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ON DENJOY'S CANONICAL CONTINUED FRACTION EXPANSION

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1. Introduction

Let x be a real non-integer number with (regular) continued fraction expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = [a_0;\ a_1,\ a_2,\ \ldots], \tag{1}$$

where $a_0 \in \mathbb{Z}$ is such that $x - a_0 \in [0, 1)$, and $a_n \in \mathbb{N}$ for $n \geq 1$. As is well-known, the regular continued fraction (RCF) expansion of x is finite if and only if $x \in \mathbb{Q}$. In this case there are two possible expansions, otherwise the expansion is unique.

Apart from the RCF expansion there are very many other continued fraction expansions: the continued fraction expansion to the nearest integer, Nakada's α-expansions, Bosma's optimal expansion ... in fact too many to mention (see [6] and [3] for some background information).

One particular expansion, which attracted no attention whatsoever, and which is quite different from the continued fraction expansions mentioned above, is Denjoy's *canonical continued fraction expansion* (see [2], or [1], p. 275–6 for the original paper by Denjoy). In [2], Denjoy stated that every real number x has continued fraction expansions of the form

$$x = [d_0;\ d_1,\ d_2,\ \ldots], \tag{2}$$

where $d_0 \in \mathbb{Z}$ is such that $x - d_0 \geq 0$, and the digits d_n are either 0 or 1. Such a continued fraction expansion of x is called a *canonical continued fraction* (CCF) *expansion* of x. Since

$$a + \cfrac{1}{0 + \cfrac{1}{b}} = a + b, \tag{3}$$

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Denjoy noted that the RCF expansion (1) can be changed into a CCF expansion (2).

In this note we will prove Denjoy's claims, and also obtain the ergodic system underlying Denjoy's CCF expansion.

2. Insertions

Denjoy's remark (3) can be 'translated' into two 'operations' (called *insertions of type* i, where $i = 1, 2$) on the digits of any continued fraction expansion. They are based on the following two equations. If $a, b \in \mathbb{Z}$, $b \geq 2$, and $\xi \geq 0$, then

$$a + \frac{1}{b+\xi} = a + \cfrac{1}{1 + \cfrac{1}{0 + \cfrac{1}{b - 1 + \xi}}}$$

and

$$a + \frac{1}{b+\xi} = a - 1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{b + \xi}}}.$$

In the first case we inserted $1/(1 + 1/(0+))$ into $a + 1/(b + \xi)$, while in the second case $1/(0 + 1/(1+))$ was inserted.

Now let $x \in \mathbb{R} \setminus \mathbb{Z}$, with RCF expansion (1), and let $d_0 \in \mathbb{Z}$ be such that $x - d_0 \geq 0$. Setting $k = a_0 - d_0$, in case $k > 0$ we can apply the second insertion to (1) just before a_1. Doing so, we get

$$x = [a_0 - 1;\ 0,\ 1,\ a_1,\ a_2,\ \ldots]$$

as a continued fraction expansion of x. Repeating this procedure $k - 1$ times we find

$$x = [a_0 - k;\ (0, 1)^k,\ a_1,\ a_2,\ \ldots],$$

where $(0, 1)^k$ is an abbreviation for the string $0, 1, \ldots, 0, 1$ of 0's and 1's of length $2k$. For $k = 0$ this string is empty, i.e., we have

$$x = [a_0 - 0;\ (0, 1)^0,\ a_1,\ a_2,\ \ldots] = [d_0;\ a_1,\ a_2,\ \ldots].$$

(This would be the case if $a_0 = d_0$; note that $d_0 > a_0$ is impossible since $a_0 = \lfloor x \rfloor$.) Next let $i \geq 1$ be the first index for which $a_i > 1$. Applying the first insertion before a_i yields

$$x = [d_0;\ (0, 1)^k,\ 1^{i-1},\ 1,\ 0,\ a_i - 1,\ a_{i+1},\ \ldots],$$

where 1^k is an abbreviation for the string $1, \dots, 1$ consisting of k 1's, which is empty if $k = 0$. Repeating this procedure $a_i - 2$ times we find

$$x = [d_0;\ (0,1)^k,\ 1^{i-1},\ (1,0)^{a_i-1},\ 1,\ a_{i+1},\ \dots].$$

Note that we would have obtained the same result if the second insertion was used $a_i - 1$ times 'behind' a_i.

Repeating this procedure, we find for any $d_0 \in \mathbb{Z}$ with $d_0 \leq x$ the following CCF expansion of x:

$$(1) \qquad x = [d_0;\ (0,1)^{a_0-d_0},\ (1,0)^{a_1-1},\ 1,\ (1,0)^{a_2-1},\ 1,\ (1,0)^{a_3-1},\ 1,\ \dots].$$

In this expansion, never two consecutive digits will both equal 0. It follows from (1), that if x is irrational, then any CCF expansion of x is infinite and unique once d_0 is given. In case x is rational, any CCF expansion of x is finite. However, with d_0 given, two possible CCF expansions exist in this case.

Note that the first n RCF digits $a_1, \dots, a_n$ of x yield $a_1 + \dots + a_n$ CCF digits equal to 1 and $a_1 + \dots + a_n - n$ CCF digits equal to 0. Let Z_k be the number of 0's among the first k CCF digits $d_1, \dots, d_k$ of x, i.e., $Z_k = \#\{1 \leq i \leq k \colon d_i = 0\}$, and let W_k be the number of 1's. Then due to Khintchine's classical result (see [7]), that for almost all x (with respect to Lebesgue measure λ):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j = \infty,$$

we deduce that

$$\lim_{n\to\infty} \frac{Z_k}{W_k} = \lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j - n}{\sum_{j=1}^{n} a_j} = 1 \qquad \text{(a.e.)}$$

So in spite of the fact that the CCF expansion of $x \in \mathbb{R} \setminus \mathbb{Z}$ always has more 1's than 0's, we see that for almost all x there are asymptotically as many 0's as 1's. We conclude this section by noting that the CCF expansion of an $x \in \mathbb{Z}$ is

$$(4') \qquad x = [d_0;\ (0,1)^{x-d_0}]$$

for any $d_0 \in \mathbb{Z}$ with $d_0 \leq x$.

3. On quadratic irrationalities and Hurwitzian numbers

An old and classical result states that a number x is a quadratic irrationality (that is, an irrational root of a polynomial of degree 2 with integer coefficients) if and only if x has an RCF expansion which is eventually periodic, i.e., x is of the form

$$(1) \qquad x = [a_0;\ a_1, \dots, a_p,\ \overline{a_{p+1}, \dots, a_{p+l}}], \quad p \geq 0,\ l \geq 1,$$

where the bar indicates the period, see [4], [8] or [9] for various classical proofs of this result. It follows from (1) that x has an eventually periodic RCF expansion of the form (1) if and only if x has a CCF expansion of the form

$$\begin{aligned} x &= [d_0;\ (0,1)^{a_0-d_0},\ (1,0)^{a_1-1},\ 1,\dots,\ (1,0)^{a_p-1},\ 1,\\ &\quad \underbrace{(1,0)^{a_{p+1}-1},\ 1,\dots,\ (1,0)^{a_{p+l}-1},\ 1}_{\text{period}},\underbrace{(1,0)^{a_{p+1}-1},\ 1,\dots,\ (1,0)^{a_{p+l}-1},\ 1}_{\text{period}},\dots],\\ &= [d_0;\ (0,1)^{a_0-d_0},\ (1,0)^{a_1-1},\ 1,\dots,\ (1,0)^{a_p-1},\ 1,\ \overline{(1,0)^{a_{p+1}-1},\ 1,\dots,(1,0)^{a_{p+l}-1},\ 1}], \end{aligned}$$

where $d_0 \in \mathbb{Z}$ is such that $x - d_0 \geq 0$. Again the bar indicates the period. Thus we see that x is a quadratic irrationality if and only if the CCF expansion (1) of x is eventually periodic for every $d_0 \in \mathbb{Z}$ with $d_0 \leq x$.

A nice generalization of the concept of eventually periodic expansions are the so-called *Hurwitzian numbers*. A number x is called Hurwitzian if and only if x has an RCF expansion of the form

$$x = [a_0;\ a_1,\dots,\ a_p,\ \overline{K_1(k),\dots,\ K_l(k)}]_{k=0}^{\infty},\quad p \geq 0,\ l \geq 1,$$

where a_0 is an integer, the a_i's are positive integers, and $K_1(k),\dots,K_l(k)$ are polynomials with rational coefficients which take positive integral values for $k = 0, 1, \dots$, and at least one of these polynomials is non-constant, see [9]. A well-known example of a Hurwitzian number is $e = [2;\ \overline{1,\ 2k+2,\ 1}]_{k=0}^{\infty}$. Again it is immediate from (1) that a number x is Hurwitzian if for every $d_0 \leq x$ the CCF expansion of x is given by

$$\begin{aligned} x = [d_0;\ &(0,1)^{a_0-d_0},\ (1,0)^{a_1-1},\ 1,\dots,\ (1,0)^{a_p-1},\ 1,\\ &\overline{(1,0)^{K_1(k)-1},\ 1,\dots,\ (1,0)^{K_l(k)-1},\ 1}]_{k=0}^{\infty}. \end{aligned}$$

4. Canonical continued fraction convergents

Let $x \in \mathbb{R}$, and let $d_0 \in \mathbb{Z}$ be such that $x - d_0 \geq 0$. Furthermore, let (1) (or (4′)) be a CCF expansion of x. Finite truncation yields the sequence of CCF convergents $(C_n)_{n\geq 0}$ of x:

$$C_n := d_0 + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \ddots + \cfrac{1}{d_n}}} = [d_0;\ d_1,\ d_2,\dots,\ d_n],\ n \geq 0.$$

The value of C_n is computed using the rules $1/0 = \infty$ and $1/\infty = 0$. For $n \geq 2$ this implies that C_n equals C_{n-2} when $d_n = 0$. This means that C_n can equal ∞ (= $1/0$). In order to study the CCF convergents C_n of x, we define matrices A_n, M_n, for $n \geq 0$

by

$$A_0 := \begin{pmatrix} 1 & d_0 \\ 0 & 1 \end{pmatrix}, \quad A_n := \begin{pmatrix} 0 & 1 \\ 1 & d_n \end{pmatrix}, \quad M_n := A_0 A_1 \cdots A_n.$$

Setting

$$M_n := \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix},$$

it follows from $M_n = M_{n-1}A_n$ that

$$M_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix},$$

whence $q_n \geq 0$, p_n and q_n are relatively prime, and

$$\begin{aligned} p_{-1} &= d_0, \ \ p_0 = 0, \ \ p_n = d_n p_{n-1} + p_{n-2}, \\ q_{-1} &= 0, \quad q_0 = 1, \ \ q_n = d_n q_{n-1} + q_{n-2}. \end{aligned}$$

These recurrence relations show again that $d_n = 0$, for some $n \geq 1$, implies that $p_n = p_{n-2}$ and $q_n = q_{n-2}$. In particular, if $a_0 - d_0 > 0$, then

$$\begin{aligned} &p_1 = p_3 = \cdots = p_{2(a_0-d_0)-1} = 1, \\ &p_2 = d_0 + 1, \ \ p_4 = d_0 + 2, \ldots, \ \ p_{2(a_0-d_0)} = a_0, \end{aligned}$$

and

$$\begin{aligned} q_1 &= q_3 = \cdots = q_{2(a_0-d_0)-1} = 0 \\ q_2 &= q_4 = \cdots = q_{2(a_0-d_0)} = 1. \end{aligned}$$

Defining the Möbius-transformations $M_n \colon \mathbb{R}^* \to \mathbb{R}^*$ by

$$M_n(t) := \frac{p_{n-1}t + p_n}{q_{n-1}t + q_n}, \ \ n \geq 1,$$

we see by induction that

$$C_n = M_n(0) = \frac{p_n}{q_n}.$$

For the RCF expansion matrices similar to A_n and M_n can be defined. For $x \in \mathbb{R} \setminus \mathbb{Z}$ with RCF expansion (1), setting

$$B_0 := \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}, \quad B_n := \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}, \quad N_n := B_0 B_1 \cdots B_n,$$

one has that

$$N_n = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix},$$

where $Q_n > 0$, P_n and Q_n are relatively prime, and

$$\frac{P_n}{Q_n} = [a_0;\ a_1, \ldots, a_n],$$

see [6]. Since

$$\begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d_0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right)^{a_0 - d_0}$$

and

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{a_i - 1} = \begin{pmatrix} 1 & 0 \\ a_i - 1 & 1 \end{pmatrix},$$

we conclude (see also (1)) that

$$M_{k(n)} = N_n,$$

where $k(n) = a_0 - d_0 + 2(a_1 - 1) + 1 + \cdots + 2(a_n - 1) + 1$, which implies that the sequence $\left(P_n/Q_n\right)_{n \geq 0}$ of RCF convergents of x is a subsequence of the sequence $(C_n)_{n \geq 0}$ of CCF convergents of x.

Now let $a_{n+1} > 1$ for some $n \geq 1$. Since $d_{k(n)+2j} = 0$ for $1 \leq j \leq a_{n+1} - 1$, we already saw that $C_{k(n)} = C_{k(n)+2j} = P_n/Q_n$ for these values of j. This also follows from the fact that

$$\begin{aligned} M_{k(n)+2j} &= N_n \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{j} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \\ &= \begin{pmatrix} jP_n + P_{n-1} & P_n \\ jQ_n + Q_{n-1} & Q_n \end{pmatrix}. \end{aligned}$$

What can we say about $C_{k(n)+2j-1}$ for $1 \leq j \leq a_{n+1} - 1$? Since

$$M_{k(n)+2j-1} = M_{k(n)+2j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} P_n & jP_n + P_{n-1} \\ Q_n & jQ_n + Q_{n-1} \end{pmatrix},$$

we see that $C_{k(n)+2j-1}$ is a mediant convergent of x, i.e.,

$$C_{k(n)+2j-1} = \frac{jP_n + P_{n-1}}{jQ_n + Q_{n-1}}.$$

Thus we see that the collection $\{C_n : n \geq -1\}$ consists of the integers $d_0, \ldots, a_0$, of $1/0$, and of all RCF and mediant convergents of x. Note that every RCF convergent P_n/Q_n of x appears a_{n+1} times as a CCF convergent of x.

5. The Denjoy map T_d

One way of finding the RCF expansion (1) of x is by using the so-called *Gauss-map* $T : [0, 1) \to [0, 1)$, defined by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, x \in (0, 1);\ T(0) := 0.$$

For $x \in \mathbb{R}$ given by (1) or (4′), let $d_0 \in \mathbb{Z}$ be such that $x - d_0 \geq 0$. Setting $\xi = x - d_0$, it is clear that

$$\xi = [0; d_1, d_2, \ldots].$$

Similarly to the RCF case, this CCF expansion of ξ can easily be obtained from a suitable map T_d, which we call the *Denjoy-map*. Let $T_d : [0, \infty) \to [0, \infty)$ be defined by

$$T_d(x) := \begin{cases} \frac{1}{x} - 1, & x \in (0, 1], \\ \frac{1}{x} - 0, & x \in (1, \infty), \\ 0, & x = 0. \end{cases}$$

Furthermore, setting

$$d_1 = d_1(\xi) := \begin{cases} 1, & \xi \in (0, 1], \\ 0, & \xi \in (1, \infty), \end{cases}$$

and

$$d_n = d_n(\xi) := d_1\left(T_d^{n-1}(\xi)\right), \qquad n > 1,$$

we find in case $T_d^k(\xi) \neq 0$ for $k = 0, 1, \ldots, n-1$, that

$$\xi = \frac{1}{d_1 + T_d(\xi)} = \frac{1}{d_1 + \cfrac{1}{d_2 + T_d^2(\xi)}} = \cdots$$

$$= \cfrac{1}{d_1 + \cfrac{1}{d_2 + \ddots + \cfrac{1}{d_n + T_d^n(\xi)}}}.$$

There are several algorithms yielding the RCF convergents and mediants, see for instance [3] or [5], where such algorithms together with the underlying ergodic systems are described. In [5], for any $x \in [0,1)$, the RCF convergents and mediants of x are 'generated' in the same order — but without the duplication of the RCF convergents — as in the case of the CCF expansion of x. The underlying map $S\colon [0,1] \to [0,1]$ in [5] is given by

$$S(x) = \begin{cases} \dfrac{x}{1-x}, & x \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1-x}{x}, & x \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

and ν is a σ-finite, infinite S-invariant measure with density g, given by

$$g(x) = \frac{1}{x}, \qquad \text{for } x \in (0,1).$$

Moreover, Ito showed in [5] that the dynamical system $([0,1), S, \nu)$ is ergodic.

It is easy to find by direct calculation that

$$S(x) = \begin{cases} T_d^2(x), & x \in \left[0, \dfrac{1}{2}\right), \\ T_d(x), & x \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

i.e., S can be seen as a *jump transformation* of T_d. Due to this, the ergodic properties of S can easily be carried over to T_d. Note that T_d^2 is used to avoid duplication of RCF convergents. Of course, since

$$T(x) = T_d^{2(k-1)+1}(x), \quad \text{for } x \in \left[\frac{1}{k+1}, \frac{1}{k}\right), \quad k \in \mathbb{N},$$

which follows from (1) or by direct calculation, the ergodic properties of T_d can also be obtained from the ergodic properties of the RCF expansion.

We have the following result.

Theorem 1. *The Denjoy-map T_d has a σ-finite, infinite invariant measure μ with density f, given by*

$$f(x) = \frac{1}{x} 1_{(0,1]}(x) + \frac{1}{1+x} 1_{(1,\infty)}(x), \qquad x \in [0,\infty),$$

and the dynamical system $([0,\infty), T_d, \mu)$ is ergodic.

Proof. By Theorem 1.1 from [10], to prove that T_d is μ-measure preserving, it is enough to show that

$$\mu\left(T_d^{-1}(A)\right) = \mu(A),$$

for every interval $A \subset [1, \infty)$. Let us first assume that $A \subset [0, 1]$. In this case $\mu(A) = \log(b/a)$ if $A = [a, b]$, and

$$\begin{aligned}\mu\left(T_d^{-1}(A)\right) &= \int_{1/(b+1)}^{1/(a+1)} \frac{dx}{x} + \int_{1/b}^{1/a} \frac{dx}{1+x}\\ &= \log\frac{b}{a} = \mu(A).\end{aligned}$$

Next assume that $A \subset (1, \infty)$. In this case

$$\mu(A) = \int_a^b \frac{dx}{1+x} = \log\frac{b+1}{a+1},$$

and

$$\mu\left(T_d^{-1}(A)\right) = \int_{1/(b+1)}^{1/(a+1)} \frac{dx}{x} = \log\frac{b+1}{a+1} = \mu(A).$$

Let A be a T_d-invariant Borel set, i.e., $T_d^{-1}(A) = A$. In order to show that T_d is ergodic with respect to Lebesgue measure λ (and therefore also ergodic with respect to μ, since λ and μ are equivalent), we should show that either $\lambda(A) = 0$ or $\lambda(A^c) = 0$, where $A^c = [0, \infty) \setminus A$.

Setting $A_1 = A \cap [0, 1]$, $A_2 = A \cap (1, \infty)$, we have

$$A_1 = \left(T_d^{-1}(A_1) \cap [0, 1]\right) \cup T_d^{-1}(A_2),$$

and

$$A_2 = T_d^{-1}(A_1) \cap (1, \infty).$$

Then

$$S^{-1}(A_1) = A_1,$$

i.e., A_1 is an S-invariant set. Since $([0, 1), S, \nu)$ is an ergodic dynamical system we see that either $\nu(A_1) = 0$ or $\nu([0, 1) \setminus A_1) = 0$, which implies that either $\lambda(A_1) = 0$ or $\lambda([0, 1) \setminus A_1) = 0$. In case $\lambda(A_1) = 0$ we clearly have $\lambda(A_2) = 0$, hence $\lambda(A) = 0$. In case $\lambda([0, 1) \setminus A_1) = 0$, it follows from $\left(T_d^{-1}([0, 1) \setminus A_1)\right) \cap (1, \infty) = (1, \infty) \setminus A_2$ that $\lambda((1, \infty) \setminus A_2) = 0$, hence $\lambda([0, \infty) \setminus A) = 0$. □

References

[1] C. Brezinski: History of continued fractions and Padé approximants, Springer Series in Computational Mathematics, **12**. Springer-Verlag, Berlin, 1991. MR 92c:01002.

[2] A. Denjoy: Complément à la notice publiée en 1934 sur les travaux scientifiques de M. Arnaud Denjoy, Hermann, Paris, 1942.

[3] K. Dajani and C. Kraaikamp: *The Mother of All Continued Fractions*, Coll. Math. **84/85** (2000), 109–123. MR 2001h:11100.

[4] G.H. Hardy and E.M. Wright: An introduction to the theory of numbers, Fifth edition. The Clarendon Press, Oxford University Press, New York, 1979. MR 81i:10002.

[5] Sh. Ito: *Algorithms with mediant convergents and their metrical theory*, Osaka J. Math. **26** (1989), 557–578. MR 90k:11101.

[6] C. Kraaikamp: *A new class of continued fraction expansions*, Acta Arith. **57** (1991), 1–39. MR 92a:11090.

[7] A.Ya. Khintchine: *Metrische Kettenbruchprobleme*, Compositio Math. **1** (1935), 361–382.

[8] C.D. Olds: Continued fractions, Random House, New York 1963. MR 26#3672.

[9] O. Perron: Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche, (German) 3te Aufl. B. G.
Teubner Verlagsgesellschaft, Stuttgart, 1954. MR 16,239e.

[10] P. Walters: An Introduction to Ergodic Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1982. MR 84e:28017.

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