Namba, M. and Takai, M.
Osaka J. Math.
40 (2003), 139–170

DEGENERATING FAMILIES OF FINITE BRANCHED COVERINGS

MAKOTO NAMBA and MAKI TAKAI

(Received May 1, 2001)

1. Introduction

The category of finite branched coverings of a given complex projective manifold M is equivalent to the category of finite extensions $K/\mathbb{C}(M)$ of the rational function field $\mathbb{C}(M)$ of M. Hence the study of finite branched coverings of M is nothing but a geometric study of extensions of algebraic function fields. In Namba [8], we constructed and studied the moduli space of equivalence classes of finite branched coverings of the complex projective line $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. If we want to compactify the moduli space, we are obliged to consider degenerations of branched coverings.

In this paper, we study degenerating families of finite branched coverings of $\mathbb{P}^1$ and $\mathbb{P}^m = \mathbb{P}^m(\mathbb{C})\,(m \geq 2)$ the m-dimensional complex projective space. In order to observe the degeneration, it is useful to introduce a picture which topologically represents a finite branched covering of the complex projective line. In §3, we call such a picture a Klein picture, since we can find such pictures in Klein [5]. In §5 (resp. §7), we assert that the topological type of the central fiber of a degenerating family of finite branched coverings of $\mathbb{P}^1$ (resp. $\mathbb{P}^m$ $(m \geq 2)$) is completely determined by that of the central branch divisor and the permutation monodromy of the general fiber. In §6, we prove (Theorem 8) that the topological structure of a degenerating family of finite branched coverings of $\mathbb{P}^1$ can be determined by the permutation monodromy of the general fiber and the braid nomodromy of the family. Some results of this paper were anounced in Namba [10].

2. Terminology

For a given connected complex manifold M, a finite branched covering of M is by definition a finite proper holomorphic mapping

$$f\colon X \longrightarrow M$$

of an irreducible normal complex space X onto M. A ramification point of f is a point of X such that f is not biholomorphic around the point. The image by f of a ramification point is called a branched point of f. The set of all ramification points (resp. branch points) is denoted by R_f (resp. B_f). This is a hypersurface of

X (resp. M). The mapping

$$f\colon X - f^{-1}(B_f) \longrightarrow M - B_f$$

is a finite unbranched covering. Its mapping degree is denoted by $\deg(f)$ and is called the degree of f. For a hypersurface B of M, a finite branched covering f is said to branch at most at B if B_f is contained in B. Finite branched coverings $f\colon X \longrightarrow M$ and $f'\colon X' \longrightarrow M$ are said to be isomorphic if there is a biholomorphic mapping $\psi\colon X \longrightarrow X'$ such that $f = f' \cdot \psi$. In this case, we denote $f \simeq f'$. Finite branched coverings $f\colon X \longrightarrow M$ and $f'\colon X' \longrightarrow M'$ are said to be equivalent (resp. topologically equivalent) if there are biholomorphic mappings (resp. orientation preserving hemeomorphisms) $\psi\colon X \longrightarrow X'$ and $\varphi\colon M \longrightarrow M'$ such that $\varphi \cdot f = f' \cdot \psi$. In this case, we denote $f \sim f'$ (resp. $f \sim f'$ (top.)).

Theorem 1 (Grauert-Remmert [4]). *Let B be a hypersurface of a connected complex manifold M and $f'\colon X' \longrightarrow M - B$ be a finite unbranched covering. Then there exists a unique* (*up to isomorphisms*) *finite covering $f\colon X \longrightarrow M$ which branches at most at B and is an extension of f'.*

A topological version of Theorem 1 is given in Fox [3]. Theorem 1 asserts that the correspondence $f \longleftrightarrow f'$ gives a categorical equivalence between finite unbranched coverings of $M - B$ and finite coverings of M branching at most at B. Thus we can apply terminology of finite unbranched coverings of $M - B$ to finite coverings of M branching at most at B; for example, covering transformations, Galois coverings, abelian coverings, cyclic coverings, etc.

Corollary 1. *There is a one-to-one correspondence between the set of all isomorphism classes of finite coverings of M branching at most at B and the set of all conjugacy classes of subgroups of finite index of the fundamental group $\pi_1(M - B, q_0)$ of $M - B$.*

3. Monodromy representations and Klein pictures

Let $f\colon X \longrightarrow M$ be a finite branched covering of a connected complex manifold M of degree d branching at most at a hypersurface B of M. Take a reference point q_0 of $M - B$ and put $f^{-1}(q_0) = \{p_1, \ldots, p_d\}$. The homotopy class $[\gamma]$ of a loop γ in $M - B$ starting from q_0 gives the homotopy class of the pull-back over f of γ starting from every point p_j, ($j = 1, \ldots, d$). Hence its end point $p_{j'}$ is determined. Thus we obtain a mapping

$$\Phi_f\colon \pi_1(M - B, q_0) \longrightarrow S_d,$$

which maps $[\gamma]$ to the permutation $j \to j'$, where S_d is the d-th symmetric group. We define the product of pathes α and β as $\alpha\beta$, where the end point of α is the initial point of β. We also define the product of permutations as in the following example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

The mapping Φ_f is then a homomorphism and is called the (permutation) monodromy representation of the covering f. Note that the representation class $[\Phi_f]$ of Φ_f does not depend on the choice of the arrangement of the points $p_1, \ldots, p_d$, nor the choice of the reference point q_0. That is, if one changes the arrangement of the points $p_1, \ldots, p_d$ or one chooses another reference point, then Φ_f is changed to $A^{-1}\Phi_f A$ for a fixed permutation A. Note also that the image of Φ_f is a transitive subgroup of S_d, for $X - f^{-1}(B)$ is connected. The image is called the monodromy group of the covering f. Monodromy groups of finite branched coverings correspond to Galois groups of algebraic equations. By the theorem of Grauert-Remmert and its corollary, we easily have the following 2 theorems:

Theorem 2. (1) *Finite branched coverings f and f' of M are isomorphic if and only if $B_f = B_{f'}$ and $[\Phi_f] = [\Phi_{f'}]$. ($[\Phi_f]$ is the representation class of Φ_f.)*
(2) *Finite branched coverings f of M and f' of M' are equivalent (resp. topologically equivalent) if and only if there is a biholomorphic mapping (resp. orientation preserving homeomorphism) $\varphi\colon M \longrightarrow M'$ such that $\varphi(B_f) = B_{f'}$ and $[\Phi_{f'} \cdot \varphi_*] = [\Phi_f]$.*

Theorem 3. *For a given homomorphism $\Phi\colon \pi_1(M - B, q_0) \longrightarrow S_d$ whose image is transitive, there exists a unique (up to isomorphisms) covering $f\colon X \longrightarrow M$ of degree d branching at most at B such that $\Phi_f = \Phi$.*

However it is a difficult problem in general to construct covering $f\colon X \longrightarrow M$ in the theorem from a given Φ concretely (analytically or algebraically). The problem for the case $M = \mathbb{P}^1$ the complex projective line and $B = \{0, 1, \infty\}$ is studied in number theory (see Schneps [11]).

We construct branched coverings of the complex projective line $\mathbb{P}^1$ topologically for any given Φ, by drawing a picture which we call a Klein picture, the idea of which comes from Klein [5]. Let $B = \{q_1, \ldots, q_n\}$ be a set of n distinct points of $\mathbb{P}^1$. Let $f\colon X \longrightarrow \mathbb{P}^1$ be a covering of degree d branching at most at B. We draw a simple loop in $\mathbb{P}^1$ passing through all points q_j, $j = 1, \ldots, n$, oriented in this order which bounds a domain (the inside area) clockwisely (see Fig. 1). We regard the inside area of the loop as a continent and the outside area as an ocean. We assume that the reference point q_0 is contained in the continent. We then pull them back over the covering f. Then we get a checked pattern of d continents and d oceans on X. We call such a

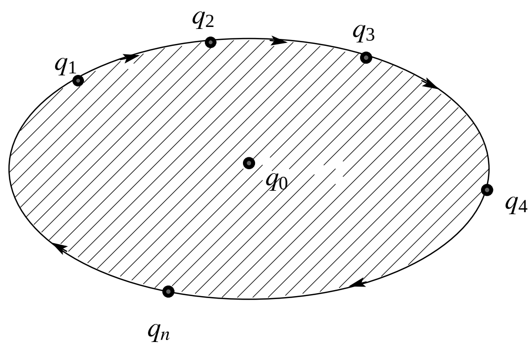

Fig. 1.

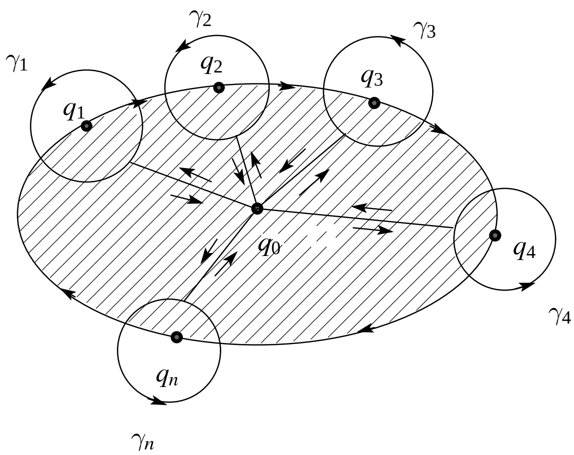

Fig. 2.

pattern the Klein picture of the covering f. The Klein picture represents the branched covering f topologically. Starting from a homomorphism $\Phi\colon \pi_1(\mathbb{P}^1 - B,\, q_0) \longrightarrow S_d$ such that $\operatorname{Im}\Phi$ is transitive, we construct the branched covering f in Theorem 2 topologically by drawing its Klein picture as follows: Put

$$A_j = \Phi(\gamma_j) \in S_d,\ j = 1,\, \ldots,\, n,$$

where γ_j are lassos surrounding the points q_j as in Fig. 2. Note that

$$\begin{aligned} \pi_1(\mathbb{P}^1 - B,\, q_0) &= \langle \gamma_1,\, \ldots,\, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle\,, \\ A_n \cdots A_1 &= 1 \in S_d. \end{aligned}$$

Thus the representation Φ is determined by the permutations A_j. Decompose each A_j into mutually prime cyclic permutations A_{j_k} whose length are e_{j_k}. Put (by Riemann-

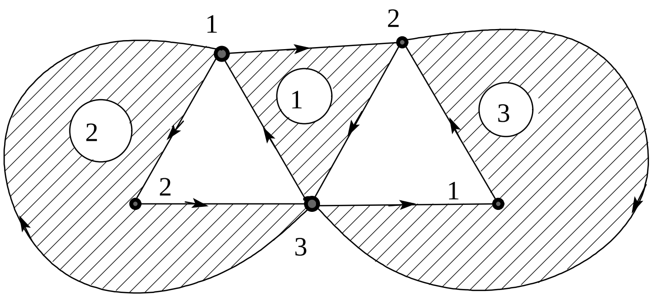

Fig. 3.

Hurwitz formula)

$$g = \frac{1}{2}\left[\sum_{j,k}(e_{j_k} - 1) - 2d\right] + 1.$$

We prepare an oriented compact surface X of genus g. We then draw the Klein picture, that is, a checked pattern of d continents and d oceans on X which is compatible with Φ. Here, the compatibility means that, for the point p_{j_k} of $f^{-1}(q_j)$ which corresponds to A_{j_k}, e_{j_k} continents and oceans are arranged alternately and counterclockwisely around p_{j_k}.

EXAMPLE 1. Put $n = 3$, $d = 3$ and

$$A_1 = \Phi(\gamma_1) = (1\ 2),\ \ A_2 = \Phi(\gamma_2) = (1\ 3),\ \ A_3 = \Phi(\gamma_3) = (1\ 2\ 3).$$

The genus of X is 0. The Klein picture in this case is as in Fig. 3, in which the points j denote the points in $f^{-1}(q_j)$ and the circled number ⓘ denotes the i-th continent. Observe that the points 1, 2 and 3 are seaside cities (vertices) of every continents arranged colckwisely in this order, while for example the continents ①, ② and ③ are arranged counterclockwisely in this order around the city 3, which means $A_3 = (1\ 2\ 3)$. (Conversely, we can read the monodromy from the Klein picture.) Put

$$f : X \longrightarrow \mathbb{P}^1,\ (z,\ w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3w - z = 0$. Then $q_1 = -2$, $q_2 = 2$, $q_3 = \infty$ and $\Phi_f = \Phi$.

EXAMPLE 2. Put $n = 3$, $d = 3$ and

$$A_j = \Phi(\gamma_j) = (1\ 2\ 3), \quad j = 1,\ 2,\ 3.$$

The genus of X is 1. The Klein picture in this case is as in Fig. 4. Put

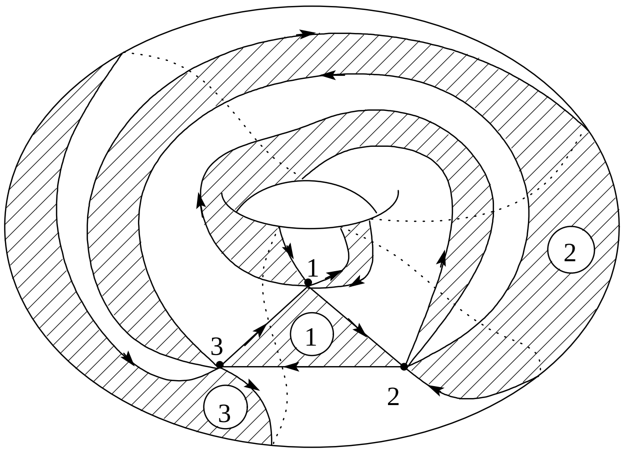

Fig. 4.

$$f : X \longrightarrow \mathbb{P}^1, \ (z,\ w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - z^3 + 1 = 0$. Then f is a cyclic covering such that $\Phi_f = \Phi$.

EXAMPLE 3. Put $n = 4$, $d = 3$ and

$$A_1 = \Phi(\gamma_1) = (1\ 3\ 2), \quad A_2 = \Phi(\gamma_2) = (1\ 3\ 2),$$
$$A_3 = \Phi(\gamma_3) = (1\ 2\ 3), \quad A_4 = \Phi(\gamma_4) = (1\ 2\ 3).$$

The genus of X is 2. The Klein picture in this case is as in Fig. 5. Put

$$f : X \longrightarrow \mathbb{P}^1, \ (z,\ w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - z^2(z-1)^2(z-2) = 0$. Then f is a cyclic covering such that $q_1 = 0$, $q_2 = 1$, $q_3 = 2$, $q_4 = \infty$ and $\Phi_f = \Phi$.

4. Families of finite branched coverings

Let T be a connected complex manifold. A family of connected complex manifolds with the parameter space T is by definition a smooth holomorphic mapping

$$\pi : M \longrightarrow T$$

of a connected complex manifold M onto a connected complex manifold T such that every fiber is connected. Here the smoothness means that the Jacobian matrix $d\pi$ is of

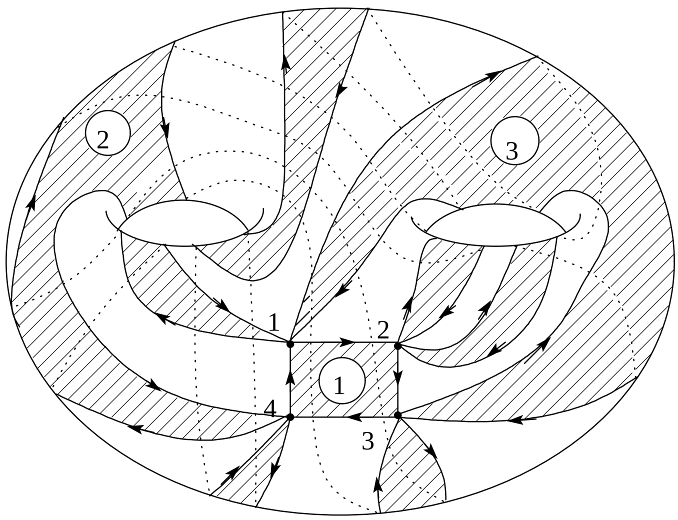

Fig. 5.

maximal rank at every point of M. Every fiber $M_t = f^{-1}(t)$ of $t \in M$ is a connected complex manifold. We denote

$$M = \{M_t\}_{t \in T}\,.$$

Let $M = \{M_t\}_{t \in T}$ be a family of connected complex manifolds. A family of finite branched coverings of $M = \{M_t\}_{t \in T}$ is by definition a finite branched covering

$$f\colon X \longrightarrow M$$

such that
(i) $M_t \not\subset B_f$ for every $t \in T$,
(ii) there is a hypersurface V of T such that

$$f_t = f\colon X_t = f^{-1}(M_t) \longrightarrow M_t$$

is a finite branched covering of M_t for every $t \in T - V$.
(iii) For any t and t' in $T - V$, f_t and $f_{t'}$ are topologically equivalent.

We denote $f = \{f_t\}$. In particular if $\pi\colon M \longrightarrow T$ is a holomorphic $\mathbb{P}^m$-bundle, then we call $f\colon X \longrightarrow M$ a family of finite branched coverings of $\mathbb{P}^m$.

REMARK. X and $X_t\,(t \in T - V)$ have only normal singularity, while $X_t\,(t \in V)$, the degenerated coverings, may not be normal. In this sense, our definition of degenerations is different from the usual one.

We are interested in X_t for $t \in V$, that is, degenerated coverings. In the subsequent sections, we restrict our consideration to degenerating families of finite branched coverings of $\mathbb{P}^m$ and a disc in $\mathbb{C}$.

EXAMPLE 4. Put $Y = \{((a_0 : a_1 : a_2 : a_3), (x_0 : x_1)) \in \mathbb{P}^3 \times \mathbb{P}^1 \mid a_0 x_1^3 + a_1 x_1^2 x_0 + a_2 x_1 x_0^2 + a_3 x_0^3 = 0\}$, $g\colon ((a_0 : a_1 : a_2 : a_3), (x_0 : x_1)) \in Y \longmapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3$, where $(a_0 : a_1 : a_2 : a_3)$ and $(x_0 : x_1)$ are homogeneous coordinate systems of $\mathbb{P}^3$ and $\mathbb{P}^1$, respectively. Then Y is non-singular and g is a branched covering of degree 3 whose branch locus is the discriminant locus

$$B_f = \{ (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3 \mid a_1^2 a_2^2 - 4a_0 a_2^3 + 18 a_0 a_1 a_2 a_3 - 4a_1^3 a_3 - 27 a_0^2 a_3^2 = 0 \}.$$

Let $\mathbb{P}^{3*}$ be the dual projective space of $\mathbb{P}^3$ and put

$$\begin{aligned} M &= \{ ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in \mathbb{P}^{3*} \times \mathbb{P}^3 \mid t_0 a_0 + t_1 a_1 + t_2 a_2 + t_3 a_3 = 0 \}, \\ &\pi\colon ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \longmapsto (t_0 : t_1 : t_2 : t_3) \in \mathbb{P}^{3*}, \\ &\pi'\colon (t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \longmapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3, \end{aligned}$$

where $(t_0 : t_1 : t_2 : t_3)$ is a homogeneous coordinate system of $\mathbb{P}^{3*}$. Then π is a $\mathbb{P}^2$-bundle on $\mathbb{P}^{3*}$. Let X be the normalization of the fiber product $M \times_{\mathbb{P}^3} Y$ of $\pi'\colon M \longrightarrow \mathbb{P}^3$ and $g\colon Y \longrightarrow \mathbb{P}^3$. Let

$$f\colon X \longrightarrow M$$

be the composition of the normalization

$$X \longrightarrow M \times_{\mathbb{P}^3} Y$$

and the projection

$$M \times_{\mathbb{P}^3} Y \longrightarrow M.$$

Then $f = \{f_t\}_{t \in \mathbb{P}^{3*}}$ is a family of branched coverings of $\mathbb{P}^2$. ($f_t\colon X_t \longrightarrow \pi^{-1}(t)$ is the restriction to the plane $\pi^{-1}(t)$ in $\mathbb{P}^3$ of g.)

We explain this as follows: Let

$$C_3 = \{ (1 : u : u^2 : u^3) \in \mathbb{P}^{3*} \mid u \in \mathbb{P}^1\}$$

be the rational normal curve, which is the image curve of the holomorphic imbedding

$$\Phi_{|D|}\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{3*}$$

of the unique complete linear system $|D|$ of degree 3 (D is a divisor on $\mathbb{P}^1$ of degree 3). B_f is then the dual variety of C_3. That is, B_f is the set of all planes in $\mathbb{P}^{3*}$

which contain tangent lines to C_3. Every divisor in $|D|$ is the intersection of C_3 with a (unique) plane in $\mathbb{P}^{3*}$. In this sense, $|D|$ is identified with $\mathbb{P}^3 = (\mathbb{P}^{3*})^*$. By the uniqueness of the complete linear system $|D|$ of degree 3, every automorphism of $\mathbb{P}^1$ acts on $|D| = \mathbb{P}^3$ (resp. on $\mathbb{P}^{3*}$) as a projective transformation, which maps B_f to B_f (resp. C_3 to C_3). Let V be the ruled surface in $\mathbb{P}^{3*}$ consisting of tangent lines to C_3.

For any two points t and t' in $\mathbb{P}^{3*} - V$, there is an automorphism φ of $\mathbb{P}^1$ such that $\varphi(t) = t'$. In fact, there are just 3 points p_1, p_2 and p_3 in C_3 (resp. p_1', p_2' and p_3' in C_3) such that the osculating plane at p_j (resp. at p_j') to C_3 passes through t (resp. t') for $j = 1, 2, 3$. Then $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\varphi(p_j) = p_j'$ $(j = 1, 2, 3)$ maps t to t'. Thus $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathbb{P}^{3*} - V$ transitively. (The orbits of the group action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^{3*}$ are $\mathbb{P}^{3*} - V$, V and C_3.) The projection π_t with the center t maps C_3 onto a rational plane cubic curve C_t with a node and 3 flexes $\pi_t(p_1)$, $\pi_t(p_2)$ and $\pi_t(p_3)$. The plane projective transformation induced by φ maps C_t to $C_{t'}$. The branch locus B_t (resp. $B_{t'}$) of f_t (resp. $f_{t'}$) is the dual curve of C_t (resp. $C_{t'}$) which is a rational plane quartic curve with 3 simple cusps. Hence the plane projective transformation induced by φ maps B_t to $B_{t'}$. Now, φ induces an automorphism of the projective manifold Y which, by the above discussion, induces an equivalence of f_t and $f_{t'}$.

A similar discussion shows that if $t \in V - C_3$ (say $t = (0 : 1 : 0 : 0)$), then B_t is the union of a rational plane cubic curve with 1 simple cusp and a line passing through a flex of the curve. For any points t and t' in $V - C_3$, f_t and $f_{t'}$ are equivalent.

If $t \in C_3$ (say $t = (0 : 0 : 0 : 1)$), then B_t is the union of an irreducible conic and a double tangent line to the conic. In this case, X_t is not irreducible.

5. Degenerating families of finite branched coverings of $\mathbb{P}^1$

Let

$$\Delta = \Delta(0, \epsilon) = \{t \in \mathbb{C} \mid |t| < \epsilon\}$$

be a disc and $\Delta^* = \Delta - \{0\}$ be the punctured disc. A finite branched covering

$$f : X \longrightarrow \Delta \times \mathbb{P}^1$$

is called a degenerating family of finite branched coverings of $\mathbb{P}^1$ and is denoted by $f = \{f_t\}$, if the following three conditions are satisfied:

(1) $t \times \mathbb{P}^1 \not\subseteq B_f$ for every $t \in \Delta$.

(2) For every $t \in \Delta^*$, $t \times \mathbb{P}^1$ meets at n points transversally with B_f. (n is constant for $t \in \Delta^*$.)

(3) For every $t \in \Delta^*$,

$$f_t = f : X_t = f^{-1}(t \times \mathbb{P}^1) \longrightarrow t \times \mathbb{P}^1$$

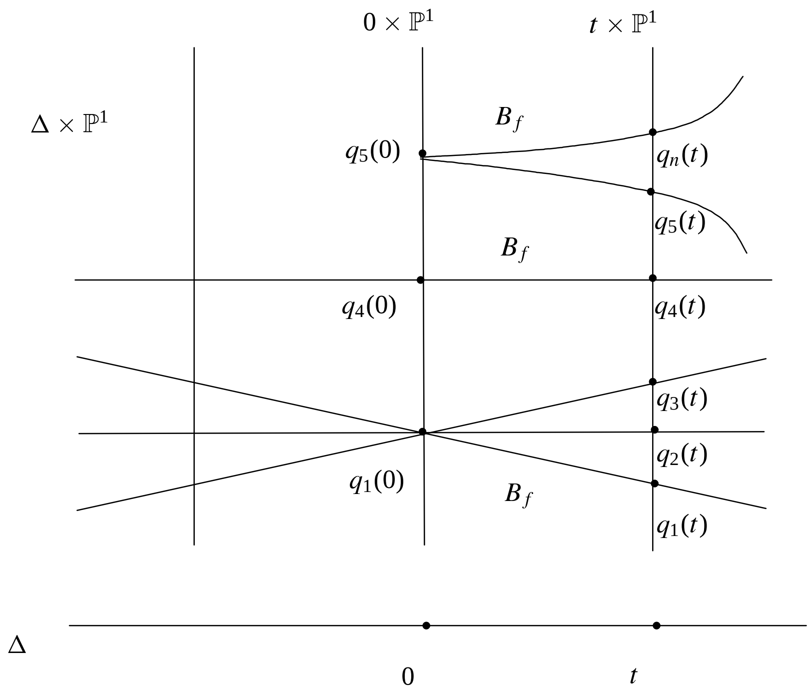

Fig. 6.

is a covering of $\mathbb{P}^1$ of degree $d = \deg(f)$ branching at $B_t = B_f \cap (t \times \mathbb{P}^1) = \{q_1(t), \ldots, q_n(t)\}$ (see Fig. 6).

The central fiber $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ is a degeneration of a general fiber X_t for $t \neq 0$. The Klein picture of f_t degenerates to a picture on X_0, which we call the Klein picture of f_0. This represents (X_0, f_0) topologically.

EXAMPLE 5. Let X_t be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3tw - z = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, \ (z, w) \longmapsto z.$$

Then $f = \{f_t\}$ is a degenerating family of branched coverings of $\mathbb{P}^1$. For a fixed non-zero t, the monodromy representation Φ_t and the Klein picture of f_t are given as same as in Example 1. Note that $q_1(t) = -2t^{3/2}$, $q_2(t) = 2t^{3/2}$ and $q_3(t) = \infty$. As $t \longrightarrow 0$, both branch points $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 3 converge to the point 1 = 2, and we get the Klein picture of f_0 as in Fig. 7. In fact $X_0 : w^3 - z = 0$.

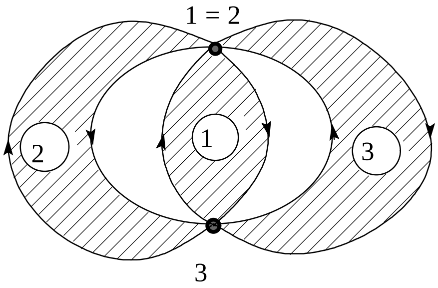

Fig. 7.

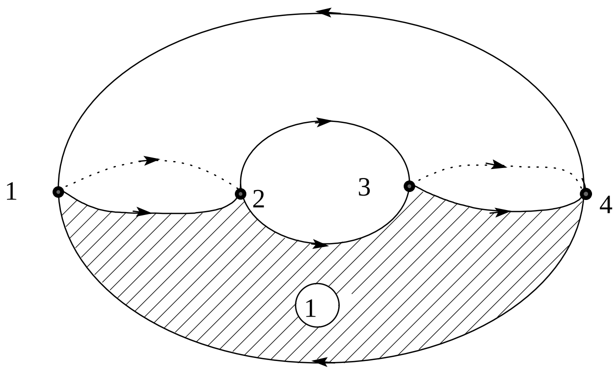

Fig. 8.

EXAMPLE 6. Let X_t be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z-t)(z-1) = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, \ (z, w) \longmapsto z.$$

Then $f = \{f_t\}$ is a degenerating family of branched double coverings of $\mathbb{P}^1$. Note that $q_1(t) = 0$, $q_2(t) = t$, $q_3(t) = 1$ and $q_4(t) = \infty$. For a fixed non-zero t, the monodromy representation Φ_t is given by $A_j = \Phi_t(\gamma_j) = (1\ 2)$ for $j = 1, 2, 3, 4$. The Klein picture of f_t is as in Fig. 8 in which the continent ② is the upper backside of the torus. As $t \longrightarrow 0$, both $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 8 converge to the point 1 = 2, and we get the Klein picture of f_0 as in Fig. 9 in which the continent ② is also the upper backside. In fact, $X_0 : w^2 - z^2(z-1) = 0$.

EXAMPLE 7. Let X_t be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z-t)(z-1)(z-1-t) = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, \quad (z, w) \longmapsto z.$$

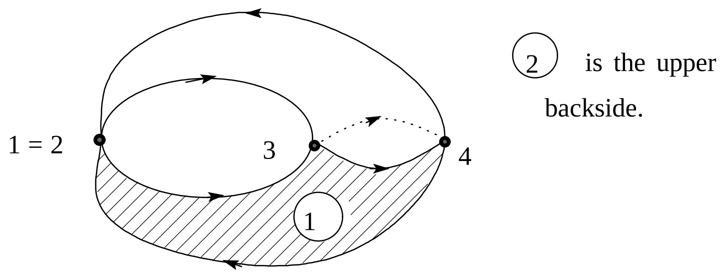

Fig. 9.

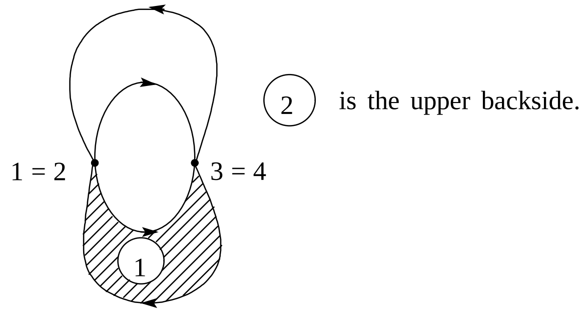

Fig. 10.

Then $f = \{f_t\}$ is a degenerating family of branched double covering of $\mathbb{P}^1$. Note that

$$q_1(t) = 0, \ q_2(t) = t, \ q_3(t) = 1, \ q_4(t) = 1 + t.$$

For fixed t with $0 < |t| < 1$, the monodromy representation Φ_t is a given by $A_j = \Phi_t(\gamma_j) = (1\,2)$ for $j = 1, 2, 3, 4$. The Klein picture of f_t is as same as that in Fig. 8 for Example 6. As $t \longrightarrow 0$, $q_2(t)$ and $q_4(t)$ converge to $q_1(0) = 0$, $q_3(0) = 1$, respectively, so the pathes connecting the points 1 to 2 and 3 to 4 in Fig. 8 converge to 1 = 2 and 3 = 4, respectively. Hence we get the Klein picture of f_0 as in Fig. 10 in which the continent ② is also the upper backside. In fact

$$X_0 : w^2 - z^2(z-1)^2 = 0,$$

which is not globally irreducible.

EXAMPLE 8. Let X_t be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z-t)(z-2t) = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, \quad (z, w) \longmapsto z.$$

As $t \longrightarrow 0$, the Klein picture of f_t converges to that of f_0 as in Fig. 11, in which ②

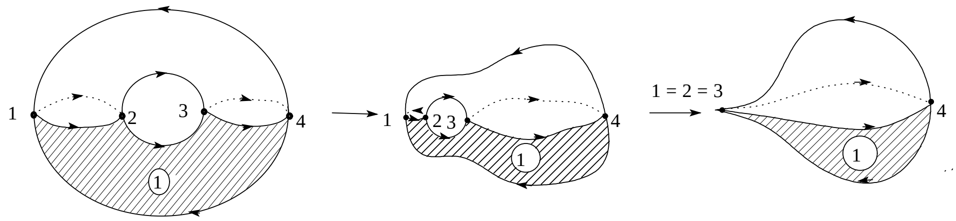

Fig. 11.

are the upper backside.

X_0 has a cusp singularity at the point $1 = 2 = 3$. In fact

$$X_0 : w^2 - z^3 = 0.$$

Now we assume and put

$$\begin{aligned} &q_1(0) = \cdots = q_{k_1}(0) = q_1^0, \\ &q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0, \\ &\qquad \cdots\cdots \\ &q_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q_{k_1+\cdots+k_r}(0) = q_r^0 \end{aligned}$$

where $k_\rho \geq 1$ ($\rho = 1, \ldots, r$), $k_1 + \cdots + k_r = n$ and $q_1^0, q_2^0, \ldots, q_r^0$ are mutually distinct. We regard

$$B_0 = B_f \cap (0 \times \mathbb{P}^1) = \{ q_1^0, q_2^0, \ldots, q_r^0 \}$$

not as a point set but as a divisor on $\mathbb{P}^1$:

$$B_0 = k_1 q_1^0 + k_2 q_2^0 + \cdots + k_r q_r^0.$$

We draw a simple loop in $\mathbb{P}^1$ passing through all points $q_1^0, \ldots, q_r^0$ oriented in this order which bounds a domain clockwisely as in Fig. 1. We call the Klein picture of f_0 for the checked patern on X_0 which is the pull-back of the picture over f_0.

Now, we show that topologically, the degenerating curve $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ can be described by the divisor B_0 and the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point.

Let $\gamma_j(t)\,(1 \leq j \leq n)$ be the lasso around $q_j(t)$ as in Fig. 2 and put

$$A_1 = \Phi_t(\gamma_1), \ \ldots, \ A_n = \Phi_t(\gamma_n).$$

Let H_ρ $(1 \leq \rho \leq r)$ be the subgroup of S_d generated by

$$A_{k_1+\cdots+k_{\rho-1}+1}, \ \ldots, \ A_{k_1+\cdots+k_{\rho-1}+k_\rho}.$$

H_ρ may not be a transitive subgroup of S_d. We denote

$$\mathfrak{A}_1^\rho, \ldots, \mathfrak{A}_{v_\rho}^\rho$$

the orbits of H_ρ on $\{1, 2, \ldots, d\}$. v_ρ is the number of orbits. Put

$$\begin{aligned} A_1^0 &= A_{k_1} A_{k_1-1} \cdots A_1, \\ A_2^0 &= A_{k_1+k_2} A_{k_1+k_2-1} \cdots A_{k_1+1}, \\ &\cdots\cdots \\ A_r^0 &= A_{k_1+\cdots+k_r} A_{k_1+\cdots+k_r-1} \cdots A_{k_1+\cdots+k_{r-1}+1} \end{aligned}$$

and

$$K = \langle A_1^0, \ldots, A_r^0 \rangle$$

the subgroup of S_d generated by $A_1^0, \ldots, A_r^0$. Let

$$\gamma_1^0, \ldots, \gamma_r^0$$

be lassos around the points

$$q_1^0, \ldots, q_r^0,$$

respectively in $0 \times \mathbb{P}^1$ as in Fig. 2. Put

$$\Phi_0(\gamma_\rho^0) = A_\rho^0 \quad (1 \leq \rho \leq r).$$

Then

$$\Phi_0 \colon \pi_1(0 \times \mathbb{P}^1 - \{q_1^0, \ldots, q_r^0\}, q_0) \longrightarrow S_d$$

is a homomorphism.

DEFINITION 1. For a permutation $A \in S_d$, if A is written as $A = A_1 \cdots A_w$, the product of mutually prime cyclic permutations, then we call the number $w = w(A)$ the weight of A. ($w(A)$ depends also on d. For example, if $d = 4$ and $A = (1\ 2\ 3)$, then $w(A) = w((1\ 2\ 3)(4)) = 2$.)

Let $\chi(X_t)$ denote the Euler characteristic of X_t.

Theorem 4. *Let* $t \neq 0$. *Then the following* (1)–(4) *hold*:
(1) $\chi(X_t) = 2 - 2g = 2d - nd + \sum_{j=1}^n w(A_j)$.
(2) $\chi(X_0) = 2d - \{nd - \sum_{\rho=1}^r (k_\rho - 1)d\} + \sum_{\rho=1}^r v_\rho$.
(3) $\chi(X_0) - \chi(X_t) = \sum_{\rho=1}^r (k_\rho - 1)d + \sum_{\rho=1}^r v_\rho - \sum_{j=1}^n w(A_j)$.

(4) $\chi(X_0) \geq \chi(X_t)$.

Proof. (1) The Klein picture of the covering $f_t\colon X_t \longrightarrow \mathbb{P}^1$ gives a cellular decomposition of X_t. The number of vertices is $\sum_{j=1}^{n} w(A_j)$, the number of sides is nd and the number of faces is $2d$. Hence

$$\chi(X_t) = 2d - nd + \sum_{j=1}^{n} w(A_j).$$

(2) Let $\hat{G}$ be the (oriented) graph on X_t of the pull-back by f_t of the cycle

$$q_1(t) \longrightarrow q_2(t) \longrightarrow \cdots \longrightarrow q_n(t) \longrightarrow q_1(t).$$

Then $\hat{G}$ is the graph whose points and lines are vertices and sides, respectively, of the Klein picture of f_t. Every point of $\hat{G}$ has been numbered as a vertix of the Klein picture. We put the circled number ⓙ on every sides of j-th continent. Thus we get a graph $\hat{G}$ with numbered points and circle numbered lines.

Let $G_\rho\,(1 \leq \rho \leq r)$ be the (oriented) graph on X_t of the pull-back by f_t of the tree

$$q_{k_1+\cdots+k_{\rho-1}+1}(t) \longrightarrow q_{k_1+\cdots+k_{\rho-1}+2}(t) \longrightarrow \cdots \longrightarrow q_{k_1+\cdots+k_{\rho-1}+k_\rho}(t).$$

Then every G_ρ is a subgraph of $\hat{G}$. Let

$$k_1 + \cdots + k_{\rho-1} + 1 \leq i < i + 1 \leq k_1 + \cdots + k_{\rho-1} + k_\rho.$$

If the permutaion A_i and A_{i+1} are witten as, say,

$$A_i = \begin{pmatrix} \cdots & a & \cdots \\ \cdots & b & \cdots \end{pmatrix}, \qquad A_{i+1} = \begin{pmatrix} \cdots & a & \cdots \\ \cdots & c & \cdots \end{pmatrix},$$

then there are lines ⓐ and ⓑ in G_ρ which have the starting point i (a point of $G_\rho \cap f_t^{-1}(q_i)$), and so are connected at the point i. Moreover there is a line ⓒ in G_ρ such that the lines ⓐ and ⓒ have the same end point $i+1$, and so are connected at the point $i+1$. Hence the lines ⓐ, ⓑ and ⓒ are connected in G_ρ.

Now the Klein picture of f_0 gives a cellular decomposition of X_0 which can be obtained from that of X_t by converging every connected component of the graphs $G_\rho\,(1 \leq \rho \leq r)$ to a point. Hence the number of vertices is $\sum_{\rho=1}^{r} v_\rho$, the number of sides is $nd - \sum_{\rho=1}^{r}(k_\rho - 1)d$ and the number of faces is $2d$. Hence

$$\chi(X_0) = 2d - \left\{nd - \sum_{\rho=1}^{r}(k_\rho - 1)d\right\} + \sum_{\rho=1}^{r} v_\rho.$$

(3) This follows from (1) and (2).

(4) For a graph G, the following inequality holds:

$$b \geq a - c,$$

where a is the number of points of G, b is the number of lines of G and c is the number of connected components of G. Here the equality holds if and only if every component is a tree, that is, a graph without cycles.
We apply this to every graph G_ρ. Then

$$\begin{aligned} a &= a_\rho = w(A_{k_1+\cdots+k_{\rho-1}+1}) + \cdots + w(A_{k_1+\cdots+k_{\rho-1}+k_\rho}), \\ b &= b_\rho = (k_\rho - 1)d, \\ c &= c_\rho = v_\rho. \end{aligned}$$

Note that

$$\sum_{\rho=1}^{r} a_\rho = \sum_{j=1}^{n} w(A_j).$$

Hence by (3)

$$\chi(X_0) - \chi(X_t) = \sum_{\rho=1}^{r} b_\rho + \sum_{\rho=1}^{r} c_\rho - \sum_{\rho=1}^{r} a_\rho = \sum_{\rho=1}^{r} (b_\rho + c_\rho - a_\rho) \geq 0. \qquad \square$$

Theorem 5. (1) *$f_0^{-1}(q_\rho^0)$ consists of v_ρ points, which can be identified with $\mathfrak{A}_1^\rho, \ldots, \mathfrak{A}_{v_\rho}^\rho$.*
(2) Every $A_\rho^0 (1 \leq \rho \leq r)$ induces a permutation $A_{\rho j}^0 : \mathfrak{A}_j^\rho \longrightarrow \mathfrak{A}_j^\rho$. X_0 has local $w(A_{\rho j}^0)$ irreducible components at the point corresponding to $\mathfrak{A}_j^\rho$.
(3) There is a natural one-to-one correspondence between the set of global irreducible components of X_0 and the set of orbits of $K = \langle A_1^0, \ldots, A_r^0 \rangle$ on $\{1, \ldots, d\}$. Φ_0, regarded as the representation to permutations on an orbit of K on $\{1, \ldots, d\}$, gives the monodromy representation of the branched covering

$$f_0 \cdot \eta \colon \hat{X}_0' \longrightarrow 0 \times \mathbb{P}^1,$$

where $\eta \colon \hat{X}_0' \longrightarrow X_0'$ is the normalization of the global irreducible component X_0' of X_0 corresponding to the orbit of K.

Proof. (1) follows from the proof (2) of Theorem 4. For a sufficiently small $|t|$,

$$q_{k_1+\cdots+k_{\rho-1}+1}(t), \ \ldots, \ q_{k_1+\cdots+k_{\rho-1}+k_\rho}(t)$$

are in a small neighborhood of q_ρ^0. Hence the lasso γ_ρ^0 is homotopic to the product

$$\gamma_{k_1+\cdots+k_{\rho-1}+k_\rho}(t) \cdots \gamma_{k_1+\cdots+k_{\rho-1}+1}(t).$$

Let

$$i\colon N = 0 \times \mathbb{P}^1 \longrightarrow M = \Delta \times \mathbb{P}^1$$

be the inclusion mapping. Then the fiber product $N \times_M X$ can be identified with X_0. Now, (2) and (3) of Theorem 5 follow from the following lemma, whose proof is straightforward and is omitted. □

Lemma 1. *Let M and N be connected complex manifolds and $h\colon N \longrightarrow M$ be a holomorphic mapping. Let $f\colon X \longrightarrow M$ be a finite unbranched covering of M of degree d and Φ_f be its monodromy representation. Let $f'\colon N \times_M X \longrightarrow N$ be the projection of the fiber product $N \times_M X$ onto N. Then the followings hold*:
(1) *There is a one-to-one correspondence between the set of orbits of $\Phi_f \cdot h_*(\pi_1(N,\ p_0))$ on $\{1,\ \ldots,\ d\}$ and the set of connected components of $N \times_M X$.*
(2) *For a connected component Y of $N \times_M X$, $f'\colon Y \longrightarrow N$ is a finite unbranched covering of N whose monodromy representation is equal to $\Phi_f \cdot h_*$ regarded as the representation to permutations on the orbit of $\Phi_f \cdot h_*(\pi_1(N,\ p_0))$ on $\{1,\ \ldots,\ d\}$ corresponding to Y.*

Theorem 6. *The following four conditions are mutually equivalent*:
(1) *X_0 is homeomorphic to X_t for $t \neq 0$.*
(2) *$\chi(X_0) = \chi(X_t)$ for $t \neq 0$.*
(3) $\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho$.
(4) $\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0)$.

Proof. If X_0 is homeomorphic to X_t, then $\chi(X_0) = \chi(X_t)$. If $\chi(X_0) = \chi(X_t)$, then every connected component of the graphs $G_\rho\,(1 \le \rho \le r)$ is a tree as is shown in the proof of Theorem 4. When t converges to 0, every connected component of the graphs $G_\rho\,(1 \le \rho \le r)$ converges to a point. This means that X_0 is homeomorphic to X_t. Next, note that

$$A_\rho^0 = A_{\rho 1}^0 \cdots A_{\rho v_\rho}^0,$$

where $A_{\rho j}^0$ is the permutation on the orbit $\mathfrak{A}_j^\rho$ induced by A_ρ^0. Hence

$$w(A_\rho^0) = w(A_{\rho 1}^0) + \cdots + w(A_{\rho v_\rho}^0).$$

In particular

$$w(A_\rho^0) \ge v_\rho.$$

Here the equality holds if and only if every $A_{\rho j}^0$ is a cyclic permutation. Hence, by (2) of Theorem 5, the equality holds if and only if X_0 is locally irreducible at every point $\mathfrak{A}_j^\rho\,(1 \le j \le v_\rho)$. Now, by (4) of Theorem 4, the following inequality holds:

$$\sum_{\rho=1}^{r}(k_\rho - 1)d \ge \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho \ge \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0).$$

If

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{k} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0),$$

then

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho.$$

Hence $\chi(X_0) = \chi(X_t)$ by Theorem 4.

Conversely, if $\chi(X_0) = \chi(X_t)$, then X_0 is homeomorphic to X_t. In particular, X_0 is locally irreducible at every point $\mathfrak{A}_j^\rho\,(1 \le j \le v_\rho,\ 1 \le \rho \le r)$. Thus

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0). \qquad \square$$

REMARK. If one of the conditions of Theorem 6 is satisfied, then X_0 is non-singular. In fact, if one of the conditions of Theorem 6 is satisfied, then every connected component of every graph $G_\rho\,(1 \le \rho \le r)$ is a tree. X_0 is obtained from X_t converging every tree to a point. Hence X_0 is still a manifold. (The total space X is also non-singular.)

This can be also shown in the following way: The arithmetic genus is constant. In particular, the arithmetic genus of X_0 is equal to the geometric genus of $X_t\,(t \ne 0)$. If X_0 is singular, then the geometric genus of X_0 is less than the arithmetic genus, a contradiction to the assumption $\chi(X_0) = \chi(X_t)$. On the other hand, if $\chi(X_0) > \chi(X_t)$, then the graph G contains a cycle. As $t \longrightarrow 0$, such a cycle Γ converges to a point p, while, for a connected open neighborhood U of Γ, $U - \Gamma$, which has two connected components, moves homeomorphically. Hence X_0 is locally a cone with the vertex p. Thus X_0 can not be a manifold, so X_0 is singular.

6. Topological equivalence of families

In this section we show that the topologial structure of the degenerating family $f = \{f_t\}$ of finite branched coverings of $\mathbb{P}^1$ is not determined by Φ_t alone, but depends also on the braid monodromy $\theta(\delta)$. Here

$$\delta\colon u \longmapsto t = t_0 e^{iu},\ (0 \le u \le 2\pi)$$

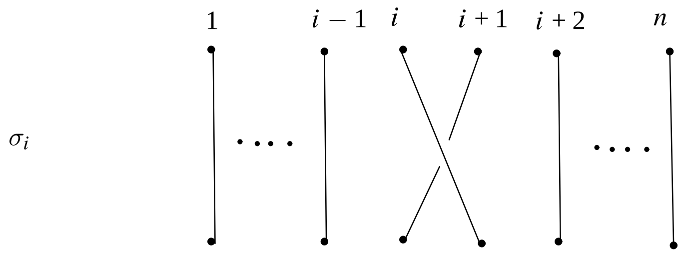

Fig. 12.

is the loop around $t = 0$. ($t_0 \in \Delta^*$ is a fixed point.) In this section we assume for simplicity that $q_j(t) \neq \infty$ for every $t \in \Delta$ and $1 \leq j \leq n$. Then

$$\{q_1(t_0 e^{iu}), \dots, q_n(t_0 e^{iu})\}_{0 \leq u \leq 2\pi}$$

gives an (Artin) braid of n strings, which is called the braid monodromy of the curve B_f around $t = 0$ and is denoted by $\theta(\delta)$. The braid $\theta = \theta(\delta)$ can not be arbitrary. It is given by a complex analytic curve B_f. So such a braid we call a complex analytic braid. We fix a reference point $t_0 \in \Delta^*$ and put

$$q_j = q_j(t_0) \quad \text{for} \quad 1 \leq j \leq n.$$

Then the Artin braid group B_n naturally acts on the fundamental group $\pi_1(\mathbb{P}^1 - \{q_1, \dots, q_n\}, q_0)$ as follows:

$$\begin{aligned} \sigma_i(\gamma_i) &= \gamma_i^{-1}\gamma_{i+1}\gamma_i, \\ \sigma_i(\gamma_{i+1}) &= \gamma_i, \\ \sigma_i(\gamma_j) &= \gamma_j \quad (j \neq i,\ i+1), \end{aligned}$$

where γ_j ($j = 1, \dots, n$) are the lassos as in Fig. 2 and σ_i ($i = 1, \dots, n-1$) are the generators of B_n defined as in Fig. 12.

A theorem of Zariski-van Kampen (see e.g. Dimca [2]) asserts

Theorem 7 (Zariski-van Kampen).

$$\begin{aligned} &\pi_1(\Delta \times \mathbb{P}^1 - B_f, q_0) \\ &\quad = \langle \gamma_1, \dots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1,\ \theta(\delta)\gamma_j = \gamma_j,\ (1 \leq j \leq n) \rangle, \end{aligned}$$

where γ_j *are lassos as in* Fig. 2 *for* $f_{t_0} : X_{t_0} \longrightarrow \mathbb{P}^1$.

The monodromy representation Φ_f of $f : X \longrightarrow \Delta \times \mathbb{P}^1$ is equal to $\Phi_{t_0} = \Phi_{f_{t_0}}$.

By Theorem 7, Φ_{t_0} satisfies

$$\Phi_{t_0} \cdot \theta(\delta) = \Phi_{t_0}.$$

DEFINITION 2. $f = \{f_t\}$ and $f' = \{f'_{t'}\}$ are said to be topologically equivalent if there are orientation preserving homeomorphisms ψ, φ and η which make the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\eta} & \Delta' \end{array}$$

Using fundamental results in the theory of fiber bundles (see Steenrod [12]), we get the following theorem, which can be regarded as a branched covering version of a theorem in Matsumoto-Montesinos [6]:

Theorem 8. *There exists a one to one correspondence between* $\{$*topological equivalence class of* $f = \{f_t\}$, *where* f_{t_0} $(t_0 \neq 0)$ *has the degree* d *and* n *branched points* $q_1, \ldots, q_n\}$ *and* $\{([\Phi], \theta) \mid [\Phi]$ *is the representation class of* $\Phi\colon \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \longrightarrow S_d$ *such that* $\operatorname{Im}\Phi$ *is transitive, and* $\theta \in B_n$ *is a complex analytic braid such that* $\Phi \cdot \theta = \Phi\}/B_n$. *Here* $\sigma \in B_n$ *acts on* $([\Phi], \theta)$ *as follows*:

$$\sigma([\Phi], \theta) = ([\Phi \cdot \sigma^{-1}], \sigma\theta\sigma^{-1}).$$

Proof. For two families

(1) $$f = \{f_t\}\colon X \longrightarrow \Delta \times \mathbb{P}^1, \quad f' = \{f'_t\}\colon X' \longrightarrow \Delta' \times \mathbb{P}^1,$$

with the assumption

$$(\Delta \times \{\infty\}) \cap B_f = \emptyset, \quad (\Delta' \times \{\infty\}) \cap B_{f'} = \emptyset,$$

we may assume that there are $q_0 \in \mathbb{C}$ and $q'_0 \in \mathbb{C}$ such that

$$(\Delta \times \{q_0\}) \cap B_f = \emptyset, \quad (\Delta' \times \{q'_0\}) \cap B_{f'} = \emptyset.$$

(For example, take q_0 and q'_0 such that $|q_0|$ and $|q'_0|$ are sufficiently large.)

Take reference points $t_0 \in \Delta^*$ and $t_0^{'} \in \Delta^{'*}$. Put

$$\begin{aligned}(t_0 \times \mathbb{C}) \cap B_f &= \{q_1 = q_1(t_0),\ \ldots,\ q_n = q_n(t_0)\},\\ (t_0^{'} \times \mathbb{C}) \cap B_{f'} &= \{q_1^{'} = q_1^{'}(t_0^{'}),\ \ldots,\ q_n^{'} = q_n^{'}(t_0^{'})\}.\end{aligned}$$

There is an orientation preserving homeomorphism

$$\xi\colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C} = \mathbb{C} \tag{2}$$

such that

$$\xi(q_j) = q_j^{'} \quad (j = 0,\ 1,\ \ldots,\ n).$$

We identify $q_j^{'}$ with q_j ($j = 0, 1, \ldots, n$) through ξ.

Now $\Delta^* \times \mathbb{C} - B_f$ is a topological fiber bundle with the base space Δ^* and the standard fiber $\mathbb{C} - \{\, n$ points$\,\}$ (see Dimca [2] and Matsuno [7]). Put

$$\begin{aligned}G = \{\, \alpha\colon \mathbb{C} \longrightarrow \mathbb{C} \mid \alpha\ &\text{ is an orientation preserving homoemorphism such that}\\ &\alpha(q_0) = q_0,\ \alpha(\{q_1,\ \ldots,\ q_n\}) = \{q_1,\ \ldots,\ q_n\}\,\}.\end{aligned}$$

G is then a topological group with compact-open topology. Let G_e be its connected component of the identity. Put

$$\pi_0(G) = G/G_e.$$

Then $\pi_0(G)$ can be naturally identified with the Artin braid group B_n of n strings (see Birman [1, p. 165]).

Now assume that the above two families

$$\begin{aligned}f &= \{f_t\}\colon X \longrightarrow \Delta \times \mathbb{P}^1,\\ f' &= \{f_t^{'}\}\colon X' \longrightarrow \Delta^{'} \times \mathbb{P}^1\end{aligned}$$

are topologically equivalent. We may assume that

$$\begin{aligned}\eta(t_0) &= t_0^{'},\\ \varphi\colon t_0 \times \mathbb{C} = \mathbb{C} &\longrightarrow t_0^{'} \times \mathbb{C} = \mathbb{C},\\ \varphi(q_0) &= q_0^{'}.\end{aligned}$$

Let

$$\begin{aligned}\chi\ &:\ \pi_1(S^1) \longrightarrow \pi_0(G)\\ (\text{resp. } \chi^{'}\ &:\ \pi_1(S^1) \longrightarrow \pi_0(G))\end{aligned}$$

be the characteristic homomorphism of the bundle

$$\Delta^* \times \mathbb{C} - B_f \longrightarrow \Delta^*$$
$$(\text{resp. } \Delta'^* \times \mathbb{C} - B_f \longrightarrow \Delta'^*)$$

(see Steenrod [12, p. 96]). Let δ (resp. δ') be the loop around $t = 0$ as before.

Two bundles

$$\Delta^* \times \mathbb{C} - B_f \quad \text{and} \quad \Delta'^* \times \mathbb{C} - B_f$$

over the base space Δ^* (which is homeomorphic to $(0, 1) \times S^1$) and Δ'^* are weakly equivalent in the sence of Steenrod [12, p. 99]. Hence by Steenrod [12, p. 100], the characteristic $\chi(\delta)$ and $\chi'(\delta')$ of these bundles satisfy either

$$\chi(\delta) = \chi'(\delta') \quad \text{or} \quad \chi(\delta) = \chi'(\delta')^{-1}$$

in $\pi_0(G)$. The equality here is up to conjugacy in $\pi_0(G)$. But the last equality does not occur by Steenrod [12, p. 100], for η is orientation preserving. Hence

(3) $$\chi(\delta) = \chi'(\delta') \quad \text{(up to conjugacy)}.$$

But $\pi_0(G)$ can be identified with B_n as noted above, Under the identification, $\chi(\delta)$ (resp. $\chi'(\delta')$) is equal to $\theta(\delta)$ (resp. $\theta'(\delta')$), the braid monodromy. Hence by (3), there is $\sigma \in B_n$ such that

(4) $$\theta'(\delta') = \sigma\theta(\delta)\sigma^{-1}.$$

Now the restriction

$$\varphi\colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t'_0 \times \mathbb{C} = \mathbb{C}$$

of φ is an orientation preserving homeomorphism. By the assumption of topological equivalence,

(5) $$[\Phi_{f_{t_0}} \cdot \varphi_*^{-1}] = [\Phi_{f'_{t'_0}}].$$

Consider an isotopy φ_t $(0 \leq t \leq 1)$ on $\mathbb{C}$ such that φ_0 = the identity and $\varphi_1 = \varphi$. This gives a braid σ. We may write

$$\varphi = \sigma.$$

Then by (5)

(6) $$[\Phi_{f_{t_0}} \cdot \sigma^{-1}] = [\Phi_{f'_{t'_0}}].$$

Now the braid σ in (4) and σ in (6) are the same. In fact, the braid σ in the relation

$$\theta^{'}(\delta^{'}) = \sigma\theta(\delta)\sigma^{-1}$$

is nothing but

$$\sigma = \varphi\colon t_0 \times \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C}$$

if we regard $\theta^{'}(\delta^{'})$ and $\theta(\delta)$ as elements of $\pi_0(G)$ (see Steenrod [12, p. 97–p. 98, p. 9–p. 12]). On the other hand, σ in (6) is also

$$\sigma = \varphi\colon t_0 \times \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C}.$$

Hence the braid σ in (4) and σ in (6) are the same. Thus there is $\sigma \in B_n$ such that

$$([\Phi_{f'}],\ \theta^{'}(\delta^{'})) = ([\Phi_f \cdot \sigma^{-1}],\ \sigma\theta(\delta)\sigma^{-1}).$$

Conversely, for two families in (1), we identify q_j' with q_j ($j = 0, 1, \dots, n$) through ξ in (2) and suppose that there is $\sigma \in B_n$ such that

$$([\Phi_{f'}],\ \theta^{'}(\delta^{'})) = ([\Phi_f \cdot \sigma^{-1}],\ \sigma\theta(\delta)\sigma^{-1}).$$

Since $\theta^{'}(\delta^{'}) = \sigma\theta(\delta)\sigma^{-1}$, the above discussion shows that two bundles

$$\Delta^* \times \mathbb{C} - B_f \quad \text{and} \quad \Delta^{'*} \times \mathbb{C} - B_{f'}$$

over Δ^* and $\Delta^{'*}$ respectively are weakly equivalent. That is, there are orientation preserving homeomorphism φ and η such that (i) the following diagram commutes:

$$\begin{array}{ccc} \Delta^* \times \mathbb{C} - B_f & \xrightarrow{\ \varphi\ } & \Delta^{'*} \times \mathbb{C} - B_{f'} \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\ \eta\ } & \Delta^{'*} \end{array}$$

(ii) $\eta(t_0) = t_0'$ and
(iii) $\varphi = \sigma\colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0' \times \mathbb{C} = \mathbb{C}$.

Now the fiber bundle structures on $\Delta^* \times \mathbb{C} - B_f$ and $\Delta^{'*} \times \mathbb{C} - B_{f'}$ can be naturally extended to those on $\Delta^* \times \mathbb{C}$ and $\Delta^{'*} \times \mathbb{C}$ respectively (see Lemma 2 in Matsuno [7]). Hence φ can be extended to an orientation preserving homeomorphism

$$\varphi\colon \Delta^* \times \mathbb{P}^1 \longrightarrow \Delta^{'*} \times \mathbb{P}^1$$

such that the following diagram commutes:

$$\begin{array}{ccc} \Delta^* \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta'^* \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\eta} & \Delta'^* \end{array}$$

We show that φ and η can be extended so that the following diagram commutes:

$$\begin{array}{ccc} \Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\eta} & \Delta' \end{array}$$

We assume and put as in §5

$$\begin{aligned} & q_1(0) = \cdots = q_{k_1}(0) = q_1^0, \\ & (\text{resp. } q'_1(0) = \cdots = q'_{k_1}(0) = q_1'^0), \\ & q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0, \\ & (\text{resp. } q'_{k_1+1}(0) = \cdots = q'_{k_1+k_2}(0) = q_2'^0), \\ & \qquad \cdots\cdots \\ & q_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q_{k_1+\cdots+k_{r-1}+k_r}(0) = q_r^0, \\ & (\text{resp. } q'_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q'_{k_1+\cdots+k_{r-1}+k_r}(0) = q_r'^0), \end{aligned}$$

where $k_\nu \geq 1$ $(\nu = 1, \ldots, r)$, $k_1 + \cdots + k_r = n$ and $q_1^0, \ldots, q_r^0$ (resp. $q_1'^0, \ldots, q_r'^0$) are mutually distinct.

We may assume that there is a continuous function $\rho(|t|)$ of $|t|$ such that

(i) $\rho(|t|) > 0$ for $|t| > 0$,

(ii) $\rho(0) = 0$,

(iii) $\Delta(q_\nu^0, \rho(|t|))$ (resp. $\Delta(q_\nu'^0, \rho(|t|))$) $(\nu = 1, \ldots, r)$ are mutually disjoint,

(iv) each $\Delta(q_\nu^0, \rho(|t|))$ (resp. $\Delta(q_\nu'^0, \rho(|t|))$) $(\nu = 1, \ldots, r)$ contains

$$\begin{aligned} & q_{k_1+\cdots+k_{\nu-1}+1}(t), \cdots, q_{k_1+\cdots+k_{\nu-1}+k_\nu}(t) \\ (\text{resp. } & q'_{k_1+\cdots+k_{\nu-1}+1}(t), \cdots, q'_{k_1+\cdots+k_{\nu-1}+k_\nu}(t)). \end{aligned}$$

Now Lemma 2 in Matsuno [7] implies that the bundle structure on $\Delta^* \times \mathbb{C} - B_f$ coincides with that of the product bundle $\Delta^* \times \mathbb{C}$ outside

$$T = \bigcup_{0<|t|<\epsilon} \bigcup_{\nu=1}^{r} \Delta(q_\nu^0, \rho(|t|)).$$

Similar assertion holds for the bundle structure on $\Delta'^* \times \mathbb{C} - B_{f'}$. Hence we may assume that φ does not depend on t outside T. Thus φ can be extended to an orientation preserving homeomorphism

$$\varphi\colon \Delta \times \mathbb{C} - \{q_1^0, \ldots, q_r^0\} \longrightarrow \Delta' \times \mathbb{C} - \{q_1'^0, \ldots, q_r'^0\}.$$

Moreover if we define

$$\varphi(q_\nu^0) = q_\nu'^0 \quad (\nu = 1, \ldots, r),$$

then φ is extended to an orientation preserving homeomorphism

$$\varphi\colon \Delta \times \mathbb{P}^1 \longrightarrow \Delta' \times \mathbb{P}^1.$$

Put also $\eta(0) = 0$. Then η is extended to an orientation preserving homeomorphism

$$\eta\colon \Delta \longrightarrow \Delta'$$

and the following diagram commutes:

$$\begin{array}{ccc} \Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\eta} & \Delta' \end{array}$$

Next, note that

$$\varphi(B_f) = B_{f'},$$
$$\varphi = \sigma\colon t_0 \times \mathbb{C} \longrightarrow t_0' \times \mathbb{C}.$$

Note also that

$$\varphi \cdot f\colon X \longrightarrow \Delta' \times \mathbb{P}^1$$

is unbranched on $\Delta' \times \mathbb{P}^1 - B_{f'}$. By Theorem 1, $\varphi \cdot f$ can be extended to a branched covering

$$f''\colon X'' \longrightarrow \Delta' \times \mathbb{P}^1.$$

$\varphi \cdot f$ and f'' coincides on $\Delta' \times \mathbb{C} - B_{f'}$ and both are Fox completions of the same unbranched coverings of $\Delta' \times \mathbb{C} - B_{f'}$. Hence by the uniqueness of the Fox completion (see Fox [3]), there is a homeomorphism

$$\psi'\colon X \longrightarrow X''$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi'} & X'' \\ {\scriptstyle \varphi\cdot f}\downarrow & & \downarrow{\scriptstyle f''} \\ \Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1 \end{array}$$

Note that ψ' is orientation preserving. Now, the representaion class of the monodromy of f'' is equal to that of $\varphi \cdot f$, which is clealy equal to $[\Phi_f \cdot \varphi_*^{-1}]$. By the assumption

$$[\Phi_f \cdot \varphi_*^{-1}] = [\Phi_f \cdot \sigma^{-1}] = [\Phi_{f'}],$$

we have

$$[\Phi_{f''}] = [\Phi_{f'}].$$

Hence there is a biholomorphic mapping

$$\psi'' : X'' \longrightarrow X'$$

which makes the following diagram commutative:

$$\begin{array}{ccc} X'' & \xrightarrow{\psi''} & X' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} \\ \Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1 \end{array}$$

Now put $\psi = \psi'' \cdot \psi'$. Then

$$\psi : X \longrightarrow X'$$

is an orientation preserving homeomorphism which makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \end{array}$$

Hence $f = \{f_t\}$ and $f' = \{f_t'\}$ are topologically equivalent. □

Considering a trivial family, we get the following corollary, which can be also derived directly from Theorem 2.

Corollary 2 (cf. Wajnryb [13]). *There exists a one to one correspondence between* $\{$*topological equivalence class of* $f: X \longrightarrow \mathbb{P}^1$ *of degree* d *with* n *branched points* $q_1, \ldots, q_n\}$ *and* $\{[\Phi] \mid [\Phi]$ *is the representation class of* $\Phi: \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \longrightarrow S_d$ *such that* $\operatorname{Im}\Phi$ *is transitive*$\}/B_n$.

REMARK. As one can see in the proof of Theorem 8, we do not need to mention the points $\{q_1, \ldots, q_n\}$ in the statement of Theorem 8 and its corollary, if we replace $\pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0)$ by the abstract group

$$\langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle .$$

7. Degenerating families of finite branched coverings of $\mathbb{P}^m$

Let $\Delta = \Delta(0, \epsilon)$ be a disc and

$$f: X \longrightarrow \Delta \times \mathbb{P}^m$$

be a finite branched covering. As in the case of $\mathbb{P}^1$, f is called a degenerating family of finite branched covering of $\mathbb{P}^m$ and is denoted by $f = \{f_t\}$ if the following 4 conditions are satisfied

(1) $t \times \mathbb{P}^m \not\subset B_f$ for every $t \in \Delta$.

(2) For every $t \in \Delta^*$, $t \times \mathbb{P}^m$ meets transversally with B_f and putting $(t \times \mathbb{P}^m) \cap B_f = t \times B_t$, B_t is a hypersurface of $\mathbb{P}^m$ of degree n. (n is constant for $t \in \Delta^*$.)

(3) For every $t \in \Delta^*$,

$$f_t = f: X_t = f^{-1}(t \times \mathbb{P}^m) \longrightarrow t \times \mathbb{P}^m$$

is a covering of $\mathbb{P}^m$ of degree $d = \deg(f)$ branching at B_t.

(4) For any points t and t' in Δ^*, f_t and $f_{t'}$ are topologically equivalent.

The central fiber $X_0 = f^{-1}(0 \times \mathbb{P}^m)$ is a degeneration of a general fiber X_t for $t \neq 0$.

We show that, topologically, the central fiber X_0 can be described by the central branch divisor B_0, where $0 \times B_0 = (0 \times \mathbb{P}^m) \cap B_f$ and by the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point. We explain this as follows:

Let L be a general line in $\mathbb{P}^m$. We may assume that L meets transversally with every B_t for $t \in \Delta$. Consider the restriction

$$f^L: X^L = f^{-1}(\Delta \times L) \longrightarrow \Delta \times L$$

of f to $X^L = f^{-1}(\Delta \times L)$. Then

Lemma 2. (1) *Every point of* $(X - f^{-1}(B_f)) \cap X^L$ *is a non-singular point of* X^L.

(2) *For* $t \neq 0$, *every point of* $f^{-1}(\mathrm{Reg}(B_f) \cap (t \times L))$ *is non-singular point of* X^L. ($\mathrm{Reg}(B_f)$ *is the set of non-singular points of the branch locus* B_f.)
(3) *For* $t \neq 0$, *the restriction*

$$f_t^L : X_t^L = f^{-1}(t \times L) \longrightarrow t \times L$$

of f^L *is a branched covering of degree* $d = \deg(f)$

Proof. (1) Let $p \in (X - f^{-1}(B_f)) \cap X^L$. Then there are local coordinate systems $(t, x_1, \ldots, x_m)$ and $(t, y_1, \ldots, y_m)$ around p in X and $q = f(p)$ in $\Delta \times \mathbb{P}^m$ such that (i) t is a local coordinate system in Δ and $(y_1, \ldots, y_m)$ is that in $\mathbb{P}^m$, (ii) L is locally given by the equation $y_2 = \cdots = y_m = 0$ and (iii) f is locally given by

$$f : (t, x_1, \ldots, x_m) \longmapsto (t, y_1, \ldots, y_m) = (t, x_1, \ldots, x_m).$$

Then f^L is locally given by

$$f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1).$$

In particular, p is a non-singular point of X^L.
(2) Let $t_0 \neq 0$ and $p \in f^{-1}(\mathrm{Reg}(B_f) \cap (t_0 \times L))$. Then there are local coordinate systems $(t, x_1, \ldots, x_m)$ and $(t, y_1, \ldots, y_m)$ around p in X and $q = f(p)$ in $\Delta \times \mathbb{P}^m$ such that (i) t is a local coordinate system in Δ around t_0 and $(y_1, \ldots, y_m)$ is that in $\mathbb{P}^m$, (ii) L is locally given by the equation $y_2 = \cdots = y_m = 0$ and (iii) f is locally given by

$$f : (t, x_1, x_2, \ldots, x_m) \longmapsto (t, y_1, y_2, \ldots, y_m) = (t, x_1^e, x_2, \ldots, x_m).$$

Then f^L is locally given by

$$f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1^e).$$

In particular, p is a non-singular point of X^L. Moreover p is a ramification point of $f_{t_0}^L$ with the ramification index e.
(3) For $t \neq 0$, the branched covering

$$f_t : X_t \longrightarrow \mathbb{P}^m$$

gives a linear system on X_t. By Bertini's theorem, X_t^L is non-singular and globally irreducible. Hence, by the proof of (2),

$$f_t^L : X_t^L \longrightarrow t \times L$$

is a branched covering of degree $d = \deg(f)$. □

This lemma shows that the singular locus $\mathrm{Sing}(X^L)$ of X^L is contained in $f^{-1}(0 \times (B_0 \cap L))$, which is a finite set. X^L is globally irreducible. Let

$$\mu\colon \tilde{X}^L \longrightarrow X^L$$

be the normalization of X^L. Since $\mathrm{Sing}(X^L)$ is a finite set, μ is a bijective holomorphic mapping. In fact, suppose that there are distinct points p_1 and p_2 in $\tilde{X}^L$ such that

$$p = \mu(p_1) = \mu(p_2) \in f^{-1}(0 \times (B_0 \cap L)).$$

Then there are disjoint connected open neighborhoods W_1 and W_2 of p_1 and p_2 respectively such that

$$\mu(W_1) = \mu(W_2) = W$$

and W is connected open neighborhood of p in X^L. Since $f^{-1}(0 \times (B_0 \cap L))$ is a finite set, we may assume that

$$f^{-1}(0 \times (B_0 \cap L)) \cap W = \{ p \}.$$

We may assume that $X_t^L \cap W$ is connected for non-zero t with $|t|$ sufficiently small. Hence $W - \{ p \}$ is a connected 2-dimensional complex manifold. Since μ is the normalization of X^L,

$$W_1 - \{ p_1 \} = W_2 - \{ p_2 \}$$

and

$$\mu\colon W_1 - \{ p_1 \} = W_2 - \{ p_2 \} \longrightarrow W - \{ p \}$$

is biholomorphic, a contradiction. Thus μ is bijective. The composition

$$f^L \cdot \mu\colon \tilde{X}^L \longrightarrow \Delta \times L$$

is a degenerating family of finite branched coverings of $L = \mathbb{P}^1$, which we denote

$$f^L \cdot \mu = \{f_t^L\}$$

by abuse of notation.

Lemma 3. (1) *Let $X_0 = X_{01} \cup \cdots \cup X_{0u}$ be the global irreducible decomposition of X_0. Then*

$$X_0^L = (X_{01} \cap X_0^L) \cup \cdots \cup (X_{0u} \cap X_0^L)$$

is the global irreducible decomposition of X_0^L.
(2) *Let* $\mathrm{Sing}^{m-1}(X_0)$ *be the union of global irreducible components of* $\mathrm{Sing}(X_0)$ *which are hypersurfaces of* X_0. *Then* (i) $\mathrm{Sing}^{m-1}(X_0) \subset f_0^{-1}(B_0)$ *and* (ii) $(\mathrm{Sing}^{m-1}(X_0)) \cap X_0^L = \mathrm{Sing}(X_0^L)$.
(3) *For a point* $p \in \mathrm{Sing}(X_0^L)$, *let*

$$(X_0)_p = Z_1 \cup \cdots \cup Z_v$$

be the local irreducible decompoition of X_0 *at* p. *Then the local irreducible decomposition of* X_0^L *at* p *is given by*

$$(X_0^L)_p = (Z_1 \cap X_0^L) \cup \cdots \cup (Z_v \cap X_0^L).$$

Proof. (1) Let

$$\mu_j : \hat{X}_{0j} \longrightarrow X_{0j}$$

$(1 \leq j \leq u)$ be the normalization of X_{0j}. By the proof of (1) of Lemma 2,

$$f_{0j} \cdot \mu_j : \hat{X}_{0j} \longrightarrow 0 \times \mathbb{P}^m \quad (f_{0j} = f_0 \mid X_{0j})$$

is a finite branched covering. By Bertini's theorem, $(f_{0j} \cdot \mu_j)^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{X}_{0j}$. Hence $f_{0j}^{-1}(0 \times L)$ is a global irreducible component of X_0^L and

$$X_0^L = f_0^{-1}(0 \times L) = \bigcup_{j=1}^{u} f_{0j}^{-1}(0 \times L) = \bigcup_{j=1}^{u} (X_{0j} \cap X_0^L)$$

is the irreducible decomposition of X_0^L.
(2) By (2) of Lemma 2, every component of $\mathrm{Sing}^{m-1}(X_0)$ is a global irreducible component R_0 of $f_0^{-1}(B_{01})$, where B_{01} is a global irreducible component of B_0. Let p be a point of $X_0^L \cap R_0$. Then p is clearly a singular point of X_0^L. Conversely, if p is a singular point of X_0^L, then $f_0(p) = q$ is on $L \cap B_{01}$ for an irreducible component B_{01} of B_0. Since L is a general line, every point on a global irreducible component R_0 with $p \in R_0$ of $f_0^{-1}(B_{01})$ is a singular point of X_0. Hence R_0 is a component of $\mathrm{Sing}^{m-1}(X_0)$. This shows (i) and (ii) of (2).
(3) We use the same notation as in the proof of (1). Every Z_k is an open set of some X_{0j}. Hence

$$\mu_{jk} : \hat{Z}_k = \mu_j^{-1}(Z_k) \longrightarrow Z_k \quad (\mu_{jk} = \mu_j \mid \hat{Z}_k)$$

is the normalization of Z_k. $(f_{0jk} \cdot \mu_{jk})^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{Z}_k$, where $f_{0jk} = f_{0j} \mid Z_k$. Hence $f_{0jk}^{-1}(0 \times L) = Z_k \cap X_0^L$ is a local irreducible component of X_0^L at p and

$$(X_0^L)_p = (Z_1 \cap X_0^L) \cup \cdots \cup (Z_v \cap X_0^L)$$

is the local irreducible decomposition of X_0^L. at p. □

Now we refer a theorem of Zariski-van Kampen. Let B be a hypersurface of degree n in $\mathbb{P}^m$. Take a general point q_0 in $\mathbb{P}^m - B$ and let

$$\pi\colon \mathbb{P}^m - \{q_0\} \longrightarrow \mathbb{P}^{m-1}$$

be the projection with the center q_0. Put

$$\hat{\pi} = \pi \mid B\colon B \longrightarrow \mathbb{P}^{m-1}$$

be the restriction. Let D be the branch locus of $\hat{\pi}$. A theorem of Zariski-van Kampen in this case can be described as follows (cf. Matsuno [7]).

Theorem 9 (Zariski-van Kampen).

$$\begin{aligned}&\pi_1(\mathbb{P}^m - B, q_0)\\ &\quad = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1,\ \theta(\delta_k)\gamma_j = \gamma_j\ (1 \le j \le n,\ 1 \le k \le s) \rangle,\end{aligned}$$

where γ_j are lassos as in Fig. 2 *on $\pi^{-1}(q_0)$, the line deleted the point $\{q_0\}$, δ_k are the generators of $\pi_1(\mathbb{P}^{m-1} - D, r_0)$ for a reference point $r_0 \in \mathbb{P}^{m-1} - D$, and $\theta(\delta_k)$ are the braid monodromy along δ_k.*

This theorem shows in particular that the monodromy Φ_{f_t} of f_t is equal to the monodromy $\Phi_{f_t}^L$ of f_t^L for a general line L passing through q_0. Hence, we conclude by Theorems 4, 5, 6 and Lemma 3 that topologically, the central fiber X_0 can be determined by the central branched divisor B_0 and by the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point.

REMARK. If $\deg B_0 = \deg B_t$ $(t \neq 0)$, then there is a surjective homomorphism

$$\pi_1(\mathbb{P}^m - B_0, q_0) \longrightarrow \pi_1(\mathbb{P}^m - B_t, q_0) \longrightarrow 0$$

(see Zariski [14]). In this case, X_0 is irreducible and

$$\dim \operatorname{Sing}(X_0) \le m - 2.$$

Hence degenerations such that

$$\dim \operatorname{Sing}(X_0) = m - 1$$

happen only if $\deg B_0 < \deg B_t$ $(t \neq 0)$, that is, only if B_0 has a multiple component as a divisor.

References

[1] J.S. Birman: Braids, Links, and Mapping Class Group, Ann. Math. Studies, **82**, Princeton, 1974.

[2] A. Dimca: Singurarities and Topology of Hypersurfaces, Berlin-Heidelberg-New York, Springer-Verlag, 1992.

[3] R.H. Fox: *Covering spaces with singularities*, Lefschetz Symposium, Princeton Univ. Press (1957), 243–262.

[4] H. Grauert and R. Remmert: *Komplexe Räume*, Math. Ann. **136** (1958), 245–318.

[5] F. Klein: Gesammelte Mathematische Abhandlungen, Band III, Berlin-Heidelberg-New York, Springer-Verlag, 1973.

[6] Y. Matsumoto, J.M. Montesinos-Amilibia: *Pseudoperiodic homeomorphisms and degenerations of Riemann surfaces*, Bull. AMS, **30** (1994), 70–75.

[7] T. Matsuno: *On a theorem of Zariski-Van Kampen type and its applications*, Osaka J. Math. **32** (1995), 645–658.

[8] M. Namba: Families of Meromorphic Functions on Compact Riemann Surfaces, Lec. Notes in Math. **767**, Springer, 1979.

[9] M. Namba: Branched Coverings and Algebraic Functions, Pitman Research Note in Math. **161**, Longman, 1987.

[10] M. Namba: *Degenerationg families of meromorphic functions*, Proc. Internat. Conf. "Geometry and Analysis in Several Complex Variables", Kyoto Univ., (1997) RIMS Kokyuroku **1058** (1998), 77–94.

[11] L. Schneps (editor): The Grothendieck Theory of Dessins d'Enfants, Lec. Notes **200**, London Math. Soc., Cambridge Univ. Press, 1994.

[12] N. Steenrod: The Topology of Fibre Bundles, Princeton Univ. Press, 1951.

[13] B. Wajnryb: *Orbits of Hurwitz action for coverings of a sphere with two special fibers*, Indag. Mathem., N.S. **7** (1996), 549–558.

[14] O. Zariski: Collected papers, Vol. III, 1978.

Makoto Namba and Maki Takai
Department of Mathematics
Osaka University
Toyonaka City, 560-0043, Japan
e-mail: namba@math.wani.osaka-u.ac.jp;
takai@gaia.math.wani.osaka-u.ac.jp