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ON THE ESSENTIAL SPECTRUM OF THE RELATIVISTIC MAGNETIC SCHRÖDINGER OPERATOR

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1. Introduction

In a series of papers ([9], [14], [15], [16]) one has studied the Weyl quantized Hamiltonian of a relativistic spinless particle with a magnetic vector potential a:

$$[Op^w(\lambda_a)u](x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \lambda_a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy d\xi,$$

where $\lambda_a(x,\xi) = (|\xi - a(x)|^2 + 1)^{1/2}$. For simplicity we suppose here that the mass of the particle is equal to 1. All the differential and pseudodifferential operators considered in this paper are, possibly unbounded, operators in $L^2(\mathbf{R}^n)$, defined on the Schwartz space $\mathcal{S}$ of rapidly decreasing smooth functions on $\mathbf{R}^n$.

In [14] it was proved that if the derivatives of any positive order of $a \in C^\infty(\mathbf{R}^n;\mathbf{R}^n)$ are bounded, then $Op^w(\lambda_a)$ is essentially selfadjoint on $\mathcal{S}$. Let h_a be its unique selfadjoint extension. In [16] the authors proved that if a itself is bounded and if all its derivatives converge to zero at infinity, then the essential spectrum of h_a is equal to the essential spectrum of $\sqrt{H_a+1}$, where H_a is the quantum nonrelativistic magnetic Hamiltonian with vector potential a, i.e. the selfadjoint operator generated by the differential operator $(D - a(X))^2$.

We shall prove in this paper that the essential spectra of h_a and of $\sqrt{H_a+1}$ are still equal if we drop the condition of boundedness of a. Thus, vector potentials a which behave at infinity as $|x|^{1-\varepsilon}$, ε positive and arbitrary small, are allowed.

More precisely, the main result of the paper is the following theorem.

Theorem 1.1. *Suppose that*:

(i) *the vector potential $a \in C^\infty(\mathbf{R}^n;\mathbf{R}^n)$ is such that*

$$\lim_{|x|\to\infty} \partial^\beta a(x) = 0, \quad \forall\, \beta \in \mathbf{N}^n,\ \beta > 0;$$

(ii) *the scalar potential V is a continuous function such that* $\lim_{|x|\to\infty} V(x) = 0$.

If h_a is the unique selfadjoint extension of $Op^w(\lambda_a)$ and H_a is the unique selfad-

joint extension of $(D - a(X))^2$, *then*

$$\sigma_{\mathrm{ess}}(h_a + V) = \sigma_{\mathrm{ess}}[(H_a + 1)^{1/2}].$$

Let us also mention, that as a by-product of our proof, we not only recover some results from [13] and [15] concerning the essential selfadjointness of some pseudodifferential operators, but we can also obtain the domain of definition of the generated self-adjoint operator (Theorem 3.4).

We now give the plan of the paper.

In the second section we recall some results on the calculus of pseudodifferential operators. These results will be used in the next two sections.

In the third one, we introduce what we call the a-magnetic Sobolev spaces, a particular case of weighted Sobolev spaces defined in [2]. As a corollary of the results proved in this section, we shall obtain that the domain of h_a is equal to the form domain of H_a. If all the derivatives of a are bounded, it is also proved that the a-magnetic Sobolev space of order m is equal to the domain of $H_a^{m/2}$. We think that this kind of results are already known, but we never saw them explicitly stated.

The last section contains the proof of the main theorem.

Always in this paper $\mathrm{Dom}(H)$ denotes the domain of the operator H and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. All the functions which appear are defined on the whole space $\mathbf{R}^n$. Therefore we shall write, for example, L^2 instead of $L^2(\mathbf{R}^n)$. $\mathbf{B}(X_1, X_2)$ denotes the space of bounded linear operators from the locally convex topological vector space X_1 to the locally convex topological vector space X_2.

2. A class of pseudodifferential operators

DEFINITION 2.1. A vector valued function $a \in C^\infty(\mathbf{R}^n; \mathbf{R}^n)$ is called an *admissible vector potential* if $\partial^\alpha a$ is bounded for any $\alpha \in \mathbf{N}^n \setminus \{0\}$.

For an admissible vector potential a we define a weight function $\lambda_a \colon \mathbf{R}^{2n} \to [1; \infty)$ by the formula

$$\lambda_a(x, \xi) = \langle \xi - a(x) \rangle.$$

When there is no risk of confusion, we shall omit the subscript a from this notation.

Lemma 2.2. *If a is an admissible vector potential then*:
(i) *the weight function λ is a basic weight function in the sense of Kumano-go and Taniguchi* [10];
(ii) *the pair $(\lambda; 1)$ is a pair of smooth weight functions in the sense of Beals* [2].

For the convenience of the reader we recall here the definition of basic weight

functions. A smooth function $\lambda\colon \mathbf{R}^{2n} \to [1;\infty)$ is called a basic weight function if there exist positive constants C, τ and $C_{\alpha\beta}$, $\alpha,\ \beta \in \mathbf{N}^n$ such that:
(i) $\lambda(x,\xi) \le C\langle x\rangle^{\tau}\langle\xi\rangle$, $\forall\,(x,\xi)\in\mathbf{R}^{2n}$;
(ii) $|\partial_\xi^\alpha\partial_x^\beta\lambda| \le C_{\alpha\beta}\lambda^{1-|\alpha|}$;
(iii) $\lambda(x+y,\xi) \le C\langle y\rangle^{\tau}\lambda(x,\xi)$, $\forall,\ x,\ y,\ \xi \in \mathbf{R}^n$.

It is easy to verify that the first assertion of Lemma 2.2 is true and this verification was already made in [14]. The fact that if λ is a basic weight function, then the pair $(\lambda;1)$ is a pair of smooth weight functions was pointed out in [2].

DEFINITION 2.3. The space

$$S_a^m = \left\{q \in C^\infty(\mathbf{R}^{2n});\ \forall\,\alpha,\ \ \beta\in\mathbf{N}^n,\ \ \exists\, C_{\alpha\beta}>0 \text{ s.t. } |q_{(\beta)}^{(\alpha)}| \le C_{\alpha\beta}\lambda^{m-|\alpha|}\right\}$$

is called the *space of symbols of order* $m \in \mathbf{R}$ associated to the weight function λ_a.

We have used the notation $q_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha\partial_x^\beta q$. We shall also use the following notation:
– $\|q\|_{a;m,l} = \max_{|\alpha+\beta|\le l}\sup_{\mathbf{R}^{2n}}|q_{(\beta)}^{(\alpha)}|\lambda^{-m+|\alpha|}$, $\forall\, q\in S_a^m$, $\forall\, l\in\mathbf{N}$;
– $S_a^{-\infty} = \bigcap_m S_a^m$.

In the following remark we list some simple properties of the spaces S_a^m and clarify their relation with the spaces of standard symbols.

Remarks 2.4. (i) If a is an admissible vector potential, then $\lambda_a \in S_a^1$.
(ii) If a is an admissible vector potential and $-\infty \le m_1 \le m_2 < \infty$, then $S_a^{m_2} \subset S_a^{m_1}$.
(iii) If a and b are admissible vector potentials, then the following assertions are equivalent:

(a) there exists a real number m such that $S_a^m = S_b^m$;
(b) $S_a^m = S_b^m$ for any $m \in \mathbf{R}$;
(c) $a - b \in L^\infty(\mathbf{R}^n;\mathbf{R}^n)$.

In particular, if a is bounded, then $S_a^m = S_0^m$, the space of standard symbols of order m, usually denoted by $S_{1,0}^m$.
(iv) A function q is in S_a^m if and only if there exists a (unique) symbol $p \in S_0^m$ such that $q(x,\xi) = p_a(x,\xi) \equiv p(x,\xi-a(x))$; for any admissible vector potential a and every $m,\ l\in\mathbf{N}$, there exists a positive constant $C = C(a,l,m)$ such that

$$\|p_a\|_{a;m,l} \le C\,\|p\|_{0;m,l}\,, \quad \forall\, p\in S_0^m.$$

Proof. The proof of the first property consists in a straightforward verification, based on the definition of our basic weight functions. It is essential that the derivatives of a are bounded. The second assertion follows directly from Definition 2.3. The proof of (iii) is based on the following remark: if we put $\xi = a(x)$ in the inequality $\langle\xi - b(x)\rangle \le C\langle\xi - a(x)\rangle$, then we obtain $\langle a(x) - b(x)\rangle \le C$. The proofs of (iv) and (v) consist in a little tedious, but again straightforward computation, which we shall not perform here. The fact that all the derivatives of a are bounded is also crucial for this points. $\square$

In what follows, we shall always suppose that a is a fixed admissible vector potential.

Let us introduce the class of a-pseudodifferential operators.

Definition 2.5. For q in S_a^m and u in $\mathcal{S}$ we define

$$Op^1(q)u(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\, d\xi,$$

$$Op^w(q)u(x) = (2\pi)^{-n}\,\mathrm{Osc} - \iint e^{i\langle x-y,\xi\rangle} q\left(\frac{x+y}{2},\xi\right) u(y)\, dyd\xi.$$

$Op^1(q)$ is called the right quantization or Kohn-Nirenberg quantization (following [5]) of q and $Op^w(q)$ is called the Weyl quantization of q. Osc in front of an integral means that the integral is defined as an oscillatory integral.

Proposition 2.6 ([2, Proposition 3.11 and Corollary 4.8]). *If $q \in S_a^m$, then $Op^1(q)$, $Op^w(q) \in \mathbf{B}(\mathcal{S},\mathcal{S})$ and they have continuous extensions from $\mathcal{S}'$ to $\mathcal{S}'$.*

It is also known that

$$\{Op^1(q);\, q \in S_a^m\} = \{Op^w(q);\, q \in S_a^m\}. \tag{1}$$

We shall denote this space with OpS_a^m. It is the space of pseudodifferential operators of order m associated to the weight function λ_a. If $A \in OpS_a^m$, then there exists a unique $q \in S_a^m$ such that $A = Op^1(q)$. The symbol q is called the Kohn-Nirenberg symbol of A and is denoted with $\sigma^1(A)$. Analogously, A has a unique Weyl symbol $\tilde{q} \in S_a^m$ such that $A = Op^w(\tilde{q})$.

Due to Lemma 2.2, the results proved in [2] and [10] are applicable to our symbols and pseudodifferential operators. We shall state now some of them in a form which is more convenient for our purposes.

Theorem 2.7 ([2, Theorem 4.13]). *If $q_j \in S_a^{m_j}$, $\forall\, j \in \mathbf{N}$ and if $m_j \downarrow -\infty$, then there exists a symbol $q \in S_a^{m_0}$ such that*

$$q - \sum_{j=0}^{N-1} q_j \in S_a^{m_N}, \quad \forall\, N \in \mathbf{N}$$

In this case we write $q \sim \sum_{j\geq 0} q_j$.

Theorem 2.8 ([2, Theorem 4.1]; [10, Theorem 2.3]). *If* $q_j \in S_a^{m_j}$, $j = 1, 2$, *then* $Op^1(q_1)Op^1(q_2) \in OpS_a^{m_1+m_2}$ *and*

$$\begin{aligned}\sigma^1\big(Op^1(q_1)Op^1(q_2)\big)(x,\xi) \\ &= (2\pi)^{-n}\,\mathrm{Osc}-\iint e^{-i\langle y,\eta\rangle} q_1(x,\xi+\eta)q_2(x+y,\xi)\,dyd\eta \\ &= q_1(x,\xi)q_2(x,\xi)+r(x,\xi)\end{aligned}$$

where

$$r(x,\xi) = (2\pi)^{-n}\int_0^1 \sum_{|\gamma|=1} \mathrm{Osc}-\iint e^{-i\langle y,\eta\rangle} q_1^{(\gamma)}(x,\xi+\theta\eta)q_{2,(\gamma)}(x+y,\xi)\,dyd\eta d\theta.$$

For any $l \in \mathbf{N}$ *there exist* $l_1, l_2 \in \mathbf{N}$ *and* $C > 0$ *such that*

$$\|r\|_{a;m_1+m_2-1,l} \le C\,\|\partial_\xi q_1\|_{a;m_1-1,l_1}\,\|\partial_x q_2\|_{a;m_2,l_2}\,.$$

For Weyl symbols there exists also a result similar to Theorem 2.8. Since our goal is the study of some selfadjoint operators, it seems more natural to use the theorem of multiplication of pseudodifferential operators in its version for Weyl symbols instead of Theorem 2.8. But we prefer to work with Kohn-Nirenberg symbols because the formulas are simpler in this case.

Theorem 2.9 ([2, Theorem 5.1]; [10, Theorem 2.7]). *If* $q \in S_a^0$, *then* $Op^1(q)$ *can be extended to a bounded operator in* $L^2(\mathbf{R}^n)$. *Moreover, there exist* $C > 0$ *and* $l \in \mathbf{N}$ *which do not depend on* q *such that*

$$\|Op^1(q)\|_{\mathbf{B}(L^2)} \le C\,\|q\|_{a;0,l}\,.$$

DEFINITION 2.10. A symbol $q \in S_a^m$ is called *elliptic* if there exist two positive constants c and R such that if $|\xi - a(x)| \ge R$, then

$$|q(x,\xi)| \ge c\lambda^m(x,\xi).$$

If the Kohn-Nirenberg symbol of a pseudodifferential operator is elliptic, then the operator is said to be elliptic.

REMARK 2.11. If $q \in S_a^m$, then $Op^w(q) - Op^1(q) \in OpS_a^{m-1}$. Therefore if the Weyl symbol of an operator is elliptic, then the operator is still elliptic.

Theorem 2.12. *If* $q \in S_a^m$ *is elliptic, then there exists a symbol* $\tilde q \in S_a^{-m}$ (*also elliptic*) *such that*

$$Op^w(q)Op^w(\tilde q) - I,\; Op^w(\tilde q)Op^w(q) - I \in OpS_a^{-\infty}.$$

The operator $Op^w(\tilde{q})$ is unique modulo $OpS_a^{-\infty}$ and is called a parametrix of $Op^w(q)$.

Results of this type are standard in the theory of pseudodifferential operators, but we prefer to sketch a proof.

Sketch of proof of Theorem 2.12. If we take into consideration Remark 2.11 and (1), we see that it is sufficient to prove the theorem for the case when Op^w is replaced with Op^1.

Let R be as in Definition 2.10 and let χ be a smooth function such that $\chi(\eta) = 0$ if $|\eta| < R$ and $\chi(\eta) = 1$ if $|\eta| > 2R$. We put

$$\tilde{q}_0(x,\xi) = \chi\big(\xi - a(x)\big)q(x,\xi)^{-1}.$$

Then $\tilde{q}_0 \in S_a^{-m}$ and $Op^1(q)Op^1(\tilde{q}_0) = I - Op^1(r_{-1})$, where $r_{-1} \in S_a^{-1}$. Therefore

$$Op^1(q)Op^1(\tilde{q}_0)\sum_{i=0}^{N}[Op^1(r_{-1})]^i - I = [Op^1(r_{-1})]^{N-1} \in OpS_a^{N-1}.$$

We define

$$\tilde{q}_i = \sigma^1\left\{Op^1(\tilde{q}_0)[Op^1(r_{-1})]^i\right\}.$$

According to Theorem 2.7, we can choose a symbol $\tilde{q} \in S_a^{-m}$ such that $\tilde{q} \sim \sum_{i\geq 0}\tilde{q}_i$. The symbol $\tilde{q}$ has all the desired properties. For the verification of this statement one also uses the fact that the Kohn-Nirenberg symbol of an operator from OpS_a^m is uniquely defined.

Theorem 2.15 below can be regarded as a particular case of Theorem 3.1 from [13]. But we shall give its proof because it is not very complicated and part of it will be used also for the proof of other theorems of our paper.

We shall need a result concerning the existence of approximate parametrices of elliptic pseudodifferential operators which depend on a parameter. A first step in this direction is the following lemma.

Lemma 2.13. *Let $q \in S_a^m$ be an elliptic symbol and let $\Lambda \subset \mathbf{C}\setminus\{0\}$ be such that*

$$|q(x,\xi) + \mu| \geq c\max(\lambda^m(x,\xi), |\mu|), \quad \forall\,(x,\xi) \in \mathbf{R}^{2n},\ \forall\,\mu \in \Lambda,$$

for some positive constant c. If $\tilde{q}_\mu(x,\xi) = (q(x,\xi)+\mu)^{-1}$, then for any α, $\beta \in \mathbf{N}^n$ there exists a constant $C_{\alpha\beta}$ such that

$$|(\tilde{q}_\mu)^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha\beta}\lambda^{-|\alpha|}(x,\xi)\min(\lambda^{-m}(x,\xi), |\mu|^{-1}), \quad \forall\,(x,\xi) \in \mathbf{R}^{2n},\ \forall\,\mu \in \Lambda.$$

Proof. The proof of the lemma consists in a straightforward verification. □

Theorem 2.14. *Let $q = q_m + q_{m-1}$, $q_m \in S_a^m$, $q_{m-1} \in S_a^{m-1}$, $m > 0$, be an elliptic symbol and let $\Lambda \subset \mathbf{C}\setminus\{0\}$ be such that the pair (q_m, Λ) satisfies the hypothesis of* Lemma 2.13. *Then there exist symbols $\tilde{q}_\mu \in S_a^m$, $r_\mu \in S_a^{-1}$ and $\mu \in \Lambda$ such that*

$$\big(Op^1(q) + \mu\big)Op^1(\tilde{q}_\mu) = 1 + Op^1(r_\mu)$$

and such that r_μ satisfy the estimates

$$\|r_\mu\|_{a;0,l} \leq C_l|\mu|^{-\min(1/m,1)}$$

with some positive constants C_l which depend on $l \in \mathbf{N}$ and do not depend on $\mu \in \Lambda$.

Proof. We take $\tilde{q}_\mu(x,\xi) = (q_m(x,\xi) + \mu)^{-1}$. Then

$$\big(Op^1(q) + \mu\big)Op^1(\tilde{q}_\mu) = \big(Op^1(q_m) + \mu\big)Op^1(\tilde{q}_\mu) + \big(Op^1(q_{m-1})Op^1(\tilde{q}_\mu)\big).$$

The conclusion of the theorem follows from Theorem 2.8 and 2.13. □

Theorem 2.15. *If $q \in S_a^m$, $m > 0$ is real and elliptic, then $Op^w(q)$ is essentially selfadjoint on $\mathcal{S}$.*

Proof. According to Remark 2.11, $Op^w(q) = Op^1(q) + Op^1(q_{m-1})$ for some $q_{m-1} \in S_a^{m-1}$. Therefore we can apply Theorem 2.14 with $\Lambda = i\mathbf{R}\setminus(-i;i)$. Next, applying Theorem 2.9, we deduce that $(Op^w(q) + i\mu)Op^1(\tilde{q}_\mu)$ can be extended to a bounded invertible operator on $L^2(\mathbf{R}^n)$, for μ real and sufficiently large. Therefore, for such values of μ, $(Op^w(q) + i\mu)(\mathcal{S})$ ($\supset (Op^w(q) + i\mu)Op^1(\tilde{q}_\mu)(\mathcal{S})$) is a dense subspace of $L^2(\mathbf{R}^n)$. The corollary of Theorem VIII.3 in [17] completes the proof of Theorem 2.15. □

3. Magnetic Sobolev spaces

The magnetic Sobolev spaces which we define below are a particular case of weighted Sobolev spaces defined in [2]. We shall use the same notation for various extensions of a pseudodifferential operator to subspaces of $\mathcal{S}'$. A distinct notation will be used only for the selfadjoint extension in $L^2(\mathbf{R}^n)$, if it exists, of a pseudodifferential operator.

DEFINITION 3.1. For $m \in \mathbf{R}$,

$$\mathcal{H}_a^m = \operatorname{span}\{Au;\ u \in L^2(\mathbf{R}^n),\ A \in OpS_a^{-m}\}.$$

The space $\mathcal{H}_a^m$ is endowed with the finest topology with respect to which each mapping $A\colon L^2 \to \mathcal{H}_a^m$, $A \in OpS_a^{-m}$ is continuous.

Theorem 3.2 ([2, Theorem 6.1]). (i) *For any* $m \in \mathbf{R}$, $\mathcal{S} \subset \mathcal{H}_a^m \subset \mathcal{S}'$ *continuously and densely.*
(ii) $(\mathcal{H}_a^m)' = \mathcal{H}_a^{-m}$ *topologically*, $\forall\, m \in \mathbf{R}$.
(iii) *If* $A \in OpS_a^\nu$, $\nu \in \mathbf{R}$, *then* $A \colon \mathcal{H}_a^{m+\nu} \to \mathcal{H}_a^m$ *is continuous*, $\forall\, m \in \mathbf{R}$.
(iv) *For any* m, $\nu \in \mathbf{R}$ *there exists an elliptic operator* $A \in OpS_a^\nu$ *such that* $A \colon \mathcal{H}_a^{m+\nu} \to \mathcal{H}_a^m$ *is a topological isomorphism. Therefore* $\mathcal{H}_a^m$ *is a Hilbertizable topological vector space.*

The most important result we shall prove in this section is that $\mathcal{H}_a^{2m} = \mathrm{Dom}(H_a^m)$, where H_a is the magnetic Hamiltonian, i.e. the unique selfadjoint extension of the operator defined on $\mathcal{S}$ by the differential expression $(D - a(X))^2$, $D = -i\partial$. We shall also prove that if $q \in S_a^m$, $m > 0$ is real and elliptic and if Q is the unique selfadjoint extension of $Op^w(q)$, then $\mathrm{Dom}(Q) = \mathcal{H}_a^m$.

We start with a theorem which gives an alternative characterization of $\mathcal{H}_a^m$. We recall that a is always an admissible vector potential.

Theorem 3.3. *If* $B \in OpS_a^m$, $m > 0$ *is elliptic, then*

$$\mathcal{H}_a^m = \{u \in L^2;\ Bu \in L^2\}.$$

Proof. The hypothesis implies that there exists an elliptic symbol q such that $B = Op^w(q)$.

"$\subset$" Let $A \in OpS_a^{-m}$ be such that $A \colon L^2 \to \mathcal{H}_a^m$ is a topological isomorphism and let u be an arbitrary function from $\mathcal{H}_a^m$. Then there exists a function $v \in L^2$ such that $u = Av$. Therefore $Bu = Op^w(q)Av \in L^2$, since $Op^w(q)A \in OpS_a^0$.

"$\supset$" Suppose that u is in L^2 and that $v = Op^w(q)u \in L^2$. If $Op^w(\tilde{q})$ is a parametrix for $Op^w(q)$, then

$$Op^w(\tilde{q})Op^w(q)u = u + Op^w(r)u,$$

for some $r \in S_a^{-\infty}$. From Theorem 3.2 (iii), it follows that $u = Op^w(\tilde{q})v - Op^w(r)u$ is in $\mathcal{H}_a^m$. $\square$

Theorem 3.4. *If* $q \in S_a^m$, $m > 0$ *is real and elliptic and if* Q *is the unique selfadjoint extension of* $Op^w(q)$, *then* $\mathrm{Dom}(Q) = \mathcal{H}_a^m$.

Proof. "$\supset$" If u is in $\mathcal{H}_a^m$, then there exists a sequence of functions $\{\varphi_j\}_{j \in \mathbf{N}} \subset \mathcal{S}$ such that $\varphi_j \to u$, $j \to \infty$, in $\mathcal{H}_a^m$. Applying Theorem 3.2 (iii), we obtain that $Op^w(q)\varphi_j \to Op^w(q)u$, $j \to \infty$, in L^2.

"$\subset$" If u is in $\mathrm{Dom}(Q)$, then there exists a function v in L^2 and a sequence of functions $\{\varphi_j\}_{j \in \mathbf{N}} \subset \mathcal{S}$ such that $\varphi_j \to u$, $j \to \infty$, in L^2 and $Op^w(q)\varphi_j \to v$,

$j \to \infty$ in L^2. Let $Op^w(\tilde{q})$ be a parametrix for $Op^w(q)$. Then

$$Op^w(\tilde{q})Op^w(q)\varphi_j = \varphi_j + Op^w(r)\varphi_j$$

for some $r \in S_a^{-\infty}$,

$$Op^w(\tilde{q})Op^w(q)\varphi_j \to Op^w(\tilde{q})v, \quad j \to \infty$$

in $\mathcal{H}_a^m$ and

$$Op^w(r)\varphi_j \to Op^w(r)u, \quad j \to \infty$$

in $\mathcal{H}_a^m$. Hence

$$\varphi_j = Op^w(\tilde{q})Op^w(q)\varphi_j - Op^w(r)\varphi_j \to Op^w(\tilde{q})v - Op^w(r)u, \quad j \to \infty$$

in $\mathcal{H}_a^m$. Therefore $u = Op^w(\tilde{q})v - Op^w(r)u$ is in $\mathcal{H}_a^m$. □

Corollary 3.5. *If a is an admissible symbol, then* $\mathrm{Dom}(H_a) = \mathcal{H}_a^2$.

For the proof of the equality $\mathcal{H}_a^{2m} = \mathrm{Dom}(H_a^m)$ we shall need the following lemma.

Lemma 3.6. *If*

$$\tilde{q}_\mu(x, \xi) = [[\xi - a(x)]^2 + \mu]^{-1}, \quad \mathrm{Re}\,\mu > 0,$$

then

$$\left[\left(D - a(X)\right)^2 + \mu\right] Op^1(\tilde{q}_\mu) = 1 + Op^1(r_\mu),$$

where $r_\mu \in S_a^{-2}$ depends continuously on μ in any seminorm $\|\cdot\|_{a;k,l}$, $k = 0, 1, 2$, $l \in \mathbf{N}$. For any $l \in \mathbf{N}$ there exists a constant C_l which does not depend on μ such that

$$\|r_\mu\|_{a;-k,l} \le C_l |\mu|^{-(2-k)/2}, \quad k = 0, \ 1, \ 2.$$

The symbol $\tilde{q}_\mu$ has the same properties as r_μ.

Proof. We have

$$\left(D - a(X)\right)^2 + \mu = Op^1(\tilde{q}_\mu^{-1}) + i \,\mathrm{div}\, a(X).$$

Therefore, if we apply Theorem 2.8, we obtain that

$$((D - a(X))^2 + \mu)Op^1(\tilde{q}_\mu) = I + i \,\mathrm{div}\, a(X)Op^1(\tilde{q}_\mu) + Op^1(r_{\mu,1}) \equiv I + Op^1(r_\mu),$$

where

$$r_\mu(x,\xi) = \frac{i\,\operatorname{div} a(x)}{(\xi - a(x))^2 + \mu} + (2\pi)^{-n}\int_0^1 \left\{ \sum_{j=1}^n \operatorname{Osc} -4 \iint e^{-i\langle y,\eta\rangle}\big(\xi_j + \theta\eta_j - a_j(x)\big) \cdot \frac{\langle \xi - a(x+y), \partial_j a(x+y)\rangle}{[(\xi - a(x+y))^2 + \mu]^2}\, dy d\eta \right\} d\theta.$$

We put

$$b_\mu(x,y,\xi,\eta) = 4\sum_{j=1}^n \int_0^1 \big(\xi_j + \theta\eta_j - a_j(x)\big)\frac{\langle \xi - a(x+y), \partial_j a(x+y)\rangle}{[(\xi - a(x+y))^2 + \mu]^2}.$$

Then

$$r_\mu(x,\xi) = \frac{i\,\operatorname{div} a(x)}{(\xi - a(x))^2 + \mu} + (2\pi)^{-n}\iint e^{-i\langle y,\eta\rangle}\langle y\rangle^{-2N}(1-\Delta_\eta)\langle\eta\rangle^{-2L}(1-\Delta_y)^L \cdot b_\mu(x,y,\xi,\eta)\, dy d\eta$$

for N and L sufficiently large. We can apply Lemma 2.13 to the first term and conclude that it satisfies the required estimates. For the estimation of the second one, applying again Lemma 2.13 and the inequalities

$$\langle \xi + \eta\rangle \le 2\langle\xi\rangle\langle\eta\rangle, \quad |a(x+y) - a(x)| \le C|y|$$

we obtain that

$$\partial_\xi^\alpha \partial_x^\beta \partial_\eta^\gamma \partial_y^\delta b_\mu(x,y,\xi,\eta) \le C_{\alpha\beta\gamma\delta}\langle \xi - a(x)\rangle^{-|\alpha|}\langle\eta\rangle\langle y\rangle^{|\alpha|}|(\xi - a(x+y))^2 + \mu|^{-1}.$$

Taking L and N eventually larger, we get that the second term of r_μ also satisfies the required estimates.

The proof of the continuity proceeds in the same manner. □

Theorem 3.7. $\operatorname{Dom}[(H_a)^m] = \mathcal{H}_a^{2m}$ *for any* $m > 0$.

Proof. If $m \in \mathbf{N}$, then the stated equality follows from Theorem 3.4. It remains to prove the equality for $m \in \mathbf{R}\backslash\mathbf{N}$.

(i) We prove first that the theorem is valid for $m \in (0;1)$. Let Γ be the path

$$\Gamma = \left\{ s e^{i\pi}; -\infty < s < -\frac{1}{2}\right\} \cup \left\{ \left(\frac{1}{2}\right) e^{i\theta}; -\pi \le \theta \le \pi \right\}$$

$$\cup \left\{ s e^{-i\pi}; -\infty < s < -\frac{1}{2} \right\}.$$

Then

$$(H_a + 1)^{m-1} = \frac{i}{2\pi} \oint_\Gamma \mu^{m-1} (H_a + 1 + \mu)^{-1} \, d\mu.$$

We also define an elliptic pseudodifferential operator of order $2m - 2$

$$Op^1(q_{m-1}) = \frac{i}{2\pi} \oint_\Gamma \mu^{m-1} Op^1(\widetilde{q}_{1-\mu}) \, d\mu$$

where

$$\tilde{q}_{1-\mu}(x, \xi) = \left[\left(\xi - a(x) \right)^2 + 1 - \mu \right]^{-1}$$

is the symbol defined in Lemma 3.6. In fact, using the theorem of residues, we obtain that

$$q_{m-1}(x, \xi) = \left[\left(\xi - a(x) \right)^2 + 1 \right]^{m-1} = (\lambda_a)(x, \xi)^{2m-2}.$$

The two contour integrals converge in $\mathbf{B}(L^2)$ in the uniform operator topology. The following identity holds on $\mathcal{H}_a^2$:

$$(H_a + 1)^m - (H_a + 1) Op^1(q_{m-1}) = (H_a + 1) \frac{i}{2\pi} \oint_\Gamma \mu^{m-1} (H_a + 1 + \mu)^{-1} Op^1(r_{1-\mu}) \, d\mu.$$

Here $r_{1-\mu}$ is the remainder obtained in Lemma 3.6. Hence the last contour integral defines an operator from $\mathbf{B}(\mathcal{H}_a^0, \mathcal{H}_a^2)$. Therefore $(H_a+1)^m - (H_a+1)Op^1(q_{m-1}) \in \mathbf{B}(L^2)$. This, combined with the ellipticity of $(H_a + 1)Op^1(q_{m-1})$ and Corollary 3.5 gives that $\mathrm{Dom}[(H_a)^m] = \mathcal{H}_a^{2m}$ for $m \in (0; 1)$.
(ii) The general case can be proved by using a bootstrap argument. Suppose that $\mathrm{Dom}[(H_a)^{m'}] = \mathcal{H}_a^{2m'}$ for any $m' \in [0; N]$ for some $N \in \mathbf{N}$ and that $u \in \mathrm{Dom}[(H_a)^m]$ for an $m \in (N; N + 1)$. This is equivalent with the fact that $u \in \mathrm{Dom}[(H_a)^N]$ and $v = (H_a + 1)u \in \mathrm{Dom}[(H_a)^{m-N}]$, i.e. with the fact that u is in $\mathcal{H}_a^{2N}$ and $(H_a + 1)Op^1(q_{m-N-1})\ (H_a + 1)^N u \in L^2$, where q_{m-N-1} was previously defined. But $(H_a + 1)Op^1(q_{m-N-1})\ (H_a + 1)^N$ is an elliptic pseudodifferential operator of order $2m$. Theorem 3.3 completes the proof. □

Another proof of Theorem 3.7 could be based on an appropriate interpolation theorem for weighted Sobolev spaces. Such results are proved in [3]. But our weight function does not satisfy the conditions imposed in that paper. Therefore we had to give a direct proof of Theorem 3.7 based on some ideas from [18]. Evidently, the interpolation theorem for magnetic Sobolev spaces is now a consequence of Theorem 3.7. But we do not know how to prove it directly.

4. The essential spectrum of the relativistic magnetic Hamiltonian

We now prove Theorem 1.1. The proof will be made in several steps and uses the next compactness criterion, which is a particular case of Theorem 6.11 from [2].

Theorem 4.1. *If $q \in S_a^\nu$ is such that*

$$\lim_{|x|+|\xi|\to\infty} \lambda_a(x,\xi)^{-\nu+|\alpha|} q_{(\beta)}^{(\alpha)}(x,\xi) = 0, \quad \forall\, \alpha,\ \beta \in \mathbf{N}^n,$$

then $Op^1(q)$ is a compact operator from $\mathcal{H}_a^{m+\nu}$ to $\mathcal{H}_a^m$ for any $m \in \mathbf{R}$.

REMARK 4.2. If a is an admissible vector potential, then $|x| + |\xi| \to \infty$ if and only if $|x| + |\xi - a(x)| \to \infty$.

From now on, it is always assumed that the vector potential a satisfies hypothesis (i) of Theorem 1.1.

Lemma 4.3. *If $p \in S_0^m$ does not depend on x, then*

$$\lim_{|x|+|\xi|\to\infty} \lambda_a(x,\xi)^{-m+|\alpha|} \left\{\sigma^1\left[Op^w(p_a)\right] - p_a\right\}_{(\beta)}^{(\alpha)}(x,\xi) = 0, \quad \forall\, \alpha,\ \beta \in \mathbf{N}^n.$$

Recall that $p_a(x,\xi) = p(\xi - a(x))$. The following elementary lemma will be useful for the proof of Lemma 4.3.

Lemma 4.4. *Let $f \in C^0(\mathbf{R}^n; \mathbf{R}_+)$be such that $f(x) \to 0$, $|x| \to \infty$. Then*

$$\lim_{|x|\to\infty} \int_{\mathbf{R}^n} \langle y\rangle^{-N} f(x+\theta y)\, dy = 0, \quad \forall\, N > n,$$

uniformly with respect to $\theta \in [0;1]$.

Proof. For any $\varepsilon > 0$ we can choose $R > 0$ such that

$$\int_{|y|>R} \langle y\rangle^{-N}\, dy < \varepsilon$$

and $f(x) < \varepsilon$, $\forall\, |x| > R$.

Let $x \in \mathbf{R}^n$, $|x| > 2R$. Then $|x + \theta y| > R$, $\forall\, |y| > R$, $\forall\, \theta \in [0;1]$. It follows that

$$\int_{|y|>R} \langle y\rangle^{-N} f(x+\theta y)\, dy < \varepsilon \left[\sup_{\mathbf{R}^n} f + \int_{\mathbf{R}^n} \langle y\rangle^{-N}\, dy\right]$$

for any $|x| > 2R$ and any $\theta \in [0;1]$. □

Proof of Lemma 4.3. Let q be equal to $\sigma^1[Op^w(p_a)] - p_a$. Then

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-n}\,\mathrm{Osc}-\iint e^{-i\langle y,\eta\rangle}\left[p\left(\xi+\eta-a\left(x+\frac{y}{2}\right)\right)-p\big(\xi+\eta-a(x)\big)\right]dyd\eta \\ &= (2\pi)^{-n}\,\mathrm{Osc}-\iint e^{-i\langle y,\eta\rangle}\left[a(x)-a\left(x+\frac{y}{2}\right)\right] \\ &\quad\cdot\int_0^1(\partial p)\left(\xi+\eta-a(x)+\theta\left(a(x)-a\left(x+\frac{y}{2}\right)\right)\right)d\theta dyd\eta \end{aligned}$$

and

$$a\left(x+\frac{y}{2}\right)-a(x)=\frac{1}{2}\left[\int_0^1(\partial a)\left(x+\frac{\theta y}{2}\right)d\theta\right]y\equiv\chi(x,y)y.$$

All the derivatives of the matrix valued function χ are bounded. Moreover, from the hypothesis and Lemma 4.4 it follows that

$$\lim_{|x|\to\infty}\int\langle y\rangle^{-2N}|\partial_x^\alpha\partial_y^\beta\chi(x,y)|\,dy=0$$

for any $\alpha,\ \beta\in\mathbf{N}^n$.

On the other hand, if we put

$$g(x,y,\xi,\eta)=\int_0^1(\partial p)\left(\xi+\eta-a(x)+\theta\left(a(x)-a\left(x+\frac{y}{2}\right)\right)\right)d\theta,$$

then, for any $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbf{N}^n$, there exists a positive constant $C_{\alpha\beta\gamma\delta}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_\eta^\gamma\partial_y^\delta g(x,y,\xi,\eta)|\le C_{\alpha\beta\gamma\delta}\langle\xi-a(x)\rangle^{m-1-|\alpha|}(\langle\eta\rangle\langle y\rangle)^{|m-1-|\alpha||}.$$

Using the definition of the oscillatory integrals, we see that

$$\begin{aligned} q_{(\alpha)}^{(\beta)}(x,\xi) = (2\pi)^{-n}\iint & e^{-i\langle y,\eta\rangle}\langle y\rangle^{-2N}(I-\Delta_\eta)^N\langle\eta\rangle^{-2L} \\ &\cdot(I-\Delta_y)^L\partial_\xi^\alpha\partial_x^\beta[\chi(x,y)y\cdot g(x,y,\xi,\eta)]\,dyd\eta, \end{aligned}$$

for N and L sufficiently large. Therefore $|q_{(\alpha)}^{(\beta)}(x,\xi)|$ can be dominated by a finite sum of terms of the form

$$\begin{aligned} &\mathrm{const}\iint\langle y\rangle^{-2N+1}\langle\eta\rangle^{-2L}|\partial_x^{\beta'}\partial_y^{\delta'}\chi(x,y)||\partial_\xi^\alpha\partial_x^{\beta''}\partial_\eta^\gamma\partial_y^{\delta''}g(x,y,\xi,\eta)|\,dyd\eta \\ &\quad\le\mathrm{const}\iint\langle y\rangle^{-2N+1+|m-1-|\alpha||}\langle\eta\rangle^{-2L+(m-1-|\alpha|)} \\ &\qquad\cdot\langle\xi-a(x)\rangle^{m-1-|\alpha|}|\partial_x^{\beta'}\partial_y^{\delta'}\chi(x,y)|\,dxdy \end{aligned}$$

$$\leq \text{const}\langle \xi - a(x)\rangle^{m-1-|\alpha|} \int \langle y\rangle^{-2N+1+|m-1-|\alpha||} |\partial_x^{\beta'} \partial_y^{\delta'} \chi(x, y)| \, dy$$

for L sufficiently large. Now the conclusion of the lemma follows if we take N sufficiently large and use the decaying properties of χ and Remark 4.2. □

In the statements of Lemmas 4.5 and 4.6 and in their proofs we shall use the same notation as in the proof of Theorem 3.7.

Lemma 4.5. *$(H_a + 1)^m - (H_a + 1)Op^1(q_{m-1})$ is a compact operator in L^2 for any $m \in (0; 1)$.*

Proof. Recall that

$$(H_a + 1)^m - (H_a + 1)Op^1(q_{m-1}) = (H_a + 1)\frac{i}{2\pi} \oint_\Gamma \mu^{m-1}(H_a + 1 + \mu)^{-1} Op^1(r_{1-\mu}) \, d\mu$$

where the integral is convergent in the uniform operator topology in $\mathbf{B}(L^2, \mathcal{H}_a^2)$.

Using the fact that $\lim_{|x|\to\infty} \partial^\beta a(x) = 0$, $\forall |\beta| > 0$, it can be checked, as in the proof of Lemma 4.3, that $r_{1-\mu}$ satisfies the hypothesis of Theorem 4.1. All that is important is that any term of $r_{1-\mu}$ is in S_a^{-1} and contains a factor which is a derivative of a. Therefore $Op^1(r_{1-\mu})$ is a compact operator in L^2 for any μ such that $\operatorname{Re} \mu < 1/2$. Hence the above contour integral defines a compact operator from L^2 to $\mathcal{H}_a^2$. □

Lemma 4.6. *$(H_a + 1)Op^1(q_{m-1}) - Op^1(q_m)$ is a compact operator in L^2 for any $m \in (0; 1)$.*

Proof. The Kohn-Nirenberg symbol of $(H_a + 1)Op^1(q_{m-1}) - Op^1(q_m)$ is

$$\begin{aligned} q(x, \xi) = {} & \frac{i \operatorname{div} a(x)}{\langle \xi - a(x)\rangle^{2-2m}} \\ & + (2\pi)^{-n} \int_0^1 \left\{ \sum_{j=1}^n \text{Osc} - \iint 4(1-m)e^{-i\langle y,\eta\rangle} \left(\xi_j + \theta\eta_j - a_j(x)\right) \right. \\ & \left. \cdot \frac{\langle \xi - a(x+y), \partial_j a(x+y)\rangle}{\langle (\xi - a(x+y))^2\rangle^{4-2m}} \, dy d\eta \right\} d\theta. \end{aligned}$$

Again, as in the proof of Lemma 4.3, it can be verified that q satisfies the hypothesis of Remark 4.2 with $\nu = 0$. □

Proof of Theorem 1.1. First we remark that

$$(H_a + 1)^{1/2} - Op^w(\lambda_a) = \left[(H_a + 1)^{1/2} - (H_a + 1)Op^1(\lambda_a^{-1})\right]$$

$$+ \left[(H_a + 1)Op^1(\lambda_a^{-1}) - Op^1(\lambda_a)\right]$$
$$+ \left[Op^1(\lambda_a) - Op^w(\lambda_a)\right]$$

is, according to Lemmas 4.3, 4.5, 4.6 and to Theorems 4.1 and 3.4 a relatively compact perturbation of h_a. Therefore

$$\sigma_{\mathrm{ess}}(h_a) = \sigma_{\mathrm{ess}}\left[(H_a + 1)^{1/2}\right].$$

Next, it is known that if V is an operator of multiplication as in the hypothesis of the theorem, then V is relatively compact with respect to $(H_a + 1)^{1/2}$ (see, e.g., the proof of Lemma 4.4 from [12]). Since $\mathrm{Dom}[(H_a+1)^{1/2}] = \mathrm{Dom}(h_a)$, V is relatively compact with respect to h_a and

$$\sigma_{\mathrm{ess}}(h_a + V) = \sigma_{\mathrm{ess}}(h_a). \qquad \square$$

Corollary 4.7. *Suppose that the hypotheses of* Theorem 4.1 *are satisfied and that* $n = 2$ *or* 3. *Then* $\sigma_{\mathrm{ess}}(h_a + V) = [1; \infty)$.

Proof. Theorem 6.1 from [4] states that in this case $\sigma_{\mathrm{ess}}(H_a) = [0; \infty)$. $\square$

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