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ON HAEFLIGER'S OBSTRUCTIONS TO EMBEDDINGS AND TRANSFER MAPS

Dedicated to the memory of Professor Katsuo Kawakubo

YOSHIYUKI KURAMOTO and TSUTOMU YASUI

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1. Introduction and statement of results

Throughout this article, n-manifolds mean compact differentiable (or topological) manifolds of dimension n. The (co-)homology is understood to have $\boldsymbol{Z}_2$ for coefficients.

For a manifold V, we denote by $w(V)$ and $\bar{w}(V)(= w(V)^{-1})$, the total Stiefel-Whitney class and the total normal Stiefel-Whitney class of V, respectively. Furthermore, we denote by $U_V \in H^{\dim V}(V \times V)$ the $\boldsymbol{Z}_2$-Thom class (or $\boldsymbol{Z}_2$-diagonal cohomology class) of V [10, p. 125]. For a (continuous) map $f\colon M^n \to N^{n+k}$ between closed manifolds M and N, we define the total Stiefel-Whitney class $w(f) = \sum_{i\geq 0} w_i(f)$ by the equation

$$w(f) = \bar{w}(M) f^* w(N).$$

For a map $f\colon M^n \to N^{n+k}$, the transfer map (or Umkehr homomorphism) $f_!\colon H^i(M) \to H^{i+k}(N)$ is defined by the commutative diagram below:

$$\begin{array}{ccc}
H^i(M) & \xrightarrow{f_!} & H^{i+k}(N) \\
\cong \big\downarrow \cap[M] & & \cong \big\downarrow \cap[N] \\
H_{n-i}(M) & \xrightarrow{f_*} & H_{n-i}(N).
\end{array}$$

Here $[V] \in H_{\dim V}(V)$ denotes the fundamental homology class of a manifold V.

Our main theorem is the following

Theorem 1.1. *For a continuous map $f\colon M^n \to N^{n+k}$ between closed topological manifolds, $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$ if and only if $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.*

The cohomology elements, appearing in this theorem, are related to the embeddability of f. A. Haefliger [7, Théorèm 5.2] proved the following

Theorem (Haefliger). *If a map $f\colon M^n \to N^{n+k}$ between topological manifolds is homotopic to a topological embedding, then $w_i(f) = 0$ for $i > k$ and*

$$U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0 \in H^{n+k}(M \times M).$$

Thus we have immediately the following

Corollary 1.2. *If a map $f\colon M^n \to N^{n+k}$ between closed topological manifolds is homotopic to a topological embedding, then $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.*

REMARK 1. It is well-known, e.g., [4, p. 246], that if f is homotopic to a differentiable embedding then $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.

REMARK 2. As we will see in §3, the assumption '*homotopic*' in Haefliger's theorem or Corollary 1.2 can be weakened to '*R-bordant*'.

R.L.W. Brown [4] established the conditions that a map $f\colon M^n \to N^{n+k}$ is cobordant to a differentiable embedding in the sense of Stong [12]. Here a map $f_1\colon M_1^n \to N_1^{n+k}$ between differentiable closed manifolds is said to be cobordant to $f_2\colon M_2^n \to N_2^{n+k}$ if there exist two cobordisms (W, M_1^n, M_2^n), $(V, N_1^{n+k}, N_2^{n+k})$ and a map $F\colon W \to V$ such that $F \mid M_i = f_i$ $(i = 1, 2)$.

From Theorem 1.1 and Brown's theorem [4], we infer immediately a result which means the converse of Haefligar's theorem up to cobordism of maps in the sense of Stong [12].

Corollary 1.3. *Let $k > 0$. Then a map $f\colon M^n \to N^{n+k}$ between differentiable manifolds is cobordant to a differentiable embedding if $w_i(f) = 0$ $(i > k)$ and $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$.*

For an n-manifold M, we use the same symbol M as the generator of $H^n(M) \cong \mathbf{Z}_2$, i.e., $H^n(M) = \mathbf{Z}_2\langle M\rangle$, and denote the $H^p(M) \times H^q(M)$-component of $u \in H^{p+q}(M \times M)$ by $[u]_{p,q}$. To prove Theorem 1.1 we use the following

Proposition 1.4. *For a map $f\colon M^n \to N^{n+k}$ and two elements $x, y \in H^*(M)$ with $\dim x + \dim y = r \leq n - k$,*

$$[(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(x \times y)]_{n,k+r} = M \times (x w_k(f) + f^* f_!(x)) y.$$

Using this proposition, we can reformulate Brown's theorem [4] in case $k > n/2$.

Theorem 1.5. *Let $k > n/2$. Then a differentiable map $f\colon M^n \to N^{n+k}$ is cobordant to a differentiable embedding if and only if the following two conditions hold*:

(1) $\langle w_I(M)w_J(f)w_i(f), [M]\rangle = 0$ *for any integer* $i(i > k)$ *and sequences* I, J *of non-negative integers such that* $|I| + |J| + i = n$.
(2) $(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(w_I(M) \times f^*(w_J(N))w_K(M)) = 0$ *for any sequences* I, J, K *of non-negative integers such that* $|I| + |J| + |K| = n - k$.

Here, $w_I(M) = w_{i_1}(M) \cdots w_{i_r}(M)$ and $|I| = \sum_{1 \leq j \leq r} i_j$ for a finite sequence $I = (i_1, \ldots, i_r)$ of non-negative integers.

The rest of this article is organized as follows: In §2, we will prove Theorem 1.1, Proposition 1.4 and Theorem 1.5. §3 will be devoted to the study of the relation between R-bordism and Haefliger's obstruction. In §4, we will give some examples of maps $f: M^n \to N^{n+k}$, e.g., a map which is cobordant to a differentiable embedding but not R-bordant to a topological embedding.

2. Proofs

To prove Theorem 1.1 and Proposition 1.4, we use the following two lemmas, the first of which is a slight generalization of [8, Lemma 2].

Lemma 2.1. *For a map* $f: M^n \to N^{n+k}$ *and an element* $x \in H^r(M)$, *we have*

$$[(f \times f)^* U_N(x \times 1)]_{n,k+r} = M \times f^* f_!(x).$$

Proof. We can choose bases $\{u_i \mid i \in I\}$ and $\{v_i \mid i \in I\}$ for $H^*(N)$ such that $\langle u_i v_j, [N]\rangle = \delta_{ij}$. Then the Thom class U_N of N can be described as $U_N = \sum_{i \in I} u_i \times v_i$ by, e.g., [10, Theorem 11.11]. The element $f_!(x)$ can be described as $f_!(x) = \sum_{i \in I} \alpha_i v_i (\alpha_i \in \mathbf{Z}_2)$. Let $I_0 = \{i \in I \mid f^*(u_i)x = M\}$. Then

$$\begin{aligned}
\alpha_i = \langle \alpha_i u_i v_i, [N]\rangle &= \left\langle u_i \sum_{i \in I} \alpha_i v_i, [N] \right\rangle = \langle u_i f_!(x), [N]\rangle \\
&= \langle u_i, f_!(x) \cap [N]\rangle = \langle u_i, f_*(x \cap [M])\rangle = \langle f^*(u_i)x, [M]\rangle \\
&= \begin{cases} 1 & i \in I_0 \\ 0 & i \notin I_0. \end{cases}
\end{aligned}$$

Thus, $f_!(x) = \sum_{i \in I_0} v_i$ and so $f^* f_!(x) = \sum_{i \in I_0} f^*(v_i)$. Hence, we have

$$\begin{aligned}
[(f \times f)^* U_N \cdot (x \times 1)]_{n,k+r} &= \left[\left(\sum_{i \in I} f^*(u_i) \times f^*(v_i) \right) (x \times 1) \right]_{n,k+r} \\
&= \left[\sum_{i \in I} f^*(u_i)x \times f^*(v_i) \right]_{n,k+r}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \in I_0} M \times f^*(v_i) = M \times \sum_{i \in I_0} f^*(v_i) \\
&= M \times f^* f_!(x).
\end{aligned}
$$

This completes the proof. $\square$

Lemma 2.2. *For an n-manifold M^n, and an element $x \in H^r(M)$, we have*

$$[U_M(x \times 1)]_{n,r} = M \times x.$$

Proof. The Thom class U_M can be described as $U_M = M \times 1 + \sum_j a_j \times b_j$, $(\dim a_j < n)$ and there is a relation $U_M(x \times 1) = U_M(1 \times x)$ (e.g., [10, Lemma 11.8]). Thus the lemma follows immediately. $\square$

Proof of Proposition 1.4. Let $x, y \in H^*(M)$ with $\dim x + \dim y = r$. Then, we have

$$
\begin{aligned}
&[(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(x \times y)]_{n,k+r} \\
&\qquad = [U_M(1 \times w_k(f))(x \times 1) + (f \times f)^* U_N(x \times 1)]_{n,k+\dim x}(1 \times y) \\
&\qquad = M \times x w_k(f) y + M \times f^* f_!(x) y \quad \text{by Lemmas 2.1–2.2} \\
&\qquad = M \times (x w_k(f) + f^* f_!(x)) y.
\end{aligned}
$$

Thus, the proposition follows. $\square$

Proof of Theorem 1.1. First we assume that $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$. Take any $a \in H^r(M)$. Then

$$
\begin{aligned}
0 &= [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times 1)]_{n,k+r} \\
&= M \times (a w_k(f) + f^* f_!(a)) \quad \text{by Proposition 1.4}
\end{aligned}
$$

Thus we get $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.

Conversely, suppose that $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$. Since $U_M(1 \times w_k(f)) + (f \times f)^* U_N \in H^{n+k}(M \times M)$, it is sufficient for our purpose to show that $(U_M(1 \times w_k(f)) + (f \times f)^* U_N)u = 0$ for all $u \in H^{n-k}(M \times M)$. By the Künneth formula, we may assume that $u = a \times b$ with $\dim a + \dim b = n - k$. Then by Proposition 1.4, we have

$$
\begin{aligned}
&(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times b) \\
&\qquad = [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times b)]_{n,n} \\
&\qquad = M \times (a w_k(f) + f^* f_!(a)) b = 0.
\end{aligned}
$$

Hence we get $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$. $\square$

Proof of Theorem 1.5. The condition (1) of Theorem 1.5 is just a restatement of the condition (i) of Brown's theorem. On the other hand, by the assumption that $k > n/2$, we have only to consider the case $r = 2$ in the condition (ii) of Brown's theorem, which is reduced to

$$\langle f^*(w_J(N))f^* f_!(w_I(M))w_K(M), [M]\rangle = \langle f^*(w_J(N))w_I(M)w_K(M)w_k(f), [M]\rangle.$$

Applying Proposition 1.4 for $x = w_I(M)$ and $y = f^*(w_J(N))w_K(M)$, we see that this equality is equivalent to the condition (2) of Theorem 1.5. □

3. Relations between *R*-bordisms and Haefliger's obstructions

The concept of R-bordism of maps is introduced in [3, §3]. Let $f_i \colon M_i^n \to N^{n+k}$ ($i = 1, 2$) be maps between topological manifolds, where M_i's are closed (while N is not necessarily closed). The two maps are said to be R-bordant if there exist a topological cobordism (W, M_1, M_2) and a continuous map $F \colon W \to N$ such that (1) $F \mid M_i = f_i$ ($i = 1, 2$) and (2) there exsist retractions $r_i \colon W \to M_i$ ($i = 1, 2$).

Let $j_i \colon M_i \to W$ be the natural inclusion ($i = 1, 2$). Then by [6, Theorem 1.2],

$$(r_2 j_1)_* \colon H_*(M_1) \to H_*(M_2)$$

is an isomorphism, and by [3, §3]

$$f_{1*} = f_{2*}(r_2 j_1)_* \colon H_*(M_1) \to H_*(N).$$

In this section, we will prove

Theorem 3.1. *Let $f \colon M^n \to N^{n+k}$ be a map between closed topological manifolds. If f is R-bordant to a topological embedding, then $w_i(f) = 0$ ($i > k$) and*

$$U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0.$$

This theorem, together with Corollary 1.3, leads to the following

Corollary 3.2. *Let $f \colon M^n \to N^{n+k}$ be a map between closed differentiable manifolds. If f is R-bordant to a topological embedding, then f is cobordant to a differentiable embedding.*

REMARK 3. If we consider *cobordism* and *embeddings* in topological category, the conclusion of this corollary is rather trivial.

Theorem 3.1 follows from Proposition 3.3 (or Corollary 3.4) below and Haefliger's theorem.

Proposition 3.3. *Let* $f_i\colon M_i^n \to N^{n+k} (i = 1, 2)$ *and* $g\colon M_1^n \to M_2^n$ *be maps such that* $g_*\colon H_*(M_1) \to H_*(M_2)$ *is an isomorphism and* $f_{1*} = f_{2*}g_*\colon H_*(M_1) \to H_*(N)$. *Then* $w(f_1) = g^* w(f_2)$ *and*

$$\begin{aligned} &U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^* U_N \\ &\qquad = (g \times g)^*(U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^* U_N). \end{aligned}$$

Proof. Let $\{u_i \mid i \in I\}$ and $\{v_i \mid i \in I\}$ be two bases for $H^*(M_2)$ such that $\langle u_i v_j, [M_2]\rangle = \delta_{ij}$. Then the Thom class U_{M_2} of M_2 can be described as $U_{M_2} = \sum_{i\in I} u_i \times v_i$ (see [10, Theorem 11.11]). Since $g_*[M_1] = [M_2]$ and g^* is an isomorphism, because so is g_*, we have the two bases $\{g^*u_i \mid i \in I\}$ and $\{g^*v_i \mid i \in I\}$ for $H^*(M_1)$ with $\langle (g^*u_i)(g^*v_j), [M_1]\rangle = \delta_{ij}$. Hence,

$$U_{M_1} = \sum_{i\in I} g^*u_i \times g^*v_i = (g \times g)^* \sum_{i\in I} u_i \times v_i = (g \times g)^* U_{M_2}.$$

Since $f_{1*} = f_{2*}g_*$, we have $f_1^* = g^* f_2^*$ and $w(f_1) = g^* w(f_2)$ by [3, Theorem 4.2]. Hence we have

$$\begin{aligned} &U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^* U_N \\ &\qquad = (g \times g)^* U_{M_2}(1 \times g^* w_k(f_2)) + (g \times g)^*(f_2 \times f_2)^* U_N \\ &\qquad = (g \times g)^*(U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^* U_N). \end{aligned}$$

This completes the proof. □

Corollary 3.4. *Let* $f_i\colon M_i^n \to N^{n+k} (i = 1, 2)$ *be maps between closed topological manifolds. If* f_1 *is* R*-bordant to* f_2, *then,* $w_i(f_1)$ $(i \geq 0)$ *and* $U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^* U_N$ *correspond to* $w_i(f_2)$ $(i \geq 0)$ *and* $U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^* U_N$, *respectively, by the canonical isomorphisms.*

REMARK 4. By virtue of Proposition 1.4 and the fact that for $f\colon M^n \to N^{n+k}$, $w_k(f) + f^* f_!(1)$ is the Poincaré dual to the element $\theta(f) \in H_{n-k}(M)$ in [3], the results in Theorem 3.1, Proposition 3.3 and Corollary 3.4 are, respectively, somewhat stronger than those in [3, Corollary 4.4, Theorem 4.2 and Corollary 4.3] in case N is a closed manifold.

4. Relations among obstructions to embeddings

For a map $f\colon M^n \to N^{n+k}$, we describe conditions (0)–(3) below:
(0) $w_i(f) = 0$ for $i > k$.
(1) $f^* f_!(a) + a w_k(f) = 0$ for all $a \in H^*(M)$.
(or equivalently, $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$ by Theorem 1.1.)

(2) $f^*f_!(w_I(M)) + w_I(M)w_k(f) = 0$ for all sequences I of non-negative integers, where $w_I(M) = w_{i_1}(M)\cdots w_{i_r}(M)$ if $I = (i_1, \dots, i_r)$.
(3) $f^*f_!(1) + w_k(f) = 0$.

So far, for a map $f\colon M^n \to N^{n+k}$ between closed differentiable manifolds, we know

$$\begin{array}{lll}
f \text{ is homotopic to a topological embedding} & & \\
\qquad\qquad\qquad\qquad\Downarrow & & \\
f \text{ is } R\text{-bordant to a topological embedding} & \Rightarrow & (0) + (1) \\
 & & \qquad\Downarrow \\
f \text{ is cobordant to a differentiable embedding} & \Leftarrow & (0) + (2) \\
 & & \qquad\Downarrow \\
f \text{ is cobordant to a differentiable embedding} & \not\Rightarrow & (0) + (3)
\end{array}$$

REMARK 5. If $k \geq n - 4$, $2k > n$ and if f satisfies the conditions (0) and (3), then f is cobordant to a differentiable embedding ([1, Theorems (3.6) and (3.9)] and [9, Corollary 1.3]).

REMARK 6. Even if f is cobordant to an embedding, the conditions (0) and (3) do not necessarily hold ([8, Remark 2]).

In this section, we will show that
(a) even if f is R-bordant to an embedding, f is not necessarily homotopic to an embedding (see Example 1 below),
(b) the conditions (0) and (2) do not imply the conditions (1) (see Example 2),
(c) the condition (3) does not lead to the condition (2) (see Example 3), and
(d) the conditions (0) and (3) induce the relation (see Proposition 4.1)

$$f^*f_!(v_i(M)) = v_i(M)w_k(f),$$

where $v_i(M)$ stands for the i-th Wu class of M defined by
$Sq(\sum_{0\leq i} v_i(M)) = w(M)$.

EXAMPLE 1. Let $S^1 = \{z \in \boldsymbol{C}^1 \mid |z| = 1\}$ be the circle, and let $f\colon S^1 \to S^1 \times S^1$ be a map defined by $f(z) = (f_1(z), f_2(z)) = (z^2, 1)$. Then f is not homotopic to an embedding. But f is R-bordant to an embedding.

REMARK 7. This example is a modification of an example appearing in earlier versions of [3], but omitted in the final one.

Proof. Suppose that f is homotopic to a topological embedding $g = (g_1, g_2)\colon S^1 \to S^1 \times S^1$. Then g_2 is homotopic to the constant map f_2. Hence, g_2 has a lifting $g_2'\colon S^1 \to \boldsymbol{R}^1$. If we put $g' = (g_1, g_2')\colon S^1 \to S^1 \times \boldsymbol{R}^1$, then g' is also an embedding.

Identifying $S^1\times\boldsymbol{R}^1$ with $\boldsymbol{C}^1-\{0\}$, we have a topological embedding $g'\colon S^1\to\boldsymbol{C}^1-\{0\}$. From now on, the authors owe C. Biasi, J. Daccach and O. Saeki for the proof. Note that $g'_*\colon H_1(S^1,\boldsymbol{Z})(\cong\boldsymbol{Z})\to H_1(\boldsymbol{C}^1-\{0\},\boldsymbol{Z})(\cong\boldsymbol{Z})$ maps $a\in\boldsymbol{Z}$ to $2a$. By the Schoenflies theorem, $g'(S^1)$ bounds a region U in $\boldsymbol{C}^1$ homeomorphic to the closed 2-dimensional disk. If $0\notin U$, then g' is null-homotopic in $\boldsymbol{C}^1-\{0\}$, which is a contradiction. If $0\in U$, then g' represents a generator of $H_1(\boldsymbol{C}^1-\{0\})$, which is also a contradiction. Thus f is not homotopic to an embedding. On the other hand, f is R-bordant to an embedding by [3, Example 4.8]. □

EXAMPLE 2. We denote by P^m the real projective m-space. Furthermore, $\pi\colon P^3\to P^3/P^2=S^3$ and $j\colon P^l\subset P^{l+k}$ stand for the natural projection and inclusion, respectively. Let $M^n=P^3\times P^l$, $N=S^3\times P^{l+k}$ and let $f=\pi\times j\colon M^n\to N^{n+k}$. Then f satisfies (0) and (2), but f does not satisfy (1).

Proof. Put

$$H^1(P^3)=\boldsymbol{Z}_2\langle x_1\rangle,\ \ H^1(P^l)=\boldsymbol{Z}_2\langle x_2\rangle,\ \ H^3(S^3)=\boldsymbol{Z}_2\langle s\rangle,\ \ H^1(P^{l+k})=\boldsymbol{Z}_2\langle y\rangle.$$

Then

$$f^*(s)=x_1^3,\quad f^*(y)=x_2,\quad w(f)=(1+x_2)^{-l-1}(1+x_2)^{l+k+1}=(1+x_2)^k.$$

Therefore

$$w_i(f)=0\quad\text{for } i>k,\quad w_k(f)=x_2^k.$$

The Thom classes of M and N are given by

$$\begin{aligned}U_M=&\sum_{0\le i\le l}x_1^3x_2^i\times x_2^{l-i}+\sum_{0\le i\le l}x_1^2x_2^i\times x_1x_2^{l-i}\\&\qquad+\sum_{0\le i\le l}x_1x_2^i\times x_1^2x_2^{l-i}+\sum_{0\le i\le l}x_2^i\times x_1^3x_2^{l-i},\\U_N=&\sum_{0\le i\le l+k}sy^i\times y^{l+k-i}+\sum_{0\le i\le l+k}y^i\times sy^{l+k-i}.\end{aligned}$$

Hence, and because $f^*(y^{l+1})=x_2^{l+1}=0$, we have

$$[U_M(1\times w_k(f))+(f\times f)^*U_N]_{n,k}=M\times(w_k(f)+f^*f_!(1))=0,$$

$$[U_M(1\times w_k(f))+(f\times f)^*U_N]_{n-1,k+1}=x_1^2x_2^l\times x_1x_2^k,$$

$$M\times f^*f_!(x_2^i)=[((f\times f)^*U_N)(x_2^i\times 1)]_{n,k+i}=M\times x_2^{k+i}.$$

Thus f does not satisfy the condition (1). But f satisfies (2), because $w_i(M) = \binom{l+1}{i} x_2^i$ and $f^* f_!(x_2^r) = x_2^{r+k} = x_2^r w_k(f)$. □

REMARK 8. The above example shows that a map f satisfying the conditions (0) and (2) is not necessarily R-bordant to an embedding, in particular that a map which is cobordant to a differentiable embedding is not necessarily R-bordant to a topological embedding.

EXAMPLE 3. Let $\pi\colon P^2 \to P^2/P^1 = S^2$ and $j\colon P^l \subset P^{l+k}$ be the natural projection and inclusion, respectively and let $f = \pi \times j\colon M = P^2 \times P^l \to S^2 \times P^{l+k}$. Then, if k is even, the relation $f^* f_!(1) = w_k(f)$ holds, however (2) does not hold.

Proof. As in Example 2, put

$$H^1(P^2) = \mathbf{Z}_2\langle x_1\rangle,\ \ H^1(P^l) = \mathbf{Z}_2\langle x_2\rangle,\ \ H^2(S^2) = \mathbf{Z}_2\langle s\rangle,\ \ H^1(P^{l+k}) = \mathbf{Z}_2\langle y\rangle.$$

Then

$$w_1(M) = x_1 + (l+1)x_2, \quad f^*(s) = x_1^2, \quad f^*(y) = x_2, \quad w_k(f) = x_2^k.$$

Just as in Example 2, we have

$$M \times (w_k(f) + f^* f_!(1)) = [U_M(1 \times w_k(f)) + (f \times f)^* U_N]_{n,k} = 0,$$

$$\begin{aligned} M \times (w_1(M)w_k(f) + f^* f_!(w_1(M))) \\ &= [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(w_1(M) \times 1)]_{n,k+1} \\ &= M \times x_1 x_2^k. \end{aligned}$$

Thus the relation $f^* f_!(1) = w_k(f)$ holds, however $f^* f_!(w_1(M)) \neq w_1(M)w_k(f)$. □

Proposition 4.1. *Assume that $f\colon M^n \to N^{n+k}$ satisfies the conditions that $w_i(f) = 0$ $(k < i)$ and $f^* f_!(1) = w_k(f)$, then*

$$f^* f_!(v_i(M)) = v_i(M)w_k(f) \quad (0 < i).$$

Proof. For each $x \in H^{n-k-i}(M)$, we have

$$\begin{aligned} x f^* f_!(v_i(M)) &= v_i(M) f^* f_!(x) \quad \text{by, e.g., [9, Lemma 2.1, (4)]} \\ &= Sq^i f^* f_!(x) \quad \text{because } \dim f^* f_!(x) = n - i \\ &= [Sq f^* f_!(x)]_n \end{aligned}$$

$$
\begin{aligned}
&= [f^* f_!(Sq(x)w(f))]_n \quad \text{by, e.g., [9, Lemma 2.1, (2)]} \\
&= f^* f_! \left(\sum_{0 \leq j} Sq^j(x) w_{i-j}(f) \right) \\
&= \sum_{0 \leq j} Sq^j(x) w_{i-j}(f) f^* f_!(1) \quad \text{by, e.g., [9, Lemma 2.1, (4)]} \\
&= \sum_{0 \leq j} Sq^j(x) w_{i-j}(f) w_k(f) \quad \text{because } f^* f_!(1) = w_k(f) \\
&= \sum_{0 \leq j} Sq^j(x) Sq^{i-j} w_k(f) \quad \text{because } w_i(f) = 0 \ (k < i) \\
&= Sq^i(x w_k(f)) = v_i(M) x w_k(f).
\end{aligned}
$$

Here, $[y]_j$ for $y \in \sum_{0 \leq i} H^i(M)$ means the j-dimensional component of y. Thus $x f^* f_!(v_i(M)) = x v_i(M) w_k(f)$ for all $x \in H^{n-k-i}(M)$. Hence $f^* f_!(v_i(M)) = v_i(M) w_k(f)$ by the Poincaré duality. □

For $k = 1$, the conditions (0) and (3) imply the condition (2), i.e. we have

Proposition 4.2. *Assume that* $f: M^n \to N^{n+1}$ *satisfies the conditions that* $w_i(f) = 0$ $(1 < i)$ *and* $f^* f_!(1) = w_1(f)$, *then for all sequences* I *of non-negative integers, we have*

$$f^* f_!(w_I(M)) = w_I(M) w_1(f).$$

Proof. By the assumption we have $\bar{w}(M) f^* w(N) = w(f) = 1 + w_1(f) = 1 + f^* f_!(1)$. Hence $w(M) = f^* w(N)(1 + f^* f_!(1))^{-1} = f^*(w(N)(1 + f_!(1))^{-1}) \in f^* H^*(N)$. Thus $w_I(M) \in f^* H^*(N)$ for all I, and therefore we obtain the result since $f_!(f^* y) = y f_!(1)$ for all $y \in H^*(N)$. □

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Yoshiyuki Kuramoto
Faculty of Education
Kagoshima University
Kagoshima 890-0065, Japan
e-mail: kuramoto@edu.kagoshima-u.ac.jp

Tsutomu Yasui
Faculty of Education
Kagoshima University
Kagoshima 890-0065, Japan
e-mail: yasui@edu.kagoshima-u.ac.jp