and $g_j \in \mathcal{H}(j = 1, 2, \dots, N)$ such that $Ag_j = \lambda_j g_j$ and

$$\left\| f_2 - \sum_{j=1}^{N} c_j g_j \right\|_{\mathcal{H}} < \varepsilon.$$

Hence we have for any $h \in \mathcal{H}$,

$$|\langle U_0(-t)V(t)f_2, h\rangle_{\mathcal{H}}| \leqq C\varepsilon\|h\|_{\mathcal{H}} + \sum_{j=1}^{N} |c_j||\langle U_0(-t)g_j, h\rangle_{\mathcal{H}}|.$$

Since (A1) implies

$$\text{w} - \lim_{t\to\infty} U_0(-t)g_j = 0,$$

we have

$$\overline{\lim_{t\to\infty}} |\langle U_0(-t)V(t)f_2, h\rangle_{\mathcal{H}}| \leqq C\varepsilon\|h\|_{\mathcal{H}}. \tag{2.5}$$

Hence we obtain (2.3) by (2.4) and (2.5). □

Below we shall show $W \not\equiv 0$ (cf. Mochizuki [13] §3).

Proof of $W \not\equiv 0$. For any $f \in \mathcal{H}$ and $g \in \mathcal{H}_0$, note that

$$\begin{aligned} &\langle U_0(-t)V(t)(A-i)^{-1}f, (A_0+i)^{-1}g\rangle_{\mathcal{H}} \\ &\quad = \langle (A-i)^{-1}f, (A_0+i)^{-1}g\rangle_{\mathcal{H}} + i\int_0^t \langle V(\tau)f, K^*U_0(\tau)g, \rangle_{\mathcal{H}}\, d\tau. \end{aligned} \tag{2.6}$$

We assume that $W \equiv 0$, i.e., for any $f \in \mathcal{H}_b^{\perp}$,

$$\|Wf\|_{\mathcal{H}} = \lim_{t\to\infty} \|V(t)f\|_{\mathcal{H}} = 0. \tag{2.7}$$

(2.7) means

$$\mathcal{H}_b^{\perp} \ominus \boldsymbol{D} = \{0\}.$$

Hence Proposition 2.1 and (2.6) imply

$$\langle (A-i)^{-1}f, (A_0+i)^{-1}g\rangle_{\mathcal{H}} = -i\int_0^\infty \langle V(\tau)f, K^*U_0(\tau)g\rangle_{\mathcal{H}}\, d\tau.$$

Putting

$$f = (A_0-i)U_0(s)\psi_M(A_0)P_+h \quad \text{and} \quad g = (A_0+i)U_0(s)\psi_M(A_0)P_+h$$

for any $h \in \mathcal{H}$, we have

$$\begin{aligned} &\langle ((A-i)^{-1} - (A_0-i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h, U_0(s)\psi_M(A_0)P_+h\rangle_{\mathcal{H}} \\ &\quad + \|\psi_M(A_0)P_+h\|_{\mathcal{H}}^2 \end{aligned}$$

$$= -i \int_0^\infty \langle V(\tau)U_0(s)\tilde{\psi}_M(A_0)P_+h,\, K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+h,\, \rangle_{\mathcal{H}}\, d\tau.$$

It follows from the above identity that

$$\begin{aligned}\|\psi_M(A_0)P_+h\|_{\mathcal{H}}^2 \leqq \|h\|_{\mathcal{H}} \Big\{ &\|((A-i)^{-1} - (A_0-i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} \\ &+ C_M \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}\, d\tau \Big\}.\end{aligned}$$

(A1) and (A2) imply

$$\lim_{s\to\infty} \|((A-i)^{-1} - (A_0-i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} = 0.$$

(A3.2) implies

$$\lim_{s\to\infty} \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+\|\, d\tau = 0.$$

Therefore we have

$$\|\psi_M(A_0)P_+h\|_{\mathcal{H}} = 0, \tag{2.8}$$

for any $h \in \mathcal{H}_0$ and any $M > 0$.

(2.8) implies $P_+ \equiv 0$. This is a contradiction with (A3). Now the proof of $W \not\equiv 0$ is completed. □

In the remainder of this section we show Corollary 2 and Theorem 3.

Proof of Corollary 2. First we consider the case $k = 0$. Since $U_0(t)$ is unitary in $\mathcal{H}$, it follows from Theorem 1 that there exist non-trivial initial data $f \in \mathcal{H}$ such that $Wf \neq 0$ and

$$\lim_{t\to\infty} \|V(t)f - U_0(t)Wf\|_{\mathcal{H}} = 0.$$

Setting $f_+ = Wf$ we have Corollary 2 for $k = 0$. Next we consider $k = 1$. Corollary 2 for $k = 0$ and (A1) imply

$$\text{w} - \lim_{t\to\infty} V(t)f = 0. \tag{2.9}$$

Noting

$$\begin{aligned}&\|V(t)(A-\zeta_0)^{-1}f - U_0(t)(A_0-\zeta_0)^{-1}f_+\|_{\mathcal{H}} \\ &\quad\leqq \|((A-\zeta_0)^{-1} - (A_0-\zeta_0)^{-1})V(t)f\|_{\mathcal{H}} + \|(A_0-\zeta_0)^{-1}(V(t)f - U_0(t)f_+)\|_{\mathcal{H}},\end{aligned}$$

we have Corollary 2 for $k = 1$ by (A2), (2.9) and Corollary 2 for $k = 0$.

The cases $k \geqq 2$ can be proved by induction, which is omittied. □

The proof of Theorem 3 is similar to that of Theorem 1. So we shall give a brief sketch of the proof only.

Proof of Theorem 3. Assume that there exist $f_n \in D(A)$, λ_n, $\lambda \in \mathbf{R}\setminus 0$ ($n = 1, 2, 3, \ldots$) such that

$$Af_n = \lambda_n f_n, \ \|f_n\|_{\mathcal{H}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \lambda.$$

The argument as in [15] (see the proof of Theorem 3.6.4) implies

$$\mathrm{w} - \lim_{n\to\infty} f_n = 0.$$

Noting $V(t)f_n = e^{-it\lambda_n} f_n$, we have

$$\|F(A)^2 f_n\|_{\mathcal{H}} = \|V(t)F(A)^2 f_n\|_{\mathcal{H}}.$$

In a similar decomposition and proof as in the estimate of (2.1) we can obtain

$$\lim_{n\to\infty} \|F(A)^2 f_n\|_{\mathcal{H}} = 0.$$

But this yields a contradiction with

$$\lim_{n\to\infty} \|F(A)^2 f_n\|_{\mathcal{H}} = \frac{\lambda^2}{(\lambda^2+1)^2} \neq 0.$$

Therefore the proof is complete. □

3. Applications

In this section we apply our framework to elastic wave equation with dissipative boundary condition in a half space of $\mathbf{R}^3$, Schrödinger equation with complex valued potential and acoustic wave equation with dissipative term. We also use C as positive constants.

Application 1 (Elastic wave equation with dissipative boundary condition in a half space of $\mathbf{R}^3$)

Let $\mathcal{H}$ be the Hilbert space with inner product:

$$\langle f, g\rangle_{\mathcal{H}} = \int_{\mathbf{R}^3_+} \left(\sum_{h,j,k,l=1}^{3} a_{hjkl}\varepsilon_{kl}(f_1)\overline{\varepsilon_{hj}(g_1)} + f_2\overline{g_2}\rho_0 \right) dx,$$

where $a_{hjkl} = \lambda_0\delta_{hj}\delta_{kl} + \mu_0(\delta_{hk}\delta_{jl} + \delta_{hl}\delta_{jk})$ and $f = {}^t(f_1, f_2)$, $g = {}^t(g_1, g_2)$. Korn's inequality (cf. Ito [5]) implies that $\mathcal{H}$ is equivalent to $\dot{H}^1(\mathbf{R}^3_+, \mathbf{C}^3) \times L^2(\mathbf{R}^3_+, \mathbf{C}^3)$ as Banach space.

We set $f = {}^t(u(x,t), u_t(x,t))$, where $u(x,t)$ is the solution to (1.3) (resp. (1.4)) with initial data $f_0 = {}^t(u(x,0), u_t(x,0)) \in \mathcal{H}$. Then (1.3) (resp. (1.4)) can be written as

$$\partial_t f = -iAf \qquad (\text{resp.}\quad \partial_t f = -iA_0 f),$$

where

$$A = i\begin{pmatrix} 0 & I_{3\times 3} \\ -\tilde{L}_0 & 0 \end{pmatrix}, \qquad A_0 = i\begin{pmatrix} 0 & I_{3\times 3} \\ -\tilde{L}_0 & 0 \end{pmatrix}, \tag{3.1}$$

$$\begin{aligned} D(A) = \{ f = {}^t(f_1, f_2) \in \mathcal{H}; \tilde{L}_0 f_1 \in L^2(\mathbf{R}^3_+, \mathbf{C}^3),\ f_2 \in H^1(\mathbf{R}^3_+, \mathbf{C}^3), \\ {}^t(\sigma_{13}(f_1), \sigma_{23}(f_1), \sigma_{33}(f_1))\,|_{x_3=0} = B(y) f_2\,|_{x_3=0} \} \end{aligned}$$

and

$$\begin{aligned} D(A_0) = \{ f = {}^t(f_1, f_2) \in \mathcal{H}; \tilde{L}_0 f_1 \in L^2(\mathbf{R}^3_+, \mathbf{C}^3),\ f_2 \in H^1(\mathbf{R}^3_+, \mathbf{C}^3), \\ \sigma_{h3}(f_1)\,|_{x_3=0} = 0 (h = 1,\ 2,\ 3) \} \end{aligned}$$

According to Lax-Phillips [11, P210–P211] or Petkov [15, Corollary 1.1.4] we can show that A (resp. A_0) generates a contraction semi-group $\{V(t)\}_{t\geqq 0}$ (resp. a unitary group $\{U_0(t)\}_{t\in\mathbf{R}}$) in $\mathcal{H}$. Using $\{V(t)\}_{t\geqq 0}$ (resp. $\{U_0(t)\}_{t\in\mathbf{R}}$) we solve $\partial_t f = -iAf$ (resp. $\partial_t f = -iA_0 f$) as follows:

$$f = V(t) f_0 \quad (\text{resp. } f = U_0(t) f_0).$$

Moreover we have the following.

Theorem 3.1. *For A_0 and A as in* (3.1)*, one has the conclusions as in* Theorem 1, Corollary 2 *and* Theorem 3.

To show this theorem, we prove (A1), (A2) and (A3) in Lemma 3.2, 3.3 and Proposition 3.4, respectively.

Lemma 3.2. *A_0 from* (3.1) *satisfies* (A1).

Proof. Lemma A of the Appendix implies $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ (see also [1]). □

Lemma 3.3. *A_0 and A from* (3.1) *satisfy* (A2).

Proof. Note that

$$\begin{aligned}&\langle((A-i)^{-1}-(A_0-i)^{-1})f, g\rangle_{\mathcal{H}}\\ &\quad=\langle A_0(A_0-i)^{-1}f,(A^*+i)^{-1}g\rangle_{\mathcal{H}}-\langle(A_0-i)^{-1}f, A^*(A^*+i)^{-1}g\rangle_{\mathcal{H}}\end{aligned}$$

for f, $g\in\mathcal{H}$. By easy calculation we have

$$\begin{aligned}&\langle((A-i)^{-1}-(A_0-i)^{-1})f, g\rangle_{\mathcal{H}}\\ &\quad=i\int_{\mathbf{R}^2}B(y)\Gamma_0((A_0-i)^{-1}f)_2\overline{\Gamma_0((A^*+i)^{-1}g)_2}\,dy,\end{aligned}\tag{3.2}$$

where Γ_0 is a trace operator defined by

$$(\Gamma_0 u)(y)=u(y,0).$$

For any $s\in(1/2,1)$, Korn's inequality implies that $\Gamma_0((A_0-i)^{-1}f)_2$ and $\Gamma_0((A^*+i)^{-1}f)_2$ belong to $H^{1-s}(\mathbf{R}^3_+,\mathbf{C}^3)$. Since $B(y)\Gamma_0\Pi_2(A_0-i)^{-1}$ is a compact operator from $\mathcal{H}$ to $L^2(\mathbf{R}^2,\mathbf{C}^3)$ by Rellich's theorem, where $\Pi_j{}^t(f_1,f_2)=f_j$ ($j=1,2$), the form $(A-i)^{-1}-(A_0-i)^{-1}$ can be extended to a compact operator, $(\Gamma_0\Pi_2(A^*+i)^{-1})^*B(y)\Gamma_0\Pi_2(A_0-i)^{-1}$, in $\mathcal{H}$.

□

Finally we show (A3). Using F_j ($j=P, S, SH, R$) (see the Appendix), we construct $P_\pm$ as follows:

$$\begin{aligned}P_\pm=T^{-1}\Bigg\{&\sum_{j=P,S,SH}\begin{pmatrix}F_j^*P_\mp^{(3)}I_{3\times3}F_j & O_{3\times3}\\ O_{3\times3} & F_j^*P_\pm^{(3)}I_{3\times3}F_j\end{pmatrix}\\ &+\begin{pmatrix}F_R^*P_\mp^{(2)}I_{3\times3}F_R & O_{3\times3}\\ O_{3\times3} & F_R^*P_\pm^{(2)}I_{3\times3}F_R\end{pmatrix}\Bigg\}T\end{aligned}\tag{3.3}$$

where

$$T=\frac{1}{\sqrt{2}}\begin{pmatrix}L_0^{1/2} & iI_{3\times3}\\ L_0^{1/2} & -iI_{3\times3}\end{pmatrix}$$

and $P_-^{(3)}$ (resp. $P_+^{(3)}$) and $P_-^{(2)}$ (resp. $P_+^{(2)}$) are negative (resp. positive) spectral projections of

$$D^{(3)}=\frac{1}{2i}(k\cdot\nabla_k+\nabla_k\cdot k)\quad\text{and}\quad D^{(2)}=\frac{1}{2i}(p\cdot\nabla_p+\nabla_p\cdot p),\quad\text{respectively,}$$

where $k=(p,p_3)\in\mathbf{R}^2\times\mathbf{R}_+$.

Using the representation of the generalized eigenfunction of L_0 and the Mellin transformation we show (A3.1)$\sim$(A3.4) (cf. Perry [14] and Kadowaki [6]). The Mellin transformations for $D^{(3)}$, $D^{(2)}$ are given as

$$(M^{(3)}u)(\lambda, \omega) = (2\pi)^{-1/2} \int_0^{+\infty} r^{1/2-i\lambda} u(r\omega)\, dr$$

and

$$(M^{(2)}v)(\lambda, \nu) = (2\pi)^{-1/2} \int_0^{+\infty} r^{-i\lambda} v(r\nu)\, dr,$$

where $u(k) \in C_0^\infty(\mathbf{R}_+^3 \backslash \{0\})$, $v(p) \in C_0^\infty(\mathbf{R}^2 \backslash \{0\})$, $\omega \in \mathbf{S}_+^2 = \{(\omega_1, \omega_2, \omega_3) = (\overline{\omega}, \omega_3) \in \mathbf{S}^2 : \omega_3 > 0\}$ and $\nu \in \mathbf{S}^1$.

Then $M^{(3)}$ (resp. $M^{(2)}$) is extended to a unitary operator from $L^2(\mathbf{R}_+^3)$ (resp. $L^2(\mathbf{R}^2)$) to $L^2(\mathbf{R} \times \mathbf{S}_+^2)$ (resp. $L^2(\mathbf{R} \times \mathbf{S}^1)$) (cf. [14, Lemma 2]).

Proposition 3.4. *$P_\pm$ from* (3.3) *satisfy* (A3).

To show Proposition 3.4 we prepare

Lemma 3.5. *Let $\psi(\lambda)$ be as in* (A3) *and $0 < \delta < c_R$ (for c_R, see the Appendix). Then for any positive integer N and $t \in \mathbf{R}_\pm$, there exists a positive constant $C_{N,\psi}$ which is independent of t such that*

$$\|\nabla_x (e^{-itA_0}\psi(A_0)P_\pm f)_1\|_{L^2(\mathbf{R}_+^3, \mathbf{C}^3)}^{|x| \leqq \delta|t|} \leqq C_{N,\psi}(1 + |t|)^{-N} \|f\|_{\mathcal{H}}, \tag{3.4}$$

$$\|(e^{-itA_0}\psi(A_0)P_\pm f)_2\|_{L^2(\mathbf{R}_+^3, \mathbf{C}^3)}^{|x| \leqq \delta|t|} \leqq C_{N,\psi}(1 + |t|)^{-N} \|f\|_{\mathcal{H}} \tag{3.5}$$

and

$$\|\Gamma_0 (e^{-itA_0}\psi(A_0)P_\pm f)_2\|_{L^2(\mathbf{R}^2, \mathbf{C}^3)}^{|y| \leqq \delta|t|} \leqq C_{N,\psi}(1 + |t|)^{-N} \|f\|_{\mathcal{H}} \tag{3.6}$$

for any $f \in \mathcal{H}_0$, where

$$\|u\|_{L^2(\mathbf{R}_+^3, \mathbf{C}^3)}^B = \left(\int_B |u|^2\, dx \right)^{1/2} \quad \textit{and} \quad \|v\|_{L^2(\mathbf{R}^2, \mathbf{C}^3)}^B = \left(\int_B |v|^2\, dy \right)^{1/2}.$$

Proof. For the sake of simplicity, we shall restrict ourselves to the case $t \in \mathbf{R}_+$ and $\operatorname{supp}\psi \subset \mathbf{R}_+$ only. Moreover we prove the case (3.5) only. Other cases are proven in the same way as below.

Concerning new notations, for instance $\Psi_j (j = P, S, SH, R)$, see the Appendix. By Lemma A of the Appendix, we have

$$(e^{-itA_0}\psi(A_0)P_+ f)_2$$

$$= -\frac{i}{2}\left\{\sum_{j=P,S} I_j(x,t) + I_R(x,t) + I_{SH}(x,t)\right\},$$

where

$$I_j(x,t) = \frac{i}{2}\int_{\mathbf{R}^3_+} \exp(-itc_j|k|)\psi(c_j|k|)\Psi_j(x,k)P_-^{(3)}F_j(L_0^{1/2}f_1 + if_2)\,dk,$$
$$I_R(x,t) = \frac{i}{2}\int_{\mathbf{R}^2} \exp(-itc_R|p|)\psi(c_R|p|)\Psi_R(x,p)P_-^{(2)}F_R(L_0^{1/2}f_1 + if_2)\,dp$$

and

$$I_{SH}(x,t) = \frac{i}{2}\int_{\mathbf{R}^3_+} \exp(-itc_S|k|)\psi(c_S|k|)\Psi_{SH}(x,k)P_-^{(3)}F_{SH}(L_0^{1/2}f_1 + if_2)\,dk.$$

If we show

(3.7) $$|I_j(x,t)| \leqq C_\psi(1+t)^{-N'}\|f\|_{\mathcal{H}} \quad (j = P,\ S,\ SH,\ R)$$

for any positive integer N', we can obtain (3.5). Here we deal with $I_S(x,t)$ and $I_R(x,t)$ only. Other terms can be dealt with in the same way.

We set

$$K^S_{x,t}(k) = \exp(itc_j|k|)\overline{\psi(c_S|k|)\Psi_S(x,k)}$$

and

$$K^R_{x,t}(p) = \exp(itc_R|p|)\overline{\psi(c_R|p|)\Psi_R(x,p)}.$$

Note that

$$I_S(x,t) = \langle F_S(L_0^{1/2}f_1 + if_2),\, P_-^{(3)}K_S\rangle_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}$$

and

$$I_R(x,t) = \langle F_R(L_0^{1/2}f_1 + if_2),\, P_-^{(2)}K_R\rangle_{L^2(\mathbf{R}^2,\mathbf{C}^3)}.$$

Noting the partially isometric property of F_S and F_R (cf. the Appendix) we have

(3.8) $$|I_S(x,t)| \leqq C\|\chi_{(-\infty,0)}M^{(3)}K^S_{x,t}\|_{L^2(\mathbf{R}\times\mathbf{S}^2_+)} \times \|f\|_{\mathcal{H}}$$

and

(3.9) $$|I_R(x,t)| \leqq C\|\chi_{(-\infty,0)}M^{(2)}K^R_{x,t}\|_{L^2(\mathbf{R}\times\mathbf{S}^1)} \times \|f\|_{\mathcal{H}},$$

where $\chi_{(-\infty,0)} = \chi_{(-\infty,0)}(\lambda)$.

$M^{(3)}K^S_{x,t}(\lambda,\omega)$ and $M^{(2)}K^R_{x,t}(\lambda,\nu)$ are represented as follows:

$$
\begin{aligned}
&M^{(3)}K^S_{x,t}(\lambda,\omega)\\
&\quad=\frac{1}{(2\pi)^2\rho_0^{1/2}}\int_0^\infty \exp i(-r\varpi\cdot y+tc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\chi_{\tilde{e}_{SV}}(\omega)\\
&\quad\{e^{ir\omega_3x_3}\overline{S_{V1}(\omega)}+e^{-ir\omega_3x_3}\overline{S_{V2}(\omega)}+e^{-ir\gamma_{P,S}(\varpi)x_3}\overline{S_{V3}(\omega)}\}\,dr\\
&\quad+\frac{1}{(2\pi)^2\rho_0^{1/2}}\int_0^\infty \exp i(-r\varpi\cdot y+tc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\chi_{\tilde{e}^0_{SV}}(\omega)\\
&\quad\{e^{ir\omega_3x_3}\overline{S^0_{V1}(\omega)}+e^{-ir\omega_3x_3}\overline{S^0_{V2}(\omega)}+e^{-r\gamma'_{P,S}(\varpi)x_3}\overline{S^0_{V3}(\omega)}\}\,dr,
\end{aligned}
$$

and

$$
\begin{aligned}
&M^{(2)}K^R_{x,t}(\lambda,\nu)\\
&=\frac{\tilde{C}}{(2\pi)^{3/2}}\left\{\int_0^\infty \exp i(-r\nu\cdot y+ir\tilde{\gamma}_{R,P}x_3+tc_Rr-\lambda\log r)r^{1/2}\psi(c_Rr)\left(2-\frac{c_R^2}{c_S^2}\right)\overline{R_1(\nu)}\,dr\right.\\
&\qquad\left.+\int_0^\infty \exp i(-r\nu\cdot y+ir\tilde{\gamma}_{R,S}x_3+tc_Rr-\lambda\log r)r^{1/2}\psi(c_Rr)(-2\tilde{\gamma}_{R,P})\overline{R_2(\nu)}\,dr\right\},
\end{aligned}
$$

where

$$
\tilde{e}_{SV}=\left\{\omega\in\mathbf{S}^2;\omega_3>\left(\frac{c_P^2}{c_S^2}-1\right)^{1/2}|\varpi|\right\}
$$

and

$$
\tilde{e}^0_{SV}=\left\{\omega\in\mathbf{S}^2;0<\omega_3<\left(\frac{c_P^2}{c_S^2}-1\right)^{1/2}|\varpi|\right\}.
$$

Since $S_{Vj}(\omega)$, $S^0_{Vj}(\omega)$ ($j=1,2,3$) and $R_k(\nu)$ ($k=1,2$) are bounded for ω and ν (see the Appendix), respectively, we have

$$
\begin{aligned}
(3.10)\quad &|M^{(3)}K^S_{x,t}(\lambda,\omega)|\\
&\leqq C\left\{\left|\int_0^\infty \exp i(-r\varpi\cdot y+r\omega_3x_3+tc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\,dr\right|\right.\\
&\quad+\left|\int_0^\infty \exp i(-r\varpi\cdot y-r\omega_3x_3+tc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\,dr\right|\\
&\quad+\left|\chi_{\tilde{e}_{SV}}(\omega)\int_0^\infty \exp i(-r\varpi\cdot y-r\gamma_{P,S}(\varpi)x_3+tc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\,dr\right|\\
&\quad\left.+\left|\chi_{\tilde{e}^0_{SV}}(\omega)\int_0^\infty \exp i(-r\varpi\cdot y+ir\gamma'_{P,S}(\varpi)x_3+itc_Sr-\lambda\log r)r^{1/2}\psi(c_Sr)\,dr\right|\right\}
\end{aligned}
$$

and

$$(3.11)\ |M^{(2)}K^R_{x,t}(\lambda,\nu)| \leqq C\left\{\left|\int_0^\infty \exp i(-r\nu\cdot y + ir\tilde{\gamma}_{R,P}x_3 + tc_R r - \lambda\log r)r^{1/2}\psi(c_R r)\,dr\right| + \left|\int_0^\infty \exp i(-r\nu\cdot y + ir\tilde{\gamma}_{R,S}x_3 + tc_R r - \lambda\log r)r^{1/2}\psi(c_R r)\,dr\right|\right\}$$

Therefore it is sufficient to estimate RHS of (3.10) and (3.11).

Note that $\operatorname{supp}\psi$ is compact and does not contain 0. Since $\lambda < 0$, $t > 0$, $|y| \leqq |x| \leqq \delta t$ and $c_R < c_S < c_P$ (cf. the Appendix) we have

$$\begin{cases} |\pm\omega_3 x_3 - \overline{\omega}\cdot y + c_S t - \dfrac{\lambda}{r}| \geqq c_S t + \dfrac{|\lambda|}{r} - |x| \geqq (c_S-\delta)t + C_\psi|\lambda| \\ |\gamma_{S,P}(\overline{\omega})x_3 - \overline{\omega}\cdot y + c_S t - \dfrac{\lambda}{r}| \geqq c_S t + \dfrac{|\lambda|}{r} - \dfrac{c_S}{c_P}|x| \geqq \left(c_S - \dfrac{c_S}{c_P}\delta\right)t + C_\psi|\lambda| \\ |i\gamma'_{S,P}(\overline{\omega})x_3 - \overline{\omega}\cdot y + c_S t - \dfrac{\lambda}{r}| \geqq c_S t + \dfrac{|\lambda|}{r} - |y| \geqq (c_S-\delta)t + C_\psi|\lambda| \\ |i\tilde{\gamma}_{S,j}x_3 - \nu\cdot y + c_R t - \dfrac{\lambda}{r}| \geqq c_R t + \dfrac{|\lambda|}{r} - |y| \geqq (c_R-\delta)t + C_\psi|\lambda| \quad (j = P,\ S) \end{cases},$$

where C_ψ is a positive constant which depends on ψ only.

Thus the stationary phase method implies

$$(3.12)\qquad |M^{(3)}K^S_{x,t}(\lambda,\omega)| \leqq C'_{\psi,m}(1+|\lambda|+t)^{-m}$$

and

$$(3.13)\qquad |M^{(2)}K^R_{x,t}(\lambda,\nu)| \leqq C'_{\psi,m}(1+|\lambda|+t)^{-m}$$

for any positive integer m, where $C'_{\psi,m}$ is a positive constant which depends on ψ and m.

Thus (3.12) and (3.13) imply

$$(3.14)\qquad \|\chi_{(-\infty,0)}M^{(3)}K^S_{x,t}\|_{L^2(\mathbf{R}\times\mathbf{S}^2_+)} \leqq C''_{\psi,m}(1+t)^{-m+1}$$

and

$$(3.15)\qquad \|\chi_{(-\infty,0)}M^{(2)}K^R_{x,t}\|_{L^2(\mathbf{R}\times\mathbf{S}^1)} \leqq C''_{\psi,m}(1+t)^{-m+1},$$

respectively, where $C''_{\psi,m}$ is a positive constant which depends on ψ and m.

(3.8), (3.14) and (3.9), (3.15) imply (3.7) for $j = S$ and R, receptively. □

Proof of Proposition 3.4. Lemma A of Appendix implies that P_+ and P_- are projection operators and satisfy $P_+ + P_- = I_d$ in $\mathcal{H}$. Below we show (A3.1)∼(A3.4).

For any $f,\ g \in \mathcal{H}$ we have by (3.2)

$$\begin{aligned} &|\langle Ke^{-itA_0}\psi(A_0)P_+ f, g\rangle_{\mathcal{H}}| \\ &\qquad \leqq CI(t) \times (\|A^*(A^*+i)^{-1}g\|_{\mathcal{H}} + \|(A^*+i)^{-1}g\|_{\mathcal{H}}), \end{aligned}$$

where

$$\begin{aligned} I(t) = &\left(\int_{\mathbf{R}^2} |B(y)\Gamma_0(e^{-itA_0}(A_0-i)^{-1}\psi(A_0)f)_2|^2\, dy\right)^{1/2} \\ &\times \left(\|A^*(A^*+i)^{-1}g\|_{\mathcal{H}} + \|(A^*+i)^{-1}g\|_{\mathcal{H}}\right). \end{aligned}$$

Decomposing $I(t)$ as follows:

$$\begin{aligned} I(t) \leqq C\Bigg\{ &\left(\int_{\mathbf{R}^2\cap\{|y|\leqq\delta t\}} |\Gamma_0(e^{-itA_0}(A_0-i)^{-1}\psi(A_0)P_+f)_2|^2\, dy\right)^{1/2} \\ &+\left(\int_{\mathbf{R}^2\cap\{|y|\geqq\delta t\}} |B(y)\Gamma_0(e^{-itA_0}(A_0-i)^{-1}\psi(A_0)P_+f)_2|^2\, dy\right)^{1/2}\Bigg\}, \end{aligned}$$

we have by (3.6) of Lemma 3.5 and (1.5)

$$I(t) \leqq C_{N,\psi}\{(1+t)^{-N} + \varphi(\delta t)\}\|f\|_{\mathcal{H}}.$$

Therefore (A3.1) is proven.

To prove (A3.2) and (A3.3) we note

$$\langle f, K^*g\rangle_{\mathcal{H}} = \langle((A-i)^{-1} - (A_0-i)^{-1})f, g\rangle_{\mathcal{H}}$$

for any $f,\ g \in \mathcal{H}$.

By easy calculation we have

$$\begin{aligned} (3.16)\qquad &\langle((A-i)^{-1} - (A_0-i)^{-1})f, g\rangle_{\mathcal{H}_0} \\ &\quad = i\int_{\mathbf{R}^2} \Gamma_0((A-i)^{-1}f)_2\overline{B(y)\Gamma_0((A_0+i)^{-1}g)_2}\, dy. \end{aligned}$$

Then using (3.16) and arguing the same way as in the proof of (A3.1), we obtain (A3.2) and (A3.3). Here we omit the detail.

We show (A3.4). For any $g \in \mathcal{H}$ and any positive integer N, we can estimate $\langle e^{itA_0}\psi(A_0)P_- f_t, g\rangle_{\mathcal{H}}$ as follows:

$$|\langle e^{itA_0}\psi(A_0)P_- f_t, g\rangle_{\mathcal{H}}|$$

$$\leqq C\Big\{ \|\nabla_x(e^{itA_0}\psi(A_0)P_- f_t)_1\|^{|x|\leqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)} \times \|\nabla_x g_1\|_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}$$
$$+ \|\nabla_x(e^{itA_0}\psi(A_0)P_- f_t)_1\|_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)} \times \|\nabla_x g_1\|^{|x|\geqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}$$
$$+ \|(e^{itA_0}\psi(A_0)P_- f_t)_2\|^{|x|\leqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)} \times \|g_2\|_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}$$
$$+ \|(e^{itA_0}\psi(A_0)P_- f_t)_2\|_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)} \times \|g_2\|^{|x|\geqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}\Big\}.$$

Thus Lemma 3.5 implies

$$|\langle e^{itA_0}\psi(A_0)P_- f_t, g\rangle_{\mathcal{H}}|$$
$$\leqq C_{N,\psi}\left\{(1+t)^{-N}\|g\|_{\mathcal{H}} + \|\nabla_x g_1\|^{|x|\geqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)} + \|g_2\|^{|x|\geqq\delta t}_{L^2(\mathbf{R}^3_+,\mathbf{C}^3)}\right\}\|f_t\|_{\mathcal{H}}.$$

Thus, noting $\sup_{t\in\mathbf{R}}\|f_t\|_{\mathcal{H}} < \infty$, we have (A3.4). □

REMARK 3.6. We can also include the elastic wave equation with dissipative boundary condition in a half space of $\mathbf{R}^2_+$ in our example by using the generalized eigenfunction expansion theorem proved by [1].

Application 2 (Schrödinger equation with complex valued potential)

Let $x \in \mathbf{R}^n$ and $\mathcal{H} = L^2(\mathbf{R}^n)$, where $n \geqq 3$. We assume that $B(x)$ is a measurable function which satisfies

(3.17) $$\operatorname{Im} B(x) \leqq 0$$

(3.18) $$|B(x)| = O(|x|^{-1}) \quad (|x| \to 0)$$

(3.19) $$|B(x)| = O(\varphi(|x|)) \quad (|x| \to \infty)$$

where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbf{R}_+)$.

Define $A_0 = -\triangle_x$, $D(A_0) = H^2(\mathbf{R}^n)$, where $\triangle_x$ is the n-dimensional Laplacian. Then A_0 is a self-adjoint operator. Using

$$\int_{\mathbf{R}^n} \frac{|u(x)|^2}{|x|^2}\,dx \leqq C\int_{\mathbf{R}^n} |\nabla_x u(x)|^2\,dx,$$

where $\nabla_x = (\partial/\partial_1, \partial/\partial_2, \ldots, \partial/\partial_n)$, we have

(3.20) $$\|Bf\|_{\mathcal{H}} \leqq a\|A_0 f\|_{\mathcal{H}} + b\|f\|_{\mathcal{H}}$$

for some a, $0 < a < 1$ and $b > 0$.

Thus $A = A_0 + B$ generates a contraction semi-group $\{V(t)\}_{t\geqq 0}$ in $\mathcal{H}$ and $D(A) = D(A_0)$ (see Theorem X-50 in Reed-Simon [16]). Of course, A_0 generates a unitary group $\{U_0(t)\}_{t\in\mathbf{R}}$ in $\mathcal{H}$. Moreover we have the following.

Theorem 3.7. *For A_0 and A as above, one has the conclusions as in* Theorem 1, Corollary 2 *and* Theorem 3.

Proof. It suffices to make a check on (A1), (A2) and (A3).

It is well-known that

$$\sigma(A_0) = \sigma_{ac}(A_0) = [0, \infty).$$

Thus we have (A1).

Sobolev's lemma,

$$H^l(\mathbf{R}^n) \hookrightarrow L^q(\mathbf{R}^n) \quad \left(\frac{1}{2} - \frac{l}{n} = \frac{1}{q}\right),$$

Rellich's theorem and (3.18), (3.19) imply that

$$(A - i)^{-1} - (A_0 - i)^{-1} \quad \text{is a compact operator in} \quad \mathcal{H}.$$

Thus (A2) is satisfied.

Finally we give a brief sketch of the proof of (A3). Choosing the positive (resp. negative) spectral projection of $D = 1/(2i)(x \cdot \nabla_x + \nabla_x \cdot x)$ as P_+ (resp. P_-) of (A3) and using Perry's lemma ([14, Lemma 1]) together with (3.20) we show (A3.1)$\sim$(A3.4) in the same way as in the proof in Application 1. □

Application 3 (Acoustic wave equation with dissipative term)

Let $x \in \mathbf{R}^n$, where $n \geqq 1$. We consider the following equation:

$$\partial_t^2 u(x,t) - \triangle_x u(x,t) + b(x)\partial_t u(x,t) = 0, (x,t) \in \mathbf{R}^n \times [0,\infty). \tag{3.21}$$

We consider (3.21) as a perturbed system of

$$\partial_t^2 u(x,t) - \triangle_x u(x,t) = 0, (x,t) \in \mathbf{R}^n \times \mathbf{R}. \tag{3.22}$$

We assume that $b(x)$ is a measurable function which satisfies

$$0 \leqq b(x) \leqq \varphi(|x|),$$

where $\varphi(r)$ is as in Application 2.

For (3.21) and (3.22) with a formulation similar to that in Application 1 (for the details, see Mochizuki [13]), we can also derive the conclusions in Theorem 1, Corollary 2 and Theorem 3. We omit the details and content ourselves by giving a brief comment on (A1), (A2) and (A3).

Due to [13] the operators associated with (3.21) and (3.22) satisfy (A1) and (A2) in suitable Hilbert spaces, and they generate a contraction semi-group and unitary group, respectively. Moreover $P_\pm$ in (A3) are defined as follows.

Let $\mathfrak{F}$ be the Fourier transformation. Let $k \in \mathbf{R}^n$ be the dual variable of x. Then we write $P_+^{(n)}$ (resp. $P_-^{(n)}$) as the positive (resp. negative) spectral projection of $1/(2i)(k \cdot \nabla_k + \nabla_k \cdot k)$. We set

$$P_\pm = T^{-1} \begin{pmatrix} \mathfrak{F}^{-1} P_\mp^{(n)} \mathfrak{F} & 0 \\ 0 & \mathfrak{F}^{-1} P_\pm^{(n)} \mathfrak{F} \end{pmatrix} T,$$

where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} (-\triangle_x)^{1/2} & i \\ (-\triangle_x)^{1/2} & -i \end{pmatrix}.$$

Then $P_\pm$ as above satisfy (A3). We omit to show that they satisfy (A3.1)~(A3.4).

Appendix

We state results which follow from Dermenjian-Guillot [1].

Let $k = (p, p_3) \in \mathbf{R}^2 \times \mathbf{R}_+ = \mathbf{R}^3_+$ be the deal variable of $x = (y, x_3) \in \mathbf{R}^2 \times \mathbf{R}_+ = \mathbf{R}^3_+$. By the polar coordinates we write k and p as

$$k = |k|\omega = |k|(\overline{\omega}, \omega_3) = |k|(\omega_1, \omega_2, \omega_3)$$

and

$$p = |p|\nu = |p|(\nu_1, \nu_2),$$

where $\omega \in \mathbf{S}^2_+ = \{(\omega_1, \omega_2, \omega_3) = (\overline{\omega}, \omega_3) \in \mathbf{S}^2 : \omega_3 > 0\}$ and $\nu \in \mathbf{S}^1$. We prepare some notations as follows:

$$c_P^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \qquad c_S^2 = \frac{\mu_0}{\rho_0},$$

$$\xi_{j,l}(k) = \left(\frac{c_j^2}{c_l^2}|k|^2 - |p|^2 \right)^{1/2}, \gamma_{j,l}(\overline{\omega}) = \left(\frac{c_j^2}{c_l^2} - |\overline{\omega}|^2 \right)^{1/2}, \quad \text{for} \quad j,\ l = P,\ S,$$

$$\xi'_{j,l}(k) = \left(|p|^2 - \frac{c_j^2}{c_l^2}|k|^2 \right)^{1/2}, \gamma'_{j,l}(\overline{\omega}) = \left(|\overline{\omega}|^2 - \frac{c_j^2}{c_l^2} \right)^{1/2} \quad \text{for} \quad j,\ l = P,\ S,$$

$$\tilde{\gamma}_{R,j} = \left(1 - \frac{c_R^2}{c_j^2} \right) \quad \text{for} \quad j = P,\ S$$

and

$$\tilde{E}_{SV} = \left\{ k \in \mathbf{R}^3_+;\, p_3 > \left(\frac{c_P^2}{c_S^2} - 1 \right)^{1/2} |p| \right\},$$

$$\tilde{E}^0_{SV} = \left\{ k \in \mathbf{R}^3_+; 0 < p_3 < \left(\frac{c_P^2}{c_S^2} - 1 \right)^{1/2} |p| \right\},$$

where c_R^2 is the unique solution in $(0, \mu_0/\rho_0)$ of the following equation with respect to α:

$$\left(1 - \frac{\alpha^2 \rho_0}{2\mu_0} \right)^{1/2} - \left(1 - \frac{\alpha^2 \rho_0}{\mu_0} \right)^{1/2} \left(\frac{\alpha^2 \rho_0}{\lambda_0 + 2\mu_0} \right)^{1/2} = 0.$$

Using the above notations we define functions as follows.
(1) For every $k \in \mathbf{R}^3_+$ let

$$\Psi_P(x,k) = \frac{1}{(2\pi)^{3/2} \rho_0^{1/2}} e^{ip\cdot y} \{ e^{-ip_3 x_3} P_1(\omega) + e^{i\xi_{S,P}(k) x_3} P_2(\omega) - e^{ip_3 x_3} P_3(\omega) \},$$

where

$$P_1(\omega) = {}^t(\omega_1, \omega_2, -\omega_3)$$

$$P_2(\omega) = \frac{4|\overline{\omega}| \left(\frac{c_P^2}{c_S^2} - 2|\overline{\omega}|^2 \right) \omega_3}{\left(\frac{c_P^2}{c_S^2} - 2|\overline{\omega}|^2 \right)^2 + 4|\overline{\omega}|^2 \omega_3 \gamma_{P,S}(\overline{\omega})} \, {}^t\!\left(\frac{\omega_1 \gamma_{P,S}(\overline{\omega})}{|\overline{\omega}|}, \frac{\omega_2 \gamma_{P,S}(\overline{\omega})}{|\overline{\omega}|}, |\overline{\omega}| \right)$$

$$P_3(\omega) = \frac{\left(\frac{c_P^2}{c_S^2} - 2|\overline{\omega}|^2 \right)^2 - 4|\overline{\omega}|^2 \omega_3 \gamma_{P,S}(\overline{\omega})}{\left(\frac{c_P^2}{c_S^2} - 2|\overline{\omega}|^2 \right)^2 + 4|\overline{\omega}|^2 \omega_3 \gamma_{P,S}(\overline{\omega})} \, {}^t(\omega_1, \omega_2, \omega_3).$$

(2) For every $k \in \tilde{E}_{SV}$ let

$$\Psi_{SV}(x,k) = \frac{1}{(2\pi)^{3/2} \rho_0^{1/2}} e^{ip\cdot y} \{ e^{-ip_3 x_3} S_{V1}(\omega) + e^{ip_3 x_3} S_{V2}(\omega) + e^{i\xi_{P,S}(k) x_3} S_{V3}(\omega) \},$$

where

$$S_{V1}(\omega) = {}^t\!\left(\frac{\omega_1 \omega_3}{|\overline{\omega}|}, \frac{\omega_2 \omega_3}{|\overline{\omega}|}, |\overline{\omega}| \right)$$

$$S_{V2}(\omega) = \frac{(1 - 2|\overline{\omega}|^2)^2 - 4|\overline{\omega}|^2 \omega_3 \gamma_{S,P}(\overline{\omega})}{(1 - 2|\overline{\omega}|^2)^2 + 4|\overline{\omega}|^2 \omega_3 \gamma_{S,P}(\overline{\omega})} \, {}^t\!\left(\frac{\omega_1 \omega_3}{|\overline{\omega}|}, \frac{\omega_2 \omega_3}{|\overline{\omega}|}, -|\overline{\omega}| \right)$$

$$S_{V3}(\omega) = \frac{4|\overline{\omega}|(1 - 2|\overline{\omega}|^2)\omega_3}{(1 - 2|\overline{\omega}|^2)^2 + 4|\overline{\omega}|^2 \omega_3 \gamma_{S,P}(\overline{\omega})} \, {}^t\!\left(\omega_1, \omega_2, \gamma_{S,P}(\overline{\omega}) \right).$$

(3) For every $k \in \tilde{E}^0_{SV}$ let

$$\Psi^0_{SV}(x,k) = \frac{1}{(2\pi)^{3/2} \rho_0^{1/2}} e^{ip\cdot y} \{ e^{-ip_3 x_3} S^0_{V1}(\omega) + e^{ip_3 x_3} S^0_{V2}(\omega) + e^{-\xi'_{P,S}(k) x_3} S^0_{V3}(\omega) \},$$

where

$$S^0_{V1}(\omega) = {}^t\left(\frac{\omega_1\omega_3}{|\overline{\omega}|}, \frac{\omega_2\omega_3}{|\overline{\omega}|}, |\overline{\omega}|\right)$$

$$S^0_{V2}(\omega) = \frac{(1-2|\overline{\omega}|^2)^2 - 4i|\overline{\omega}|^2\omega_3\gamma'_{S,P}(\overline{\omega})}{(1-2|\overline{\omega}|^2)^2 + 4i|\overline{\omega}|^2\omega_3\gamma'_{S,P}(\overline{\omega})}\ {}^t\left(\frac{\omega_1\omega_3}{|\overline{\omega}|}, \frac{\omega_2\omega_3}{|\overline{\omega}|}, -|\overline{\omega}|\right)$$

$$S^0_{V3}(\omega) = \frac{4|\overline{\omega}|(1-2|\overline{\omega}|^2)\omega_3}{(1-2|\overline{\omega}|^2)^2 + 4i|\overline{\omega}|^2\omega_3\gamma'_{S,P}(\overline{\omega})}\,{}^t\left(\omega_1, \omega_2, i\gamma'_{S,P}(\overline{\omega})\right).$$

(4) For every $k \in \mathbf{R}^3_+$ let

$$\Psi_{SH}(x,k) = \frac{1}{(2\pi)^{3/2}\rho_0^{1/2}} e^{ip\cdot y}(e^{ip_3x_3} + e^{-ip_3x_3})S_H(\omega),$$

where

$$S_H(\omega) = {}^t\left(-\frac{\omega_2}{|\overline{\omega}|}, -\frac{\omega_1}{|\overline{\omega}|}, 0\right).$$

(5) For every $p \in \mathbf{R}^2$ let

$$\Psi_R(x,p) = \frac{\tilde{C}}{2\pi}|p|^{1/2}e^{ip\cdot y}\left\{\left(2 - \frac{c_R^2}{c_S^2}\right)e^{-|p|\tilde{\gamma}_{R,P}x_3}R_1(\nu) - 2\tilde{\gamma}_{R,P}e^{-|p|\tilde{\gamma}_{R,S}x_3}R_2(\nu)\right\},$$

where

$$R_1(\nu) = {}^t\left(-i\frac{\nu_1}{|\nu|}, i\frac{\nu_2}{|\nu|}, \tilde{\gamma}_{R,P}\right),\ R_2(\nu) = {}^t\left(-i\frac{\nu_1}{|\nu|}\tilde{\gamma}_{R,S}, i\frac{\nu_2}{|\nu|}\tilde{\gamma}_{R,S}, -1\right)$$

and $\tilde{C}$ is a strictly positive constant such that

$$4\pi^2\int_0^\infty |\Psi_R(x,p)|^2\rho_0\,dx_3 = 1.$$

The relation between the above functions and the generalized eigenfunctions (ψ_P, ψ_{SV}, ψ^0_{SV}, ψ_{SH} and ψ_R) of [1] is the following

$$\Psi_P(x,k) = \left(\frac{c_Pp_3}{|k|}\right)^{1/2}\psi_P(x;p,c_P|k|), \quad \Psi_{SV}(x,k) = \left(\frac{c_Sp_3}{|k|}\right)^{1/2}\psi_{SV}(x;p,c_S|k|)$$

$$\Psi^0_{SV}(x,k) = \left(\frac{c_Sp_3}{|k|}\right)^{1/2}\psi^0_{SV}(x;p,c_S|k|), \quad \Psi_{SH}(x,k) = \left(\frac{c_Sp_3}{|k|}\right)^{1/2}\psi_{SH}(x;p,c_S|k|)$$

and $\Psi_R(x,p) = \psi_R(x;p)$.

By [1, Theorem 3.6], the following operators, F_P, F_S, F_{SH} and F_R,

$$F_P u(k) = L^2(\mathbf{R}^3_+, \mathbf{C}^3) - \lim_{N\to\infty} \int_{\mathbf{R}^3_+ \cap \{|x| \leqq N\}} \overline{\Psi_P(x,k)} u(x) \rho_0 \, dx,$$

$$F_S u(k) = L^2(\mathbf{R}^3_+, \mathbf{C}^3) - \lim_{N\to\infty} \int_{\mathbf{R}^3_+ \cap \{|x| \leqq N\}} \{\chi_{\tilde{E}_{SV}}(k) \overline{\Psi_{SV}(x,k)} + \chi_{\tilde{E}^0_{SV}}(k) \overline{\Psi^0_{SV}(x,k)}\} u(x) \rho_0 \, dx,$$

$$F_{SH} u(k) = L^2(\mathbf{R}^3_+, \mathbf{C}^3) - \lim_{N\to\infty} \int_{\mathbf{R}^3_+ \cap \{|x| \leqq N\}} \overline{\Psi_{SH}(x,k)} u(x) \rho_0 \, dx$$

and

$$F_R u(p) = L^2(\mathbf{R}^2, \mathbf{C}^3) - \lim_{N\to\infty} \int_{\mathbf{R}^3_+ \cap \{|x| \leqq N\}} \overline{\Psi_R(x,p)} u(x) \rho_0 \, dx$$

are partially isometric from $\mathcal{G} = L^2(\mathbf{R}^3_+, \mathbf{C}^3; \rho_0 \, dx)$ onto $L^2(\mathbf{R}^3_+, \mathbf{C}^3)$ and $L^2(\mathbf{R}^2, \mathbf{C}^3)$, respectively. Defining the operator F as follows:

$$Fu = (F_P u, F_S u, F_{SH} u, F_R u) \quad \text{for} \quad u \in \mathcal{G},$$

we have by [1, Theorem 3.6]

Lemma A. *F is unitary from $\mathcal{G}$ to*

$$\hat{\mathcal{H}} = \bigoplus_{j=1}^{3} L^2(\mathbf{R}^3_+, \mathbf{C}^3) \bigoplus L^2(\mathbf{R}^2, \mathbf{C}^3)$$

and for every $u \in D(L_0)$

$$F L_0 u = (c_P^2 |k|^2 F_P u, c_S^2 |k|^2 F_S u, c_S^2 |k|^2 F_{SH} u, c_R^2 |p|^2 F_R u).$$

References

[1] Y. Dermenjian and J.C. Guillot: *Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle,* Math. Meth. in the Appl. Sci. **10** (1988), 87–124.

[2] V. Enss: *Asymptotic completeness for quantum mechanical potential scattering*, Comm. Math. Phys **61** (1978), 285–291.

[3] V. Georgiev: *Disappearing solutions for dissipative hyperbolic systems of constant multiplicity*, Hokkaido Math. J. **15** (1986), 357–385.

[4] H. Isozaki and H. Kitada: *Modified wave operators with time-independent modifiers,* J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **32** (1985), 77–104.

[5] H. Ito: *Extended Korn's inequalities and the associated best possible constants*, J. Elasticity, **24** (1990), 43–78.

[6] M. Kadowaki: *Asymptotic completeness for acoustic propagators in perturbed stratified media*, Integral Eq. Operator Th. **26** (1996), 432–459.
[7] M. Kadowaki: *Low and high energy resolvent estimates for wave propagation in stratified media and their applications*, J. Differential Equations, **179** (2002), 246–277.
[8] M. Kadowaki: *Resolvent estimates and scattering states for dissipative systems*, Publ. RIMS, Kyoto Univ. **38** (2002), 191–209.
[9] T. Kato: *Wave operator and similarity for some non-selfadjoint operators*, Math. Annalen, **162** (1966), 258–279.
[10] S.T. Kuroda: Spectral theory II, Iwanami, Tokyo, 1979, Japanese.
[11] P.D. Lax and R.S. Phillips: *Scattering theory for dissipative systems*, J. Funct. Anal. **14** (1973), 172–235.
[12] A. Majda: *Disappearing solutions for the dissipative wave equations*, Indiana Univ. Math. J. **24** (1975), 1119–1138.
[13] K. Mochizuki: *Scattering theory for wave equations with dissipative terms*, Publ. RIMS, Kyoto Univ. **12** (1976), 383–390.
[14] P. Perry: *Mellin transforms and scattering theory* I. *Short range potentials*, Duke Math. J. **47** (1980), 187–193.
[15] V. Petkov: Scattering theory for hyperbolic operators, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1989.
[16] M. Reed and B. Simon: Methods of Modern Mathematical Physics, Vol. 2, Academic Press, New York, San Francisco, London, 1975.
[17] S. Shimizu: *Eigenfunction expansions for elastic wave propagation problem in stratified media* $\mathbf{R}^3$, Tsukuba. J. Math. **18** (1994), 283–350.
[18] B. Simon: *Phase space analysis of simple scattering systems*: *extensions of some work of Enss*, Duke Math. J. **46** (1979), 119–168.
[19] H. Soga: *Formulation of the Lax-Phillips scattering theory for the Rayleigh wave in a half-space*, Seminar Notes of Mathematical Sciences, Ibaraki Univ. **3** (2000), 62–67.
[20] P. Stefanov and V. Georgiev: *Existence of the scattering operator for dissipative hyperbolic systems with variable multiplicities*, J. Operator Theory **19** (1988), 217–241.
[21] R. Weder: Spectral and scattering theory for wave propagation in perturbed stratified media, Applied Mathematical Sciences 87, Springer-Verlag, New York, Berlin, Heidelberg, 1991.
[22] C. Wilcox: Sound propagation in stratified fluids, Applied Mathematical Sciences 50, Springer-Verlag, New York, Berlin, Heidelberg, 1984.

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