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NORMAL EMBEDDING OF SPHERES INTO $\mathbb{C}^n$

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1. Introduction

The notion of normal submanifold was introduced by J.C. Sikorav ([6]) as a weaker version of Lagrangian submanifold.

Polterovich ([5]) showed that if L is a closed normal non-Lagrangian submanifold of a symplectic manifold M and the Euler characteristic of L vanishes then its displacement energy $e(L)$ vanishes.

The basic notions such as 'normal', 'symplectic', 'weakly Lagrangian' etc. are explained in Section 2 below and the definition of the displacement energy is provided in the later part of this section.

It is well known that S^1 and S^3 are totally real submanifolds of $\mathbb{C}^1$ and $\mathbb{C}^3$, respectively. L. Polterovich ([5]) showed that if L is a totally real submanifold of a symplectic manifold (V, ω) and L is parallelizable then L is normal. So S^1 and S^3 are normal submanifold of $\mathbb{C}^1$ and $\mathbb{C}^3$, respectively. In fact S^1 is a Lagrangian submanifold of $\mathbb{C}^1$ and it follows that it is a normal submanifold. As for S^3 we consider the standard embedding and explicitly construct in Section 4 below the Lagrangian subbundle of $T\mathbb{C}^3|_{S^3}$ which is transverse to the tangent bundle.

The following two theorems are our main results which respectively answer the two questions: (a) Which S^n admits a normal embedding into $\mathbb{C}^n$? (b) When the product of spheres admits a normal embedding into the complex Euclidean space?

Theorem 1.1. *S^n admits a normal embedding into $\mathbb{C}^n$ if and only if n is* 1 *or* 3.

Theorem 1.2. $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_i \geq 1$, $i = 1, 2, \ldots, k$, $k \geq 2$, *admits a normal embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if and only if some n_i is odd.*

Note that H. Hofer ([3]) defined the *displacement energy* of a subset A of a sym-

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plectic manifold M as

$$\inf\left\{\operatorname*{Max}_{M\times I} H - \operatorname*{Min}_{M\times I} H \mid H\in\mathcal{C} \text{ such that } g_H^1 A\cap A=\emptyset\right\}$$

where $\mathcal{C}$ is the set of all smooth real valued functions which attain both maximum and minimum on the product $M\times I$ of M with the closed unit interval I and g_H^1 is the Hamiltonian flow at time 1 determined by H.

The normal embeddings of Theorem 1.2 are not necessarily the product of the standard embeddings (see [7]) and therefore their images may not be contained in a codimension 1 plane. Also the embedding is not Lagrangian unless some n_i is 1. Even if some n_i is 1 and the embedding is Lagrangian, we recall the fact that any Lagrangian embedding of a manifold of dimension greater than 1 and with vanishing Euler characteristic can be C^l-approximated for any $l\geq 1$, by non-Lagrangian normal embeddings ([5]). Therefore, Theorem 1.12 in [5] by L. Polterovich implies:

Corollary 1.3. *Assume $k>1$. If some n_i, $i=1,2,\dots,k$, is odd, the product of spheres, $S^{n_1}\times S^{n_2}\times\cdots\times S^{n_k}$, $n_i\geq 1$, $i=1,2,\dots,k$, $k\geq 2$, admits a normal embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$, for which the displacement energy vanishes.*

2. Basic notions and facts

A smooth manifold M is called *symplectic* if there is a nondegenerate closed 2-form ω on M. Such a 2-form is called a *symplectic form* or a *symplectic structure* on M. It follows that dim M should be even if M is symplectic.

On the other hand a vector bundle of finite rank is referred to as a *symplectic vector bundle* if it is considered with a fixed symplectic two form. A subbundle η of a symplectic vector bundle ξ is a *Lagrangian subbundle* if 2 (rank η) = rank ξ and the restriction of the symplectic form to η is the zero form.

Let M be a symplectic manifold of dimension $2n$ with a symplectic structure ω. Let L be a smooth manifold of dimension n and let $f\colon L\to M$ be an embedding (resp. immersion). We call f a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle TL of L is a Lagrangian subbundle of the symplectic vector bundle f^*TM with the symplectic form $f^*\omega$. We call f a *normal embedding* (resp. *immersion*) if there is a Lagrangian subbundle $\mathbb{L}$ of f^*TM which is transverse to TL. Note that every Lagrangian submanifold L of M is normal.

We say that an embedding $f\colon L\to M$ is weakly Lagrangian if $TL\subset f^*TM$ is homotopic through n-dimensional subbundles to a Lagrangian subbundle ([4]) in f^*TM.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or normal embedding must be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

3. Proofs

First of all we need the following.

Lemma 3.1. *Let f be a normal embedding of a smooth oriented n-dimensional manifold L into a symplectic $2n$-dimensional manifold M. Then*

$$TL \cong \nu_f$$

where TL is the tangent bundle of L and ν_f, the normal bundle of f.

Proof. Since f is a normal embedding, there is a Lagrangian subbundle $\mathbb{L}^n \subset f^*TM$ which is transverse to $TL \subset f^*TM$. In particular, we have: $f^*TM = TL + \mathbb{L}$. Since the quotient bundle f^*TM/TL is none other than ν_f, we have $\mathbb{L} \cong \nu_f$.

Now let J be an almost complex structure on M compatible with the symplectic structure. Then we have $TL + \mathbb{L} = f^*TM = J\mathbb{L} + \mathbb{L}$ and it follows that $TL \cong f^*TM/\mathbb{L} \cong J\mathbb{L}$. Thus we conclude that

$$TL \cong J\mathbb{L} \cong \mathbb{L} \cong \nu_f \ . \qquad \square$$

Corollary 3.2. *If a smooth oriented closed n-manifold L admits a normal embedding into $\mathbb{C}^n$, then we have*

$$\chi(L) = 0$$

where $\chi(L)$ is the Euler number of L.

Proof. Regard L as a normal submanifold of $\mathbb{C}^n$ and let ν denote the normal bundle. Consider the normal neighborhood N of L. Let $D\nu$ and $S\nu$ denote respectively the disk and the sphere bundles of ν. Then one of the generator U of the integral cohomolgy group

$$H^n(D\nu, S\nu; \mathbb{Z}) \cong H^n(N, \partial N : \mathbb{Z}) \cong Z$$

pulled back to $H^n(N;\mathbb{Z}) \cong H^n(L;\mathbb{Z})$ is the Euler class of TL, presuming a suitable orientation of L, since $\nu \cong TL$ by Lemma 3.1 above. The Euler class evaluated at the fundamental class of L is the Euler number of L. However U when pulled back to $H^n(N)$ is the zero element, for we have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathbb{C}^n, \mathbb{C}^n - \mathrm{int}N; \mathbb{Z}) & \longrightarrow & H^n(N, \partial N; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^n(\mathbb{C}^n; \mathbb{Z}) & \longrightarrow & H^n(N; \mathbb{Z}) \end{array}$$

where all the arrows come from the inclusions. $\square$

Proof of Theorem 1.1 If $f\colon S^n \to \mathbb{C}^n$ is an embedding, then the normal bundle ν_f of f must be trivial since it is stably trivial and its Euler class vanishes. So by Lemma 3.1 the tangent bundle TS^n is trivial. Thus if $n \neq 1, 3, 7$, S^n does not admit any normal embedding into $\mathbb{C}^n$.

On the other hand, S^1 admits a Lagrangian embedding. Also by applying an observation of Polterovich ([5]), S^3 has a normal embedding since S^3 admits totally real embedding and it is parallelizable.

It remains to show that S^7 does not admit any normal embedding, which is the assertion of Corollary 3.4 below.

The following is needed to show that S^7 admits no normal embedding into $\mathbb{C}^7$, which however seems worth an observation on its own right.

Theorem 3.3. *Let M be a symplectic $2n$-manifold and L be a smooth n-manifold which admits a normal embedding into M. If L is parallelizable, then the embedding is weakly Lagrangian.*

Proof. We regard L as a normal submanifold of M. Let $\mathbb{L}$ be a Lagrangian subbundle of $TM|_L$ which is transverse to $TL \subset TM|_L$. Let J denote an almost complex structure of M compatible with the symplectic structure.

Then we have that $TL \cong TM|_L/\mathbb{L}$ and $J\mathbb{L} \cong TM|_L/\mathbb{L}$ (See the proof of Lemma 3.1). Thus we have: $TL \cong J\mathbb{L} \cong \mathbb{L}$.

In particular, $\mathbb{L}$ is trivial.

Let $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ be global frames respectively of TL and $\mathbb{L}$. Then define a homotopy $\mathbb{L}_t$, $0 \le t \le 1$, in $TM|_L$ from TL to $\mathbb{L}$ by defining $\mathbb{L}_t$ as the subbundle generated by the frame:

$$\{\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\}, \ \ \gamma_i(t) = (1-t)e_i + tf_i, i = 1,\ 2, \dots, n.$$

It is straightforward to see that $\{\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\}$ is indeed a frame, that is, $\gamma_1(t),\ \gamma_2(t), \dots, \gamma_n(t)$ are linearly independent at any point of L for all $t \in [0,1]$. □

Corollary 3.4. *S^7 does not admit any normal embedding into $\mathbb{C}^7$.*

Proof. Assume that S^7 admits a normal embedding into $\mathbb{C}^7$. Then, since S^7 is parallelizable, the normal embedding is weakly Lagrangian by Theorem 3.3 above. But according to Kawashima ([4]), S^n admits a weakly Lagrangian embedding if and only if $n = 1, 3$. This means that S^7 does not admit any normal embedding. □

REMARK. (i) A totally real submanifold L of a symplectic manifold which is parallelizable is normal according to Polterovich. Theorem 3.3 further means that L is weakly Lagrangian.

(ii) Note that Theorem 3.3 together with our explicit construction in the next section of the Lagrangian subbundle transverse to TS^3 for the standard embedding of S^3 into $\mathbb{C}^3$ proves that the standard embedding is weakly Lagrangian (cf. [4]).

Proof of Theorem 1.2. We prove the case when $k = 2$ and the general case follows by an inductive argument.

If both m and n are even, then $\chi(S^m \times S^n) \neq 0$, by Corollary 3.2, $S^m \times S^n$ does not admit any normal embedding.

If m or n is odd, then $S^m \times S^n$ admits a totally real embedding into $\mathbb{C}^{m+n}$ (cf. Example 1, [7]) and $S^m \times S^n$ is parallelizable. Therfore, according to Polterovich ([5]), $S^m \times S^n$ admits a normal embedding into $\mathbb{C}^{m+n}$. $\square$

4. A Lagrangian subbundle transverse to the tangent bundle of S^3

Three linearly independent tangent vector fields X_1, X_2, X_3 of

$$S^3 = \{(x_1, x_2, x_3, x_4, 0, 0) \in \mathbb{C}^3 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

are defined as follows:

$$\begin{aligned} X_1(x) &= (-x_2, x_1, -x_4, x_3, 0, 0)\\ X_2(x) &= (-x_3, x_4, x_1, -x_2, 0, 0)\\ X_3(x) &= (-x_4, -x_3, x_2, x_1, 0, 0)\ . \end{aligned}$$

Now the three linearly independent normal vector fields on S^3 are defined as follows:

$$\begin{aligned} N_1(x) &= (x_1, x_2, x_3, x_4, 0, 0)\\ N_2(x) &= (-x_1 - x_4, -x_2 - x_3, x_2 - x_3, x_1 - x_4, 1, 0)\\ N_3(x) &= (-x_1 + x_3, -x_2 - x_4, -x_1 - x_3, x_2 - x_4, 0, 1)\ . \end{aligned}$$

Then clearly N_1, N_2, N_3 are not in the tangent space T_xS^3. In fact, we have that the determinant of the matrix $(X_1, X_2, X_3, N_1, N_2, N_3)$ is -1 and the standard symplectic form vanishes on the subspace generated by N_1, N_2, N_3. Thus the subbundle of $T\mathbb{C}^3|_{S^3}$ generated by N_1, N_2, N_3 is a Lagrangian subbundle transverse to the tangent bundle.

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