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CONSTRUCTION OF AFFINE PLANE CURVES WITH ONE PLACE AT INFINITY

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1. Introduction

Let C be an irreducible algebraic curve in complex affine plane $\mathbf{C}^2$. We say that C has *one place at infinity*, if the closure of C intersects with the ∞-line in $\mathbf{P}^2$ at only one point P and C is locally irreducible at that point P.

The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbf{C}^2$ has been studied by many mathematicians since Suzuki [17] and Abhyankar-Moh [2] proved independently that the canonical model of C is a line when C is non-singular and simply connected. Zaidenberg-Lin [19] proved that C has the canonical model of type $y^q = x^p$, where p and q are coprime integers > 1, when C is singular and simply connected. A'Campo-Oka [5] studied the case of genus $g \leq 3$ as an application of a resolution tower of toric modifications. For the case $g \leq 4$ Neumann [12] studied from the viewpoint of the link at infinity, and Miyanishi [9] studied from the algebrico-geometric viewpoint. Nakazawa-Oka [11] gave the classifications of all the canonical models for the case $g \leq 7$ using the result of A'Campo-Oka, and gave the classifications for the case $g \leq 16$ without proof. Jaworski [8] studied normal forms of irreducible germs of functions of two variables with given Puiseux pairs. Oka [14, 15] gave the normal form of plane curves which are locally irreducible at the origin and with a given sequence of weight vectors corresponding to the Tschirnhausen-good resolution tower, and showed that the moduli space of such curves is of the form $(\mathbf{C}^*)^a \times \mathbf{C}^b$. Furthermore, Oka translated this result to the case of affine curves with one place at infinity.

Also, Abhyankar-Moh [1, 3, 4] investigated properties of δ*-sequences* which are sequences of pole orders of *approximate roots* of C. This result is called Abhyankar-Moh's semigroup theorem. Sathaye-Stenerson [16] proved that if a sequence S of natural numbers satisfies Abhyankar-Moh's condition then there exists a curve with one place at infinity of the δ-sequence S. Suzuki [18] made it clear the relationship between the δ-sequence and the dual graph of the minimal resolution of the singularity of the curve C at infinity, and gave an algebrico-geometric proof of semigroup theorem and its inverse theorem due to Sathaye-Stenerson.

In this paper, we develop Suzuki's result and give an algebrico-geometric proof of Oka's result (Theorem 7 and Corollary 1). We shall also give an algorithm to compute the normal form and the moduli space of the curve with one place at infinity from a given δ-sequence[1].

Our construction method of normal forms is different from [8, 14, 15] in the following respects. First, this method uses δ-sequences generating semigroups of affine plane curves with one place at infinity. Second, this method directly generates defining polynomials at the origin of curves with one place at infinity.

2. Preparations

In this section, we introduce some definitions and facts which is needed to describe our theorem.

Let C be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbf{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. By the consideration of the Newton boundary, we can get

$$f(x, y) = (ux^p + vy^q)^d + \sum_{q\alpha+p\beta<pqd} c_{\alpha\beta} x^\alpha y^\beta,$$

where $u, v \in \mathbf{C}^*$, $m = pd$ and $n = qd$. By a finitely many times of the coordinate transformations of the form

$$\begin{cases} x_1 = x \\ y_1 = y + cx^p \end{cases}$$

and the exchange of the coordinates x and y, we can reduce the polynomial f into one of the following two types:

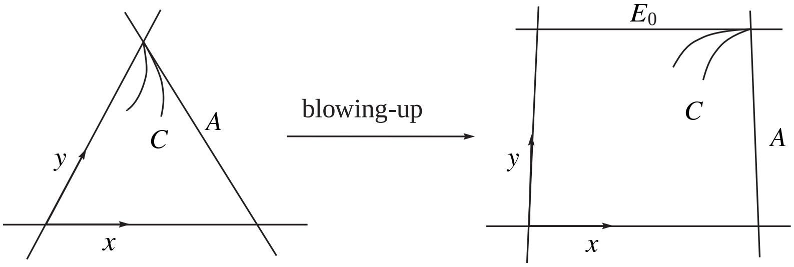
(A) $m = 1, n = 0$

(B) $m = pd, n = qd, \gcd(p, q) = 1, p > q > 1$.

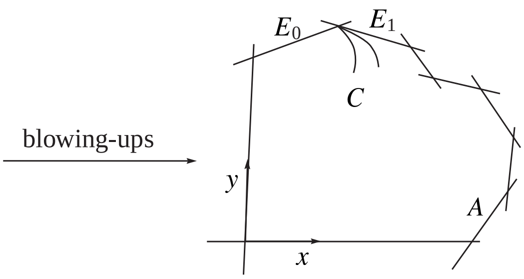
A curve of type (A) is a line. We call the curve of type (B) *non-linearlizable*. In this paper, we shall consider only the curves of the type (B) from now on. The closure $\overline{C}$ of C in the projective plane $\mathbf{P}^2$ passes through the intersection point O of the ∞-line A and the line $x = 0$ by the assumption $p > q$.

Let us denote by E_0 the (-1)-curve appeared by the blowing-up of the point O, and continue to denote the proper transform of A by the same character A. Let a be the natural number satisfying $aq < p < (a + 1)q$. If $a = 1$, then the proper transform of $\overline{C}$ is tangent to A, or else is tangent to E_0.

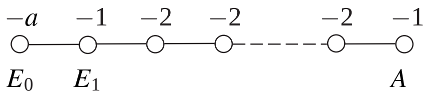
[1]The computer calculation by our algorithm verified the result of Nakazawa-Oka [11].



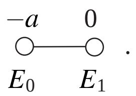
In case $a > 1$, after further $a - 1$ times of the blowing-ups of the point at infinity of the curve C, the proper transform of $\overline{C}$ is tangent to the (-1)-curve E_1 obtained by the last blowing-up. (In case $a = 1$, we set $E_1 = A$.)



Thus we get a compactification of $\mathbf{C}^2$ with the boundary curve of which the dual graph is of the following form:



By $a - 1$ times of the blowing-downs of the (-1)-curve on the right hand side from A of the above dual graph, we get the following dual graph:

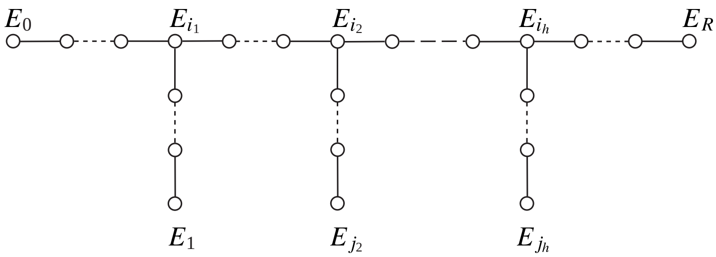


Let $(M_1,\ E_0 \cup E_1)$ be the compactification of $\mathbf{C}^2$ thus obtained.

The intersection point of E_0 and E_1 is the indetermination point of f. Now, we blow up from the surface M_1 the indetermination points of f successively, until the indetermination points of f disappear. Let M_f be the surface thus obtained. We denote the proper transform in M_f of E_0 (resp. E_1) by the same character E_0 (resp. E_1). Let $E_i\,(2 \leq i \leq R)$ be the proper transform in M_f of the (-1)-curve obtained by the $(i-1)$-th blowing-up. Furthermore, we set $E_f = E_0 \cup E_1 \cup \cdots \cup E_R$.

The following theorem about the compactification (M_f, E_f) of $\mathbf{C}^2$ is very important for the classification problem of the curves with one place at infinity.

Theorem 1 ([18]). (i) *The dual graph* $\Gamma(E_f)$ *of* E_f *has the following form*:



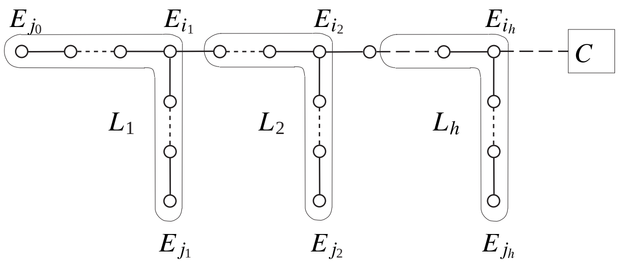
(ii) f *is non-constant only on* E_R *and has the pole on* $E_f - E_R$.
(iii) *The degree of* f *on* E_R *is* 1.
(iv) E_R *is the unique* (-1)-*curve in* E_f.

Note. There is a small gap in the proof of (i) described in [18]. Let Z (resp. P, S) be the union of the components of E_f on which $f = 0$ (resp. $f = \infty$, f =non-constant). Let T be the union of the other components of E_f. From the proof of (i) described in [18], we know that Z and P are both connected and $S = E_R$. Here, since f is non-zero constant on T, T does not intersect Z and P. If $T \neq \emptyset$, then T intersects only S. But since $S(= E_R)$ is the last (-1)-curve on M_f, the relations of intersection among Z, P, S and T is one of the following two types:

$$\text{(I) } P\text{—}S\text{—}Z \qquad \text{(II) } P\text{—}S\text{—}T.$$

If $Z \neq \emptyset$, then we get the contradiction as it is described in [18]. The similar argument applies to the case of $T \neq \emptyset$. Thus we get $Z = \emptyset$ and $T = \emptyset$. As a consequence, $\Gamma(E_f)$ has the above form.

In $\Gamma(E_f)$, let $i_1, i_2, \ldots, i_h$ (resp. $j_0, j_1, \ldots, j_h$) be the indices of the branch vertices (resp. the terminal vertices) from the left hand side, where $j_0 = 0$ and $j_1 = 1$. Let M_C be the surface obtained by the blowing-down of $E_R, E_{R-1}, \ldots, E_{i_h+1}$ from M_f. For $i\,(0 \leq i \leq i_h)$, we shall continue to denote by E_i the proper transform of E_i in M_C. Further, we set $E_C = E_0 \cup E_1 \cup \cdots \cup E_{i_h}$. We shall call the pair (M_C, E_C) the compactification of $\mathbf{C}^2$ obtained by the *minimal resolution* of the singularity of C at infinity. We set $L_k = \bigcup_{i_{k-1}<i\leq i_k} E_i$ for each $k\,(1 \leq k \leq h)$ like the following figure, where $i_0 = -1$.

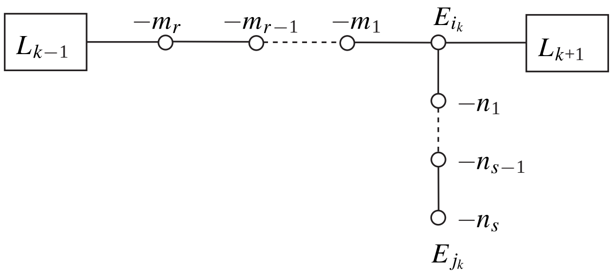


DEFINITION 1 (δ-sequence). Let δ_k $(0 \le k \le h)$ be the order of the pole of f on E_{j_k}. We shall call the sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ the *δ-sequence* of C (or of f).

We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

Fact 1. $\delta_0 = n$, $\delta_1 = m$.

DEFINITION 2 ((p, q)-sequence). Now, we assume that the weights of L_k is of the following form:



We define the natural numbers p_k, a_k, q_k, b_k satisfying

$$(p_k, a_k) = 1,\ (q_k, b_k) = 1,\ 0 < a_k < p_k,\ 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

We shall call the sequence $\{(p_1, q_1), (p_2, q_2), \dots, (p_h, q_h)\}$ the *(p, q)-sequence* of C (or of f).

We shall assume that $f(x, y)$ is monic in y. We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots). Let $f(x, y)$ be the defining polynomial, monic in y, of a curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the δ-sequence of f. We set $n = \deg_y f$, $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ and $n_k = n/d_k$ $(1 \le k \le h+1)$. Then, for each k $(1 \le k \le h+1)$, a pair of polynomials $(g_k(x, y), \psi_k(x, y))$ satisfying the following conditions is uniquely determined:
(i) g_k is monic in y and $\deg_y g_k = n_k$,
(ii) $\deg_y \psi_k < n - n_k$,
(iii) $f = g_k^{d_k} + \psi_k$.
We call this g_k the *k-th approximate root* of f.

We can easily get the following fact from the definition of approximate roots.

Fact 2. *We have*

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \quad g_{h+1} = f$$

where $c_k \in \mathbf{C}$, $p = \deg_x f/d$, $q = \deg_y f/d$, $d = \gcd\left\{\deg_x f, \deg_y f\right\}$ *and* $\lfloor p/q \rfloor$ *is the maximal integer* l *such that* $l \leq p/q$.

DEFINITION 4 (g-sequence). The sequence of polynomials $g_0 := x$, $g_1, \dots, g_{h+1}$ is called the *g-sequence* of f.

Here, we denote by C_k the curve defined by $g_k(x, y) = 0$ in $\mathbf{C}^2$. The following theorem about C_k plays a vital role in the main theorem.

Theorem 2. *For each* $k\,(0 \leq k \leq h)$, C_k *is also with one place at infinity. Further, its closure* $\overline{C}_k$ *in* M_C *intersects transversely* E_{j_k}, *and does not intersect other irreducible components of* E_C.

Suzuki [18] gave the algebrico-geometric proof of this theorem. We get the following theorem as a corollary of the above theorem.

Theorem 3. *For each* $k\,(0 \leq k \leq h)$, g_k *has the pole of order* δ_k *on* E_{i_h}.

The following lemma about approximate roots will be used in Theorem 6.

Lemma 1. *Let* f *be the defining polynomial, monic in* y, *of a curve with one place at infinity. Let* $\{\delta_0, \delta_1, \dots, \delta_h\}$ *be the* δ*-sequence of* f, *and* $g_0, g_1, \dots, g_h, g_{h+1}$ *be the* g*-sequence of* f. *Then,* $g_k\,(0 \leq k \leq h-1)$ *is also the* k*-th approximate root of* g_j *for any* j *with* $k < j < h+1$.

Proof. For example, see Proposition 2.2 in [5]. □

3. Intersection matrix and successive blow-up

Let M be a non-singular projective algebraic surface over complex number field, and E be an algebraic curve on M. We shall assume that $E_1, E_2, \dots, E_s$ are irreducible components of E, and denote by I_E the intersection matrix $((E_i \cdot E_j))_{i,j=1,\dots,s}$ of E. The following lemma about the intersection matrix is well-known by Mumford.

Lemma 2. *E is an exceptional set if and only if I_E is negative definite.*

Let E'_1 be the (-1)-curve appeared by blowing-up at a point P_0 on a surface M, and let P_1 be a point on E'_1. For $i\,(\geq 1)$, let E'_{i+1} be the (-1)-curve appeared by blowing-up at a point P_i, and let P_{i+1} be the point on E'_{i+1}. We get $\{P_i\}_{i=0,\dots,r}$ and $\{E'_i\}_{i=1,\dots,r}$ by the above finite operations. In this paper we call this finite sequence of blowing-ups a *successive blow-up from* P_0. Let M' be the surface obtained by a successive blow-up from P_0. For $i\,(1 \leq i \leq r)$, we shall continue to denote by E'_i the proper transform of E'_i in M'. Further, we set $E' = \bigcup_{i=1}^r E'_i$ and $\Delta_{E'} = \det(-I_{E'})$. We have the following fact since $\Delta_{E'}$ is invariant under the successive blow-up.

Fact 3. $\Delta_{E'} = 1$.

The following lemma is Lemma 1 in [18]. Here, we describe it because it is used many times in the next section.

Lemma 3. *Let $E_1, E_2, \dots, E_r, E_{r+1}$ be the irreducible components of E and assume that the dual graph $\Gamma(E)$ is of the following linear type*:

$$\begin{array}{ccccccc} E_1 & & E_2 & & E_r & & E_{r+1} \\ \circ & \text{------} & \circ & - - - - & \circ & \text{------} & \circ \\ -n_1 & & -n_2 & & -n_r & & \end{array} \qquad (n_i \geq 2).$$

Assume further that there exists a holomorphic function f on a neighborhood U of $\bigcup_{i=1}^r E_i$ such that the zero divisor (f) of f on U is written in the following form:

$$\sum_{i=1}^r m_i E_i + m_{r+1} E_{r+1} \cap U\,.$$

Let (p_i, p_{i+1}) be the coprime integers defined by the following continued fraction:

$$\frac{p_{i+1}}{p_i} = n_i - \cfrac{1}{n_{i-1} - \ddots - \cfrac{1}{n_1}} \quad (1 \leq i \leq r).$$

Then, $m_i = m_1 p_i \; (1 \leq i \leq r+1)$.

Now, consider a pair of natural numbers (p, q) with $\gcd(p, q) = 1$, $p > q > 0$. We can easily show that there exists a unique pair of natural numbers (a, b) with $pq - aq - bp = 1$, $0 < a < p$, $0 < b < q$.

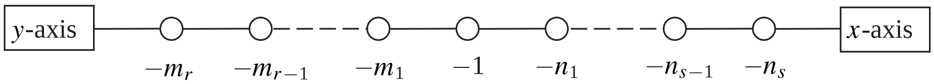
We consider the following continued fractions for the above mentioned p, q, a, b:

$$\frac{p}{a} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}}, \quad \frac{q}{b} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}},$$

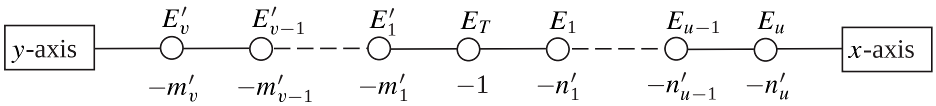
where $m_i \geq 2$ and $n_j \geq 2$.

Let (x, y) be the local coordinate for the neighborhood of a point P on M which has P as the origin. Then,

Lemma 4. *we can construct a exceptional curve with the following weights by a successive blow-up from* P.



Proof. We consider the curve C defined by $x^p + y^q = 0$. The resolution graph at origin of C is as follows:



Let I_E be the intersection matrix of the exceptional curve E corresponding to the above dual graph. Here, we set

$$\frac{p'}{a'} = m'_1 - \cfrac{1}{m'_2 - \ddots - \cfrac{1}{m'_u}}, \quad \frac{q'}{b'} = n'_1 - \cfrac{1}{n'_2 - \ddots - \cfrac{1}{n'_v}}.$$

We get $\det(-I_E) = p'q' - a'q' - b'p'$. On the other hand, E is the exceptional curve obtained by a successive blow-up from origin. Therefore, we get $\det(-I_E) = 1$ by Fact 3. Thus $p'q' - a'q' - b'p' = 1$.

As the above dual graph, let $E_i\,(1 \leq i \leq u)$, E_T, $E'_j\,(1 \leq j \leq v)$ be the irreducible components of E. We denote by $\mu_i\,(1 \leq i \leq u)$ the zero order of the function x on E_i and by μ_T the zero order of the function x on E_T. Also, we denote by $\nu_j\,(1 \leq j \leq v)$ the zero order of the function y on E'_j and by ν_T the zero order of the function y on E_T. Since $q = \mu_T$ and $\mu_u = 1$, we get $q' = \mu_T/\mu_u = q$ by Lemma 3. As the same way, we get $p = p'$. Thus $pq - a'q - b'p = 1$. Further, it must be $a = a'$, $b = b'$, since $0 < a' < p$ and $0 < b' < q$. Therefore, we get $v = r$, $m'_i = m_i\,(1 \leq i \leq r)$, $u = s$,

$n'_j = n_j\,(1 \le j \le s)$ by the uniqueness of the expansion into continued fraction. As a result, the assertion was proved. □

4. Construction of a curve with one place at infinity

We set $\mathbf{N} = \{n \in \mathbf{Z} \mid n \ge 0\}$ and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. The following theorem about δ-sequence and (p, q)-sequence is called Abhyankar-Moh's Semigroup Theorem.

Theorem 4 (Abhyankar-Moh). *Let C be a non-linearlizable affine plane curve with one place at infinity. Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ be the δ-sequence of C and $\{(p_1, q_1), \ldots, (p_h, q_h)\}$ be the (p, q)-sequence of C. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h + 1)$. We have then,*

(i) $q_k = d_k/d_{k+1},\ d_{h+1} = 1\,(1 \le k \le h)$,

(ii) $d_{k+1}p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \le k \le h) \end{cases}$,

(iii) $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}\ (1 \le k \le h)$.

The following theorem gives the converse of the above theorem.

Theorem 5 (Sathaye-Stenerson [16]). *Let $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \ge 1)$ be the sequence of $h + 1$ natural numbers. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}\,(1 \le k \le h + 1)$ and $q_k = d_k/d_{k+1}\,(1 \le k \le h)$. Furthermore, suppose that the following conditions are satisfied*:

(1) $\delta_0 < \delta_1$,

(2) $q_k \ge 2\,(1 \le k \le h)$,

(3) $d_{h+1} = 1$,

(4) $\delta_k < q_{k-1}\delta_{k-1}\,(2 \le k \le h)$,

(5) $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}\ (1 \le k \le h)$.

Then, there exists a curve with one place at infinity of the δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$.

Suzuki [18] gave an algebrico-geometric proof of the above two theorem by the consideration of the resolution graph at infinity.

DEFINITION 5 (Abhyankar-Moh's condition). We shall call the conditions (1)–(5) concerning $\{\delta_0, \delta_1, \ldots, \delta_h\}$ in Theorem 5 *Abhyankar-Moh's condition*.

Theorem 6. *Let $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \ge 1)$ be the sequence of $h + 1$ natural numbers satisfying Abhyankar-Moh's condition. Set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h + 1)$ and $q_k = d_k/d_{k+1}\,(1 \le k \le h)$. Then,*

(i) *the defining polynomial f, monic in y, of a curve with one place at infinity of* *the δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$ has the following form using the approximate roots $g_0, g_1, \ldots, g_h$ of f:*

$$f = g_h^{q_h} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \ldots, \alpha_h)\in\Lambda} c_{\alpha_0\alpha_1\cdots\alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h}$$

where $a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} \in \mathbf{C}^*$, $c_{\alpha_0\alpha_1\cdots\alpha_h} \in \mathbf{C}$, $(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{h-1})$ is the sequence of h non-negative integers satisfying

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \ \bar{\alpha}_i < q_i \ (0 < i < h)$$

and

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \ldots, \alpha_h) \in \mathbf{N}^{h+1} \;\middle|\; \alpha_i < q_i \ (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h \right\}.$$

(ii) *Conversely, let g_h be the defining polynomial, monic in y, of a curve with one place at infinity of the δ-sequence $\{\delta_0/q_h, \delta_1/q_h, \ldots, \delta_{h-1}/q_h\}$, and $g_0, g_1, \ldots, g_{h-1}$ be the approximate roots of g_h. For any non-zero complex number $a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}}$ corresponding to the sequence of h non-negative integers $(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{h-1})$ satisfying*

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \quad \bar{\alpha}_i < q_i \ (0 < i < h)$$

and any complex numbers $c_{\alpha_0\alpha_1\cdots\alpha_h}$ corresponding to the sequences of $h+1$ non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_h)$ satisfying

$$\sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h, \quad \alpha_i < q_i \ (0 < i < h), \quad \alpha_h < q_h - 1,$$

we consider

$$f = g_h^{q_h} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \ldots, \alpha_h)\in\Lambda} c_{\alpha_0\alpha_1\cdots\alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h}$$

where

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \ldots, \alpha_h) \in \mathbf{N}^{h+1} \;\middle|\; \alpha_i < q_i \ (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h \right\}.$$

Then, the curve defined by $f = 0$ is a curve with one place at infinity of the δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$, and has the approximate roots $g_0, g_1, \ldots, g_h$.

Proof of Theorem 6. We shall prove (i). By the procedure described in the proof of Proposition 10 in [18], using the approximate roots $g_0, g_1, \ldots, g_h$ of f and the set of $h+1$ non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_h)$ with $\max\{\sum_{i=0}^{h} \alpha_i \delta_i\} = q_h\delta_h$, we can write f as follows:

$$f = \sum_{\alpha_i < q_i (1 \le i \le h)} c_{\alpha_0\alpha_1\cdots\alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} + g_h^{q_h}, \quad c_{\alpha_0\alpha_1\cdots\alpha_h} \in \mathbf{C}.$$

Here, we suppose $f = g_h^{q_h} + g_h^{q_h-1}$. We have $\deg_y g_h^{q_h-1} = n_h(q_h-1) = \deg_y f - n_h = n - n_h$. But this is a contradiction, since g_h is h-th approximate root of f. Thus we get $\alpha_h < q_h - 1$. By Theorem 4(iii) and the uniqueness of $\{\alpha_i\}_{i=0,\ldots,h}$ (e.g., Lemma 7 in [18]), we have $\{\overline{\alpha}_i\}_{i=0,\ldots,h-1}$ with $\sum_{i=0}^{h-1} \overline{\alpha}_i \delta_i = q_h\delta_h$. As a result, (i) was proved.

We shall prove (ii).

CASE $h = 1$. Set $\delta_0 = q$ and $\delta_1 = p$. We can write f as follows:

$$f = y^q + ax^p + \sum_{q\alpha + p\beta < pq} c_{\alpha\beta} x^\alpha y^\beta, \quad a \in \mathbf{C}^*, \quad c_{\alpha\beta} \in \mathbf{C}.$$

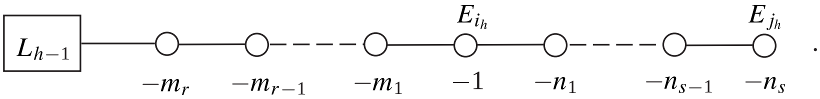
The curve defined by $f = 0$ has one place at infinity of the δ-sequence $\{q, p\}$ by the consideration of Newton boundary.

CASE $h \ge 2$. Set $\delta_i/q_h = \tilde{\delta}_i$ $(0 \le i \le h-1)$. We denote by C_k the curve defined by $g_k = 0$ for each k with $0 \le k \le h$. Further, we shall denote by $(\tilde{M}, \tilde{E})$ the compactification of $\mathbf{C}^2$ obtained by the minimal resolution of C_h at infinity. Let $\tilde{C}_k$ be the proper transform of C_k on $\tilde{M}$ and $\tilde{E}_i$ be the irreducible components of $\tilde{E}$. (The way of numbering about indices is same as Section 2.) By Theorem 2, $\tilde{C}_k$ has one place at infinity and intersects transversely $\tilde{E}_{j_k}$ $(0 \le k \le h-1)$.

Let Q be the intersection point of $\tilde{C}_h$ and $\tilde{E}_{i_{h-1}}$. Set $p_h = q_{h-1}\delta_{h-1} - \delta_h$. ($p_h > 0$ since Abhyankar-Moh's condition (4).) We have $\gcd(p_h, q_h) = 1$ from $\gcd(q_h, \delta_h) = d_{h+1} = 1$ and get a unique pair of natural numbers (a_h, b_h) with $p_h q_h - a_h q_h - b_h p_h = 1$, $0 < a_h < p_h$, $0 < b_h < q_h$. We define $\{m_i\}_{i=1,\ldots,r}$, $\{n_j\}_{j=1,\ldots,s}$ using the following expansion into continued fractions by p_h, a_h, q_h, b_h:

$$\frac{p_h}{a_h} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}}, \qquad \frac{q_h}{b_h} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

By Lemma 4, we can obtain the following branch L_h such that C_h intersects transversely E_{j_h} using the successive blow-up from Q:



Let M be the surface thus obtained, E be the total transform of $\tilde{E}$ on M. We denote by E_i (resp. $\overline{C}_k$) the proper transform of $\tilde{E}_i$ (resp. $\tilde{C}_k$).

Set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}\,(1 \le k \le h+1)$ and $q_k = d_k/d_{k+1}\,(1 \le k \le h)$. By Theorem 3, g_k has the pole of order $\tilde{\tilde{\delta}}_k$ on $E_{i_{h-1}}$ for each $k\,(0 \le k \le h-1)$. Thus g_k has the pole of order $\tilde{\tilde{\delta}}_k$ on E_{j_h} and of order $q_h\tilde{\tilde{\delta}}_k (= \delta_k)$ on E_{i_h}. On the other hand, g_h has the pole of order δ_h on E_{i_h}. In fact, we can write g_h on a neighborhood of Q as follows:

$$g_h = \frac{v}{u^{q_{h-1}\tilde{\tilde{\delta}}_{h-1}}} \times (\text{non-const}).$$

Hence g_h has the pole of order $q_h\,(q_{h-1}\tilde{\tilde{\delta}}_{h-1}) - p_h$ on E_{i_h}. This value is equal to δ_h by the assumption of p_h.

Now, we consider the curve C defined by $f = 0$. Set $\phi = f - g_h^{q_h}$ and $\Phi = \phi/g_h^{q_h}$. Since the both of $g_h^{q_h}$ and ϕ has the pole of order $q_h\delta_h$ on E_{i_h}, Φ is non-constant or constant$(\neq 0)$ on E_{i_h}.

Let A (resp. B) be the closure of the connected component of $E - E_{i_h}$ which contains E_0 (resp. E_{j_h}). Let P_{g_h} be the pole divisor of g_h on M, and D be its restriction to A. Here, let F_1 be the irreducible component of A intersecting E_{i_h}. Since g_h has the pole of order δ_h on E_{i_h}, we have $(D \cdot F_1) < 0$. Also, since $(D \cdot E_i) = 0$ for any E_i with $E_i \neq F_1$, using Proposition 2 in [6], the intersection matrix of A is negative definite. Thus it follows that A is exceptional set. Φ is holomorphic on A since $A \cap \overline{C}_h = \emptyset$. On the other hand,

$$\begin{aligned}
&\deg_y g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} \\
&\qquad = \sum_{i=0}^{h} \alpha_i n_i = \sum_{i=1}^{h} \alpha_i n_i \\
&\qquad = \alpha_1 + \alpha_2 q_1 + \alpha_3 q_2 q_1 + \cdots + \alpha_h q_{h-1} \cdots q_1 \\
&\qquad < (q_1 - 1) + (q_2 - 1)q_1 + (q_3 - 1)q_2 q_1 + \cdots + (q_h - 1)q_{h-1} \cdots q_1 \\
&\qquad = q_h q_{h-1} \cdots q_1 - 1 \\
&\qquad < q_h q_{h-1} \cdots q_1 = q_h n_h = \deg_y g_h^{q_h}.
\end{aligned}$$

Therefore, we get $\deg_y \phi < \deg_y g_h^{q_h}$. Hence, $\Phi = 0$ on E_0. Further, $\Phi = 0$ on A, since A is compact. As a result, it must be that Φ is non-constant on E_{i_h}.

Let P_Φ be the pole divisor of Φ on M. We denote by $B_1, B_2, \dots, B_s$ the irreducible components of B in order from the component intersecting E_{i_h}. Since Φ has the pole on B_1 and $\overline{C}_h$, the support of P_Φ is $B \cup \overline{C}_h$ and we can write $P_\Phi =$

$q_h\overline{C}_h + \sum_{i=1}^{s} \mu_i B_i\ (\mu_i > 0)$. By

$$n_s - \cfrac{1}{n_{s-1} - \ddots - \cfrac{1}{n_1}} = \frac{q_h}{b'}$$

and Lemma 3, we get $\mu_1 q_h = q_h$, where μ_1 is the pole order of Φ on B_1. Hence, $\mu_1 = 1$. This implies that Φ is a rational function of degree 1 on E_{i_h}. Therefore, the curve defined by $\Phi = -1$ intersects transversely E_{i_h} at only one point. Since the curve $\Phi = -1$ coincides with $\overline{C}$, we get

$$(\overline{C} \cdot E_i) = \begin{cases} 1 & (i = i_h) \\ 0 & (i \neq i_h) \end{cases} .$$

As a result, C has one place at infinity.

We have $f = g_h^{q_h}$ on A, since $\Phi = 0$ on A. Hence, f has the pole of the same order as $g_h^{q_h}$ on each irreducible component of A. In particular, f has the pole of order $q_h\tilde{\delta}_k = \delta_k$ on each $E_{i_k}\ (0 \leq k \leq h-1)$. Since Φ is non-constant on E_{i_h}, f has the pole of the same order as $g_h^{q_h}$ on E_{i_h}. Since the value of its pole order is $q_h\delta_h$, using Lemma 3, it follows that f has the pole of order δ_h on E_{j_h}. Consequently, $\{\delta_0, \delta_1, \ldots, \delta_h\}$ is the δ-sequence of f.

Finally, we show that $g_0, g_1, \ldots, g_h$ are the approximate roots of f. By

$$\begin{aligned} &\deg_y g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} \\ &\quad = n_0\alpha_0 + n_1\alpha_1 + \cdots + n_h\alpha_h \\ &\quad \leq n_1(q_1 - 1) + n_2(q_2 - 1) + \cdots + n_{h-1}(q_{h-1} - 1) + n_h(q_h - 2) \\ &\quad = -n_1 + n_h q_h - n_h < n - n_h, \end{aligned}$$

g_h is h-th approximate root of f. Therefore, by Lemma 1, $g_0, g_1, \ldots, g_h$ are the approximate roots of f. □

The following theorem is the main theorem in this paper, and is obtained by using Theorem 6 inductively.

Theorem 7. *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}\ (h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition* (*see* Definition 5). *Set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h+1)$ *and* $q_k = d_k/d_{k+1}\ (1 \leq k \leq h)$.

(1) *We define* $g_k\ (0 \leq k \leq h+1)$ *as follows*:

$$\begin{cases} g_0 = x, \\ g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbf{C},\ p = \dfrac{\delta_1}{d_2},\ q = \dfrac{\delta_0}{d_2}, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \qquad + \sum_{(\alpha_0,\alpha_1,\dots,\alpha_i)\in\Lambda_i} c_{\alpha_0\alpha_1\cdots\alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_i^{\alpha_i}, \\ \qquad a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} \in \mathbf{C}^*,\ c_{\alpha_0\alpha_1\cdots\alpha_i} \in \mathbf{C} \quad (1 \leq i \leq h), \end{cases}$$

where $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1})$ *is the sequence of* i *non-negative integers satisfying*

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i,\ \bar{\alpha}_j < q_j\ (0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_i) \in \mathbf{N}^{i+1} \,\middle|\, \alpha_j < q_j\ (0 < j < i), \alpha_i < q_i - 1, \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.$$

Then, $g_0, g_1, \dots, g_h$ *are approximate roots of* $f (= g_{h+1})$, *and* f *is the defining polynomial, monic in* y, *of a curve with one place at infinity of the* δ*-sequence* $\{\delta_0, \delta_1, \dots, \delta_h\}$.

(2) *The defining polynomial* f, *monic in* y, *of a curve with one place at infinity of the* δ*-sequence* $\{\delta_0, \delta_1, \dots, \delta_h\}$ *is obtained by the procedure of* (1), *and the values of parameters* $\{a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}}\}_{1\leq i\leq h}$ *and* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\leq i\leq h}$ *are uniquely determined for* f.

The above theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

Corollary 1. *Let* $\{\delta_0, \delta_1, \dots, \delta_h\}\ (h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition. The moduli space of the curve* C *with one place at infinity of the* δ*-sequence* $\{\delta_0, \delta_1, \dots, \delta_h\}$ *is isomorphic to*

$$(\mathbf{C}^*)^h \times \mathbf{C}^b,$$

where b *is the total number of parameters* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\leq i\leq h}$ *appeared in the defining polynomial, monic in* y, *of* C *obtained in* Theorem 7.

Proof. We consider the defining polynomial f, monic in y, of the curve C with one place at infinity of the δ-sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$. We denote by a the number of non-zero parameters in f and by b the number of others. By Theorem 7, the moduli space of C is $(\mathbf{C}^*)^a \times \mathbf{C}^b$. f has $h+2$ polynomials $g_0, g_1, \ldots, g_{h+1}$. Here, both of g_0 and g_1 do not have non-zero parameter. Also, $g_{i+1}\,(1 \leq i \leq h)$ has exactly one non-zero parameter because the sequence of $i+1$ non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_i)$ with $\sum_{j=0}^{i} \alpha_j \tilde{\delta}_j = q_i \tilde{\delta}_i$ is determined uniquely. As a result, we get $a = h$. □

By the above results, we can easily get an algorithm generating the defining polynomial and computing the moduli space from a δ-sequence. We will introduce them in the next section.

5. Algorithms

Using Theorem 7, the following algorithm generating the defining polynomial of the curve with one place at infinity from a δ-sequence is obtained.

Algorithm 1: generating polynomial
Input: δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$
Output: the defining polynomial $f(x, y)$ of the curve with one place at infinity of the δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$

$D \leftarrow [\delta_h, \delta_{h-1}, \ldots, \delta_0]$
$d_k \leftarrow \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}\,(1 \leq k \leq h+1)$
$Q \leftarrow [q_h, \ldots, q_1]$ where $q_k = d_k/d_{k+1}\,(1 \leq k \leq h)$
$DL \leftarrow cons(D, [\ ])$
$QL \leftarrow cons(Q, [\ ])$
$m \leftarrow h+1$
while $m \neq 2$ **do**
  $T \leftarrow reverse(cdr(D))$
  $D \leftarrow [\ ]$
  while $T \neq [\ ]$ **do**
    $D \leftarrow cons(car(T)/car(Q), D)$
    $T \leftarrow cdr(T)$
  end
  $DL \leftarrow cons(D, DL)$
  $Q \leftarrow cdr(Q)$
  $QL \leftarrow cons(Q, QL)$
  $m \leftarrow length(D)$
end
$AL \leftarrow [x]$
$D \leftarrow car(DL)$
$l \leftarrow \lfloor car(D)/car(cdr(D)) \rfloor$
$g_1 \leftarrow y + \sum_{j=0}^{l} c_j x^j$

$AL \leftarrow cons(g_1, AL)$
while $DL \neq [\ ]$ **do**
$D \leftarrow car(DL)$
$Q \leftarrow car(QL)$
$q_0 \leftarrow \lfloor car(Q) \times car(D)/car(reverse(D)) \rfloor + 1$
$L \leftarrow append(Q, [q_0])$
$k \leftarrow length(D) - 1$, i.e., $D = [\overline{\delta}_k, \ldots, \overline{\delta}_0]$, $L = [q_k, \ldots, q_0]$.
$(\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{k-1}) \leftarrow$ the sequence of non-negative integers with
$\sum_{i=0}^{k-1} \overline{\alpha}_i \overline{\delta}_i = \overline{\delta}_k q_k$, $\overline{\alpha}_i < q_i$ $(0 \leq i \leq k-1)$, $\overline{\delta}_i \in D$ and $q_i \in L$
$\{(\alpha_0, \alpha_1, \ldots, \alpha_k)\} \leftarrow$ the set of sequences of non-negative integers with
$\sum_{i=0}^{k} \alpha_i \overline{\delta}_i < \overline{\delta}_k q_k$, $\alpha_i < q_i$ $(0 \leq i < k)$, $\alpha_k < q_k - 1$, $\overline{\delta}_i \in D$ and $q_i \in L$
$g_{k+1} \leftarrow g_k^{q_k} + a_{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{k-1}} \prod_{i=0}^{k-1} g_i^{\bar{\alpha}_i} + \sum c_{\alpha_0, \alpha_1, \ldots, \alpha_k} \prod_{i=0}^{k} g_i^{\alpha_i}$
$AL \leftarrow cons(g_{k+1}, AL)$
$DL \leftarrow cdr(DL)$
$QL \leftarrow cdr(QL)$
end
return $car(AL)$

SUPPLEMENTATION:

- $[\ldots]$:= A list. (This is a data structure with ordered elements.)
- $\lfloor p \rfloor$:= The maximal integer n such that $n \leq p$.
- $car(L)$:= The first element of a given non-null list L.
- $cdr(L)$:= The list obtained by removing the first element of a given non-null list L.
- $cons(A, L)$:= The list obtained by adding an element A to the top of a given list L.
- $reverse(L)$:= The reversed list of a given list L.
- $append(L_1, L_2)$:= The list obtained by adding all elements in a list L_2 according to the order as it is to the last element in a list L_1.
- $length(L)$:= The number of elements of a given list L.
- $a_{*,*,\ldots,*}$ is a parameter in $\mathbf{C}^*$.
- $c_{*,*,\ldots,*}$ is a parameter in $\mathbf{C}$.

The moduli space of f is obtained by counting the numbers of $\{a_{*,*,\ldots,*}\}$ and $\{c_{*,*,\ldots,*}\}$ in f which the above algorithm outputted. But we can compute the moduli space from a δ-sequence without generating the defining polynomial. The following algorithm directly compute the moduli space from a δ-sequence.

Algorithm 2: computation of moduli space
Input: δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$
Output: $[M, N]$ (This means the moduli space $(\mathbf{C}^*)^M \times \mathbf{C}^N$.)

$D \leftarrow [\delta_h, \delta_{h-1}, \ldots, \delta_0]$

$d_k \leftarrow \gcd\{\delta_0,\ \delta_1,\ \ldots,\ \delta_{k-1}\}\ (1 \le k \le h+1)$
$Q \leftarrow [q_h,\ \ldots,\ q_1]$ where $q_k = d_k/d_{k+1}\ (1 \le k \le h)$
$Q \leftarrow cons(1, Q)$
$M \leftarrow h$
$N \leftarrow 0$
while true do
  $k \leftarrow length(D) - 1$, i.e., $D = [\bar{\delta}_k, \ldots, \bar{\delta}_0]$
  $D \leftarrow [\bar{\delta}_k/car(Q),\ \bar{\delta}_{k-1}/car(Q),\ \ldots,\ \bar{\delta}_0/car(Q)]$
  $Q \leftarrow cdr(Q)$
  $q_0 \leftarrow \lfloor car(Q) \times car(D)/car(reverse(D)) \rfloor + 1$
  $L \leftarrow append(Q, [q_0])$, i.e., $L = [q_k, \ldots, q_0]$
  $n \leftarrow$ the number of $(\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_k)$ with $\sum_{i=0}^{k} \alpha_i \bar{\delta}_i < car(Q) \times car(D)$,
    $\alpha_i < q_i\ (0 \le i \le k-1)$, $\alpha_k < q_k - 1$, $\bar{\delta}_i \in D$ and $q_i \in L$
  $N \leftarrow N + n$
  if $length(D) = 2$ **then break**
  $D \leftarrow cdr(D)$
end
$N \leftarrow N + \lfloor p/q \rfloor + 1$
return $[M, N]$

6. Polynomial curve

6.1. Abhyankar's question. In this section, we will introduce Abhyankar's question.

DEFINITION 6 (planar semigroup). Let $\{\delta_0,\ \delta_1,\ \ldots,\ \delta_h\}\ (h \ge 1)$ be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by $\{\delta_0,\ \delta_1,\ \ldots,\ \delta_h\}$ is said to be *a planar semigroup*.

DEFINITION 7 (polynomial curve). Let C be an algebraic curve defined by $f(x, y) = 0$, where $f(x, y)$ is an irreducible polynomial in $\mathbf{C}[x, y]$. We call C *a polynomial curve*, if C has a parametrisation $x = x(t)$, $y = y(t)$, where $x(t)$ and $y(t)$ are polynomials in $\mathbf{C}[t]$.

Abhyankar's Question. Let Ω be a planar semigroup. Is there a polynomial curve with δ-sequence generating Ω?

This question is still open. Moh [10] showed that there is no polynomial curve with δ-sequence $\{6, 8, 3\}$. But there is a polynomial curve $(x, y) = (t^3, t^8)$ with δ-sequence $\{3, 8\}$ which generates the same semigroup as above. Sathaye-Stenerson [16] proved that the semigroup generated by $\{6, 22, 17\}$ has no other δ-sequence generating the same semigroup, and proposed the following conjecture for this question.

Sathaye-Stenerson's Conjecture. There is no polynomial curve having the δ-sequence $\{6, 22, 17\}$.

By Algorithm 1, the defining polynomial of the curve with one place at infinity of the δ-sequence $\{6, 22, 17\}$ as follows:

$$\begin{aligned} f &= (g_2^2 + a_{2,1}x^2 g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\ &\quad + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0} \end{aligned}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

This result gives us a new approach to investigate the curve with one place at infinity of the δ-sequence $\{6, 22, 17\}$ using a computer algebra system.

6.2. Computation of moduli space. Suzuki gave an algorithm generating the list of δ-sequences of curves with one place at infinity, and implemented on a computer. From the list of δ-sequences obtained by Suzuki, we could get normal forms and moduli spaces of curves with one place at infinity of genus ≤ 100 by using the algorithm introduced in previous section. As a result, we could verify the result of Nakazawa-Oka [11].

The following is the list of moduli spaces of curves with one place at infinity for the cases genus ≤ 30.

EXAMPLE 1. The case

```
[7,[4,6,11],[2,15]]
```

means that the moduli space of the curve with one place at infinity of genus 7 and the δ-sequence $\{4, 6, 11\}$ is isomorphic to $(\mathbf{C}^*)^2 \times \mathbf{C}^{15}$.

```
[1,[2,3],[1,5]],
[2,[2,5],[1,8]],
[3,[2,7],[1,11]],
[3,[3,4],[1,9]],
[3,[4,6,3],[2,7]],
[4,[2,9],[1,14]],
[4,[3,5],[1,11]],
[4,[4,6,5],[2,9]],
[4,[6,9,2],[2,7]],
[5,[2,11],[1,17]],
[5,[4,6,7],[2,11]],
[6,[2,13],[1,20]],
[6,[3,7],[1,15]],
[6,[4,5],[1,14]],
[6,[4,6,9],[2,13]],
[6,[6,9,4],[2,10]],
[6,[4,10,5],[2,11]],
[7,[2,15],[1,23]],
[7,[3,8],[1,17]],
[7,[4,6,11],[2,15]],
[7,[6,9,5],[2,12]],
[7,[8,12,3],[2,10]],
[7,[10,15,2],[2,9]],
[7,[4,10,7],[2,13]],
[7,[6,15,2],[2,10]],
[7,[6,8,3],[2,10]],
[7,[8,12,6,3],[3,8]],
```

```
[8,[2,17],[1,26]],
[8,[4,10,9],[2,15]],
[9,[2,19],[1,29]],
[9,[3,10],[1,21]],
[9,[4,7],[1,19]],
[9,[6,9,7],[2,15]],
[9,[10,15,3],[2,11]],
[9,[4,10,11],[2,17]],
[9,[6,15,4],[2,12]],
[9,[4,14,7],[2,15]],
[9,[6,8,7],[2,13]],
[9,[6,10,3],[2,12]],
[9,[8,12,6,7],[3,11]],
[10,[2,21],[1,32]],
[10,[3,11],[1,23]],
[10,[5,6],[1,20]],
[10,[6,9,8],[2,17]],
[10,[8,12,5],[2,14]],
[10,[14,21,2],[2,11]],
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