Yamamuro, K.
Osaka J. Math.
40 (2003), 187–206

GROWTH OF SAMPLE FUNCTIONS OF SELFSIMILAR ADDITIVE PROCESSES

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(Received September 5, 2001)

1. Introduction and results

A stochastic process $\{X_t : t \geq 0\}$ on $\mathbf{R}^d$, which is defined on a probability space $(\Omega, \mathcal{F}, P)$, is said to be a selfsimilar additive process with exponent $H > 0$ if it satisfies the following conditions:
(i) $\{X_{ct}\}$ and $\{c^H X_t\}$ have the same finite-dimensional distributions for every $c > 0$,
(ii) $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent for any n and any choice of $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$,
(iii) alomost surely X_t is right continuous in $t \geq 0$ and has left limits in $t > 0$.
We can derive the sample function behavior of selfsimilar additive processes with exponent H from those with exponent 1 by using their selfsimilarity. Hence, throughout this paper we only consider a selfsimilar additive process $\{X_t\}$ on $\mathbf{R}^1$ with exponent 1. The distribution of X_1 is self-decomposable. Thus its Lévy measure is represented as $k(x)/|x|\,dx$, where $k(x)$ is nonnegative decreasing on $(0,\infty)$ and nonegative increasing on $(-\infty, 0)$, and $\int_{\mathbf{R}^1}(1 \wedge |x|^2)k(x)/|x|\,dx < \infty$. We use the words "increase" and "decrease" in the wide sense in this paper. From now on we suppose that both the Gaussian covariance and the drift of $\{X_t\}$ are zero and $\int_{|x|\leq 1} k(x)\,dx < \infty$, that is, that the characteristic function of X_1 is represented as

$$\hat{P}_{X_1}(z) = \int_{\mathbf{R}^1} e^{izx} P_{X_1}(dx) = \exp\left[\int_{\mathbf{R}^1} (e^{izx} - 1)\frac{k(x)}{|x|}\,dx\right],$$

where $k(x) \geq 0$, $k(x)$ is decreasing on $(0,\infty)$ and increasing on $(-\infty, 0)$, and $\int_{\mathbf{R}^1}(1 \wedge |x|)k(x)/|x|\,dx < \infty$. Here we denoted the distribution of X_1 by P_{X_1}.

We have investigated recurrence-transience for selfsimilar additive processes (see [11], [16], and [17]). However, the attempt has not been completely successful so far. In order to achieve this aim, we need to get imformation about their sample function behavior. Hence we study growth of their sample functions as time tends to infinity. There is a precedent for this study, but it only deals with increasing selfsimilar additive prcesses (see [14]). Even in the case of Lévy processes on $\mathbf{R}^1$, growth of sample functions is not known under general assumption, but some papers deal with the case of subordinators and of symmetric Lévy processes (for example, see [5], [6], [9], and [7]). In order to solve this kind of problems, we need to develop some techniques, and we shall suggest useful tools. One is to use the fisrt Borell-Cantelli lemma and the generalization of the second Borell-Cantelli lemma. The other is to use the distribution of the hitting times with respect to $\{X_t\}$ to apply these lemmas. We can find them being applied to strictly stable processes (see Corollary 11.3 and Theorem 11.5 in [8]). We note that strictly stable processes are only Lévy processes which are selfsimilar additive processes. These tools might be more useful in the future. In addition, we shall devise some method which does not need the Borell-Cantelli lemmas.

We state the limsup behavior of the selfsimilar additive processes $\{X_t\}$.

Theorem 1.1. *Let $h(t)$ be an increasing positive measurable function on $[1,\infty)$ and let $h(t)/t$ be increasing. Suppose that $k(x)>0$ for all $x\in\mathbf{R}^1\setminus\{0\}$,*

$$\limsup_{|x|\to\infty}\frac{k(\rho_0 x)}{k(x)}<1 \quad \text{for some } \rho_0>1, \tag{1.1}$$

and

$$\liminf_{|x|\to\infty}\frac{k(\rho x)}{k(x)}>0 \quad \text{for every } \rho>1. \tag{1.2}$$

If

$$\int_1^\infty\left\{k\left(\frac{h(t)}{t}\right)+k\left(-\frac{h(t)}{t}\right)\right\}\frac{dt}{t}<\infty \quad (\text{resp.} =\infty), \tag{1.3}$$

then we have

$$\limsup_{t\to\infty}\frac{|X_t|}{h(t)}=0 \quad (\text{resp.} =\infty) \quad \text{a.s.} \tag{1.4}$$

Corollary 1.1. *In the same setting as in* Theorem 1.1, *there does not exist a function $h(t)$ such that*

$$\limsup_{t\to\infty}\frac{|X_t|}{h(t)}=1 \quad \text{a.s.}$$

EXAMPLE. If $|x|^\alpha k(x)$ is slowly varying at $\pm\infty$ for some $\alpha\geq 0$, then (1.2) holds. In particular, if $\alpha>0$, we have $\lim_{|x|\to\infty}k(\rho_0 x)/k(x)<1$ for any $\rho_0>1$. This means that (1.1) holds. Such important examples are strictly stable processes with index α, where $0<\alpha<1$.

REMARK 1.1. The paper [14] deals with the case where $k(x)=0$ on $(-\infty,0)$ with respect to this problem. It has the assumption that $k(x)\in\mathrm{OR}$. This assumption corresponds to the condition $k(x)+k(-x)\in\mathrm{OR}$ in the case where $k(x)>0$ for all $x\in\mathbf{R}^1\setminus\{0\}$. Our conditions (1.1) and (1.2) are stronger than it.

REMARK 1.2. In [15] recently Toshiro Watanabe has shown the following: Suppose that μ is an infinitely divisible distribution with Lévy measure ν. Let $\mu^*(r) = \mu(\{x \in \mathbf{R}^d : |x| > r\})$ and $\nu^*(r) = \nu(\{x \in \mathbf{R}^d : |x| > r\})$. If $\nu^*(r) \in$ OR, then it follows that $0 < \liminf_{r\to\infty} \mu^*(r)/\nu^*(r) \le \limsup_{r\to\infty} \mu^*(r)/\nu^*(r) < \infty$.

Using this, we can show the same result as Theorem 1.1 in the case where the characteristic function of X_1 is represented as

$$Ee^{izX_1} = \exp\left[-2^{-1}az^2 + \int_{\mathbf{R}^1}(e^{izx} - 1 - izx1_{\{|x|\le 1\}}(x))\frac{k(x)}{|x|}\,dx + i\gamma z\right].$$

Here $a \ge 0$ and $\gamma \in \mathbf{R}^1$.

Let $\lambda = k(0+) + k(0-)$. If $\lambda < \infty$, then we define the function $K(x)$ on $(0, \infty)$ by

$$\begin{aligned} K(x) &= (x \wedge 1)^\lambda \exp\left[\int_{x\wedge 1}^1 \frac{\lambda - k(u) - k(-u)}{u}\,du\right] \\ &= \exp\left[-\int_{x\wedge 1}^1 \frac{k(u) + k(-u)}{u}\,du\right]. \end{aligned}$$

We state the liminf behavior of the selfsimilar additive processes $\{X_t\}$.

Theorem 1.2. *Let $0 < \lambda < \infty$. Suppose that $h(t)$ is a strictly increasing positive measurable function on $[1, \infty)$, and that $h(t)/t$ is decreasing. Furthermore, in the case where $\lambda \ge 1$, we suppose the following additional condition*: $\limsup_{t\to\infty} h(t^\alpha)/h(t)^\alpha < \infty$ *for some* $\alpha > 1$.

If

$$\int_1^\infty K\left(\frac{h(t)}{t}\right)\frac{dt}{t} = \infty \quad (\text{resp. } < \infty), \tag{1.5}$$

then we have

$$\liminf_{t\to\infty} \frac{|X_t|}{h(t)} = 0 \quad (\text{resp. } = \infty) \ \textit{a.s.} \tag{1.6}$$

Corollary 1.2. *In the same setting as in* Theorem 1.2, *there does not exist a function $h(t)$ such that*

$$\liminf_{t\to\infty} \frac{|X_t|}{h(t)} = 1 \quad \text{a.s.}$$

REMARK 1.3. With respect to this problem the paper [14] assume that $k(x) = 0$ on $(-\infty, 0)$ but does not that $\limsup_{t\to\infty} h(t^\alpha)/h(t)^\alpha < \infty$ for some $\alpha > 1$ in the case where $\lambda \geq 1$. It is important progress that we do not assume the support of $k(x)$ is contained in either $(-\infty, 0]$ or $[0, \infty)$.

REMARK 1.4. Let $h(t) = 2 - t^{-1}$. Then, from Theorem 1.2, we can obtain that $\{X_t\}$ is transient in the case where $0 < \lambda < \infty$. This fact has been already shown in [16].

REMARK 1.5. If $h(t)/t$ is increasing, then it follows that

$$\infty \geq \limsup_{t\to\infty} \frac{|X_t|}{t} \geq C_0 \limsup_{t\to\infty} \frac{|X_t|}{h(t)}.$$

If $h(t)/t$ is decreasing, then it follows that

$$\liminf_{t\to\infty} \frac{|X_t|}{h(t)} \geq C_1 \liminf_{t\to\infty} \frac{|X_t|}{t} \geq 0.$$

Here C_0 and C_1 are positive constants. Therefore our results show how different sample function behavior of a selfsimilar additive process is from the strong law of large numbers.

To give an example, let $\alpha > 1$, and let $k(x) = x^{-\alpha} \wedge 2$ if $x > 0$ and let $k(x) = |x|^{-\alpha} \wedge 1$ if $x < 0$. Choosing $h(t) = t$, we have $\limsup_{t\to\infty} |X_t|/h(t) = \infty$ a.s. from Theorem 1.1 and $\liminf_{t\to\infty} |X_t|/h(t) = 0$ a.s. from Theorem 1.2. By the way, if $\{Y_t\}$ is a Lévy process whose distribution at time 1 is identical with P_{X_1}, then we have

$$\lim_{t\to\infty} \frac{Y_t}{t} = \int_{\mathbf{R}^1} y P_{Y_1}(dy) = \int_{\mathbf{R}^1} x \frac{k(x)}{|x|}\, dx = \frac{\alpha}{\alpha - 1}(2^{1-\alpha^{-1}} - 1) \quad \text{a.s.}$$

by virtue of [10] Theorem 36.5.

2. Proof of Theorem 1.1

First, we prepare some lemmas.

Lemma 2.1. *Let $\epsilon > 0$. Then it follows that*

$$P\left(\sup_{0\leq s\leq t} |X_s| > 3\epsilon\right) \leq 3 \sup_{0\leq s\leq t} P\left(|X_s| > \epsilon\right). \tag{2.1}$$

Proof. Let $Z_1,\ Z_2, \ldots, Z_n$ be independent random variables. Put $S_k = \sum_{i=1}^k Z_i$ for $k \geq 1$. Then it follows that

$$P\left(\max_{1\leq k\leq n} |S_k| > 3\epsilon\right) \leq 3 \max_{1\leq k\leq n} P(|S_k| > \epsilon).$$

Indeed, this is shown by using [10, p. 126] Lemma 20.2. Hence, as X_t is right continuous and $\{X_t\}$ has independent increments, we can obtain (2.1). □

The following lemma is found in [4, p. 574].

Lemma 2.2. *Let F be a distribution function on $[0, \infty)$. Suppose that F is absolutely continuous with density f. Then if the function $x \mapsto \exp\{xf(x)/(1-F(x))\}f(x)$ is integrable on $[0, \infty)$, then F is subexponential.*

The following lemma is found in [3] or in [4, p. 581].

Lemma 2.3. *For F infinitely divisible on $[0, \infty)$ with Lévy measure ν, it follows that $\nu(1, x]/\nu(1, \infty)$ is subexponential if and only if $\lim_{x\to\infty}(1 - F(x))/\nu(x, \infty) = 1$.*

Lemma 2.4. *Suppose that $k(x) > 0$ for all $x \in \mathbf{R}^1 \setminus \{0\}$. If we have*

$$\liminf_{x\to\infty} \frac{k(\rho_0 x)}{k(x)} > 0 \tag{2.2}$$

for some $\rho_0 > 1$, then it follows that

$$1 \leq \liminf_{x\to\infty} \frac{P(X_1 > x)}{\int_{2x}^{\infty} k(u)u^{-1}\,du} \tag{2.3}$$

and

$$1 \leq \liminf_{x\to\infty} \frac{P(X_1 - X_{\rho_0^{-1}} > x)}{\int_{2x}^{\infty}(k(u) - k(\rho_0 u))u^{-1}\,du}. \tag{2.4}$$

In particular, if $\rho_0 = 2$, then we have

$$\limsup_{x\to\infty} \frac{P(X_1 > x)}{\int_{2^{-1}x}^{\infty} k(u)u^{-1}\,du} < \infty. \tag{2.5}$$

REMARK. Let $x > 0$. We note that the lemma holds again with $k(-u)$ in place of $k(u)$ with respect to $P(-X_1 > x)$.

Proof. Let X_1 have the decomposition $X_1 = Z_1 + Z_2 + Z_3$ such that

$$\hat{P}_{Z_1}(z) = \exp\left[\int_1^{\infty}(e^{izx} - 1)k(x)x^{-1}\,dx\right],$$
$$\hat{P}_{Z_2}(z) = \exp\left[\int_{-1}^{1}(e^{izx} - 1)k(x)|x|^{-1}\,dx\right],$$

$$\hat{P}_{Z_3}(z) = \exp\left[\int_{-\infty}^{-1} (e^{izx} - 1)k(x)|x|^{-1}\,dx\right].$$

Now we notice that

$$P(X_1 > x) \geq P(Z_1 > 2x)P(Z_2 + Z_3 > -2^{-1}x), \tag{2.6}$$

$$P(X_1 > x) \leq P(Z_1 > 2^{-1}x) + P(Z_2 > 2^{-1}x). \tag{2.7}$$

Here (2.7) was shown since $Z_3 \leq 0$ a.s. From (2.6) we obtain that

$$\begin{aligned} P(Z_2 + Z_3 > -2^{-1}x) &\leq \frac{P(X_1 > x)}{P(Z_1 > 2x)} \\ &= \frac{P(X_1 > x)}{\int_{2x}^{\infty} k(u)u^{-1}\,du} \times \frac{\int_{2x}^{\infty} k(u)u^{-1}\,du}{P(Z_1 > 2x)}. \end{aligned} \tag{2.8}$$

From (2.2) we have

$$\begin{aligned} &\int_1^{\infty} \exp\left\{\frac{k(x)}{\int_x^{\infty} k(u)u^{-1}\,du}\right\} \frac{k(x)}{x\int_1^{\infty} k(u)u^{-1}\,du}\,dx \\ &\leq \int_1^{\infty} \exp\left\{\frac{k(x)}{k(\rho_0 x)\int_x^{\rho_0 x} u^{-1}\,du}\right\} \frac{k(x)}{x\int_1^{\infty} k(u)u^{-1}\,du}\,dx \\ &\leq \text{const.} \times \int_1^{\infty} \frac{k(x)}{x\int_1^{\infty} k(u)u^{-1}\,du}\,dx < \infty. \end{aligned}$$

Hence the distribution function $\int_1^x k(u)u^{-1}\,du/\int_1^{\infty} k(u)u^{-1}\,du$ on $[1, \infty)$ is subexponential by virtue of Lemma 2.2. Therefore, as $x \to \infty$ in (2.8), we can get (2.3) by Lemma 2.3.

Next, we shall show (2.4). The Lévy measure of $X_1 - X_{\rho_0^{-1}}$ is $(k(x) - k(\rho_0 x))/|x|\,dx$. We have

$$\begin{aligned} &\int_1^{\infty} \exp\left\{\frac{k(x) - k(\rho_0 x)}{\int_x^{\infty}(k(u) - k(\rho_0 u))u^{-1}\,du}\right\} \frac{k(x) - k(\rho_0 x)}{x\int_1^{\infty}(k(u) - k(\rho_0 u))u^{-1}\,du}\,dx \\ &= \int_1^{\infty} \exp\left\{\frac{k(x) - k(\rho_0 x)}{\int_x^{\rho_0 x} k(u)u^{-1}\,du}\right\} \frac{k(x) - k(\rho_0 x)}{x\int_1^{\infty}(k(u) - k(\rho_0 u))u^{-1}\,du}\,dx \\ &\leq \int_1^{\infty} \exp\left\{\frac{k(x)}{k(\rho_0 x)\int_x^{\rho_0 x} u^{-1}\,du}\right\} \frac{k(x) - k(\rho_0 x)}{x\int_1^{\infty}(k(u) - k(\rho_0 u))u^{-1}\,du}\,dx < \infty. \end{aligned}$$

Hence, using Lemmas 2.2 and 2.3, we can get (2.4) in the same way as we showed (2.3).

Lastly, we shall prove (2.5). From (2.7) we have

$$\frac{P(X_1 > x)}{\int_{2^{-1}x}^{\infty} k(u)u^{-1}\,du} \leq \frac{P(Z_1 > 2^{-1}x)}{\int_{2^{-1}x}^{\infty} k(u)u^{-1}\,du} + \frac{P(Z_2 > 2^{-1}x)}{\int_{2^{-1}x}^{\infty} k(u)u^{-1}\,du}. \tag{2.9}$$

By virtue of Lemma 2.3 the first term in the above right-hand side converges 1 as $x \to \infty$. And the second term is caluculated as follows: Since $k(x) \in \mathrm{OR}$, we have $\int_{2^{-1}x}^{\infty} k(u)/u\,du \in \mathrm{OR}$. And we have $P\left(Z_2 > 2^{-1}x\right) = o(\exp(-\alpha x \log x))$ for some $\alpha > 0$ by virtue of Theorem 26.8 in [10]. Hence we get

$$\sup_{x>2} \frac{P\left(Z_2 > 2^{-1}x\right)}{\int_{2^{-1}x}^{\infty} k(u)/u\,du} < \infty$$

by the representation theorem for OR. Therefore, as $x \to \infty$ in (2.9), we can get (2.5). We have completed the proof of the lemma. □

The following lemma is found in [13, p. 317].

Lemma 2.5. *Let E_n be any sequence of events. If*

$$\sum_{n=1}^{\infty} P(E_n) = \infty$$

and if for some $c > 0$

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \sum_{m=1}^{n} P(E_k \bigcap E_m)}{(\sum_{k=1}^{n} P(E_k))^2} \leq c,$$

then we have

$$P(\limsup_{n\to\infty} E_n) \geq c^{-1}.$$

Lemma 2.6. *Suppose that $k(x) > 0$ for all $x \in \mathbf{R}^1$. If* (1.1) *holds, then it follows that*

$$\limsup_{a\to\infty} \left(\frac{\int_a^{\infty} k(x)x^{-1}\,dx}{k(a)} + \frac{\int_{-\infty}^{-a} k(x)|x|^{-1}\,dx}{k(-a)} \right) < \infty$$

Proof. Put $B(a) = \sup_{x\geq a} k(\rho_0 x)/k(x)$. Now we have, for large enough a,

$$\frac{\int_a^{\infty} k(x)x^{-1}\,dx}{k(a)} = \frac{1}{k(a)} \sum_{n=0}^{\infty} \int_{a\rho_0^n}^{a\rho_0^{n+1}} k(x)x^{-1}\,dx$$

$$\le \sum_{n=0}^{\infty} \frac{k(a\rho_0^n)}{k(a)} \log \rho_0$$
$$\le \sum_{n=0}^{\infty} B(a)^n \log \rho_0 < \infty.$$

Here we calculated as follows: For large enough a,

$$\frac{k(a\rho_0^n)}{k(a)} = \frac{k(a\rho_0^n)}{k(a\rho_0^{n-1})} \cdot \frac{k(a\rho_0^{n-1})}{k(a\rho_0^{n-2})} \cdot \cdots \cdot \frac{k(a\rho_0)}{k(a)} \le B(a)^n < 1.$$

In the same way we can also get

$$\limsup_{a\to\infty} \frac{\int_{-\infty}^{-a} k(x)|x|^{-1}\,dx}{k(-a)} < \infty.$$

The lemma has been proved. □

Now we shall prove Theorem 1.1.

Proof of Theorem 1.1. (i) First, we shall consider the case where the integral of (1.3) is convergent. Then we have $\lim_{t\to\infty} h(t)/t = \infty$. Put $M(t) = \sup_{0\le s\le t} |X_s|$. Let $\epsilon > 0$. By Lemma 2.1 we have

$$\sum_{n=1}^{\infty} P(M(2^n) > 3\epsilon h(2^{n-1}))$$
$$\le 3\sum_{n=1}^{\infty} \sup_{0\le s\le 2^n} P(|X_s| > \epsilon h(2^{n-1}))$$
$$\le 3\sum_{n=1}^{\infty} P\left(|X_1| > \epsilon\frac{h(2^{n-1})}{2^n}\right)$$
$$\le \text{const.} \times \sum_{n=l}^{\infty} \int_{|x|>\epsilon\frac{h(2^{n-1})}{2^{n+1}}} k(x)|x|^{-1}\,dx + 3(l-1) = \mathrm{I}, \quad \text{(say)}.$$

Here l is large enough and the last inequality was shown by Lemma 2.4. We notice that

$$\int_1^{\infty} \frac{dt}{t} \int_{|x|>\frac{\epsilon h(t)}{4t}} k(x)|x|^{-1}\,dx \ge \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} \frac{dt}{t} \int_{|x|>\epsilon\frac{h(2^{n+1})}{2^{n+3}}} k(x)|x|^{-1}\,dx$$
$$= \log 2 \sum_{n=0}^{\infty} \int_{|x|>\epsilon\frac{h(2^{n+1})}{2^{n+3}}} k(x)|x|^{-1}\,dx.$$

Hence we have, for $\epsilon < 4$,

$$
\begin{aligned}
I &\le \text{const.} \times \int_1^\infty \left\{ k\left(\epsilon \frac{h(t)}{4t}\right) + k\left(-\epsilon \frac{h(t)}{4t}\right)\right\} \frac{dt}{t} + \text{const.} \\
&\le \text{const.} \times \int_1^\infty \left\{ k\left(\frac{h(t)}{t}\right) + k\left(-\frac{h(t)}{t}\right)\right\} \frac{dt}{t} + \text{const.} < \infty,
\end{aligned}
$$

where the first ineqaulity was shown by Lemma 2.6 and the last ineqaulity from (1.2). By virtue of the first Borel-Cantelli lemma we have

$$
\begin{aligned}
1 &= P(\liminf_{n\to\infty}\{M(2^n) \le 3\epsilon h(2^{n-1})\}) \\
&\le P(\liminf_{n\to\infty}\{\sup_{2^{n-1}\le s\le 2^n} |X_s| \le 3\epsilon h(2^{n-1})\}) \\
&\le P\left(\liminf_{n\to\infty}\left\{\sup_{2^{n-1}\le s\le 2^n} \frac{|X_s|}{h(s)} \le 3\epsilon\right\}\right).
\end{aligned}
$$

Hence

$$
\limsup_{s\to\infty} \frac{|X_s|}{h(s)} \le 3\epsilon \quad \text{a.s.}
$$

As $\epsilon \to 0$, we can get (1.4).

(ii) Next, we shall consider the case where the integral of (1.3) is divergent. Let $\epsilon > 0$ and $c > 1$. Put $E_n = \{|X_{\rho_0^n}| \le \epsilon\rho_0^{n+1}, |X_{\rho_0^{n+1}} - X_{\rho_0^n}| > ch(\rho_0^{n+1}) + \epsilon\rho_0^{n+1}\}$. It suffices to show that $\lim_{\epsilon\to\infty} P(\limsup_{n\to\infty} E_n) = 1$. Indeed, as we have

$$
\limsup_{n\to\infty} E_n \subset \limsup_{n\to\infty}\{|X_{\rho_0^{n+1}}| > ch(\rho_0^{n+1})\},
$$

it follows that

$$
\limsup_{n\to\infty} \frac{|X_{\rho_0^{n+1}}|}{h(\rho_0^{n+1})} \ge c \quad \text{a.s.}
$$

As $c \to \infty$, we have

$$
\limsup_{s\to\infty} \frac{|X_s|}{h(s)} = \infty \quad \text{a.s.}
$$

First, we shall show that $\sum_{n=1}^\infty P(E_n) = \infty$. Now we have

$$
\begin{aligned}
\sum_{n=1}^\infty P(E_n) &= \sum_{n=1}^\infty P(|X_{\rho_0^n}| \le \epsilon\rho_0^{n+1}) P(|X_{\rho_0^{n+1}} - X_{\rho_0^n}| > ch(\rho_0^{n+1}) + \epsilon\rho_0^{n+1}) \\
&= P(|X_1| \le \epsilon\rho_0) \sum_{n=1}^\infty P\left(|X_1 - X_{\rho_0^{-1}}| > c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}} + \epsilon\right) = I, \quad \text{(say).}
\end{aligned}
$$

If $\lim_{n\to\infty} h(\rho_0^n)/\rho_0^n < \infty$, then $I = \infty$. Hence from now on we suppose that $\lim_{n\to\infty} h(\rho_0^n)/\rho_0^n = \infty$. By virtue of Lemma 2.4 we have, for large enough l_0 and l_1,

$$\begin{aligned} I &\geq \text{const.} \times \sum_{n=l_0}^{\infty} \int_{|x|>2\left(c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}+\epsilon\right)} \frac{k(x)-k(\rho_0 x)}{|x|}\,dx \\ &\geq \text{const.} \times \sum_{n=l_1}^{\infty} \int_{|x|>4c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}} \frac{k(x)-k(\rho_0 x)}{|x|}\,dx \\ &= \text{const.} \times \sum_{n=l_1}^{\infty} \int_{4c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}<|x|\leq 4\rho_0 c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}} \frac{k(x)}{|x|}\,dx = J, \quad \text{(say)}. \end{aligned}$$

Here, as we have

$$\int_{b<|x|\leq\rho_0 b} \frac{k(x)}{|x|}\,dx \geq (k(\rho_0 b)+k(-\rho_0 b))\log\rho_0$$

for $b>0$, it follows that, for large enough l_2,

$$\begin{aligned} J &\geq \text{const.} \times \sum_{n=l_1}^{\infty} \left\{ k\left(4\rho_0 c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}\right) + k\left(-4\rho_0 c\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}\right)\right\} \\ &\geq \text{const.} \times \sum_{n=l_2}^{\infty} \left\{ k\left(\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}\right) + k\left(-\frac{h(\rho_0^{n+1})}{\rho_0^{n+1}}\right)\right\} \\ &\geq \text{const.} \times \int_{\rho_0^{l_2+1}}^{\infty} \left\{ k\left(\frac{h(t)}{t}\right) + k\left(-\frac{h(t)}{t}\right)\right\}\frac{dt}{t} = \infty. \end{aligned}$$

Here the second inequlity was shown from (1.2).

Let $n<m$. Furthermore, we have

$$\begin{aligned} P(E_n\cap E_m) &\leq P(E_n\cap\{|X_{\rho_0^{m+1}}-X_{\rho_0^m}|>ch(\rho_0^{m+1})+\epsilon\rho_0^{m+1}\}) \\ &= P(E_n)P(|X_{\rho_0^{m+1}}-X_{\rho_0^m}|>ch(\rho_0^{m+1})+\epsilon\rho_0^{m+1}\}) \\ &= P(E_n)\frac{P(E_m)}{P(|X_{\rho_0^m}|\leq\epsilon\rho_0^{m+1})} \\ &= P(E_n)\frac{P(E_m)}{P(|X_1|\leq\epsilon\rho_0)}. \end{aligned}$$

Hence, by Lemma 2.5, we have $\liminf_{\epsilon\to\infty} P(\limsup_{n\to\infty} E_n) = 1$. The theorem has been proved. □

3. Proof of Theorem 1.2

First, we prepare some lemmas.

Lemma 3.1. *Let* $\lambda = k(0+) + k(0-) < \infty$. *There is a positive constant* C *such that*

$$P(|X_1| \le a) \ge CK(a)$$

for all small enough $a > 0$.

Proof. In the case where $k(0+) = 0$ or $k(0-) = 0$, the lemma has been already shown in [12, p. 298]. We shall prove the lemma in the case where $k(0+)k(0-) > 0$. Define two distributions μ_1 and μ_2 by

$$\hat{\mu_1}(z) = \exp\left[\int_0^\infty (e^{izx} - 1)\frac{k(x)}{x}\,dx\right] \quad \text{and} \quad \hat{\mu_2}(z) = \exp\left[\int_{-\infty}^0 (e^{izx} - 1)\frac{k(x)}{|x|}\,dx\right],$$

respectively. K. Sato and M. Yamazato proved the following (see (5.7) in [12, p. 298]): As $x \downarrow 0$, we have

$$\mu_1([0, x]) \sim C_1 x^{k(0+)} K_1(x),$$
$$\mu_2([-x, 0]) \sim C_2 x^{k(0-)} K_2(x),$$

where C_1 and C_2 are positive constants and

$$K_1(x) = \exp\left[\int_x^1 \frac{k(0+) - k(u)}{u}\,du\right],$$
$$K_2(x) = \exp\left[\int_x^1 \frac{k(0-) - k(-u)}{u}\,du\right].$$

We notice that the support of μ_1 or μ_2 is contained in $[0, \infty)$ or $(-\infty, 0]$, respectively. Hence we can obtain that

$$\begin{aligned} P(|X_1| \le a) &\ge \mu_1([0, a])\mu_2([-a, 0]) \\ &\sim C_1 C_2 K(a) \quad \text{as } a \to 0. \end{aligned}$$

The lemma has been proved. □

The following lemma was pointed out by K. Sato and M. Yamazato (see Lemma 2.4 in [12, p. 280]).

Lemma 3.2. *If* $0 < \lambda < \infty$, *then there is a constant* M *such that, for* $z \in \mathbf{R}^1$,

$$|\hat{P}_{X_1}(z)| \le MK\left(|z|^{-1}\right).$$

Here we introduce some terminology. From now on define the function $g(t)$ on $[0,\infty)$ by

$$g(t)=\begin{cases} e^t & t\geq 1 \\ te & t<1. \end{cases} \tag{3.1}$$

And we define a time-homogeneous transition function $\overline{p}(h,y,B)$ by

$$\overline{p}(h,y,B)=P(X_{g(t+h)}-X_{g(t)}+x\in\Gamma)$$

for $h\geq 0$, $y=(t,x)\in[0,\infty)\times\mathbf{R}^1$, and $B\in\mathcal{B}([0,\infty)\times\mathbf{R}^1)$, where $\Gamma=\{z\in\mathbf{R}^1:(t+h,z)\in B\}$. Let $\{Y_h\}$ be the time-homogeneous Markov process with this transition probablity $\overline{p}(h,y,B)$. The process $\{Y_h\}$ is expressed by a system of probability measures $\{\overline{P}^y : y\in[0,\infty)\times\mathbf{R}^1\}$ on the space of paths on $[0,\infty)\times\mathbf{R}^1$. The expectation with respect to $\overline{P}^y$ is denoted by $\overline{E}^y$. Furthermore denote by $\tilde{P}_h$ the transition operator of $\{Y_h\}$. Refer to the paper [16].

Lemma 3.3. *The process $\{Y_h\}$ is a Hunt process.*

Proof. Denote by C_0 the real Banach space of continuous functions on $[0,\infty)\times\mathbf{R}^1$ vanishing at infinity with the norm of uniform convergence. For any $f\in C_0$ and $y=(t,x)\in[0,\infty)\times\mathbf{R}^1$, we have

$$\tilde{P}_h f(y)=\int_\Omega P(d\omega)f(t+h,X_{g(t+h)}-X_{g(t)}+x).$$

For each t, almost surely the limit of $X_{g(s)}$ as $s\downarrow t$ is equal to the limit as $s\uparrow t$, so we have $\tilde{P}_h f\in C_0$. Furthermore $\tilde{P}_h f(y)\to f(y)$ as $h\downarrow 0$ for any $f\in C_0$. Hence, by virtue of Theorem 9.4 in [2, p. 46], the process $\{Y_h\}$ is a Hunt process. □

For $a>0$, let $f_a(x)=(a-|x|)\vee 0$. Then the Fourier transform of f_a is as follows:

$$\hat{f}_a(z)=\int_{\mathbf{R}^1}e^{izx}f_a(x)\,dx=\left(\frac{\sin 2^{-1}az}{2^{-1}z}\right)^2.$$

And let $\phi_a(t,x)=\hat{f}_a(x)$ for each $(t,x)\in[0,\infty)\times\mathbf{R}^1$.

Lemma 3.4. *Let $\lambda<\infty$. We define a projection Π by $\Pi(t,x)=x$. Let $\tau=\inf\{t>0:|\Pi(Y_t)|<a\}$ for $a>0$. Then we have*

$$\overline{P}^y(\tau<\infty)\leq C_0\overline{E}^y\left[\int_0^\infty\phi_{a^{-1}}(Y_s)\,ds\right]$$

for large enough a. Here C_0 is a positive constant.

Proof. By virtue of Lemma 3.3 the process $\{Y_h\}$ has the strong Markov property. Hence we have

$$\begin{aligned}
\bar{E}^y\left[\int_0^\infty \phi_{a^{-1}}(Y_s)\,ds\right] &\geq \bar{E}^y\left[\int_\tau^\infty \phi_{a^{-1}}(Y_s)\,ds;\tau<\infty\right]\\
&= \bar{E}^y\left[\int_0^\infty \phi_{a^{-1}}(Y_{s+\tau})\,ds;\tau<\infty\right]\\
&= \bar{E}^y\left[\bar{E}^{Y_\tau}\left[\int_0^\infty \phi_{a^{-1}}(Y_s)\,ds\right];\tau<\infty\right]\\
&\geq \bar{P}^y(\tau<\infty)\inf_{t\geq 0,|x|\leq a}\bar{E}^{(t,x)}\left[\int_0^\infty \phi_{a^{-1}}(Y_s)\,ds\right].
\end{aligned}$$

Therefore it suffices to prove that

$$\inf_{t\geq 0,|x|\leq a}\bar{E}^{(t,x)}\left[\int_0^\infty \phi_{a^{-1}}(Y_s)\,ds\right]>0.$$

Now we shall show it. For $0<\delta<1$, we have

$$\begin{aligned}
\bar{E}^{(t,x)}\left[\int_0^\infty \phi_{a^{-1}}(Y_s)\,ds\right] &= E\left[\int_0^\infty \hat{f}_{a^{-1}}(X_{g(s+t)}-X_{g(t)}+x)\,ds\right]\\
&\geq \int_0^\delta E\hat{f}_{a^{-1}}(X_{g(s+t)}-X_{g(t)}+x)\,ds\\
&= \int_0^\delta ds\int_{\mathbf{R}^1} f_{a^{-1}}(z)\exp[\varphi(z)]\,dz = I, \quad \text{(say)},
\end{aligned}$$

where

$$\varphi(z) = izx + \int_{\mathbf{R}^1}(e^{izu}-1)\frac{k(u/(g(s+t)))-k(u/(g(t)))}{|u|}\,du.$$

First, we suppose that $t\leq 1$. Let $M>0$. Then we obtain that

$$\begin{aligned}
&\left|\int_{|u|\leq g(s+t)M}(e^{izu}-1)\frac{k(u/(g(s+t)))-k(u/(g(t)))}{|u|}\,du\right|\\
&\qquad\leq |z|\int_{|u|\leq g(s+t)M} k\left(\frac{u}{g(s+t)}\right)du \leq |z|g(\delta+1)\int_{|u|\leq M}k(u)\,du.
\end{aligned}$$

And we have

$$\begin{aligned}
&\left|\int_{|u|>g(s+t)M}(e^{izu}-1)\frac{k(u/(g(s+t)))-k(u/g(t))}{|u|}\,du\right|\\
&\qquad\leq 2\int_{|u|>g(s+t)M}\frac{k(u/(g(s+t)))-k(u/g(t))}{|u|}\,du \leq 2\int_{|u|>M}\frac{k(u)}{|u|}\,du.
\end{aligned}$$

Second, we suppose that $t > 1$. Then we obtain that

$$\left|\int_{|u|\le 1}(e^{izu}-1)\frac{k(u/(g(s+t)))-k(u/g(t))}{|u|}\,du\right|$$
$$\le |z|\int_{|u|\le 1}\left(k\left(\frac{u}{g(s+t)}\right)-k\left(\frac{u}{g(t)}\right)\right)du \le \lambda|z|.$$

And we have

$$\left|\int_{|u|>1}(e^{izu}-1)\frac{k(u/(g(s+t)))-k(u/g(t))}{|u|}\,du\right| \le 2\int_{|u|>1}\frac{k(u/(g(s+t)))-k(u/g(t))}{|u|}\,du$$
$$=2\int_{\frac{1}{g(s+t)}<|u|\le\frac{1}{g(t)}}\frac{k(u)}{|u|}\,du \le 2\lambda\log\frac{g(s+t)}{g(t)}=2\lambda s.$$

Let $\epsilon > 0$. If we firstly choose large enough M, secondly small enough δ, and lastly large enough a, then we can get $|\varphi(z)| < 1+\epsilon$ for all z with $|z| \le a^{-1}$. Therefore, choosing small enough ϵ, we have

$$I \ge \int_0^\delta ds\int_{\mathbf{R}^1} f_{a^{-1}}(z)\exp[-(1+\epsilon)]\cos(1+\epsilon)\,dz > 0.$$

We have been proved the lemma. □

Lemma 3.5. *Let $h(t)$ be an increasing positive function on $[1,\infty)$ such that $h(t)/t$ is decreasing. Let $0 < \lambda < \infty$. If*

$$\int_1^\infty K\left(\frac{h(t)}{t}\right)\frac{dt}{t} < \infty$$

and $\limsup_{t\to\infty} h(t^\alpha)/h(t)^\alpha < \infty$ for some $\alpha > 1$, then we have, for any positive number β with $\beta < \lambda$,

$$\int_1^\infty\left(\frac{h(t)}{t}\right)^\beta\frac{dt}{t} < \infty.$$

Proof. Since $\int_1^\infty K\left(h(t)/t\right)t^{-1}\,dt < \infty$, we have $\lim_{t\to\infty} h(t)/t = 0$. And we have $K(h(t)/t) \ge (h(t)/t)^\lambda$ for large enough t. Hence we obtain that, for large enough M,

$$\infty > \int_1^\infty K\left(\frac{h(t)}{t}\right)\frac{dt}{t} \ge \int_M^\infty\left(\frac{h(t)^\alpha}{t^\alpha}\right)^{\lambda\alpha^{-1}}\frac{dt}{t}$$
$$\ge \text{const.}\times\int_M^\infty\left(\frac{h(t^\alpha)}{t^\alpha}\right)^{\lambda\alpha^{-1}}\frac{dt}{t}$$

$$= \text{const.} \times \alpha^{-1} \int_{M^\alpha}^{\infty} \left(\frac{h(t)}{t}\right)^{\lambda\alpha^{-1}} \frac{dt}{t}.$$

We repeate this calculation. Then we can get the lemma. □

Now we shall prove Theorem 1.2.

Proof of Theorem 1.2. (i) We shall consider the case where the integral of (1.5) is divergent. Let $\epsilon > 0$. We have

$$\begin{aligned}
1 &\geq \sum_{n=1}^{\infty} P(|X_{2^n}| \leq \epsilon h(2^n), |X_{2^{n+k}}| > \epsilon h(2^{n+k}) \quad \text{for } k = 1, 2, \ldots) \\
&\geq \sum_{n=1}^{\infty} P(|X_{2^n}| \leq \epsilon h(2^n), |X_{2^{n+k}} - X_{2^n}| > \epsilon(h(2^{n+k}) + h(2^n)) \quad \text{for } k = 1, 2, \ldots) \\
&= \sum_{n=1}^{\infty} P(|X_{2^n}| \leq \epsilon h(2^n)) P(|X_{2^{n+k}} - X_{2^n}| > \epsilon(h(2^{n+k}) + h(2^n)) \quad \text{for } k = 1, 2, \ldots) \\
&\geq \sum_{n=1}^{\infty} P\left(|X_1| \leq \epsilon \frac{h(2^n)}{2^n}\right) P(|X_{2^{k-1}} - X_{2^{-1}}| > \epsilon h(2^k) \quad \text{for } k = 1, 2, \ldots).
\end{aligned}$$

Here the last inequality was shown since $(h(2^{n+k}) + h(2^n))/2^{n+1} \leq h(2^{n+k})/2^n \leq h(2^k)$. Now we have

$$\sum_{n=1}^{\infty} P\left(|X_1| \leq \epsilon \frac{h(2^n)}{2^n}\right) = \infty. \tag{3.2}$$

Indeed, if $\lim_{t\to\infty} h(t)/t > 0$, then (3.2) holds. We suppose that $\lim_{t\to\infty} h(t)/t = 0$. By Lemma 3.1 we have, for large enough m,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(|X_1| \leq \epsilon \frac{h(2^n)}{2^n}\right) &\geq \text{const.} \times \sum_{n=m}^{\infty} K\left(\epsilon \frac{h(2^n)}{2^n}\right) \\
&\geq \text{const.} \times \sum_{n=m}^{\infty} \int_{2^n}^{2^{n+1}} K\left(\epsilon \frac{h(t)}{t}\right) \frac{dt}{t} \\
&= \text{const.} \times \int_{2^m}^{\infty} K\left(\epsilon \frac{h(t)}{t}\right) \frac{dt}{t} \\
&\geq \text{const.} \times \int_{2^m}^{\infty} K\left(\frac{h(t)}{t}\right) \frac{dt}{t}.
\end{aligned}$$

Here the third inequality was shown since $K(x)/(1 \wedge x)^\lambda$ is slowly varying at 0. Therefore, if the integral of (1.5) is divergent, then (3.2) holds.

Hence, from the calculation at the beginning and from (3.2), we obtain that, for any $\epsilon > 0$,

$$P(|X_{2^{k-1}} - X_{2^{-1}}| > \epsilon h(2^k) \quad \text{for } k = 1, 2, \ldots) = 0. \tag{3.3}$$

Now, from (3.3), we have

$$P\left(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\{|X_{2^{n-1}} - X_{2^{-1}}| > \epsilon h(2^n)\}\right) = \sum_{k=1}^{\infty} p_k,$$

where

$$p_k = P(|X_{2^{k-1}} - X_{2^{-1}}| \le \epsilon h(2^k), \quad |X_{2^{n+k-1}} - X_{2^{-1}}| > \epsilon h(2^{n+k}) \quad \text{for } n = 1, 2, \ldots).$$

Now let k be fixed. Since $h(t)$ is strictly increasing, there is a positive constant c such that $h(2^{k+n}) - h(2^k) \ge ch(2^n)$ for all positive integer n. Indeed, we have $h(2^{k+n}) - h(2^k) \ge h(2^n)(1 - h(2^k)/h(2^{k+1}))$ for $n \ge k+1$, and $h(2^{k+n}) - h(2^k) \ge h(2^n)\inf_{1\le n\le k}(h(2^{k+n}) - h(2^k))/h(2^k)$ for $n \le k$.

Hence, from (3.3), we have

$$\begin{aligned} p_k &\le P(|X_{2^{n+k-1}} - X_{2^{k-1}}| > \epsilon(h(2^{n+k}) - h(2^k)) \quad \text{for } n = 1, 2, \ldots) \\ &\le P(|X_{2^{n-1}} - X_{2^{-1}}| > \epsilon 2^{-k} ch(2^n) \quad \text{for } n = 1, 2, \ldots) = 0. \end{aligned}$$

Therefore we have

$$P\left(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\{|X_{2^{n-1}} - X_{2^{-1}}| > \epsilon h(2^n)\}\right) = 0,$$

so it follows that

$$\liminf_{n\to\infty} \frac{|X_{2^{n-1}} - X_{2^{-1}}|}{h(2^n)} \le \epsilon \quad \text{a.s.}$$

Using selfsimilarity, we have

$$\liminf_{n\to\infty} \frac{|X_{2^n} - X_1|}{h(2^n)} \le 2\epsilon \quad \text{a.s.}$$

Now, as the integral of (1.5) is divergent, we have $\lim_{t\to\infty} h(t) = \infty$. Consequently, as $\epsilon \downarrow 0$, we have

$$\liminf_{n\to\infty} \frac{|X_{2^n}|}{h(2^n)} = 0 \quad \text{a.s.}$$

We have completed the proof in the case where the integral of (1.5) is divergent.

(ii) We shall consider the case where the integral of (1.5) is convergent. It suffices to show that, for large enough $a > 0$,

$$\sum_{n=1}^{\infty} P\left(|X_t| < ah(2^n) \quad \text{for some } t \in (2^{n-1}, 2^n]\right) < \infty. \tag{3.4}$$

Indeed, by virtue of the first Borel-Cantelli lemma we have

$$\begin{aligned} 1 &= P\left(\bigcup_{l=1}^{\infty}\bigcap_{n=l}^{\infty}\{|X_t| \ge ah(2^n) \quad \text{for all } t \in (2^{n-1}, 2^n]\}\right) \\ &\le P\left(\bigcup_{l=1}^{\infty}\bigcap_{n=l}^{\infty}\left\{\frac{|X_t|}{h(t)} \ge a \quad \text{for all } t \in (2^{n-1}, 2^n]\right\}\right). \end{aligned}$$

Hence we have $\liminf_{t\to\infty}|X_t|/h(t) \ge a$ a.s. Then, as $a \to \infty$, we have $\liminf_{t\to\infty}|X_t|/h(t) = \infty$. Now we shall prove (3.4). As the integral of (1.5) is convergent, we have $\lim_{t\to\infty} h(t)/t = 0$. Here it was shown since $K(h(t)/t) \ge (1 \wedge (h(t)/t))^{\lambda}$. Recall that the function $g(t)$ is defined by (3.1). Let $b_n = g^{-1}(2^{n-1}/h(2^n))$. By Lemma 3.4 we obtain that, for large enough l,

$$\begin{aligned} &\sum_{n=l}^{\infty} P\left(|X_t| < ah(2^n) \quad \text{for some } t \in (2^{n-1}, 2^n]\right) \\ &\quad \le \sum_{n=l}^{\infty} P(\inf\{t > 2^{n-1} : |X_t| < ah(2^n)\} < \infty) \\ &\quad = \sum_{n=l}^{\infty} P\left(\inf\left\{t > \frac{2^{n-1}}{h(2^n)} : |X_t| < a\right\} < \infty\right) \\ &\quad = \sum_{n=l}^{\infty} P\left(\inf\{t > b_n : |X_{g(t)}| < a\} < \infty\right) \\ &\quad = \sum_{n=l}^{\infty} \int_{\mathbf{R}^1} \overline{P}^{(0,0)}(Y_{b_n} \in dy)\overline{P}^{y}\left(\inf\{t > 0 : |\Pi(Y_t)| < a\} < \infty\right) \\ &\quad \le C_0 \sum_{n=l}^{\infty} \int_{\mathbf{R}^1} \overline{P}^{(0,0)}(Y_{b_n} \in dy)\overline{E}^{y}\left[\int_0^{\infty} \phi_{a^{-1}}(Y_s)\,ds\right] \\ &\quad = C_0 \sum_{n=l}^{\infty} E\left[\int_{b_n}^{\infty} \hat{f}_{a^{-1}}(X_{g(s)})\,ds\right] = I, \quad \text{(say)}. \end{aligned}$$

Here, by Lemma 3.2 we have, for large enough l,

$$I = C_0 \sum_{n=l}^{\infty} \int_{b_n}^{\infty} ds \int_{\mathbf{R}^1} f_{a^{-1}}(x)\hat{P}_{X_1}(g(s)x)\,dx$$

$$
\begin{aligned}
&\le C_0 M \sum_{n=l}^{\infty} \int_{b_n}^{\infty} ds \int_{\mathbf{R}^1} f_{a^{-1}}(x) K\left(\frac{1}{g(s)|x|}\right) dx \\
&\le 2C_0 M a^{-1} \sum_{n=l}^{\infty} \int_{b_n}^{\infty} \frac{ds}{g(s)} \int_0^{g(s)a^{-1}} K\left(\frac{1}{x}\right) dx \\
&= 2C_0 M a^{-2} \sum_{n=l}^{\infty} \int_{b_n}^{\infty} \frac{ds}{g(s)} \int_0^{g(s)} K\left(\frac{a}{x}\right) dx = 2C_0 M a^{-2}(J_1 + J_2),
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \sum_{n=l}^{\infty} \int_{g(b_n)}^{\infty} \frac{dt}{t^2} \int_0^a K\left(\frac{a}{x}\right) dx, \\
J_2 &= \sum_{n=l}^{\infty} \int_{g(b_n)}^{\infty} \frac{dt}{t^2} \int_a^t K\left(\frac{a}{x}\right) dx.
\end{aligned}
$$

First, we shall calculate J_1. We have

$$
J_1 \le a \sum_{n=l}^{\infty} \frac{1}{g(b_n)} \le \frac{2a}{\log 2} \int_{2^{l-1}}^{\infty} \frac{h(t)}{t^2}\, dt.
$$

In the case where $\lambda > 1$, using Lemma 3.5, we have $J_1 < \infty$ since the integral of (1.5) converges. And, in the case where $\lambda \le 1$, we have $h(t)/t \le K(h(t)/t)$ for large enough t. Thus we have $J_1 < \infty$ again since the integral of (1.5) converges.

Next, we shall calculate J_2. Let $\lambda < 1$. Since $K(ax^{-1})$ is regularly varying of index $-\lambda$, by virtue of [1, p. 28] Karamata's Theorem in we obtain that

$$
\lim_{t\to\infty} tK\left(at^{-1}\right)\left(\int_a^t K(ax^{-1})\, dx\right)^{-1} = -\lambda + 1
$$

and

$$
\lim_{y\to\infty} K(ay^{-1})\left(\int_y^{\infty} K(at^{-1})t^{-1}\, dt\right)^{-1} = \lambda.
$$

Let $\min\{1-\lambda, \lambda\} > \epsilon > 0$. Hence we obtain that, for large enough l,

$$
\begin{aligned}
J_2 &\le \frac{1}{1-\lambda-\epsilon} \sum_{n=l}^{\infty} \int_{g(b_n)}^{\infty} \frac{K(at^{-1})}{t}\, dt \\
&\le \frac{1}{(\lambda-\epsilon)(1-\lambda-\epsilon)} \sum_{n=l}^{\infty} K\left(\frac{a}{g(b_n)}\right) \\
&\le \text{const.} \times \int_{2^{l-1}}^{\infty} K\left(2a\frac{h(t)}{t}\right) \frac{dt}{t}
\end{aligned}
$$

$$\leq \text{const.} \times \int_{2^{l-1}}^{\infty} K\left(\frac{h(t)}{t}\right) \frac{dt}{t} < \infty.$$

Here we used the fact that $K(x)/(1 \wedge x)^{\lambda}$ is slowly varying at 0.
Now we shall consider the case where $\lambda \geq 1$. Let $1 > \delta > 0$, where $\lambda - \delta > 1$ if $\lambda > 1$. Since $K(x)/(1 \wedge x)^{\lambda}$ is slowly varying at 0, we have

$$\int_a^t \frac{1}{x^{\lambda-\delta}} x^{\lambda-\delta} K\left(\frac{a}{x}\right) dx \leq \text{const.} \times \int_a^t \frac{dx}{x^{\lambda-\delta}}.$$

Hence, by virtue of Lemma 3.5, we obtain that, for large enough l,

$$\begin{aligned} J_2 &\leq \sum_{n=l}^{\infty} \left(C_0 \int_{g(b_n)}^{\infty} \frac{dt}{t^2} + C_1(\lambda) \int_{g(b_n)}^{\infty} \frac{dt}{t^{2-\delta}} \right) \\ &= C_0 \sum_{n=l}^{\infty} \frac{1}{g(b_n)} + \frac{C_1(\lambda)}{1-\delta} \sum_{n=l}^{\infty} \frac{1}{g(b_n)^{1-\delta}} \\ &\leq \frac{2C_0}{\log 2} \int_{2^{l-1}}^{\infty} \frac{h(t)}{t^2} dt + \frac{2^{1-\delta} C_1(\lambda)}{(1-\delta)\log 2} \int_{2^{l-1}}^{\infty} \left(\frac{h(t)}{t}\right)^{1-\delta} \frac{dt}{t} < \infty, \end{aligned}$$

where C_0 and $C_1(\lambda)$ are nonnegative constants, and, in particular, $C_1(\lambda) = 0$ if $\lambda > 1$. Here we used that $K(h(t)/t) \geq h(t)/t$ for large enough t in the case where $\lambda = 1$. Hence (3.4) has been shown. We have completed the theorem. □

ACKNOWLEDGEMENT. The author should like to express his gratitude to Professor K. Sato for relevant comments on the article. He also thanks the referee for his valuable and careful comments.

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