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ON LOCAL RIGHT PURE SEMISIMPLE RINGS OF LENGTH TWO OR THREE

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1. Introduction

We investigate in this paper non-commutative local rings R of the smallest length that are potential counter-examples to the pure semisimplicity conjecture.

Throughout the paper R is an associative ring with an identity element. We call R *local*, if the Jacobson radical $J(R)$ of R is a two-sided maximal ideal. We denote by $\mathrm{mod}(R)$ the category of finitely generated right R-modules. Given a right R-module X_R of finite length we denote by $l(X_R)$ the length of X_R.

We recall that a ring R is said to be of *finite representation type*, if R is artinian and the number of the isomorphism classes of finitely generated indecomposable right (and left) R-modules is finite. Following [24] we call a ring R *right pure semisimple*, if every right R-module is a direct sum of finitely generated modules.

It is well known that a ring R is of finite representation type if and only if R is right pure semisimple and R is left pure semisimple (see [2], [11], [18], [22]–[24]). It is still an open question, called the *pure semisimplicity conjecture*, if a right pure semisimple ring R is of finite representation type (see [2] and [24], [25], [28]). In [13] the question is answered in affirmative for rings R satisfying a polynomial identity and for self-injective rings R (see also [7], [19] and [31]). The reader is referred to [42] and to the author's expository papers [30] and [32] for a basic background and historical comments on the pure semisimplicity conjecture.

It was shown by the author in [28] and [33] that there is a chance to find a counter-example R to the pure semisimplicity conjecture and R might be hereditary with two simple non-isomorphic modules. The existence of a counter-example depends on a generalized Artin problem on division ring extensions.

In the present paper we are mainly interested in the existence of counter-examples R to pure semisimplicity conjecture that are local of the smallest length, that is, of length $l(R_R)$ two or three. This continues our study started in [28], [35] and [33].

It is shown in Lemma 3.1 that every such a local ring R has $J(R)^2 = 0$. Therefore we study representation-infinite right pure semisimple local rings R with $J(R)^2 = 0$ such that the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R)$ is of the form $\cdots \to \bullet \to \bullet \to \bullet \to$

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$\cdots \to \bullet \to \bullet$ by applying our recent results on right pure semisimple hereditary rings and generalized Artin problem on division ring extensions obtained in [33] and [36].

Assume that R is a local ring such that the square of the Jacobson radical $J = J(R)$ of R is zero. Then $F = R/J$ is a division ring and J is an F-F-bimodule. By applying the results of [33] and [36] we show in Theorem 3.4 that the Auslander-Reiten quiver $\Gamma(\operatorname{mod} R)$ is connected of the form $\cdots \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$ if and only if the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F)$ of the F-F-bimodule ${}_F J_F$ (see Section 2) belongs to the set $\mathcal{DS}_{pss}$ (of cardinality $2^{\aleph_0}$) of infinite pure semisimple dimension-sequences $v = (\ldots, v_{-s}, v_{-s+1}, \ldots, v_{-2}, v_{-1}, v_0, \infty)$ with $v_j \in \mathbb{N}$ constructed in [33] (see also Section 2). In this case, we show that the Auslander-Reiten translation quiver $\Gamma(\operatorname{mod} R)$ of the category $\operatorname{mod}(R)$ is connected and has any of the forms (see (3.5) and (3.6))

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & & & & & R & - - - & L_3^+ & \cdots - - - & & L_m^+ & \\
 & & & & & & & & & & & & & \nearrow & & \searrow & \nearrow & \cdots & \nearrow & & \searrow \\
\cdots & & L_6 & - - - & & L_4 & - - - & & L_2 & - - - & & L_0 & - - - & & & L_2^+ & - - - \cdots & L_{m-1}^+ & - - - & & L_{m+1}^+ \\
\cdots & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & & & & & & & & & \\
\cdots & - - - & & L_4 & - - - & & L_3 & - - - & & L_1 & & & & & & & & & & &
\end{array}$$

$$\begin{array}{cccccccccccccccccccc}
 & & & & & & & & & & & & & & R & - - - & L_3^+ & \cdots - - - & & L_{m+1}^+ \\
 & & & & & & & & & & & & & \nearrow & & \searrow & \nearrow & \cdots & \nearrow & \\
\cdots & & L_6 & - - - & & L_4 & - - - & & L_2 & - - - & & L_0 & - - - & & & L_2^+ & - - - \cdots & L_m^+ & & \\
\cdots & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & & & & & & & & \\
\cdots & - - - & & L_4 & - - - & & L_3 & - - - & & L_1 & & & & & & & & & &
\end{array}$$

Moreover, the infinite Jacobson radical $\operatorname{rad}^{\infty}(\operatorname{mod} R) = \bigcap_{m=1}^{\infty} \operatorname{rad}^m(\operatorname{mod} R)$ of the category $\operatorname{mod}(R)$ is non-zero and it is generated by all R-module homomorphisms from the ring R to L_m, for $m = 0, 1, 2, \ldots$. The square $(\operatorname{rad}^{\infty}(\operatorname{mod} R))^2$ of $\operatorname{rad}^{\infty}(\operatorname{mod} R)$ is zero.

For the notion of the Auslander-Reiten translation quiver the reader is referred to [3] and [27].

In particular, Theorem 3.4 shows how potential local counter-examples R to the pure semisimplicity conjecture of length $l(R_R)$ two or three should look like, if the Auslander-Reiten quiver of $\operatorname{mod}(R)$ is connected of the following form
$\cdots \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$.

The main results of the paper are presented in Section 3, where also two open problems are formulated. In Section 2 we collect preliminary facts and notation we need in this paper.

Throughout this paper we use freely the terminology and notation introduced in [28] and [33]. The reader is referred to [3] and [27] for a background and terminology on representation theory of finite dimensional algebras and artinian rings.

By the Auslander-Reiten quiver of the category $\operatorname{mod}(R)$ we mean the oriented graph $\Gamma(\operatorname{mod}(R))$ whose vertices are the isomorphism classes $[X]$ of indecomposable modules X in $\operatorname{mod}(R)$ and there exists an arrow $[X] \to [Y]$ in $\Gamma(\operatorname{mod}(R))$ if and only if there exists an irreducible morphism $X \to Y$ in $\operatorname{mod}(R)$ (see [3], [27]). Usually we identify the isomorphism class $[X]$ in $\Gamma(\operatorname{mod} R)$ with the module X in $\operatorname{mod}(R)$. Sometimes we view $\Gamma(\operatorname{mod} R)$ as a translation quiver (see [3, Section VII.4] and [27, 11.49]). In this case we draw a dashed edge between indecomposable modules X and Z in $\Gamma(\operatorname{mod} R)$ if there exists an almost split sequence $0 \to X \to Y \to Z \to 0$.

The Jacobson radical $\operatorname{rad}_R = \operatorname{rad}(\operatorname{mod} R)$ of the category $\operatorname{mod}(R)$ is the two-sided ideal of the category $\operatorname{mod}(R)$ such that $\operatorname{rad}_R(X, Y)$ consists of all non-invertible elements of the group $\operatorname{Hom}_R(X, Y)$ for each pair of indecomposable modules X and Y in $\operatorname{mod}(R)$ (see [3] and [27]). The two-sided ideal

$$\operatorname{rad}^{\infty}(\operatorname{mod} R) = \bigcap_{j=0}^{\infty} \operatorname{rad}^{j}(\operatorname{mod} R)$$

of the category $\operatorname{mod}(R)$ is called the infinite Jacobson radical of $\operatorname{mod}(R)$. The reader is referred to [32] and [37] for some applications of $\operatorname{rad}^{\infty}(\operatorname{mod} R)$ in the representation theory of artinian rings.

Given two indecomposable modules X and Y in $\operatorname{mod}(R)$ we view the abelian group

$$\operatorname{Irr}(X, Y) = \operatorname{rad}_R(X, Y)/\operatorname{rad}_R^2(X, Y)$$

as an $\operatorname{End}(Y)/J\operatorname{End}(Y)$-$\operatorname{End}(X)/J\operatorname{End}(X)$-bimodule, and we call it a *bimodule of irreducible morphisms* from X to Y (see [27, Section 11.1]).

Some of the results of this paper were presented on the Yamaguchi Conference "The 32nd Symposium on Ring Theory and Representation Theory" in October 1999 (see [34]).

2. Bimodules and pure semisimple dimension sequences

We start this section by recalling from [33] some definitions and notation we need throughout this paper.

Assume that F and G are division rings and ${}_F M_G$ is a non-zero F-G-bimodule. We recall that the matrix ring

$$R_M = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix} \tag{2.1}$$

is hereditary and the modules X in $\operatorname{mod}(R_M)$ can be identified with triples $X = (X'_F, X''_G, t)$, where X'_F, X''_G are finite dimensional vector spaces over F and G, respectively, and $t\colon X' \otimes_F M_G \to X''_G$ is a G-linear map. We write (X'_F, X''_G) instead of (X'_F, X''_G, t), if the choice of t is an obvious one. The vector

$$\mathbf{dim}\, X = (\dim X'_F, \dim X''_G) \in \mathbb{Z}^2$$

is called the *dimension vector* of X.

Given an F-G-bimodule ${}_F N_G$ we set $\operatorname{l.dim}(N) = \dim_F N$ and $\operatorname{r.dim}(N) = \dim N_G$ and we define the right dualisation and the left dualisation of ${}_F N_G$ to be the G-F-bimodule

$$N^{*r} = \mathrm{Hom}_G({}_F N_G, G) \quad \text{and} \quad N^{*l} = \mathrm{Hom}_F({}_F N_G, F) \tag{2.2}$$

respectively. To any bimodule ${}_F M_G$ we associate a sequence of iterated right dualisations of ${}_F M_G$ by setting $M^{(0)} = M$ and $M^{(j)} = (M^{(j+1)})^{*r}$ for $j \leq -1$. The sequence of iterated left dualisations of ${}_F M_G$ is defined by the formula $M^{(j)} = (M^{(j-1)})^{*l}$ for $j \geq 1$. We also set

$$d_j^M = \mathrm{r.dim}(M^{(j)}), \qquad R_{2j} = \begin{pmatrix} F & M^{(2j)} \\ 0 & G \end{pmatrix}, \qquad R_{2j+1} = \begin{pmatrix} G & M^{(2j+1)} \\ 0 & F \end{pmatrix} \tag{2.3}$$

for any $j \in \mathbb{Z}$.

With any F-G-bimodule ${}_F M_G$ for which there exists an integer $m \geq 0$ such that

$$d_j^M = \mathrm{r.dim}\, M^{(j)} \text{ is finite for all } j \leq m \text{ and } d_{m+1}^M = \mathrm{r.dim}\, M^{(m+1)} = \infty \tag{2.4}$$

we associate the *infinite dimension-sequence*

$$\mathbf{d}_{-\infty}({}_F M_G) = \big(\dots, d_{-j}(M), \dots, d_{-2}(M), d_{-1}(M), d_0(M), \infty\big) \tag{2.5}$$

where $d_0(M) = d_m^M = \mathrm{r.dim}\, M^{(m)}$ and $d_j(M) = d_{m-j}^M = \mathrm{r.dim}\, M^{(m-j)}$ for all $j \geq 1$. The number m is called the *iterated dimension height of* ${}_F M_G$.

Our idea is to study the indecomposable modules over any local right pure semisimple ring R with radical square zero in terms of the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F)$ of the F-F-bimodule ${}_F J_F = J(R)$, where $F = R/J(R)$.

For this purpose we recall from [5] that the set

$$\mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_3 \cup \dots \cup \mathcal{D}_m \cup \cdots \tag{2.6}$$

of *dimension-sequences* $(d_1, \dots, d_m)$, $m \geq 1$, is defined inductively to be the minimal set satisfying the following two conditions:

(i) $\mathcal{D}_2 = \{(0,0)\}$ and $\mathcal{D}_3 = \{(1,1,1)\}$.

(ii) If the set $\mathcal{D}_m$ is defined we define $\mathcal{D}_{m+1}$ to be the set of all sequences of the form

$$\boldsymbol{\xi}_{i+1}(d_1, \dots, d_m) = (d_1, \dots, d_{i-1}, d_i + 1, 1, d_{i+1} + 1, d_{i+2}, \dots, d_m),$$

where $(d_1, \dots, d_m) \in \mathcal{D}_m$ and $i = 1, \dots, m-1$.

We note that for each m the set $\mathcal{D}_m$ of dimension-sequences of length m is closed under the action of cyclic permutations.

We recall from [28] that a sequence $(d_1, \dots, d_m)$ is said to be a *simple restriction of a dimension-sequence* if it is obtained from a dimension-sequence in $\mathcal{D}$ by omitting the last coordinate.

Note that the set $\mathcal{D}^{\vee}$ of simple restriction of dimension-sequences contains the following sequences and their reversions: (0), (1, 1), (1, 2, 1), (2, 1, 2), (1, 2, 2, 1), (2, 2, 1, 3), (2, 1, 3, 1).

It was shown in [29, Proposition 3.1] that in the case the ring R_M is right pure semisimple and representation-infinite there exists an integer $m \geq 0$ such that
(a) $d^M_{m+1} = \infty$ and $d^M_j < \infty$ for all $j \leq m$, and
(b) for any pair $s \leq m$ and $t \geq 2$ the sequence $(d^M_{s-t}, d^M_{s-t+1}, \dots, d^M_{s-1}, d^M_s)$ is not a simple restriction of a dimension-sequence.

The following definition was introduced in [33] in relation with an idea of constructing a large family of potential counter-examples to the pure semisimplicity conjecture.

DEFINITION 2.7. The set of *pure semisimple infinite dimension-sequences* is the set $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$, where $\mathcal{DS}^{(1)}_{pss}$ and $\mathcal{DS}^{(2)}_{pss}$ are defined as follows.

The set $\mathcal{DS}^{(1)}_{pss}$ is a minimal set of sequences

$$v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty),$$

with $v_{-j} \in \mathbb{N}$ non-zero for any $j \in \mathbb{N}$, satisfying the following two conditions:
(i) $\boldsymbol{\omega} = (\dots, 2, 2, \dots, 2, 2, 1, \infty) \in \mathcal{DS}^{(1)}_{pss}$;
(ii) if $v = (\dots, v_{-m}, \dots, v_{-1}, v_0, \infty)$ is a sequence in $\mathcal{DS}^{(1)}_{pss}$ then all sequences of the form

$$\boldsymbol{\xi}_{-m}(v) = (\dots, v_{-m-1}, 1 + v_{-m}, 1, 1 + v_{-m+1}, v_{-m+2}, \dots, v_{-2}, v_{-1}, v_0, \infty)$$

belong to $\mathcal{DS}^{(1)}_{pss}$, for all $m \geq 1$.

Given a dimension-sequence $u = (\dots, u_{-j}, u_{-j+1}, \dots, u_{-2}, u_{-1}, u_0, \infty)$ in $\mathcal{DS}^{(1)}_{pss}$ we define the *depth* of u to be the minimal integer $l(u) \geq 0$ such that $u_{-j} = 2$ for all $j \geq 1 + l(u)$.

A sequence $v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty)$ belongs to $\mathcal{DS}^{(2)}_{pss}$ if there exists a sequence of positive integers $j(1), j(2), \dots, j(s), \dots$ such that
(a) for every $m \geq 0$ the set $\{s \in \mathbb{N};\ j(s) = m\}$ is finite,
(b) $\lim_{s\to\infty} \boldsymbol{\xi}_{-j(s)}\boldsymbol{\xi}_{-j(s-1)} \cdots \boldsymbol{\xi}_{-j(1)}(\boldsymbol{\omega}) = v$, where $\lim_{s\to\infty} w^{(s)} = w$ means that there exists a sequence $0 < r_1 < r_2 < \cdots < r_s < \cdots$ of positive integers such that $w^{(s)}_0 = w_0$, $w^{(s)}_{-1} = w_{-1}, \dots, w^{(s)}_{-r_s} = w_{-r_s}$,
(c) for every integer $s \geq 0$ there exists an integer $r_s > s$ such that $j(r_s) \geq 1 + l(\boldsymbol{\xi}_{-j(r_{s-1})}\boldsymbol{\xi}_{-j(r_{s-2})} \cdots \boldsymbol{\xi}_{-j(1)}(\boldsymbol{\omega}))$.

It was shown in [33] that the cardinality of the set $\mathcal{DS}^{(2)}_{pss}$ is continuum. The set $\mathcal{DS}^{(1)}_{pss}$ is constructed from the principal sequence

$$\boldsymbol{\omega} = (\dots, 2, 2, \dots, 2, 2, 1, \infty)$$

in a similar way as the set $\mathcal{D}$ of dimension-sequences was constructed in [5] starting from the trivial dimension-sequence (1, 1, 1). In particular, each of the countably many sequences

$$\begin{array}{l}(\ldots,2,2,\ldots,2,2,3,1,2,\infty),\\(\ldots,2,2,\ldots,2,2,3,1,5,1,2,2,\infty),\\(\ldots,2,2,\ldots,2,2,3,1,5,1,2,5,1,2,2,\infty),\\(\ldots,2,2,\ldots,2,2,3,1,5,1,2,5,1,2,5,1,2,2,\infty),\\(\ldots,2,2,\ldots,2,2,3,1,5,1,2,5,1,2,5,1,2,5,1,2,2,\infty),\\ \vdots\quad\vdots\;\vdots\;\vdots\qquad\qquad\qquad\qquad\ddots\end{array}$$

belongs to $\mathcal{DS}^{(1)}_{pss}$. The set $\mathcal{DS}^{(2)}_{pss}$ is constructed from the principal sequence $\boldsymbol{\omega}$ by applying infinitely many operations $\boldsymbol{\xi}_{-j(1)},\ldots,\boldsymbol{\xi}_{-j(s)},\ldots$ with the fast growth of the sequence $j(1),\ldots,j(s),\ldots$ described by the property (c) in Definition 2.7. Note that the sequence

$$(\ldots,2,1,5,\ldots,2,1,5,2,1,5,2,1,5,1,2,\infty)$$

belongs to the set $\mathcal{DS}^{(2)}_{pss}$.

3. Small right pure semisimple local rings

Our investigation of potential counter-examples to the pure semisimplicity conjecture of length two or three depends on the following useful observation.

Lemma 3.1. *Let R be a right pure semisimple local ring of infinite representation type. If $2 \le l(R_R) \le 3$, then $J(R)^2 = 0$.*

Proof. If $l(R_R) = 2$, then $J = J(R)$ is a simple right R-module and therefore $J^2 = 0$. Let $l(R_R) = 3$ and assume to the contrary that $J^2 \neq 0$. Let $x \in J$ be such that its square $x^2 \in J^2$ is not zero. It follows that $J^3 = 0$, J^2 is a simple right ideal, $J^2 = x^2R$ and therefore $x \notin J^2$. Since $l(R_R) = 3$ and $J^2 \neq 0$, it follows that J/J^2 is a simple module generated by the coset $\overline{x}$ of x and therefore $J = xR + x^2R = xR$. This shows that R is right serial. Since R is of infinite representation type, R is not left serial, by [8]. On the other hand, R is right pure semisimple and right serial. It then follows from [26, Theorem 2.2] that $J^2 = 0$, and we get a contradiction. This finishes the proof. □

The above lemma shows that right artinian local rings of right length two or three, that are potential counter-examples to the pure semisimplicity conjecture, are square zero radical rings. Therefore we assume throughout this section that R is a local right artinian ring such that $J(R)^2 = 0$. It follows that $F = R/J(R)$ is a division ring and $J = J(R)$ is an F-F-bimodule in a natural way.

Following Gabriel [10], we associate with R the hereditary right artinian ring

$$R_J = \begin{pmatrix} R/J & {}_{(R/J)}J_{(R/J)} \\ 0 & R/J \end{pmatrix} = \begin{pmatrix} F & {}_FJ_F \\ 0 & F \end{pmatrix}$$

and the reduction functor

$$\mathbb{F} : \operatorname{mod}(R) \longrightarrow \operatorname{mod}(R_J) \tag{3.2}$$

defined by attaching to any module Y in $\operatorname{mod}(R)$ the triple $\mathbb{F}(Y) = (Y', Y'', t)$, where $Y' = Y/YJ$ and $Y'' = YJ$ are viewed as right R/J-modules and $t\colon Y' \otimes_{R/J} J_{R/J} \to Y''_{R/J}$ is a R/J-homomorphism defined by formula $t(\overline{y} \otimes r) = y \cdot r$ for $\overline{y} = y + J$ and $r \in J$. The triple $\mathbb{F}(Y)$ is viewed as a right R_J-module in a natural way. If $f : Y \to Z$ is an R-homomorphism we set $\mathbb{F}(f) = (f', f'')$, where $f''\colon Y'' \to Z''$ is the restriction of f to $Y'' = YJ$ and $f'\colon Y' \to Z'$ is the R/J-homomorphism induced by f.

Now we collect the main properties of the functor $\mathbb{F}$ we need later.

Lemma 3.3. *Let R be a local right artinian ring such that $J(R)^2 = 0$. Let us view $J = J(R)$ as an F-F-bimodule, where $F = R/J(R)$ is a division ring. Under the notation introduced above the functor $\mathbb{F}$* (3.2) *has the following properties.*

(i) *$\mathbb{F}$ is full and establishes a representation equivalence between* $\operatorname{mod}(R)$ *and the category* $\operatorname{Im}\mathbb{F}$, *that is, a homomorphism $f : X \to Y$ is an isomorphism if and only if $\mathbb{F}(f)$ is an isomorphism.*

(ii) *A right R_J-module X belongs to* $\operatorname{Im}\mathbb{F}$ *if and only if X has no non-zero summand isomorphic to a simple projective right R_J-module.*

(iii) *The functor $\mathbb{F}$ preserves the indecomposability, projectivity and the length. Moreover, $\mathbb{F}$ defines a bijection between the isomorphism classes of indecomposable modules in* $\operatorname{mod}(R)$ *and the isomorphism classes of indecomposable modules in* $\operatorname{mod}(R_J)$, *which are not simple and projective.*

(iv) *The functor $\mathbb{F}$ carries a homomorphism $f : Y \to Z$ in* $\operatorname{mod}(R)$ *to zero if and only if* $\operatorname{Im} f \subseteq ZJ$. *For any pair Y, Z of indecomposable modules in* $\operatorname{mod}(R_J)$ *the functor $\mathbb{F}$ induces ring isomorphisms*

$$\operatorname{End}(Y)/J\operatorname{End}(Y) \cong \operatorname{End}\big(\mathbb{F}(Y)\big)/J\operatorname{End}\big(\mathbb{F}(Y)\big)$$

and

$$\operatorname{End}(Z)/J\operatorname{End}(Z) \cong \operatorname{End}\big(\mathbb{F}(Z)\big)/J\operatorname{End}\big(\mathbb{F}(Z)\big).$$

If, in addition, Y is not isomorphic to a direct summand of ZJ then $\mathbb{F}$ induces an $\operatorname{End}(Y)/J\operatorname{End}(Y)$-$\operatorname{End}(Z)/J\operatorname{End}(Z)$-*bimodule isomorphism*

$$\operatorname{Irr}(Y, Z) \cong \operatorname{Irr}\big(\mathbb{F}(Y), \mathbb{F}(Z)\big).$$

In particular, the functor $\mathbb{F}$ *carries irreducible morphisms in* $\mathrm{mod}(R)$ *to irreducible morphisms or to zero.*

(v) $\mathbb{F}$ *carries* rad_R^j *to* $\mathrm{rad}_{R_J}^j$ *for all* $j \geq 2$ *and carries* rad_R^∞ *to* $\mathrm{rad}_{R_J}^\infty$ *in such a way that*

- $\mathrm{rad}_R^\infty \neq 0$ *if and only if* $\mathrm{rad}_{R_J}^\infty \neq 0$, *and*
- $(\mathrm{rad}_R^\infty)^2 \neq 0$ *if and only if* $(\mathrm{rad}_{R_J}^\infty)^2 \neq 0$.

(vi) *The ring* R *is right pure semisimple* (*resp. of finite representation type*) *if and only if* R_J *is right pure semisimple* (*resp. of finite representation type*).

Proof. The statements (i)–(iv) are essentially proved in [10, Section 9] (see also [3, Lemma X.2.1]).

(vi) It follows easily from (iii) that R is of finite representation type if and only if R_J of finite representation type. To finish the proof of (vi) we recall from [22] and [23] that a right artinian ring S is right pure semisimple if and only if the ideal $\mathrm{rad}_R = \mathrm{rad}(\mathrm{mod}\, R)$ is right T-nilpotent, that is, for every sequence $X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$ of indecomposable modules $X_1, X_2, \ldots$ in $\mathrm{mod}\, R$ connected by non-isomorphisms $f_1, f_2, \ldots$ there exists $m \geq 2$ such that $f_m f_{m-1} \cdots f_2 f_1 = 0$ (see also [12]). Hence, in view of (iii), the ring R is right pure semisimple if and only if R_J is right pure semisimple.

(v) Apply a well-known and standard arguments used in [10, Section 9] and [3, Section X.2]). The details are left to the reader. □

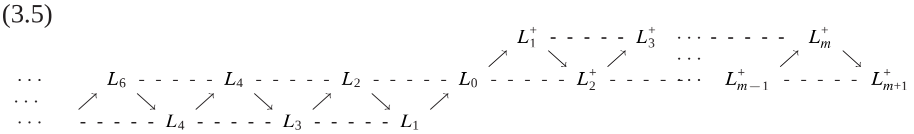
Our main result of this section is the following.

Theorem 3.4. *Assume that* R *is a local right artinian ring such that every indecomposable non-projective module* Z *in* $\mathrm{mod}(R)$ *admits an almost split sequence* $0 \to X \to Y \to Z \to 0$. *Assume that* $J(R)^2 = 0$ *and view* $J = J(R)$ *as a bimodule over the division ring* $F = R/J(R)$. *Then* $d_j^J = \mathrm{r.dim}\, J^{(j)} < \infty$ *for all* $j \leq 0$ *and the following conditions are equivalent.*

(a) *The ring* R *is of infinite representation type and the Auslander-Reiten quiver* $\Gamma(\mathrm{mod}\, R)$ *of* $\mathrm{mod}(R)$ *is connected of the form* $\cdots \to\bullet\to\bullet\to\bullet\to \cdots \to\bullet\to\bullet$

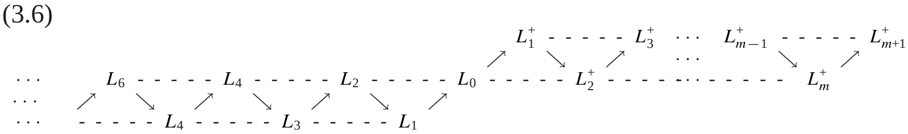
(b) *There exists an integer* $m \geq 0$ *such that* $d_{m+1}^J = \mathrm{r.dim}\, J^{(m+1)} = \infty$, $d_j^J = \mathrm{r.dim}\, J^{(j)} < \infty$ *for all* $j \leq m$ *and the Auslander-Reiten translation quiver* $\Gamma(\mathrm{mod}\, R)$ *of the category* $\mathrm{mod}(R)$ *has the form*

(3.5)



if $m \geq 1$ *is odd, and the form*

(3.6)



if $m \geq 0$ *is even, where* $L_1^+ = R$, L_0 *is a unique simple right* R*-module and* $L_1 \cong E_R(L_0)$ *is an injective envelope of* L_0*. Here we draw a dashed edge between indecomposable modules* X *and* Z *if they are connected by an almost split sequence* $0 \to X \to Y \to Z \to 0$.

(c) *There exists an integer* $m \geq 0$ *such that* $d_{m+1}^J = \mathrm{r.dim}\, J^{(m+1)} = \infty$, $d_j^J = \mathrm{r.dim}\, J^{(j)} < \infty$ *for all* $j \leq m$ *and the infinite dimension-sequence* $\mathbf{d}_{-\infty}({}_F J_F)$ *of the* F*-*F*-bimodule* ${}_F J_F$ *belongs to the set* $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$.

(d) *The infinite radical* $\mathrm{rad}_R^\infty = \mathrm{rad}^\infty(\mathrm{mod}\, R)$ *of the category* $\mathrm{mod}(R)$ *is non-zero, whereas its square* $(\mathrm{rad}_R^\infty)^2$ *is zero.*

If any of the conditions (a)–(d) *is satisfied, then the infinite Jacobson radical* rad_R^∞ *of* $\mathrm{mod}(R)$ *is generated by all* R*-module homomorphisms* $L_0 \to L_{j+1}$ *and all* R*-module homomorphisms* $L_i^+ \to L_j$ *for* $j = 0, 1, 2, \ldots$ *and* $i \geq 1$.

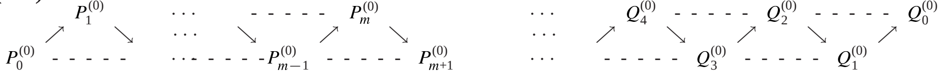
Proof. Consider the reduction functor $\mathbb{F} : \mathrm{mod}(R) \longrightarrow \mathrm{mod}(R_J)$ of (3.2) associated with R, where

$$R_J = \begin{pmatrix} F & {}_F J_F \\ 0 & F \end{pmatrix}$$

is hereditary and right artinian. We claim that every indecomposable non-projective module L in $\mathrm{mod}(R_J)$ admits an almost split sequence $0 \to M \to N \to L \to 0$. For, since L is not projective, L is in the image of $\mathbb{F}$ and according to Lemma 3.3 there exits a non-projective indecomposable module Z in $\mathrm{mod}(R)$ such that $L \cong \mathbb{F}(Z)$. By our assumption, there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}(R)$ and applying Lemma 3.3 (v) one shows that the derived sequence $0 \to \mathbb{F}(X) \to \mathbb{F}(Y) \to \mathbb{F}(Z) \to 0$ in $\mathrm{mod}(R_J)$ is almost split. In view of the isomorphism $L \cong \mathbb{F}(Z)$ our claim follows. It follows from [25, Corollary 1.9] that the number $d_j^J = \mathrm{r.dim}\, J^{(j)}$ is finite for any $j \leq 0$.

(c) $\Rightarrow$ (b) Assume (c) is satisfied. By Theorem 4.16, Proposition 4.17 and Corollary 4.18 of [33] the hereditary ring R_J is of infinite representation type, the Auslander-Reiten translation quiver $\Gamma(\mathrm{mod}\, R_J)$ of $\mathrm{mod}(R_J)$ has two connected components and is of the form

(3.7)



if $m \geq 1$ is odd, and of the form

(3.8)
$$\begin{array}{ccccccccccccccccccc} & P_1^{(0)} & & \cdots & & \text{-----} & P_{m+1}^{(0)} & & \cdots & & Q_4^{(0)} & \text{-----} & Q_2^{(0)} & \text{-----} & Q_0^{(0)} \\ \nearrow & & \searrow & \cdots & \searrow & \nearrow & & & \cdots & \nearrow & & \searrow \;\; \nearrow & & \searrow \;\; \nearrow & \\ P_0^{(0)} & \text{-----} & & \cdots\text{-----} & P_m^{(0)} & & & & \cdots & & \text{-----} & Q_3^{(0)} & \text{-----} & Q_1^{(0)} & \end{array}$$

if $m \geq 0$ is even, where the left hand component is preprojective and finite, whereas the other one is preinjective and infinite. The infinite radical $\mathrm{rad}_{R_J}^{\infty} = \mathrm{rad}^{\infty}(\mathrm{mod}\, R_J)$ of the category $\mathrm{mod}(R_J)$ is non-zero, whereas its square $(\mathrm{rad}_{R_J}^{\infty})^2$ is zero.

Since R_J is of infinite representation type, in view of Lemma 3.3 (ii), the module $Q_m^{(0)}$ is in the image of the functor $\mathbb{F}$ for any $m \geq 0$, because it follows from [6] that none of the modules $Q_m^{(0)}$ is simple projective. For any $j \in \mathbb{N}$ and $1 \leq i \leq m+1$, we denote by

$$L_j = \mathbb{F}^{-1}(Q_j^{(0)}) \text{ and } L_i^+ = \mathbb{F}^{-1}(P_i^{(0)}) \tag{3.9}$$

an indecomposable module in $\mathrm{mod}(R)$ corresponding, via the functor $\mathbb{F}$, to $Q_j^{(0)}$ and to $P_i^{(0)}$ in $\Gamma(\mathrm{mod}\, R_J)$, respectively, that is, L_j and L_i^+ are indecomposable modules in $\mathrm{mod}(R)$ such that $\mathbb{F}(L_j) \cong Q_j^{(0)}$ and $\mathbb{F}(L_i^+) \cong P_i^{(0)}$ (apply Lemma 3.3 (iii)).

By Lemma 3.3 (i)–(v), the preinjective component of $\Gamma(\mathrm{mod}(R_M))$ corresponds to the part of the Auslander-Reiten translation quiver of $\mathrm{mod}(R)$ formed by the modules $L_0, L_1, \ldots, L_s, \ldots$ shown in (3.5) and (3.6). It follows from Lemma 3.3 (iii) that the module L_0 is simple, and therefore $J(R) \cong L_0 \oplus \cdots \oplus L_0$ (a direct sum of $\dim J_F$ copies of L_0). Since the inclusion $\mathrm{soc}(R) = J(R) \hookrightarrow R$ is an irreducible morphism and L_0 is a direct summand of $J(R)$, there is an irreducible morphism $u\colon L_0 \to R$ such that $\mathbb{F}(u) = 0$. The preprojective component of $\Gamma(\mathrm{mod}(R_J))$ starts with two projective modules

$$(0, F) = P_0^{(0)} \hookrightarrow P_1^{(0)} = (F, J_F).$$

It follows from Lemma 3.3 (i)–(iii) that $\mathbb{F}(R) \cong P_1^{(0)}$ and $P_0^{(0)}$ is not in the image of $\mathbb{F}$. We recall from Lemma 3.3 (iv) that $\mathbb{F}$ carries irreducible morphisms to irreducible ones or to zero. Consequently, the Auslander-Reiten translation quiver of $\mathrm{mod}(R)$ is obtained from $\Gamma(\mathrm{mod}\, R_J)$ via $\mathbb{F}$ as a gluing of the preprojective component of $\mathrm{mod}(R_J)$ with its preinjective component by the identification of $P_0^{(0)}$ with $Q_0^{(0)}$. It follows that $\Gamma(\mathrm{mod}\, R)$ is connected and has the required shape shown in (3.5) and (3.6). This finishes the proof of the implication (c)⇒(b).

(c)⇒(d) Apply Lemma 3.3 (v) and the facts used above in the proof of the implication (c)⇒(b).

(d)⇒(c) By Lemma 3.3 (v), the infinite radical $\mathrm{rad}_{R_J}^{\infty} = \mathrm{rad}^{\infty}(\mathrm{mod}\, R_J)$ of the category $\mathrm{mod}(R_J)$ is non-zero, whereas its square $(\mathrm{rad}_{R_J}^{\infty})^2$ is zero. It follows from [32, Theorem 4.4] and [36] that there exists an integer $m \geq 0$ such that $d_{m+1}^J = \mathrm{r.dim}\, J^{(m+1)} = \infty$, $d_j^J = \mathrm{r.dim}\, J^{(j)} < \infty$ for all $j \leq m$ and the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F)$ of the F-F-bimodule ${}_F J_F$ belongs to the set $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup$

$\mathcal{DS}_{pss}^{(2)}$ and R_J is of infinite representation type. This yields (c).

The implication (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (c) Assume that (a) holds and let $f: Y \to Z$ be an irreducible morphism in mod(R) with Y and Z indecomposable modules such that $\mathbb{F}(f) = 0$. By Lemma 3.3 (iv), $\operatorname{Im} f \subseteq ZJ$ and therefore Z is projective, f is injective and the monomorphism $\operatorname{Im} f \subseteq ZJ$ splits. Hence, in view of Lemma 3.3 (iv), either $F(f)$ is irreducible, or else $\mathbb{F}(f) = 0$, $Z \cong R$ and Y is a simple direct summand of $\operatorname{soc} R_R \cong J(R)_R$. It then follows that the Auslander-Reiten quiver $\Gamma(\operatorname{mod} R_J)$ has at most two components and one of them is finite if $\Gamma(\operatorname{mod} R_J)$ is not connected, because $\Gamma(\operatorname{mod} R)$ is connected of the form $\cdots \to\bullet\to\bullet\to\bullet\to \cdots \to\bullet\to\bullet$, by our assumption. Since R is of infinite representation type, according to Lemma 3.3 (vi), the ring R_J is also of infinite representation type. We also recall that the dimension $d_j^J = \text{r.dim}\, J^{(j)}$ is finite for all $j \le 0$.

In order to prove (c), we assume to the contrary that $d_n^J = \text{r.dim}\, J^{(n)}$ is finite for all $n \ge 0$. It follows from [17], [25, Section 1] and [33, Proposition 2.6] that there exists a sequence of reflection functors

$$\cdots \leftrightarrows \operatorname{mod}(R_{-j}) \underset{\mathcal{S}_{-j}^{-}}{\overset{\mathcal{S}_{-j}^{+}}{\rightleftarrows}} \operatorname{mod}(R_{-j+1}) \leftrightarrows \cdots \leftrightarrows \operatorname{mod}(R_{-1}) \underset{\mathcal{S}_{-1}^{-}}{\overset{\mathcal{S}_{-1}^{+}}{\rightleftarrows}}$$

$$\operatorname{mod}(R_M) \underset{\mathcal{S}_{0}^{-}}{\overset{\mathcal{S}_{0}^{+}}{\rightleftarrows}} \operatorname{mod}(R_1) \leftrightarrows \cdots \leftrightarrows \operatorname{mod}(R_{m-1}) \underset{\mathcal{S}_{m-1}^{-}}{\overset{\mathcal{S}_{m-1}^{+}}{\rightleftarrows}} \operatorname{mod}(R_m) \leftrightarrows \ldots$$

which is infinite to the left and infinite to the right, and therefore the preprojective modules form an infinite connected component $\mathcal{P}_J$ of $\Gamma(\operatorname{mod} R_J)$ of the form

$$\begin{array}{cccccccccccccc}
 & & P_1^{(0)} & \text{-----} & \cdots & & \text{-----} & P_m^{(0)} & \text{-----} & & & \cdots \\
 & \nearrow & & \searrow & \nearrow \ \cdots & \searrow & & \nearrow & \searrow & & \nearrow & \cdots \\
P_0^{(0)} & \text{-----} & & P_2^{(0)} & \text{---------} & P_{m-1}^{(0)} & \text{-----} & & P_{m+1}^{(0)} & \text{-----} & & \cdots
\end{array}$$

and the preinjective modules form an infinite connected component $\mathcal{Q}_J$ of $\Gamma(\operatorname{mod} R_J)$ of the form shown in (3.7) such that $\mathcal{P}_J \neq \mathcal{Q}_J$ and $\Gamma(\operatorname{mod} R_J) = \mathcal{P}_J \cup \mathcal{Q}_J$. This is a contradiction, because we have observed above that one of the components should be finite.

Consequently, there exists an integer $m \ge 0$ such that $d_{m+1}^J = \text{r.dim}\, J^{(m+1)} = \infty$ and $d_j^J = \text{r.dim}\, J^{(j)} < \infty$ for all $j \le m$. It then follows from [33, Proposition 2.6] and the remarks made above that there exist a finite preprojective component $\mathcal{P}_J$ of the form (3.7) or (3.8), and an infinite preinjective component $\mathcal{Q}_J$ of $\Gamma(\operatorname{mod} R_J)$ such that $\Gamma(\operatorname{mod} R_J) = \mathcal{P}_J \cup \mathcal{Q}_J$, because $\Gamma(\operatorname{mod} R_J)$ has at most two components. By [32, Theorem 4.4] and [36], the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F)$ of the F-F-bimodule ${}_F J_F$ belongs to the set $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$. This finishes the proof of the implication (a) $\Rightarrow$ (c), and consequently, the statements (a)–(d) are equivalent.

Since the final statement of the theorem follows from the Proposition 3.10 (f) below, the theorem is proved. □

Proposition 3.10. *Assume that R is a local right artinian ring such that $J(R)^2 = 0$ and view $J = J(R)$ as a bimodule over the division ring $F = R/J(R)$. Assume also that there exists an integer $m \geq 0$ such that $d^J_{m+1} = \mathrm{r.dim}\, J^{(m+1)} = \infty$, $d^J_j = \mathrm{r.dim}\, J^{(j)} < \infty$ for all $j \leq m$ and the dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F) = (\ldots, d_{-j}(J), \ldots, d_{-1}(J), d_0(J), \infty)$ belongs to $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$. Then the following statements hold.*

(a) *The ring R is right pure semisimple of infinite representation type, that is, R is a counter-example to the pure semisimplicity conjecture.*

(b) *The ring R is not self-injective and the global dimension of R is infinite. The length $l(R_R)$ of the right R-module R_R is $1 + \dim J_F$.*

(c) *The Auslander-Reiten translation quiver $\Gamma(\mathrm{mod}\, R)$ of the category $\mathrm{mod}(R)$ consists of the modules L_j and L^+_i (3.9) with $j \geq 0$ and $0 \leq i \leq m+1$. It has the form (3.5) if m is odd, and the form (3.6) if m is even, where $L^+_1 = R$, L_0 is a unique simple right R-module and $L_1 \cong E_R(L_0)$ is an injective envelope of L_0.*

(d) *For any $s \geq 2$ and $0 \leq n \leq m-1$ there exist almost split sequences in $\mathrm{mod}(R)$*

$$0 \longrightarrow L_s \longrightarrow (L_{s-1})^{d^J_{-s}} \longrightarrow L_{s-2} \longrightarrow 0$$

and

$$0 \longrightarrow L^+_n \longrightarrow (L^+_{n+1})^{d^J_n} \longrightarrow L^+_{n+2} \longrightarrow 0,$$

where $d^J_j = \mathrm{r.dim}\, J^{(j)}$, L_s and L^+_n are the modules (3.9), and we set $L^+_0 = L_0$ and $L^+_1 = R$.

(e) *There is no almost split sequence in $\mathrm{mod}(R)$ starting from an indecomposable module L if and only if L is isomorphic to L_1, L^+_m or L^+_{m+1}.*

(f) *The infinite Jacobson radical rad^∞_R of $\mathrm{mod}(R)$ is generated by all R-module homomorphisms from L_0 to L_{j+1} and all R-module homomorphisms from L^+_i to L_j for $j = 0, 1, 2, \ldots$ and arbitrary $i \geq 1$.*

(g) *If $\mathbf{d}_{-\infty}({}_F J_F) = \boldsymbol{\omega} = (\ldots, 2, 2, \ldots, 2, 2, 2, 1, \infty)$, then $J(R) \cong L_0^{d^J_0}$, $l(L_j) = 2j+1$ for $j \geq 0$, $l(R) = l(L^+_1) = 1 + d^J_0$, $l(L^+_j) = 1 + j d^J_0$ for $j = 1, \ldots, m+1$, all irreducible morphisms $L_m \to L_{m-1}$ are surjective, all irreducible morphisms $L^+_n \to L^+_{n+1}$ are injective, and the number of indecomposable modules in $\mathrm{mod}(R)$ of length s is 0, 1 or 2, for every $s \geq 1$.*

Proof. Consider the reduction functor $\mathbb{F}\colon \mathrm{mod}(R) \to \mathrm{mod}(R_J)$ (3.2) with the properties collected in Lemma 3.3, where $R_J = \begin{pmatrix} F & {}_F J_F \\ 0 & F \end{pmatrix}$.

(a) Since $\mathbf{d}_{-\infty}({}_F J_F) = (\ldots, d_{-j}(J), \ldots, d_{-1}(J), d_0(J), \infty)$ belongs to the set $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$, Theorem 4.16, Proposition 4.17 and Corollary 4.18 of [33] apply to the hereditary ring R_J. In particular, it follows that R_J is right pure semisimple of infinite representation type and therefore the ring R is also right pure semisimple of in-

finite representation type, by Lemma 3.3.
(b) Let L_0 denote a unique simple right R-module. Since R is a local non-simple ring, $L_0 \cong R/J$ is not projective. It follows that the semisimple right R-module $J \cong L_0 \oplus \cdots \oplus L_0$, a direct sum of $l(J_R)$ copies of L_0, is not projective and the global dimension of R is infinite. In view of (a), we conclude that R is not self-injective, because self-injective right pure semisimple rings are finite representation type, by [13, Corollary 5.3]. The remaining statement of (b) is obvious, because $J(R)^2 = 0$.

The statement (c) is a consequence of Theorem 3.4.
(d) Fix $s \geq 2$. By Theorem 3.4, the Auslander-Reiten translation quiver of mod(R) has one of the forms (3.5) and (3.6) and is obtained via the reduction functor $\mathbb{F}$: mod(R) $\to$ mod(R_J) of (3.2) from the Auslander-Reiten translation quiver of mod(R_J) shown in (3.7) and (3.8). The ring R_J is of infinite representation type. It follows from [33, Corollary 2.11] applied to $F = G$, ${}_F M_G = {}_F J_F$ and $R_M = R_J$ that there exist ring isomorphisms $\mathrm{End}(Q^{(0)}_{2j}) \cong F$, $\mathrm{End}(Q^{(0)}_{2j+1}) \cong F$ for all $j \geq 0$, an F-F-bimodule isomorphism $\mathrm{Irr}(Q^{(0)}_s, Q^{(0)}_{s-1}) \cong \mathrm{Hom}_{R_J}(Q^{(0)}_s, Q^{(0)}_{s-1}) \cong J^{(-s-1)}$ and an almost split sequence

$$0 \longrightarrow Q^{(0)}_s \xrightarrow{\varphi_s} (Q^{(0)}_{s-1})^{d^J_{-s}} \xrightarrow{\psi_s} Q^{(0)}_{s-2} \longrightarrow 0 \tag{3.11}$$

in mod(R_J), where $d^J_{-s} = \mathrm{r.dim}\, J^{(-s)} = \mathrm{l.dim}\, J^{(-s-1)} = \mathrm{l.dim}\,\mathrm{Irr}(Q^{(0)}_s, Q^{(0)}_{s-1})$. Since $Q^{(0)}_j \cong \mathbb{F}(L_j)$ for $j \geq 0$ and the functor $\mathbb{F}$ is full, there exist R-module homomorphisms

$$L_s \xrightarrow{f_s} (L_{s-1})^{d^J_{-s}} \xrightarrow{g_s} L_{s-2}$$

such that $g_s f_s = 0$, $\varphi_s = \mathbb{F}(f_s)$ and $\psi_s = \mathbb{F}(g_s)$, that is, $\mathbb{F}$ carries the above sequence to the exact sequence (3.11), up to isomorphism. Hence, by applying the definition of the functor $\mathbb{F}$, we easily conclude that the sequence

$$0 \longrightarrow L_s \xrightarrow{f_s} (L_{s-1})^{d^J_{-s}} \xrightarrow{g_s} L_{s-2} \longrightarrow 0 \tag{3.12}$$

is exact in mod(R). By Lemma 3.3 (v) and the observation made above, there is a ring isomorphism $\mathrm{End}(L_s)/J\,\mathrm{End}(L_s) \cong \mathrm{End}(\mathbb{F}(L_s))/J\,\mathrm{End}(\mathbb{F}(L_s)) \cong \mathrm{End}(Q^{(0)}_s) \cong F$, and an F-F-bimodule isomorphisms

$$\mathrm{Irr}(L_s, L_{s-1}) \cong \mathrm{Irr}\big(\mathbb{F}(L_s), \mathbb{F}(L_{s-1})\big) \cong \mathrm{Irr}\big(Q^{(0)}_s, Q^{(0)}_{s-1}\big) \cong J^{(-s-1)},$$
$$\mathrm{Irr}(L_{s-1}, L_{s-2}) \cong \mathrm{Irr}\big(\mathbb{F}(L_{s-1}), \mathbb{F}(L_{s-2})\big) \cong \mathrm{Irr}\big(Q^{(0)}_{s-1}, Q^{(0)}_{s-2}\big) \cong J^{(-s)},$$

and $J^{(-s-1)} \cong \mathrm{Hom}_F(J^{(-s)}, F)$. It follows that

$$\begin{aligned} \mathrm{l.dim}\,\mathrm{Irr}(L_s, L_{s-1}) &= \mathrm{l.dim}\, J^{(-s-1)} \\ &= \mathrm{r.dim}\, J^{(s)} \\ &= d^J_{-s} \\ &= \mathrm{r.dim}\,\mathrm{Irr}(L_{s-1}, L_{s-2}). \end{aligned}$$

Hence, in view of [27, Proposition 11.13] applied to the category $\mathcal{A} = \mathrm{mod}(R)$, we conclude that (3.12) is an almost split sequence in $\mathrm{mod}(R)$.

The existence of the second almost split sequence in (d) can be proved in a similar way by applying the functor $\mathbb{F}$ and using an almost split sequence

$$0 \longrightarrow P_n^{(0)} \xrightarrow{\varphi_n'} (P_{n+1}^{(0)})^{d_n^J} \xrightarrow{\psi_n'} P_{n+2}^{(0)} \longrightarrow 0$$

in $\mathrm{mod}(R_J)$ for $0 \leq n \leq m-2$ (see [33, Corollary 2.11]).

(e) Apply (d) and the shape of the Auslander-Reiten translation quiver of $\mathrm{mod}(R)$ described in (3.5) and (3.6).

(f) First we show that $\mathrm{Hom}_R(L_i^+, L_s) = \mathrm{rad}_R^\infty(L_i^+, L_s)$ for all $s \geq 0$ and $i \geq 1$. Assume that $s \geq 2$ and let $h\colon L_i^+ \to L_{s-2}$ be a non-zero R-homomorphism. Note that L_j is not isomorphic to L_i^+, because $\mathbb{F}(L_i^+)$ is preprojective, while $\mathbb{F}(L_j)$ is not preprojective for all $j \geq 0$. Since (3.12) is an almost split sequence, there is an R-module homomorphism $h^{(s-1)} = (h_j^{(s-1)})\colon L_i^+ \to (L_{s-1})^{d_{-s}^J}$ of h such that $h = g_s h^{(s-1)}$ and $h_j^{(s-1)}\colon L_i^+ \to L_{s-1}$ belongs to $\mathrm{rad}(\mathrm{mod}\, R)$ for all j. It follows that $h^{(s-1)}$ also belongs to $\mathrm{rad}(\mathrm{mod}\, R)$. Since (3.12) is an almost split sequence, g_s is an irreducible morphism and therefore g_s belongs to $\mathrm{rad}(\mathrm{mod}\, R)$. Consequently, $h = g_s h^{(s-1)}$ belongs to the square of $\mathrm{rad}(\mathrm{mod}\, R)$. Applying the above arguments to each of the homomorphisms $h_j^{(s-1)}\colon L_i^+ \to L_{s-1}$, we show that $h_j^{(s-1)}$ belongs to the square of $\mathrm{rad}(\mathrm{mod}\, R)$. It follows that $h^{(s-1)}$ belongs to the square of $\mathrm{rad}(\mathrm{mod}\, R)$ and consequently $h = g_s h^{(s-1)}$ belongs to the cube of $\mathrm{rad}(\mathrm{mod}\, R)$. Continuing this way we show that h belongs to $\mathrm{rad}^j(\mathrm{mod}\ R)$ for any $j \geq 0$, and therefore $h \in \mathrm{rad}^\infty(\mathrm{mod}\, R)$ (compare with [40]).

The above arguments also yield $\mathrm{Hom}_R(L_0, L_{s+1}) = \mathrm{rad}_R^\infty(L_0, L_{s+1})$ for all $s \geq 0$. Consequently, rad_R^∞ contains the set

$$\mathcal{X} = \bigcup_{i\geq 1} \bigcup_{s\geq 0} \mathrm{Hom}_R(L_i^+, L_s) \cup \mathrm{Hom}_R(L_0, L_{s+1}).$$

Now we show that $\mathcal{X}$ generates the infinite radical rad_R^∞ of $\mathrm{mod}(R)$. For this purpose we note first that any R-module homomorphism $h \in \mathrm{rad}^\infty(L_n, L_j)$ has a factorisation through a direct sum of monomorphisms $\mathrm{soc}\, L_t \hookrightarrow L_t$ for some $t \geq 1$. Assume for simplicity that $n < j$. Then $\mathbb{F}(h) \in \mathrm{Hom}_{R_J}(\mathbb{F}(L_n), \mathbb{F}(L_j)) = 0$ and according to Lemma 3.3, h factorises through $L_j J \subseteq \mathrm{soc}\, L_j \subseteq L_j$ as we required. The remaining cases follow in a similar way. Since the monomorphism $\mathrm{soc}\, L_j \hookrightarrow L_j$ is a sum of homomorphisms $L_0 \hookrightarrow L_j$, it follows that $\mathrm{rad}^\infty(L_n, L_j)$ is contained in the two-sided ideal of $\mathrm{mod}(R)$ generated by the set $\mathcal{X}$.

Further we note that any R-module homomorphism $h \in \mathrm{rad}^\infty(L_n^+, L_s^+)$ has a factorisation through a direct sum of monomorphisms $\mathrm{soc}\, L_t^+ \hookrightarrow L_t^+$ for some $t \geq 1$, and therefore h has a factorisation through a homomorphism $L_n^+ \to \mathrm{soc}\, L_t^+$, which is a sum of homomorphisms $L_n^+ \to L_0$. It follows that $\mathrm{rad}^\infty(L_n^+, L_s^+)$ is contained in the ideal of $\mathrm{mod}(R)$ generated by the set $\mathcal{X}$.

Finally, take any homomorphism $h \in \mathrm{rad}^\infty(L_j, L_n^+)$. Since $\mathbb{F}(h) = 0$, according to Lemma 3.3, h factorises through $L_n^+ J \subseteq \mathrm{soc}\, L_n^+ \subseteq L_n^+$. It follows that there is a factorisation $h = h''h'$, where $h' \in \mathrm{rad}^\infty(L_j, \mathrm{soc}\, L_n^+)$. Consequently, h' is a sum of homomorphisms in $\mathrm{rad}^\infty(L_j, L_0)$. It follows that $\mathrm{rad}^\infty(L_j, L_n^+)$ is contained in the ideal of $\mathrm{mod}(R)$ generated by the set $\mathcal{X}$. This finishes the proof of (f).

(g) Since we assume that $\mathbf{d}_{-\infty}({}_F F_F) = \boldsymbol{\omega}$, $d_j^J = 2$ for all $j \leq m-1$, $d_m^J = 1$ and $d_{m+1}^J = \infty$, where $m \geq 0$. Recall that the Auslander-Reiten translation quiver of $\mathrm{mod}(R_J)$ has one of the forms (3.5) or (3.6), the module $Q_0^{(0)}$ is simple injective and $Q_1^{(0)}$ is the injective envelope of $P_0^{(0)} \cong (0, F)$. It follows that $Q_0^{(0)} \cong (F, 0)$, $Q_1^{(0)} \cong (J_F^{(-2)}, F)$ (see [25]) and therefore $\mathbf{dim}\, Q_0^{(0)} = (1, 0)$, $\mathbf{dim}\, Q_1^{(0)} = (d_{-2}^J, 1) = (2, 1)$. Furthermore, the almost split sequence (3.11) in $\mathrm{mod}(R_J)$ yields

$$\mathbf{dim}\, Q_s^{(0)} = d_{-s}^J\, \mathbf{dim}\, Q_{s-1}^{(0)} - \mathbf{dim}\, Q_{s-2}^{(0)} = 2\, \mathbf{dim}\, Q_{s-1}^{(0)} - \mathbf{dim}\, Q_{s-2}^{(0)}$$

for all $s \geq 2$. Hence, for $s = 2$, we get $\mathbf{dim}\, Q_2^{(0)} = 2\, \mathbf{dim}\, Q_1^{(0)} - \mathbf{dim}\, Q_0^{(0)} = (3, 2)$, and applying inductively the above equality yields $\mathbf{dim}\, Q_s^{(0)} = (s+1, s)$ and $l(Q_s^{(0)}) = 2s+1$ for any $s \geq 0$. Hence, in view of Lemma 3.3 (iii), we conclude that $l(L_s) = l(\mathbb{F}(L_s)) = l(Q_s^{(0)}) = 2s + 1$. We recall that every irreducible morphism between indecomposable modules is either injective or surjective (see [3, Lemma 5.1] and [27, Section 11.1]). It follows that any irreducible morphism $L_s \to L_{s-1}$ is surjective for $s \geq 1$, because it is not injective.

Now we note that the second almost split sequence in (d) yields

$$l(L_{n+2}^+) = d_n^J l(L_{n+1}^+) - l(L_n^+) = 2l(L_{n+1}^+) - l(L_n^+)$$

for $n = 0, 1, \ldots, m-1$. Since L_0^+ is simple and $L_1^+ \cong R$, $l(L_0^+) = 1$ and $l(L_1^+) = 1 + d_0^J \leq 3$. Hence we get $l(L_2^+) = l(L_1^+) - l(L_0^+) = 2(1 + d_0^J) - 1 = 1 + 2d_0^J$, and inductively we show that $l(L_j^+) = 1 + j d_0^J$ for $j = 1, \ldots, m+1$. Consequently the statement (g) follows. □

The following corollary shows how potential local counter-examples R to the pure semisimplicity conjecture of length two or three should look like, and gives the structure of their Auslander-Reiten translation quiver $\Gamma(\mathrm{mod}\, R)$.

Corollary 3.13. *Assume that R is a local right pure semisimple ring of infinite representation type such that $2 \leq l(R_R) \leq 3$. Then $J(R)^2 = 0$, $J = J(R)$ is a bimodule over the division ring $F = R/J(R)$, there exists an integer $m \geq 0$ such that $d_{m+1}^J = \mathrm{r.dim}\, J^{(m+1)} = \infty$, $d_j^J = \mathrm{r.dim}\, J^{(j)} < \infty$ for all $j \leq m$, the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F) = (\ldots, d_{-j}(J), \ldots, d_{-1}(J), d_0(J), \infty)$, with $d_{-j}(J) = d_{m-j}^J$, (2.5) is defined and the following conditions are equivalent*:

(a) *The Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R)$ of $\mathrm{mod}(R)$ is infinite and connected of the form* $\cdots \to\bullet\to\bullet\to\bullet\to \cdots \to\bullet\to\bullet$.

(b) *The infinite dimension-sequence* $\mathbf{d}_{-\infty}({}_F J_F)$ *of* ${}_F J_F$ *belongs to the set* $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$.
(c) *The infinite radical* $\mathrm{rad}^{\infty}_R = \mathrm{rad}^{\infty}(\mathrm{mod}\, R)$ *of the category* $\mathrm{mod}(R)$ *is non-zero, whereas its square* $(\mathrm{rad}^{\infty}_R)^2$ *is zero.*

If any of the conditions (a)–(c) *is satisfied, then* R *is a counter-example to the pure semisimplicity conjecture, the Auslander-Reiten translation quiver* $\Gamma(\mathrm{mod}\, R)$ *has one of the forms* (3.5) *or* (3.6), *and* R *has the properties presented in* Proposition 3.10.

Proof. We know from Lemma 3.1 that $J(R)^2 = 0$. Since R is right pure semisimple, according to [25, Proposition 2.4] every indecomposable non-projective module X in $\mathrm{mod}(R)$ admits an almost split sequence $0 \to X \to Y \to Z \to 0$ and Theorem 3.4 and Proposition 3.10 apply. □

In connection with [28, Remark 2.4] the following observation is useful.

Corollary 3.14. *Assume* $F \subset G$ *are division rings such that* $F \cong G$, $\dim_F G = \infty$ *and that the associated infinite dimension-sequence* $\mathbf{d}_{-\infty}({}_F G_G)$ (3.2) *of the* F-G-*bimodule* ${}_F G_G$ *belongs to* $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$. *Then*
(a) *the trivial extension* $T_G = F \ltimes {}_F G_G$ *of* F *by* ${}_F G_G$ *is a local ring and it is a counter-example to the pure semisimplicity conjecture of length two* (*that is,* $l(T_G) = 2$, *when* T_G *is viewed as a right* T_G-*module*),
(b) *the ring* T_G *is not self-injective,*
(c) *the global dimension of* T_G *is infinite, and*
(d) *the Auslander-Reiten quiver* $\Gamma(\mathrm{mod}\, T_G)$ *of* $\mathrm{mod}(T_G)$ *is connected of the form* $\cdots \to\bullet\to\bullet\to\bullet\to \cdots \to\bullet\to\bullet$.

Proof. Apply Theorem 3.4. □

REMARK 3.15. Since for any $v = (\ldots, v_{-m}, \ldots, v_{-1}, v_0, \infty) \in \mathcal{DS}_{pss}$ there exists $j \geq 1$ such that $v_{-j} = 1$, according to [28, Remark 4.5] the existence of an F-G-bimodule ${}_F M_G$ such that $\mathbf{d}_{-\infty}({}_F M_G) = v$ is an infinite version of the Artin problem for division ring extensions studied in [4], [20], [28] and [29] (see [28, Section 4]). In the situation we study in Corollary 3.14 we assume in addition that $F \cong G$.

We hope that, by applying a modification of the bimodule amalgam rings construction of Schofield [21, Chapter 13], one can construct a division ring embedding $F \subseteq G \cong F$ such that $\mathbf{d}_{-\infty}({}_F G_G) = v$ for some of the dimension-sequences $v \in \mathcal{DS}_{pss}$.

A solution of this problem is strongly related with the main problems studied in [15], [38] and [39] of finding special classes of artinian rings without self-extensions (compare with [1], [41]).

We finish the paper by raising the following problems related with the one stated in [33, Problem 4.21] for hereditary rings of the form R_M (2.1).

Problem 3.16. Assume that R is a right artinian local ring with the Jacobson radical $J = J(R)$, such that $J^2 = 0$, $F = R/J$ and the associated infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F J_F)$ of (2.5) associated to the F-F-bimodule ${}_F J_F$ belongs to the set $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$. Let $L_0, L_1, L_2, \ldots, L_s, \ldots$ be pairwise non-isomorphic indecomposable R-modules shown in (3.5) and defined by (3.9) (see Theorem 3.4).
(a) Find a decomposition of the right R-module

$$\mathcal{L}(R) = \prod_{m=0}^{\infty} L_m \Big/ \bigoplus_{m=0}^{\infty} L_m \tag{3.17}$$

in a direct sum of indecomposable modules.
(b) Give a characterization of local rings R for which the R-module $\mathcal{L}(R)$ is projective.

In [16] a partial solution of the problem [33, Problem 4.21] is presented for hereditary rings of the form R_M (2.1).

The following interesting problem stated in [31, Problem 3.2] remains unsolved.

Problem 3.18. Give a characterisation of semiperfect rings R for which every indecomposable right R-module is pure-projective or pure-injective. Is every such a ring R right artinian or right pure semisimple?

Let us finish the paper by the following open question related with Theorem 3.4.

Problem 3.19. Prove that under the assumption in Theorem 3.4 the statement (a) is equivalent to the following one:

(a′) The Auslander-Reiten quiver $\Gamma(\operatorname{mod} R)$ is infinite and connected.

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