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DEFORMABLE FLAT TORI IN S^3 WITH CONSTANT MEAN CURVATURE

To the memory of Professor Shukichi Tanno

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1. Introduction

In 1975, Yau [9, p. 87] posed the problem of the classification of flat tori in the unit 3-sphere S^3. Concerning this problem, the author established a method for constructing all the flat tori in S^3 ([2]), and obtained some results on flat tori in S^3 ([1], [3], [4], [5]). In this paper, using this method, we study isometric deformations of flat tori isometrically immersed in S^3 with constant mean curvature, and we obtain the classification of undeformable flat tori in S^3.

For positive constants R_1 and R_2 satisfying $R_1^2 + R_2^2 = 1$, let $F\colon \mathbb{R}^2 \to S^3$ be an isometric immersion given by

$$F(x_1, x_2) = \left(R_1 \cos\frac{x_1}{R_1}, R_1 \sin\frac{x_1}{R_1}, R_2 \cos\frac{x_2}{R_2}, R_2 \sin\frac{x_2}{R_2}\right), \tag{1.1}$$

and G_0 a lattice of $\mathbb{R}^2$ defined by

$$G_0 = \{(2\pi R_1 n_1, 2\pi R_2 n_2) : n_1, n_2 \in \mathbb{Z}\}. \tag{1.2}$$

If G is a lattice of $\mathbb{R}^2$ such that $G \subset G_0$, then we obtain a flat torus $\mathbb{R}^2/G$ and an isometric immersion

$$F/G\colon \mathbb{R}^2/G \to S^3 \tag{1.3}$$

with constant mean curvature. Conversely, every flat torus isometrically immersed in S^3 with constant mean curvature is obtained in this way. Note that the immersion F/G is the composition of the covering map $\mathbb{R}^2/G \to \mathbb{R}^2/G_0$ and the embedding $F/G_0\colon \mathbb{R}^2/G_0 \to S^3$. In [3] the author studied isometric deformations of F/G_0, and proved the following theorem.

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Theorem 1.1. *If* $f_t\colon \mathbb{R}^2/G_0 \to S^3$, $t \in \mathbb{R}$, *is a smooth one parameter family of isometric immersions with* $f_0 = F/G_0$, *then for each* $t \in \mathbb{R}$ *there exists an isometry* $A_t\colon S^3 \to S^3$ *such that* $f_t = A_t \circ f_0$.

This theorem says that every isometric deformation of F/G_0 is trivial. On the other hand, there are many lattices $G \subset G_0$ such that the immersion F/G is deformable. Let $W_+(n)$ and $W_-(n)$ be lattices of $\mathbb{R}^2$ defined by

$$W_\pm(n) = \{(2\pi R_1 n_1, 2\pi R_2 n_2) : n_1 \pm n_2 \in n\mathbb{Z}\} \subset G_0. \tag{1.4}$$

Then we can show that if $G \subset W_+(n)$ or $G \subset W_-(n)$ for some integer $n \geq 2$, the immersion F/G is deformable (Theorem 3.1). Here, we give the following definition.

DEFINITION. For immersions $f_1\colon M_1 \to S^3$ and $f_2\colon M_2 \to S^3$, we write $f_1 \equiv f_2$, and we say "f_1 is congruent to f_2" if there exist an isometry $A\colon S^3 \to S^3$ and a diffeomorphism $\rho\colon M_1 \to M_2$ such that $A \circ f_1 = f_2 \circ \rho$. An isometric immersion $f\colon M \to S^3$ is said to be *deformable* if it admits an isometric deformation $f_t\colon M \to S^3$ ($t \in \mathbb{R}$, $f_0 = f$) such that $f_0 \not\equiv f_1$.

The assertion of Theorem 3.1 leads us to the problem of finding all the lattices $G \subset G_0$ such that the immersion F/G is deformable. In this paper we study this problem, and prove the following theorem.

Theorem 1.2. *Let* G *be a lattice of* $\mathbb{R}^2$ *such that* $G \subset G_0$. *Then the immersion* F/G *is deformable if and only if there exists an integer* $n \geq 2$ *such that* $G \subset W_+(n)$ *or* $G \subset W_-(n)$.

Furthermore, as a corollary of this theorem, we obtain the following classification of undeformable flat tori isometrically immersed in S^3.

Theorem 1.3. *Let* $f\colon M \to S^3$ *be an isometric immersion of a flat torus* M *into the unit sphere* S^3. *Then the following statements are equivalent.*
(1) *Every isometric deformation of* f *is trivial.*
(2) *There exist positive constants* R_1 *and* R_2 *with* $R_1^2 + R_2^2 = 1$ *such that* f *is congruent to the immersion* F/G *given by* (1.3), *where the lattice* G *satisfies* $G \not\subset W_+(n)$ *and* $G \not\subset W_-(n)$ *for all integers* $n \geq 2$.

REMARK. Let G be a lattice of $\mathbb{R}^2$ generated by the following two vectors

$$u = (2\pi R_1 u_1, 2\pi R_2 u_2),\ v = (2\pi R_1 v_1, 2\pi R_2 v_2), \quad u_i, v_i \in \mathbb{Z}.$$

Then it is easy to see that the following statements are equivalent.

(1) $G \not\subset W_+(n)$ and $G \not\subset W_-(n)$ for all integers $n \geq 2$.
(2) $g.c.d(u_1 + u_2, v_1 + v_2) = g.c.d(u_1 - u_2, v_1 - v_2) = 1$.

The outline of this paper is as follows. In Section 2 we study the geometry of a flat torus $M_\gamma \subset S^3$ which is the inverse image under the Hopf fibration $S^3 \to S^2$ of a closed curve γ in S^2. In Section 3 we show that if γ is an n-fold circle in S^2 $(n \geq 2)$, then the flat torus $M_\gamma \subset S^3$ is deformable (Lemma 3.2). Using this lemma, we obtain Theorem 3.1. In Section 4 we prove Theorems 1.2 and 1.3. The key ingredient in the proof of Theorem 1.2 is Lemma 4.1 which is obtained by using a method developed in [2]. The assertion of Theorem 1.3 follows from the main result of [5] which states that every flat torus isometrically immersed in S^3 with nonconstant mean curvature is deformable. In the final section we prove Theorem 5.1. This theorem, which is used in the proof of Lemma 3.2 , ensures the existence of certain deformation of an n-fold circle in S^2 for $n \geq 2$.

2. Hopf tori in S^3

In this section we study the geometry of a flat torus in S^3 constructed by using the Hopf fibration $S^3 \to S^2$. We start with the description of the Hopf fibration by using the group structure of S^3. Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of all 2×2 skew Hermitian matrices of trace 0. We define a positive definite inner product $\langle \ , \ \rangle$ on $\mathfrak{su}(2)$ by

$$\langle u, v \rangle = -\frac{1}{2}\,\mathrm{trace}(uv), \quad u, v \in \mathfrak{su}(2).$$

The inner product $\langle \ , \ \rangle$ is invariant under the adjoint action $\mathrm{Ad}\colon SU(2) \to \mathrm{Aut}(\mathfrak{su}(2))$. We set

$$e_1 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

Then $\{e_1, \ e_2, \ e_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$ such that

$$[e_1, e_2] = 2e_3, \ [e_2, e_3] = 2e_1, \ [e_3, e_1] = 2e_2,$$

where $[\ ,\ ]$ is the Lie bracket on $\mathfrak{su}(2)$. For $i = 1, 2, 3$, we denote by E_i the left invariant vector field on $SU(2)$ corresponding to e_i, and we endow $SU(2)$ with a bi-invariant Riemannian metric $\langle \ , \ \rangle$ satisfying $\langle E_i, E_j \rangle = \delta_{ij}$. Then $SU(2)$ is a Riemannian manifold isometric to the unit sphere S^3. Henceforth, we identify S^3 with $SU(2)$. Let S^2 be the unit sphere in $\mathfrak{su}(2)$ given by $S^2 = \{u \in \mathfrak{su}(2) : |u| = 1\}$, and $p\colon S^3 \to S^2$ the Hopf fibration given by

$$p(a) = \mathrm{Ad}(a)e_3.$$

The vector field E_3 is tangent to the fibers of the Hopf fibration. For $X, Y \in T_a S^3$, it follows that

$$\langle p_* X, p_* Y\rangle = 4\{\langle X, Y\rangle - \langle X, E_3\rangle \langle Y, E_3\rangle\}, \tag{2.1}$$

$$p_*(D_X E_3) = -J(p_* X), \tag{2.2}$$

where D denotes the Riemannian connection on S^3, and J denotes the almost complex structure on S^2 defined by $J(v) = [u, v]/2$ for $v \in T_u S^2$. We identify the unit tangent bundle of S^2 with the subset $US^2 \subset S^2 \times S^2$ defined by

$$US^2 = \{(u, v) \in S^2 \times S^2 : \langle u, v\rangle = 0\}.$$

Here, the canonical projection $p_1 \colon US^2 \to S^2$ is given by $p_1(u, v) = u$. Furthermore, we define a double covering $p_2 \colon S^3 \to US^2$ by

$$p_2(a) = (\mathrm{Ad}(a)e_3, \mathrm{Ad}(a)e_1).$$

Let $\gamma \colon \mathbb{R} \to S^2$ be a 2π-periodic regular curve in S^2. Using the Hopf fibration $p \colon S^3 \to S^2$, we construct a 2-dimensional torus M_γ and an immersion $f_\gamma \colon M_\gamma \to S^3$ by

$$M_\gamma = \{(e^{is}, a) \in S^1 \times S^3 : \gamma(s) = p(a)\}, \quad f_\gamma(e^{is}, a) = a,$$

where S^1 denotes the unit circle in $\mathbb{C}$. The immersion f_γ induces a flat Riemannian metric on M_γ (see [8]). So we obtain a flat torus M_γ and an isometric immersion

$$f_\gamma \colon M_\gamma \to S^3.$$

The immersion f_γ is called the *Hopf torus* corresponding to γ.

In the rest of this section we describe the Riemannian structure of M_γ and the second fundamental form of f_γ. Let $L(\gamma)$ be the length of γ and $K(\gamma)$ the total geodesic curvature of γ, that is,

$$L(\gamma) = \int_0^{2\pi} |\gamma'(s)|\, ds, \quad K(\gamma) = \int_0^{2\pi} k_\gamma(s)|\gamma'(s)|\, ds,$$

where $k_\gamma(s)$ denotes the geodesic curvature of $\gamma(s)$ given by

$$k_\gamma(s) = \frac{\langle \gamma''(s), J(\gamma'(s))\rangle}{|\gamma'(s)|^3}.$$

We now consider the curve $\hat{\gamma} \colon \mathbb{R} \to US^2$ given by

$$\hat{\gamma}(s) = \left(\gamma(s), \frac{\gamma'(s)}{|\gamma'(s)|}\right), \tag{2.3}$$

and denote by $I(\gamma)$ the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma} \mid [0, 2\pi]$. Note that $H_1(US^2) \cong \mathbb{Z}_2$. Let $c(s)$ be a lift of the curve $\hat{\gamma}(s)$ with respect to the double covering $p_2 \colon S^3 \to US^2$. Since $p_2(-a) = p_2(a)$, we obtain

$$c(s + 2\pi) = \begin{cases} c(s) & \text{if} \quad I(\gamma) = 0, \\ -c(s) & \text{if} \quad I(\gamma) = 1. \end{cases} \tag{2.4}$$

We set

$$\Omega(\gamma) = \begin{cases} K(\gamma) & \text{if} \quad I(\gamma) = 0, \\ K(\gamma) + 2\pi & \text{if} \quad I(\gamma) = 1, \end{cases} \tag{2.5}$$

and define $W(\gamma)$ to be the lattice of $\mathbb{R}^2$ generated by the following two vectors

$$v_1 = \left(\frac{L(\gamma)}{2}, \ \frac{\Omega(\gamma)}{2} \right), \qquad v_2 = (0, 2\pi). \tag{2.6}$$

Then the Riemannian structure of M_γ is given by the following

Lemma 2.1. *The flat torus M_γ is isometric to $\mathbb{R}^2/W(\gamma)$.*

To establish the lemma we consider the covering $\varphi \colon \mathbb{R}^2 \to M_\gamma$ defined by

$$\varphi(s, \tau) = (e^{is}, \ \overline{\gamma}(s) \exp(\tau e_3)), \tag{2.7}$$

where $\overline{\gamma}(s)$ is a curve in S^3 such that $p(\overline{\gamma}(s)) = \gamma(s)$ and $\langle \overline{\gamma}'(s), E_3 \rangle = 0$. Then it follows from (2.1) that $|\gamma'(s)| = |p_* \overline{\gamma}'(s)| = 2|\overline{\gamma}'(s)|$. So we obtain

$$\varphi^* g_\gamma = \frac{1}{4} |\gamma'(s)|^2 \, ds^2 + d\tau^2, \tag{2.8}$$

where g_γ denotes the Riemannian metric on M_γ. Let $\rho \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism given by

$$\rho(s, \tau) = \left(\frac{1}{2} \int_0^s |\gamma'(s)| \, ds, \ \tau \right),$$

and $\Phi \colon \mathbb{R}^2 \to M_\gamma$ a covering map defined by

$$\Phi(x_1, x_2) = \varphi(\rho^{-1}(x_1, x_2)). \tag{2.9}$$

Since $\rho^*(dx_1^2 + dx_2^2) = \varphi^* g_\gamma$, the map Φ is a Riemannian covering, and so the assertion of Lemma 2.1 follows from the lemma below.

Lemma 2.2. *For $x, x' \in \mathbb{R}^2$, $\Phi(x) = \Phi(x')$ if and only if $x' - x \in W(\gamma)$.*

Proof. We set $x = \rho(s,\tau)$ and $x' = \rho(s',\tau')$. Then we obtain

$$\Phi(x) = (e^{is},\ \bar{\gamma}(s)\exp(\tau e_3)), \quad \Phi(x') = (e^{is'},\ \bar{\gamma}(s')\exp(\tau' e_3)). \tag{2.10}$$

Since $p(c(s)) = p(\bar{\gamma}(s))$, there exists a real valued function $\mu(s)$ such that

$$c(s) = \bar{\gamma}(s)\exp(\mu(s)e_3). \tag{2.11}$$

On the other hand, it follows from [4, Lemma2.2] that the curve $c(s)$ satisfies

$$c(s)^{-1}c'(s) = \frac{1}{2}|\gamma'(s)|(e_2 + k_\gamma(s)e_3). \tag{2.12}$$

Since $\left\langle \bar{\gamma}^{-1}(s)\bar{\gamma}'(s), e_3 \right\rangle = 0$, (2.11) and (2.12) imply $\mu'(s) = (1/2)k_\gamma(s)|\gamma'(s)|$. Hence

$$\mu(s+2\pi) - \mu(s) = \int_0^{2\pi} \frac{1}{2}k_\gamma(s)|\gamma'(s)|\,ds = \frac{1}{2}K(\gamma). \tag{2.13}$$

Using (2.4), (2.5), (2.11) and (2.13), we obtain $\bar{\gamma}(s+2\pi) = \bar{\gamma}(s)\exp\{-(1/2)\Omega(\gamma)e_3\}$. So it follows from (2.10) that $\Phi(x) = \Phi(x')$ if and only if there exist m_1, $m_2 \in \mathbb{Z}$ satisfying

$$s' - s = 2m_1\pi, \quad \tau' - \tau = \frac{m_1}{2}\Omega(\gamma) + 2m_2\pi. \tag{2.14}$$

Since $x' - x = \{(1/2)\int_s^{s'} |\gamma'(s)|\,ds,\ \tau' - \tau\}$, we see that (2.14) is equivalent to

$$x' - x = \left(\frac{m_1}{2}L(\gamma),\ \frac{m_1}{2}\Omega(\gamma) + 2m_2\pi\right).$$

This completes the proof. $\square$

We now deal with the second fundamental form of the immersion $f_\gamma\colon M_\gamma \to S^3$. Let ξ be a unit normal vector field of f_γ such that

$$p_*\xi(e^{is}, a) = 2n(s), \quad (e^{is}, a) \in M_\gamma,$$

where $n(s) = J(\gamma'(s))/|\gamma'(s)|$, and let h_γ denote the second fundamental form of the immersion f_γ with respect to ξ. Then

Lemma 2.3. $\varphi^* h_\gamma = (1/2)k_\gamma(s)|\gamma'(s)|^2\,ds^2 - |\gamma'(s)|\,ds d\tau$.

Proof. We set

$$f = f_\gamma \circ \varphi, \quad X = \frac{\partial f}{\partial s}, \quad Y = \frac{\partial f}{\partial \tau}.$$

Since $\langle \overline{\gamma}', E_3 \rangle = 0$, it follows from [2, Lemma 3.3] that

$$p_*(D_X X) = p_*(D_{\overline{\gamma}'} \overline{\gamma}') = \nabla_{\gamma'} \gamma',$$

where ∇ denotes the Riemannian connection on S^2. Hence (2.1) implies

$$\begin{aligned} h_\gamma\left(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s}\right) &= \langle D_X X, \xi(\varphi)\rangle = \frac{1}{4}\langle p_*(D_X X), p_*\xi(\varphi)\rangle \\ &= \frac{1}{4}\langle \nabla_{\gamma'}\gamma', 2n\rangle = \frac{1}{2}\langle \gamma'', n\rangle = \frac{1}{2}k_\gamma|\gamma'|^2. \end{aligned} \tag{2.15}$$

Since $Y(s,\tau) = E_3(f(s,\tau))$, it follows from (2.2) that $p_*(D_X Y) = p_*(D_X E_3) = -J(p_* X) = -J(\gamma')$. Hence

$$\begin{aligned} h_\gamma\left(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial \tau}\right) &= \langle D_X Y, \xi(\varphi)\rangle = \frac{1}{4}\langle p_*(D_X Y), p_*\xi(\varphi)\rangle \\ &= \frac{1}{4}\langle -J(\gamma'), 2n\rangle = -\frac{1}{2}|\gamma'|. \end{aligned} \tag{2.16}$$

Since the integral curves of the vector field E_3 are geodesics in S^3, we see that $D_Y Y = 0$. Hence

$$h_\gamma\left(\frac{\partial\varphi}{\partial \tau}, \frac{\partial\varphi}{\partial \tau}\right) = \langle D_Y Y, \xi(\varphi)\rangle = 0. \tag{2.17}$$

The assertion of Lemma 2.3 follows from (2.15)–(2.17). □

Using (2.8) and Lemma 2.3, we obtain

$$|H_\gamma(\varphi(s,\tau))| = |k_\gamma(s)|, \tag{2.18}$$

where H_γ denotes the mean curvature vector field of the immersion f_γ.

3. Isometric deformations of F/G

Let $W_\pm(n)$ denote the lattices of $\mathbb{R}^2$ defined by (1.4). In this section we show the following theorem.

Theorem 3.1. *Let G be a lattice of $\mathbb{R}^2$ such that $G \subset G_0$. If $G \subset W_+(n)$ or $G \subset W_-(n)$ for some integer $n \geq 2$, the isometric immersion F/G given by (1.3) is deformable.*

To establish the theorem above we need some lemmas. For each integer $n \geq 1$, let $\gamma\colon \mathbb{R} \to S^2$ be a 2π-periodic regular curve defined by

$$\gamma(s) = (\cos\theta\cos ns)e_1 + (\cos\theta\sin ns)e_2 + (\sin\theta)e_3, \tag{3.1}$$

where θ is a constant such that

$$\frac{R_2^2 - R_1^2}{2R_1R_2} = \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{3.2}$$

Note that the geodesic curvature of γ satisfies

$$k_\gamma(s) = \tan\theta. \tag{3.3}$$

We now consider the Hopf torus $f_\gamma\colon M_\gamma \to S^3$ corresponding to γ. Then

Lemma 3.2. *The immersion f_γ is deformable for $n \geq 2$.*

Proof. Since $n \geq 2$, it follows that there exists a smooth one parameter family of 2π-periodic regular curves $\gamma_t\colon \mathbb{R} \to S^2$, $t \in \mathbb{R}$, such that $\gamma_0 = \gamma$ and

$$L(\gamma_t) = L(\gamma), \quad K(\gamma_t) = K(\gamma), \tag{3.4}$$

$$k_{\gamma_t}(0) \neq \tan\theta \quad \text{for all} \quad t \neq 0. \tag{3.5}$$

The existence of γ_t as above will be established in the final section (Theorem 5.1). Let $\overline{\gamma}_t\colon \mathbb{R} \to S^3$, $t \in \mathbb{R}$ be a smooth one parameter family of curves in S^3 such that $p(\overline{\gamma}_t(s)) = \gamma_t(s)$ and $\langle \overline{\gamma}_t'(s), E_3 \rangle = 0$, and $\Phi_t\colon \mathbb{R}^2 \to M_{\gamma_t}$ the Riemannian covering map defined in the same way as (2.9). Then, by Lemma 2.2, Φ_t induces the isometry

$$\tilde{\Phi}_t\colon \mathbb{R}^2/W(\gamma_t) \to M_{\gamma_t}.$$

Since $I(\gamma_t) = I(\gamma)$, it follows from (3.4) that $W(\gamma_t) = W(\gamma)$. So, by setting $f_t = f_{\gamma_t} \circ \tilde{\Phi}_t \circ \tilde{\Phi}_0^{-1}$, we obtain a smooth one parameter family of isometric immersions $f_t : M_\gamma \to S^3$, $t \in \mathbb{R}$, such that $f_0 = f_\gamma$. Let H_t denote the mean curvature vector field of f_t. Then it follows from (2.18) and (3.5) that there exists a point $a \in M_\gamma$ such that $|H_1(a)| \neq |\tan\theta|$. On the other hand, (3.3) implies that $|H_0(x)| = |\tan\theta|$ for all $x \in M_\gamma$. Hence $f_0 \not\equiv f_1$, and so the immersion f_γ is deformable. □

Lemma 3.3. *The immersions $F/W_+(n)$ and $F/W_-(n)$ are deformable for $n \geq 2$.*

Proof. We first note that $F/W_+(n) \equiv F/W_-(n)$. So, by Lemma 3.2, it is sufficient to show that $f_\gamma \equiv F/W_+(n)$. Let $\Phi\colon \mathbb{R}^2 \to M_\gamma$ be the Riemannian covering defined by (2.9), and $\tilde{f}_\gamma\colon \mathbb{R}^2 \to S^3$ an isometric immersion given by $\tilde{f}_\gamma = f_\gamma \circ \Phi$. We denote by $\tilde{h}$ the second fundamental form of the immersion $\tilde{f}_\gamma$. Then it follows from Lemma 2.3 and (3.3) that

$$\tilde{h} = 2\tan\theta\, dx_1^2 - 2\,dx_1\,dx_2. \tag{3.6}$$

Let T be an isometry of $\mathbb{R}^2$ given by

$$T(x_1, x_2) = (R_2x_1 + R_1x_2, -R_1x_1 + R_2x_2).$$

Then it follows from (3.2) and (3.6) that

$$T^*\tilde{h} = \frac{R_2}{R_1}\,dx_1^2 - \frac{R_1}{R_2}\,dx_2^2.$$

Hence the isometric immersions $\tilde{f}_\gamma \circ T : \mathbb{R}^2 \to S^3$ and $F : \mathbb{R}^2 \to S^3$ have the same second fundamental form. So it follows from the fundamental theorem of the theory of surfaces that there exists an isometry $A : S^3 \to S^3$ satisfying

$$\tilde{f}_\gamma \circ T = A \circ F. \tag{3.7}$$

On the other hand, (3.1) implies

$$L(\gamma) = 2n\pi\cos\theta,\quad K(\gamma) = 2n\pi\sin\theta,\quad I(\gamma) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

So, by (2.6), the lattice $W(\gamma)$ is generated by

$$v_1 = (n\pi\cos\theta,\ n\pi\sin\theta + n\pi),\qquad v_2 = (0,\ 2\pi).$$

Hence we obtain $W(\gamma) = \{n_1\xi_1 + n_2\xi_2 : n_1 + n_2 \in n\mathbb{Z}\}$, where

$$\xi_1 = (\pi\cos\theta,\ \pi\sin\theta - \pi),\qquad \xi_2 = (\pi\cos\theta,\ \pi\sin\theta + \pi).$$

By (3.2) the vectors ξ_1 and ξ_2 can be written as

$$\xi_1 = 2\pi R_1(R_2,\ -R_1),\qquad \xi_2 = 2\pi R_2(R_1,\ R_2).$$

This shows that $T(W_+(n)) = W(\gamma)$, and so it follows from Lemma 2.2 that there exists a diffeomprphism $\overline{T} : \mathbb{R}^2/W_+(n) \to M_\gamma$ satisfying $\Phi \circ T = \overline{T} \circ q$, where q denotes the canonical projection of $\mathbb{R}^2$ onto $\mathbb{R}^2/W_+(n)$. Therefore (3.7) implies that $f_\gamma \circ \overline{T} = A \circ F/W_+(n)$. □

By the lemma above the assertion of Theorem 3.1 follows from the following

Lemma 3.4. *Let W be a lattice of $\mathbb{R}^2$ such that $W \subset G_0$ and the immersion F/W is deformable. Then for each lattice $G \subset W$ the immersion F/G is deformable.*

Proof. By the assumption, the isometric immersion $F/W : \mathbb{R}^2/W \to S^3$ admits an isometric deformation $f_t : \mathbb{R}^2/W \to S^3$ such that $f_0 \not\equiv f_1$. Then the mean curvature vector field of f_t, denoted by H_t, satisfies

$$\begin{cases} |H_0(x)| = |R_2^2 - R_1^2|/2R_1R_2 & \text{for all} \quad x \in \mathbb{R}^2/W, \\ |H_1(a)| \neq |R_2^2 - R_1^2|/2R_1R_2 & \text{for some} \quad a \in \mathbb{R}^2/W. \end{cases} \tag{3.8}$$

We now consider the canonical projection $q\colon \mathbb{R}^2/G \to \mathbb{R}^2/W$ and an isometric deformation of F/G given by $\overline{f_t} = f_t \circ q$. Then (3.8) implies that $\overline{f_0} \not\equiv \overline{f_1}$, and so the immersion F/G is deformable. □

4. Proof of main theorems

In this section we give the proof of Theorems 1.2 and 1.3. Consider the map $\sigma\colon G_0 \to G_0$ defined by

$$\sigma(2\pi R_1 n_1,\ 2\pi R_2 n_2) = (2\pi R_1 n_2,\ 2\pi R_2 n_1)\,.$$

The following lemma is the key ingredient in the proof of Theorem 1.2.

Lemma 4.1. *Let G be a lattice of $\mathbb{R}^2$ such that $G \subset G_0$. If $F_t\colon \mathbb{R}^2 \to S^3$, $t \in \mathbb{R}$, is a G-invariant isometric deformation of the immersion F, then the deformation F_t is $\sigma(G)$-invariant.*

Proof. Let $v \in G$. Then it is sufficient to show that

$$F_t(x + \sigma(v)) = F_t(x) \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.1}$$

Since $F_t\colon \mathbb{R}^2 \to S^3$ is an isometric immersion, it follows from [7] that there exists a diffeomorphism $T_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$|\partial_i T_t| = 1, \quad h_t\,(\partial_i T_t, \partial_i T_t) = 0 \quad \text{for} \quad i = 1,\ 2, \tag{4.2}$$

where h_t denotes the second fundamental form of F_t. We may assume that the map $(t, s_1, s_2) \mapsto T_t(s_1, s_2)$ is smooth and

$$T_t(0,0) = (0,0), \quad T_0(s_1, s_2) = (R_1(s_1 - s_2), R_2(s_1 + s_2)). \tag{4.3}$$

By (4.2) we obtain $\partial_1\partial_2 T_t = (0,0)$. So it follows from $T_t(0,0) = (0,0)$ that

$$T_t(s_1, s_2) = T_t(s_1, 0) + T_t(0, s_2). \tag{4.4}$$

We set

$$(l_1(t), l_2(t)) = T_t^{-1}(v), \quad z(t) = T_t(l_1(t), 0). \tag{4.5}$$

Then $v = T_0(l_1(0), l_2(0)) = (R_1(l_1(0) - l_2(0)), R_2(l_1(0) + l_2(0)))$, and so we obtain

$$v + \sigma(v) = 2z(0).$$

Since F_t is G-invariant, the relation above implies

$$F_t(x + \sigma(v)) = F_t(x + 2z(0)). \tag{4.6}$$

Let $p_t\colon \mathbb{R}^2 \to \mathbb{R}^2/G$ be a covering given by $p_t = p \circ T_t$, where $p\colon \mathbb{R}^2 \to \mathbb{R}^2/G$ denotes the canonical projection. Since $p(v) = p(0, 0)$, it follows from (4.3) and (4.5) that $p_t(l_1(t), l_2(t)) = p_t(0, 0)$. So there exists a diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $p_t \circ \varphi = p_t$ and $\varphi(0, 0) = (l_1(t), l_2(t))$. Then $T_t(\varphi(s_1, s_2)) - T_t(s_1, s_2) \in G$, and so it follows from (4.5) that

$$T_t(\varphi(s_1, s_2)) = T_t(s_1, s_2) + v. \tag{4.7}$$

We now consider an immersion $\tilde{F}_t\colon \mathbb{R}^2 \to S^3$ defined by $\tilde{F}_t = F_t \circ T_t$. Then, by (4.2), we see that the immersion $\tilde{F}_t$ is a FAT. Here, we refer the reader to [2, p. 460] for the definition of FAT. Furthermore, it follows from (4.7) that $\tilde{F}_t \circ \varphi = \tilde{F}_t$. Therefore, [2, Theorem 2.3] implies

$$\varphi(s_1, s_2) = (s_1 + l_1(t), s_2 + l_2(t)). \tag{4.8}$$

In particular, we obtain $\tilde{F}_t(s_1 + l_1(t), s_2 + l_2(t)) = \tilde{F}_t(s_1, s_2)$, and so it follows from [2, Theorem 3.9, Lemma 5.5] that

$$\tilde{F}_t(s_1 + 2l_1(t), s_2) = \tilde{F}_t(s_1, s_2). \tag{4.9}$$

On the other hand, combining (4.7) and (4.8), we obtain

$$T_t(s_1 + l_1(t), s_2 + l_2(t)) = T_t(s_1, s_2) + v. \tag{4.10}$$

Using (4.4), (4.5) and (4.10), we see that

$$\begin{aligned} T_t(s_1 + l_1(t), s_2) &= T_t(s_1 + l_1(t), l_2(t)) + T_t(0, s_2) - T_t(0, l_2(t)) \\ &= T_t(s_1, s_2) + v - T_t(0, l_2(t)) = T_t(s_1, s_2) + z(t). \end{aligned}$$

This implies $T_t(s_1 + 2l_1(t), s_2) = T_t(s_1, s_2) + 2z(t)$, and so it follows from (4.9) that

$$F_t(x + 2z(t)) = F_t(x). \tag{4.11}$$

By (4.6) and (4.11), we see that (4.1) follows from the assertion that $z(t) = z(0)$ for all $t \in \mathbb{R}$. To establish this assertion, suppose that there exists t_0 such that $z(t_0) \neq z(0)$.

Since the set of all points $z(t) \in \mathbb{R}^2$ is not countable, there exists $a \in \mathbb{R}$ such that $z(a)$ is not contained in the countable set $\bigcup_{n=1}^{\infty}\{x/2n : x \in G\}$. Let $f_a : \mathbb{R}^2/G \to S^3$ be an immersion defined by the relation $f_a \circ p = F_a$, and $\{y_n\}_{n=1}^{\infty}$ a sequence in $\mathbb{R}^2/G$ given by $y_n = p(2nz(a))$. Then it follows from (4.11) that $f_a(y_m) = f_a(y_n)$. Furthermore, as $2nz(a) \notin G$ for all $n \geq 1$, we obtain $y_m \neq y_n$ $(m \neq n)$. So, using the fact that f_a is locally injective, we see that the sequence $\{y_n\}_{n=1}^{\infty}$ has no convergent subsequence. This contradicts the fact that $\mathbb{R}^2/G$ is compact. Hence we obtain $z(t) = z(0)$ for all $t \in \mathbb{R}$. □

We now recall the lattices $W_{\pm}(n)$ given by (1.4), and for each lattice $G \subset G_0$ we consider the lattice

$$G + \sigma(G) = \{u + \sigma(v) : u, v \in G\}.$$

Then we obtain

Lemma 4.2. *Let G be a lattice of $\mathbb{R}^2$ such that $G \subset G_0$. If $G \not\subset W_+(n)$ and $G \not\subset W_-(n)$ for all integers $n \geq 2$, then $G + \sigma(G) = G_0$.*

Proof. Since $G + \sigma(G) \subset G_0$, it is sufficient to show that $G_0 \subset G + \sigma(G)$. Let u and v be generators of the lattice G. Since $u, v \in G_0$, we can write as

$$u = (2\pi R_1 u_1, 2\pi R_2 u_2), \quad v = (2\pi R_1 v_1, 2\pi R_2 v_2), \qquad u_i, \ v_i \in \mathbb{Z}.$$

For each integer $n \geq 2$, using the assumption $G \not\subset W_+(n)$, we see that there exist $k, \ l \in \mathbb{Z}$ such that the integer n does not divide $k(u_1 + u_2) + l(v_1 + v_2)$, and so n is not a common divisor for $u_1 + u_2$ and $v_1 + v_2$. Hence the greatest common divisor for $u_1 + u_2$ and $v_1 + v_2$ is equal to 1. Similarly, using the assumption that $G \not\subset W_-(n)$ for all $n \geq 2$, we see that the greatest common divisor for $u_1 - u_2$ and $v_1 - v_2$ is equal to 1. Hence there exist $p, \ q, \ r, \ s \in \mathbb{Z}$ such that

$$p(u_1 + u_2) + q(v_1 + v_2) = 1, \quad r(u_1 - u_2) + s(v_1 - v_2) = 1. \tag{4.12}$$

We now consider the elements $a, b \in G$ given by

$$a = pu + qv, \quad b = ru + sv.$$

Then it follows from (4.12) that

$$b - (ru_2 + sv_2)(a + \sigma(a)) = (2\pi R_1, 0),$$
$$a - (pu_1 + qv_1)(b - \sigma(b)) = (0, 2\pi R_2).$$

So the lattice $G + \sigma(G)$ contains $(2\pi R_1, 0)$ and $(0, 2\pi R_2)$. Hence $G_0 \subset G + \sigma(G)$. □

Lemma 4.3. *Let G be a lattice of $\mathbb{R}^2$ such that $G \subset G_0$. If $G + \sigma(G) = G_0$, then every isometric deformation of F/G is trivial.*

Proof. Let $f_t : \mathbb{R}^2/G \to S^3$, $t \in \mathbb{R}$, be an isometric deformation of F/G. Then we obtain a G-invariant isometric deformation of F given by $F_t = f_t \circ p$, where p denotes the canonical projection of $\mathbb{R}^2$ onto $\mathbb{R}^2/G$. Since $G + \sigma(G) = G_0$, it follows from Lemma 4.1 that each F_t is G_0-invariant, and so we obtain

$$F_t/G_0 : \mathbb{R}^2/G_0 \to S^3, \quad t \in \mathbb{R}$$

which is an isometric deformation of the embedding $F/G_0 : \mathbb{R}^2/G_0 \to S^3$. Then Theorem 1.1 implies that for each $t \in \mathbb{R}$ there exists an isometry $A_t : S^3 \to S^3$ satisfying $F_t/G_0 = A_t \circ (F/G_0)$. Let $q : \mathbb{R}^2/G \to \mathbb{R}^2/G_0$ denote the canonical projection. Since $f_t = (F_t/G_0) \circ q$, we obtain

$$A_t \circ f_0 = A_t \circ (F/G_0) \circ q = (F_t/G_0) \circ q = f_t.$$

Hence the isometric deformation f_t is trivial. □

Proof of Theorem 1.2. To establish Theorem 1.2, it is sufficient to show the converse of Theorem 3.1. Suppose that $G \not\subset W_+(n)$ and $G \not\subset W_-(n)$ for all integers $n \geq 2$. Then it follows from Lemmas 4.2 and 4.3 that every isometric deformation of F/G is trivial. In particular, the isometric immersion F/G is not deformable. This shows the converse of Theorem 3.1. □

Proof of Theorem 1.3. By Lemmas 4.2 and 4.3, it is easy to see that (2) $\Rightarrow$ (1). We now show that (1) $\Rightarrow$ (2). Recall the main result of [5] which states that every flat torus isometrically immersed in S^3 with nonconstant mean curvature is deformable. Hence, the assumption (1) implies that the mean curvature of the immersion $f : M \to S^3$ must be constant. So there exist positive constants R_1 and R_2 with $R_1^2 + R_2^2 = 1$ such that f is congruent to the immersion F/G given by (1.3). Then F/G is not deformable, and so it follows from Theorem 1.2 that the lattice G satisfies $G \not\subset W_+(n)$ and $G \not\subset W_-(n)$ for all integers $n \geq 2$. □

5. Deformations of circles in S^2

For each 2π-periodic regular curve $\gamma(s)$ in S^2, we recall the following notation.

$$L(\gamma) = \int_0^{2\pi} |\gamma'(s)|\, ds, \quad K(\gamma) = \int_0^{2\pi} k_\gamma(s)|\gamma'(s)|\, ds,$$

where $k_\gamma(s)$ denotes the geodesic curvature of $\gamma(s)$ given by

$$k_\gamma(s) = \frac{\langle \gamma''(s), J(\gamma'(s))\rangle}{|\gamma'(s)|^3}.$$

In this section we prove the following theorem which was used in the proof of Lemma 3.2.

Theorem 5.1. *For each integer $n \geq 2$, let $\gamma\colon \mathbb{R} \to S^2$ be a 2π-periodic regular curve defined by $\gamma(s) = (\cos\theta\cos ns)e_1 + (\cos\theta\sin ns)e_2 + (\sin\theta)e_3$, where θ is a constant satisfying $-\pi/2 < \theta < \pi/2$. Then there exists a smooth one parameter family of 2π-periodic regular curves $\gamma_t\colon \mathbb{R} \to S^2$, $-\delta < t < \delta$, such that*
(1) $\gamma_0 = \gamma$,
(2) $L(\gamma_t) = L(\gamma)$, $K(\gamma_t) = K(\gamma)$,
(3) $k_{\gamma_t}(0) \neq \tan\theta$ *for all* $t \neq 0$.

We first show the following lemma which proves the assertion of Theorem 5.1 in the case of $\theta = 0$.

Lemma 5.2. *For each integer $n \geq 2$, let $\alpha\colon \mathbb{R} \to S^2$ be a 2π-periodic regular curve defined by $\alpha(s) = (\cos ns)e_1 + (\sin ns)e_2$. Then there exists a smooth one parameter family of 2π-periodic regular curves $\alpha_t\colon \mathbb{R} \to S^2$, $-\epsilon < t < \epsilon$, such that*
(1) $\alpha_0 = \alpha$,
(2) $L(\alpha_t) = L(\alpha)$, $K(\alpha_t) = K(\alpha)$,
(3) $k_{\alpha_t}(0) \neq 0$ *for all* $t \neq 0$.

Proof. Let $v_1(s)$ and $v_2(s)$ be 2π-periodic functions defined by

$$v_1(s) = \cos s, \quad v_2(s) = \cos ms, \quad \text{where} \quad m = 2n + 1. \tag{5.1}$$

For each $x = (x_1, x_2) \in \mathbb{R}^2$, we consider the curve $q_x\colon \mathbb{R} \to S^2$ given by

$$q_x(s) = \cos\left(\sum_{i=1}^{2} x_i v_i(s)\right)\alpha(s) + \sin\left(\sum_{i=1}^{2} x_i v_i(s)\right)e_3.$$

Note that $q_x(s + 2\pi) = q_x(s)$, and

$$q_o(s) = \alpha(s), \quad \text{where} \quad o = (0, 0).$$

So there exists an open neighborhood V of the origin $o \in \mathbb{R}^2$ such that for each $x \in V$ the curve q_x is regular. We consider the smooth functions $\bar{L}$, $\bar{K}\colon V \to \mathbb{R}$ given by

$$\bar{L}(x) = L(q_x) = \int_0^{2\pi} |q_x'(s)|\, ds, \quad \bar{K}(x) = K(q_x) = \int_0^{2\pi} k_{q_x}(s)|q_x'(s)|\, ds.$$

Since $v_i(s+\pi) = -v_i(s)$, we obtain $|q_x'(s+\pi)| = |q_x'(s)|$ and $k_{q_x}(s+\pi) = -k_{q_x}(s)$. Therefore

$$\bar{K}(x) = 0. \tag{5.2}$$

Since $q_o\colon \mathbb{R} \to S^2$ is a geodesic, the origin $o \in \mathbb{R}^2$ is a critical point for the smooth function $\bar{L}$. The Hessian of $\bar{L}$ at the critical point o is given by

$$\frac{\partial^2 \bar{L}}{\partial x_i \partial x_j}(o) = \frac{1}{n}\int_0^{2\pi} (v_i'(s)v_j'(s) - n^2 v_i(s)v_j(s))\,ds.$$

So it follows from (5.1) that

$$\frac{\partial^2 \bar{L}}{\partial x_1 \partial x_1}(o) = \frac{1-n^2}{n}\pi, \quad \frac{\partial^2 \bar{L}}{\partial x_2 \partial x_2}(o) = \frac{m^2-n^2}{n}\pi, \quad \frac{\partial^2 \bar{L}}{\partial x_1 \partial x_2}(o) = 0. \tag{5.3}$$

Since $n > 1$ and $m = 2n+1$, the index of $\bar{L}$ at the critical point o is equal to -1. Hence the Lemma of Morse [6, p. 6] implies that there exists a local coordinate system (y_1, y_2) in a neighborhood U of the origin o such that

$$\bar{L}(x) = \bar{L}(o) - y_1(x)^2 + y_2(x)^2, \quad y_1(o) = y_2(o) = 0. \tag{5.4}$$

For a sufficiently small $\epsilon > 0$, let $x(t) = (x_1(t), x_2(t))$, $-\epsilon < t < \epsilon$, be a smooth curve in U defined by

$$y_1(x(t)) = t, \quad y_2(x(t)) = t, \tag{5.5}$$

and we consider the smooth one parameter family of 2π-periodic regular curves $\alpha_t\colon \mathbb{R} \to S^2$, $-\epsilon < t < \epsilon$ given by $\alpha_t = q_{x(t)}$. Then it follows from (5.2), (5.4) and (5.5) that

$$\alpha_0 = q_o = \alpha, \quad L(\alpha_t) = \bar{L}(x(t)) = \bar{L}(o), \quad K(\alpha_t) = \bar{K}(x(t)) = 0.$$

This implies the assertions (1) and (2). Since the geodesic curvature of α_t satisfies $k_{\alpha_t} = \langle \alpha_t'', J(\alpha_t')\rangle / |\alpha_t'|^3$, we obtain

$$k_{\alpha_t}(0) = \frac{\varphi(t)}{n^2\cos^2(x_1(t)+x_2(t))}, \tag{5.6}$$

where $\varphi(t) = n^2\cos(x_1(t)+x_2(t))\sin(x_1(t)+x_2(t)) - x_1(t) - m^2 x_2(t)$. Note that

$$\varphi(0) = 0, \quad \varphi'(0) = (n^2-1)x_1'(0) + (n^2-m^2)x_2'(0). \tag{5.7}$$

Differentiating the relation $\bar{L}(x(t)) = \bar{L}(o)$ and using (5.3), we obtain

$$(n^2-1)x_1'(0)^2 + (n^2-m^2)x_2'(0)^2 = 0. \tag{5.8}$$

If $\varphi'(0) = 0$, it follows from (5.7) and (5.8) that $x_1'(0) = x_2'(0) = 0$ which is a contradiction. Hence $\varphi'(0) \neq 0$. So the assertion (3) follows from (5.6). □

Proof of Theorem 5.1. Let $\alpha_t : \mathbb{R} \to S^2$, $-\epsilon < t < \epsilon$, be a smooth one parameter family of 2π-periodic regular curves satisfying the conditions (1)–(3) of Lemma 5.2, and n_t a unit normal vector field along α_t given by $n_t(s) = J(\alpha_t'(s))/|\alpha_t'(s)|$. Consider the curve $\gamma_t : \mathbb{R} \to S^2$ given by $\gamma_t(s) = (\cos\theta)\alpha_t(s) + (\sin\theta)n_t(s)$. Then it follows from the relation $n_t'(s) = -k_{\alpha_t}(s)\alpha_t'(s)$ that

$$\gamma_t'(s) = (\cos\theta - k_{\alpha_t}(s)\sin\theta)\alpha_t'(s). \tag{5.9}$$

Since $k_{\alpha_0}(s) = 0$ and $\cos\theta > 0$, there exists a positive number δ such that

$$\cos\theta - k_{\alpha_t}(s)\sin\theta > 0 \quad \text{for} \quad |t| < \delta.$$

So it follows that $\gamma_t : \mathbb{R} \to S^2$, $-\delta < t < \delta$, is a smooth one parameter family of 2π-periodic regular curves. Hence it is sufficient to show that the family γ_t satisfies (1)–(3) of Theorem 5.1. Using (1) of Lemma 5.2, we obtain $n_0(s) = e_3$. This implies the assertion (1). On the other hand, the geodesic curvature of γ_t satisfies

$$k_{\gamma_t}(s) = \frac{\sin\theta + k_{\alpha_t}(s)\cos\theta}{\cos\theta - k_{\alpha_t}(s)\sin\theta}\ . \tag{5.10}$$

By (5.9) and (5.10) we obtain

$$L(\gamma_t) = \cos\theta L(\alpha_t) - \sin\theta K(\alpha_t), \quad K(\gamma_t) = \sin\theta L(\alpha_t) + \cos\theta K(\alpha_t).$$

So the assertion (2) follows from (2) of Lemma 5.2. Furthermore, using (3) of Lemma 5.2 and (5.10), we see that $k_{\gamma_t}(0) \neq \tan\theta$ for all $t \neq 0$. This implies the assertion (3). □

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