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LITTLEWOOD-PALEY INEQUALITY FOR A DIFFUSION SATISFYING THE LOGARITHMIC SOBOLEV INEQUALITY AND FOR THE BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD WITH BOUNDARY

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1. Introduction

In this paper, we discuss the Littlewood-Paley inequality. Typical example is the Brownian motion on the Euclidean space and it leads to the following inequality: for any $p > 1$ there exist a positive constant C such that

$$C^{-1}\|\nabla u\|_p \leq \|\sqrt{-\Delta}u\|_p \leq C\|\nabla u\|_p. \tag{1.1}$$

$\sqrt{-\Delta}$, the square root of the minus Laplacian, is called the Cauchy operator. (1.1) is equivalent to the L^p-boundedness of the Riesz transformation.

This kind of inequality also holds for the Ornstein-Uhlenbeck process on an abstract Wiener space, which was proved by P.A. Meyer [11] in a probabilistic approach.

In this paper, we attempt to extend this inequality for a diffusion process associated with a Dirichlet form that admits a square field operator. There have been several related works, e.g., Bakry [3, 4], Shigekawa-Yoshida [16]. In these papers, they assumed that Γ_2 is positive or bounded from below. We replace this boundedness assumption with the exponential integrability of negative part of Γ_2. To handle this case, we assume that the logarithmic Sobolev inequality holds. Moreover our square field operator is of the gradient form, i.e., the Dirichlet form $\mathcal{E}$ is given as follows;

$$\mathcal{E}(u, v) = \int_M (\nabla u, \nabla v)\mu(dx). \tag{1.2}$$

We adopt a probabilistic approach which was developed by Meyer and Bakry. We will show the inequality for the Littlewood-Paley G-function. Since the square field operator is given as a gradient, we consider another semigroup that acts on vector valued functions and use the semigroup domination to estimate vector valued functions. Using this method, the estimate for vector valued functions can be reduced to the scalar case. But the unboundedness of Γ_2 causes some troubles and so we could not prove the exact inequality. We only show that the L^p-norm is dominated by L^q-norm for $1 < p < q$ (see the precise statement in §2).

We also discuss the Brownian motion on a Riemannian manifold with boundary. We impose the Neumann boundary condition on the Brownian motion. In this case, the quantity corresponding to Γ_2 is singular (i.e., it is not a function but a smooth measure). We deal with it by way of an associated additive functional. The additive functional belongs to the Kato class and we can show the exact inequality (i.e., no loss of exponent).

The organization of the paper is as follows. We give a formulation and a main result in §2. We define Γ_2 in our formulation. It is a generalization of Ricci curvature and is based on a square field operator for vector valued functions. In §3, the maximal ergodic inequality for a semigroup with a potential is given. Here the logarithmic Sobolev inequality is essential. We give a proof of the main theorem in §5. To do this, we prepare fundamental inequalities for the Littlewood-Paley G-function in §4. A proof for the Littlewood-Paley inequality is given in §5. Combining this with the intertwining property of semigroups, we can get the main result. The Brownian motion on a Riemannian manifold with boundary is dealt with in §6.

2. Symmetric diffusion

Let us introduce a diffusion process that we use in the paper. Let M be a topological space. We assume M to be Souslinian. Suppose we are given a Borel probability measure μ on M and a Dirichlet form $\mathcal{E}$ in $L^2(\mu)$. We assume that there exists a Hunt diffusion process $(X_t, P_x)_{x\in M}$ associated with $\mathcal{E}$. We denotes the generator and the semigroup by L and $\{T_t\}$, respectively. We assume that $1 \in \operatorname{Dom}(L)$ and $L1 = 0$ where 1 denotes the function that is identically equal to 1. Hence the diffusion (X_t) is conservative. We also assume that the Dirichlet form satisfies the following defective logarithmic Sobolev inequality: there exist $\alpha > 0$ and $\beta \geq 0$ such that

$$\int_M u^2 \log\left(\frac{u}{\|u\|_2}\right)\mu(dx) \leq \alpha\mathcal{E}(u,u) + \beta(u,u). \tag{2.1}$$

Here (,) denotes the inner product in L^2.

Further we assume that the square field operator Γ is well-defined. Here Γ: $\operatorname{Dom}(\mathcal{E}) \times \operatorname{Dom}(\mathcal{E}) \to L^1(\mu)$ is a continuous bilinear map which is characterized as follows:

$$2(\Gamma(v,w),u) = \mathcal{E}(vw,u) - \mathcal{E}(v,wu) - \mathcal{E}(vu,w), \quad \forall u,v,w \in \mathcal{E}\cap L^\infty. \tag{2.2}$$

A crucial assumption is as follows; there exists a 'gradient operator' ∇ such that ∇ is a closed operator from $L^2(\mu)$ to $L^2(\mu; K)$ and it satisfies $\Gamma(u,v) = (\nabla u, \nabla v)$. Here K is a (separable) Hilbert space. $L^2(\mu; K)$ may be possibly the set of all square integrable section of a vector bundle over M. But we use $L^2(\mu; K)$ for notational convention. We need another semigroup $\{\hat{T}_t\}$ in $L^2(\mu; K)$. Let $\{\hat{T}_t\}$ be a contraction symmetric semigroup associated with a bilinear form $\hat{\mathcal{E}}$. We also need a square field operator

for $\{\hat{T}_t\}$ and so we assume that
(A.1) For $\theta \in \mathrm{Dom}(\hat{L})$, it holds that $|\theta|_K^2 \in \mathrm{Dom}(L_1)$.
Here L_1 is the generator in $L^1(\mu)$. Under this condition, we define a square field operator $\hat{\Gamma}$ as

$$2\hat{\Gamma}(\theta, \eta) = L(\theta, \eta)_K - (\hat{L}\theta, \eta)_K - (\theta, \hat{L}\eta)_K. \tag{2.3}$$

We assume the following two properties: the positivity and the derivation property.
(A.2) $\hat{\Gamma}(\theta, \theta) \geq 0$ for $\theta \in \mathrm{Dom}(\hat{\mathcal{E}})$.
(A.3) For $\theta,\ \eta \in \mathrm{Dom}(\hat{\mathcal{E}}) \cap L^\infty$ and $u \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, it holds that

$$2u\hat{\Gamma}(\theta, \eta) = -\big(\nabla u, \nabla(\theta, \eta)\big) + \hat{\Gamma}(\theta, u\eta) + \hat{\Gamma}(u\theta, \eta). \tag{2.4}$$

Then, by the semigroup domination theorem (see [14]), we have

$$|\hat{T}_t\theta| \leq T_t|\theta|. \tag{2.5}$$

Let $\mathcal{S}_b(K)$ be the space of all self-adjoint operator on K that is bounded from below. Let R be a function on M taking values in $\mathcal{S}_b(K)$. Define a bilinear form $\hat{\mathcal{E}}^R$ by

$$\hat{\mathcal{E}}^R(\theta, \eta) = \hat{\mathcal{E}}(\theta, \eta) + \int_M \big(R(x)\theta(x), \eta(x)\big)_K \mu(dx). \tag{2.6}$$

The associated semigroup will be denoted by $\hat{T}_t^R$. We assume the following intertwining property, which is crucial in the paper.

$$\nabla T_t u = \hat{T}_t^R \nabla u, \qquad \text{for } u \in \mathrm{Dom}(\nabla). \tag{2.7}$$

R plays the role of so called Γ_2.

We take a scalar function V such that

$$\big(R(x)k, k\big)_K \geq V(x)(k, k)_K. \tag{2.8}$$

The semigroup generated by $L-V$ is denoted by $\{T_t^V\}$. The generator of $\hat{T}_t^R$ is $\hat{L}-R$. Again by the domination theorem, it holds that

$$|\hat{T}_t^R\theta| \leq T_t^V|\theta|. \tag{2.9}$$

V can be decomposed as $V = V_+ - V_-$ where $V_+ = V \vee 0$ and $V_- = (-V) \vee 0$. The last assumption is that
(A.4) $\quad e^{V_-} \in L^{\infty -} = \bigcap_{p \geq 1} L^p$.

For scalar functions, we can define two kinds of norms: $\|\nabla u\|_p$ and $\|\sqrt{1-L}u\|_p$. It is a fundamental question whether these norms are equivalent or not. For example, if the generator L is the Ornstein-Uhlenbeck operator on an abstract Wiener space, then the equivalence of two norms are known as the Meyer equivalence.

Under our conditions, we can get the following result.

Theorem 2.1. *For any* $1 < p < q < \infty$, *we have*

$$\|\nabla u\|_p \lesssim \|\sqrt{1-L}u\|_q, \tag{2.10}$$

$$\|\sqrt{1-L}u\|_p \lesssim \|\nabla u\|_q + \|u\|_q. \tag{2.11}$$

In the above theorem, the notation $A \lesssim B$ stands for $A \leq kB$ for a positive constant k. Further, in (2.10) for example, the constant depends only on p but is independent of u. We use this convention in the sequel without mentioning.

To prove the theorem, we use the Littlewood-Paley G-function. We introduce it in §4 and give a proof of the theorem in §5.

3. Maximal ergodic inequality

In this section, we discuss the maximal ergodic inequality. This inequality is known for a symmetric Markov semigroup (see e.g., Stein [17]). Here we consider a semigroup with a potential. To show the inequality, we adopt a probabilistic method due to Rota [13].

We consider an additive functional A_t associated to a smooth signed measure ρ under the Revuz correspondence. We define a Dirichlet form by

$$\mathcal{E}^\rho(u, v) = \mathcal{E}(u, v) + \int_M \tilde{u}\tilde{v}\rho(dx) \tag{3.1}$$

where $\tilde{u}$ denotes the quasi-continuous modification of u. The associated semigroup is denoted by $\{T_t^\rho\}$, which is expressed as

$$T_t^\rho u(x) = E_x\left[u(X_t)e^{-A_t}\right] \tag{3.2}$$

where E_x denote the expectation under the measure P_x.

Theorem 3.1. *Assume that for any* $q \geq 1$, *there exist constants* c_q, β_q *such that*

$$E_x[e^{-qA_t}]^{1/q} \leq c_q e^{\beta_q t} \qquad \forall t \geq 0,\ \text{q.e.-}x. \tag{3.3}$$

Here "q.e." *means that it holds except for a set of capacity* 0. *Then for any* $p > 1$ *there exist constants* λ, c *such that*

$$\left\|\sup_{t\geq 0}|e^{-\lambda t}T_t^\rho u|\right\|_p \leq c\|u\|_p, \qquad \forall u \in L^p. \tag{3.4}$$

In particular, if ρ is non-negative (i.e., A_t is non-negative), we can take $\lambda = 0$.

Proof. We note that $|T_t^\rho u| \leq T_t^{-\rho_-}|u|$, where $\rho = \rho_+ - \rho_-$ is the Hahn decomposition of ρ. Without loss of generality, we may assume that ρ is non-positive.

Set

$$M_t = T^\rho_{T-t}u(X_t)e^{-A_t}. \tag{3.5}$$

Here θ_t is the shift operator. We show first that $\{M_t\}$ is a martingale under $P_\mu := \int_M P_x\mu(dx)$. In fact,

$$\begin{aligned} E_\mu\left[u(X_T)e^{-A_T} \mid \mathcal{F}_t\right] &= E_\mu\left[u(X_{T-t}\circ\theta_t)e^{-A_{T-t}\circ\theta_t - A_t} \mid \mathcal{F}_t\right] \\ &= e^{-A_t}E_\mu\left[u(X_{T-t}\circ\theta_t)e^{-A_{T-t}\circ\theta_t} \mid \mathcal{F}_t\right] \\ &= e^{-A_t}E_{X_t}\left[u(X_{T-t})e^{-A_{T-t}}\right] \qquad \text{(Markov property)} \\ &= e^{-A_t}T^\rho_{T-t}u(X_t). \end{aligned}$$

We note, by the Markov property,

$$\begin{aligned} T^\rho_{T-t}u(X_T) &= E_{X_T}\left[u(X_{T-t})e^{-A_{T-t}}\right] \\ &= E_\mu\left[u(X_{T-t}\circ\theta_T)e^{-A_{T-t}\circ\theta_T} \mid X_T\right] \\ &= E_\mu\left[u(X_{2T-t})e^{-A_{2T-t}+A_T} \mid X_T\right]. \end{aligned}$$

Now, using the reversibility of (X_t), i.e., $(X_{2T-t})_{0\leq t\leq 2T}$ has the same law as $(X_t)_{0\leq t\leq 2T}$, we have

$$T^\rho_{T-t}u(X_T) = E_\mu\left[u(X_t)e^{-A_T+A_t} \mid X_T\right].$$

Hence

$$\begin{aligned} T^\rho_{2(T-t)}u(X_T) &= T^\rho_{T-t}T^\rho_{T-t}u(X_T) \\ &= E_\mu\left[T^\rho_{T-t}u(X_t)e^{-A_T+A_t} \mid X_T\right] \\ &= E_\mu\left[M_t e^{-A_T+2A_t} \mid X_T\right]. \end{aligned}$$

Noting that we have taken A to be non-positive, we have

$$\begin{aligned} \sup_{0\leq t\leq T}|T^\rho_{2(T-t)}u(X_T)| &\leq E_\mu\left[\sup_{0\leq t\leq T}|M_t|e^{-A_T} \,\middle|\, X_T\right] \\ &\leq E_\mu\left[\sup_{0\leq t\leq T}|M_t|^p \,\middle|\, X_T\right]^{1/p} E_\mu\left[e^{-qA_T} \mid X_T\right]^{1/q}. \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \end{aligned}$$

On the other hand,

$$E_\mu\left[e^{-qA_T} \mid X_T\right]^{1/q} = E_\mu\left[e^{-q(A_{2T}-A_T)} \mid X_T\right]^{1/q} \qquad \text{(reversibility)} \tag{3.6}$$

$$
\begin{aligned}
&= E_\mu\left[e^{-qA_T\circ\theta_T)} \mid X_T\right]^{1/q} \qquad \text{(additivity)}\\
&= E_{X_T}\left[e^{-qA_T}\right]^{1/q} \qquad \text{(Markovian property)}\\
&\le c_q e^{\beta_q T}. \qquad (\because\ (3.3))
\end{aligned}
$$

Thus we have

$$
\sup_{0\le t\le T} |T^\rho_{2(T-t)} u(X_T)| \le c_q e^{\beta_q T} E_\mu\left[\sup_{0\le t\le T} |M_t|^p \;\middle|\; X_T\right]^{1/p}
$$

Hence, by the Doob inequality

$$
\begin{aligned}
&\left\|\sup_{0\le t\le T} |T^\rho_{2(T-t)} u|\right\|_p\\
&\qquad\le c_q e^{\beta_q T} E_\mu\left[E_\mu\left[\sup_{0\le t\le T} |M_t|^p \;\middle|\; X_T\right]\right]^{1/p}\\
&\qquad= c_q e^{\beta_q T} E_\mu\left[\sup_{0\le t\le T} |M_t|^p\right]^{1/p}\\
&\qquad\le C' c_q e^{\beta_q T} E_\mu[|M_T|^p]^{1/p} \qquad \text{(Doob's inequality)}\\
&\qquad= C' c_q e^{\beta_q T} E_\mu[|u(X_T)|^p e^{-pA_T}]^{1/p}\\
&\qquad= C' c_q e^{\beta_q T} E_\mu\left[E_\mu[|u(X_T)|^p e^{-pA_T} \mid X_T]\right]^{1/p}\\
&\qquad= C' c_q e^{\beta_q T} E_\mu\left[|u(X_T)|^p E_\mu[e^{-pA_T} \mid X_T]\right]^{1/p}\\
&\qquad\le C' c_q e^{\beta_q T} c_p e^{\beta_p T} E_\mu[|u(X_T)|^p]^{1/p} \qquad (\because\ (3.6))\\
&\qquad= C' c_p c_q e^{(\beta_p+\beta_q)T} \|u\|_p.
\end{aligned}
$$

Thus we can find constants $k > 0$ and $C > 0$ which are independent of T and u such that

$$
\left\|\sup_{0\le t\le 2T} |T^\rho_t u|\right\|_p \le Ce^{2kT}\|u\|_p
$$

We take $\lambda > k$. Note that for any integer n,

$$
\begin{aligned}
\left\|\sup_{n\le t\le n+1} e^{-\lambda t}|T^\rho_t u|\right\|_p &\le e^{-\lambda n}\left\|\sup_{0\le t\le n+1} |T^\rho_t u|\right\|_p\\
&\le Ce^{-\lambda n} e^{k(n+1)}\|u\|_r.
\end{aligned}
$$

Summing up in n,

$$
\sum_{n=0}^\infty \left\|\sup_{n\le t\le n+1} e^{-\lambda t}|T^\rho_t u|\right\|_p \le Ce^k \sum_{n=0}^\infty e^{-(\lambda-k)n}\|u\|_p
$$

$$\le \frac{C}{e^{-k} - e^{-\lambda}} \|u\|_p.$$

Clearly this leads us to

$$\begin{aligned}
\left\| \sup_{0 \le t < \infty} e^{-\lambda t} |T_t^\rho u| \right\|_p &\le \left\| \sum_{n=0}^{\infty} \sup_{n \le t \le n+1} e^{-\lambda t} |T_t^\rho u| \right\|_p \\
&\le \sum_{n=0}^{\infty} \left\| \sup_{n \le t \le n+1} e^{-\lambda t} |T_t^\rho u| \right\|_p \\
&\le \frac{C}{e^{-k} - e^{-\lambda}} \|u\|_p.
\end{aligned}$$

This completes the proof. $\square$

The assumption (3.3) is rather strong. We replace it with the assumption (A.4). In this case, we set $\rho = Vm$. Hence, the associated additive functional is given by

$$A_t = \int_0^t V(X_s)\, ds. \tag{3.7}$$

Here we denote the semigroup $\{T_t^\rho\}$ by $\{T_t^V\}$. Since $\mathcal{E}$ satisfies the logarithmic Sobolev inequality (2.1), we have (see, e.g., [14]),

$$\|T_t^V u\|_p \le \|e^{-V}\|^t_{\alpha p^2/4(p-1)} e^{4\beta t/\alpha} \|u\|_p. \tag{3.8}$$

This means that there exists a constant γ_p such that

$$\|T_t^V\|_{p \to p} \le e^{\gamma_p t}. \tag{3.9}$$

E.g., set $\gamma_p = (4\beta/\alpha) \log \|e^{-V}\|_{\alpha p^2/4(p-1)}$.

In particular, when $p = 2$,

$$\|T_t^V u\|_2 \le \|e^{-V}\|^t_\alpha e^{4\beta t/\alpha} \|u\|_2 \tag{3.10}$$

In this case, taking $u = 1$, we have

$$\begin{aligned}
E_\mu[e^{-A_t}] &= \int_X E_x[1 e^{-A_t}] \mu(dx) \\
&= \|T_t^V 1\|_1 \\
&\le \|T_t^V 1\|_2 \\
&\le \|e^{-V}\|^t_\alpha e^{4\beta t/\alpha} \|1\|_2 \\
&\le \|e^{-V}\|^t_\alpha e^{4\beta t/\alpha}.
\end{aligned}$$

Hence, for any $\gamma > 0$, it holds that

$$E_\mu[e^{-\gamma A_t}] \leq \|e^{-\gamma V}\|_\alpha^t e^{4\beta t/\alpha} \tag{3.11}$$

Noticing this inequality, we can get the following maximal ergodic inequality.

Theorem 3.2. *Take any $1 < p < r < \infty$. If we take $\lambda > 0$ to be sufficiently large, then there exists a constant $c > 0$ such that*

$$\left\| \sup_{t \geq 0} |e^{-\lambda t} T_t^V u| \right\|_p \leq c\|u\|_r. \tag{3.12}$$

Proof. By the same proof as in Theorem 3.1, we have

$$\sup_{0 \leq t \leq T} |T_{2(T-t)}^V u(X_T)| \leq E_\mu\left[\sup_{0 \leq t \leq T} |M_t| e^{-A_T} \;\middle|\; X_T \right].$$

Hence,

$$\begin{aligned}
&\left\| \sup_{0 \leq t \leq T} |T_{2(T-t)}^V u| \right\|_p \\
&\quad \leq E_\mu\left[E_\mu\left[\sup_{0 \leq t \leq T} |M_t| e^{-A_T} \;\middle|\; X_T \right]^p \right]^{1/p} \\
&\quad \leq E_\mu\left[\sup_{0 \leq t \leq T} |M_t|^p e^{-pA_T} \right]^{1/p} \\
&\quad \leq E_\mu\left[\sup_{0 \leq t \leq T} |M_t|^{pq/p} \right]^{(p/q)\cdot(1/p)} E_\mu[e^{-puA_T}]^{(q-p)/pq} \qquad \left(\frac{1}{q/p} + \frac{1}{u} = 1 \right) \\
&\quad \leq C E_\mu[|M_T|^q]^{1/q} E_\mu[e^{-puA_T}]^{(q-p)/pq} \qquad \text{(Doob's inequality)} \\
&\quad = C E_\mu[|u(X_T) e^{-A_T}|^q]^{1/q} E_\mu[e^{-puA_T}]^{(q-p)/pq} \\
&\quad \leq C E_\mu[|u(X_T)|^{rq/q}]^{(q/r)\cdot(1/q)} E_\mu[e^{-qvA_T}]^{(r-q)/rq} E_\mu[e^{-puA_T}]^{(q-p)/pq} \\
&\qquad \left(\frac{1}{r/q} + \frac{1}{v} = 1 \right) \\
&\quad \leq C\|u\|_r \|e^{-qvV}\|_\alpha^{(r-q)T/rq} e^{4\beta(r-q)T/\alpha rq} \|e^{-puV}\|_\alpha^{(q-p)T/pq} e^{4\beta(q-p)T/\alpha pq}.
\end{aligned}$$

Thus we can find a constant $k > 0$ which is independent of T and u such that

$$\left\| \sup_{0 \leq t \leq 2T} |T_t^V u| \right\|_p \leq C e^{2kT} \|u\|_r.$$

The rest is the same as Theorem 3.1. This completes the proof. $\square$

4. Littlewood-Paley G-functions

Let us introduce the Littlewood-Paley G-functions. To do this, we recall the subordination of a semigroup. Set $T_t^\lambda = e^{-\lambda t}T_t$ ($\lambda \geq 0$). We take λ to be large enough. For any $t \geq 0$, define a measure μ_t on $[0, \infty)$ by

$$\mu_t(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2}\, ds. \tag{4.1}$$

In terms of the Laplace transform, this measure is characterized as

$$\int_0^\infty e^{-\alpha s}\mu_t(ds) = e^{-\sqrt{\alpha}\, t} \quad \text{for } \alpha > 0.$$

Then the subordination $\{Q_t^\lambda\}$ of $\{T_t^\lambda\}$ is defined by

$$Q_t^\lambda = \int_0^\infty T_s^\lambda\, \mu_t(ds). \tag{4.2}$$

The generator of $\{Q_t^\lambda\}$ in $L^2(\mu)$ is $-\sqrt{\lambda - L}$.

We recall that $\{T_t^V\}$ is the semigroup with the potential V. We set $T_t^{\lambda+V} = e^{-\lambda t}T_t^V$ and we also define the subordination of $\{T_t^V\}$ as

$$Q_t^{\lambda+V} = \int_0^\infty T_s^{\lambda+V}\, \mu_t(ds). \tag{4.3}$$

The operator norm of $\{Q_t^{\lambda+V}\}$ in L^p is estimated as

$$\begin{aligned}
\|Q_t^{\lambda+V}\|_{p\to p} &\leq \int_0^\infty \|T_s^{\lambda+V}\|_{p\to p}\, \mu_t(ds) \\
&\leq \int_0^\infty e^{-\lambda s+\gamma_p s}\, \mu_t(ds) \\
&= e^{-\sqrt{\lambda-\gamma_p}\, t}.
\end{aligned}$$

Here γ_p is the constant in (3.9). Moreover, by the semigroup domination $|\hat{T}_t^{\lambda+R}\theta| \leq T_t^{\lambda+V}|\theta|$, we have

$$\|\hat{T}_t^{\lambda+R}\|_{p\to p} \leq \|T_t^{\lambda+V}\|_{p\to p} \leq e^{-(\lambda-\gamma_p)t}. \tag{4.4}$$

Similarly we have

$$\|\hat{Q}_t^{\lambda+R}\|_{p\to p} \leq e^{-\sqrt{\lambda-\gamma_p}\, t}. \tag{4.5}$$

For any real valued function u, define

$$g^{\rightarrow}(x, t) = |\partial_t Q_t^\lambda u(x)|^2, \tag{4.6}$$

$$g^{\uparrow}(x,t) = |\nabla Q_t^{\lambda} u(x)|_K^2, \tag{4.7}$$

$$g(x,t) = g^{\rightarrow}(x,t) + g^{\uparrow}(x,t). \tag{4.8}$$

Here $\partial_t = \partial/\partial t$. Then, the Littlewood-Paley G-function is defined by

$$G^{\rightarrow}u(x) = \left\{\int_0^\infty t g^{\rightarrow}(x,t)\,dt\right\}^{1/2}, \tag{4.9}$$

$$G^{\uparrow}u(x) = \left\{\int_0^\infty t g^{\uparrow}(x,t)\,dt\right\}^{1/2}, \tag{4.10}$$

$$Gu(x) = \left\{\int_0^\infty t g(x,t)\,dt\right\}^{1/2}. \tag{4.11}$$

Moreover, we define the H-functions by

$$H^{\rightarrow}u(x) = \left\{\int_0^\infty t Q_t g^{\rightarrow}(x,t)\,dt\right\}^{1/2}, \tag{4.12}$$

$$H^{\uparrow}u(x) = \left\{\int_0^\infty t Q_t g^{\uparrow}(x,t)\,dt\right\}^{1/2}, \tag{4.13}$$

$$Hu(x) = \left\{\int_0^\infty t Q_t g(x,t)\,dt\right\}^{1/2}. \tag{4.14}$$

For vector valued function θ, we define G-function and H-function, similarly. That is, e.g.,

$$\hat{g}^{\rightarrow}(x,t) = |\partial_t \hat{Q}_t^{\lambda+R}\theta(x)|^2, \tag{4.15}$$

$$\hat{G}^{\rightarrow}\theta(x) = \left\{\int_0^\infty t \hat{g}^{\rightarrow}(x,t)\,dt\right\}^{1/2} \tag{4.16}$$

$$\hat{H}^{\rightarrow}\theta(x) = \left\{\int_0^\infty t Q_t \hat{g}^{\rightarrow}(x,t)\,dt\right\}^{1/2}. \tag{4.17}$$

Notice that, in this case, we use the semigroup $\{\hat{Q}_t^{\lambda+R}\}$ that is the subordination of $\{\hat{T}_t^{\lambda+R}\}$. $\hat{G}^{\uparrow}\theta$, $\hat{H}^{\uparrow}\theta$, $\hat{G}\theta$, and $\hat{H}\theta$ are defined similarly. For example,

$$\hat{g}^{\uparrow}(x,t) = \hat{\Gamma}\left(\hat{Q}_t^{\lambda+R}\theta, \hat{Q}_t^{\lambda+R}\theta\right)(x)$$

(see (2.3) for the definition of $\hat{\Gamma}$).

The following proposition is easily obtained by the spectral decomposition:

Proposition 4.1. *It holds that*

$$\|G^{\rightarrow}u\|_2 = \frac{1}{2}\|u\|_2, \tag{4.18}$$

and

$$\|\hat{G}^{\rightarrow}\theta\|_2 = \frac{1}{2}\|\theta\|_2. \tag{4.19}$$

Later we need the interrelationship between G and H functions and so we first prepare the following.

Lemma 4.2. *We have the following estimate*:

$$|T_t^{\lambda+V}u(x)|^2 \leq \left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}T_t|u|^2(x), \tag{4.20}$$

$$|Q_t^{\lambda+V}u(x)|^2 \leq \left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}Q_t|u|^2(x). \tag{4.21}$$

Proof. By the Feynman-Kac formula, we have

$$\begin{aligned} |T_t^{\lambda+V}u(x)|^2 &= \left|E_x\left[\exp\left\{-\lambda t - \int_0^t V(X_s)\,ds\right\}u(X_t)\right]\right|^2 \\ &\leq E_x\left[\exp\left\{-2\lambda - 2\int_0^t V(X_s)\,ds\right\}\right]E_x[|u(X_t)|^2] \\ &= T_t^{2(\lambda+V)}1(x)\cdot T_t|u|^2(x) \\ &\leq \left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}\cdot T_t|u|^2(x). \end{aligned}$$

Further we have,

$$\begin{aligned} |Q^{\lambda+V}u(x)|^2 &= \left|\int_0^\infty T_s^{\lambda+V}u(x)\lambda_t(ds)\right|^2 \\ &\leq \int_0^\infty |T_s^{\lambda+V}u(x)|^2\lambda_t(ds) \\ &\leq \int_0^\infty \left\{\sup_{r\geq 0} T_r^{2(\lambda+V)}1(x)\right\}\cdot T_s|u|^2(x)\lambda_t(ds) \\ &= \left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}\cdot Q_t|u|^2(x). \end{aligned}$$

This completes the proof. □

Now we can show the following estimate between G-functions and H-functions.

Proposition 4.3. *We have that*

$$\hat{G}^{\rightarrow}\theta \leq 2\Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\}^{1/2} \cdot \hat{H}^{\rightarrow}\theta. \tag{4.22}$$

For scalar function, we have

$$G^{\rightarrow}u \leq 2H^{\rightarrow}u, \tag{4.23}$$

$$G^{\uparrow}u \leq 2\Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\}^{1/2} \cdot H^{\uparrow}u. \tag{4.24}$$

Proof. We have,

$$|\hat{Q}_t^{\lambda+R}\theta(x)| \leq \int_0^\infty |\hat{T}_s^{\lambda+R}\theta(x)|\mu_t(ds) \leq \int_0^\infty T_s^{\lambda+V}|\theta|(x)\mu_t(ds) = Q_t^{\lambda+V}|\theta|(x).$$

Using Lemma 4.2, we have

$$|\hat{Q}_t^{\lambda+R}\theta(x)|^2 \leq \{Q_t^{\lambda+V}|\theta|(x)\}^2 \leq \Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\} \cdot Q_t|\theta|^2(x).$$

Therefore

$$\begin{aligned}
\hat{g}^{\rightarrow}(x, 2t) &= |\partial_s \hat{Q}_s^{\lambda+R}\theta(x)|^2\Big|_{s=2t} \\
&= \Big|\sqrt{\lambda - \hat{L} + R}\,\hat{Q}_{2t}^{\lambda+R}\theta(x)\Big|^2 \\
&= \Big|\hat{Q}_t^{\lambda+R}\sqrt{\lambda - \hat{L} + R}\,\hat{Q}_t^{\lambda+R}\theta(x)\Big|^2 \\
&\leq \Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\} Q_t\Big|\sqrt{\lambda - \hat{L} + R}\,\hat{Q}_t^{\lambda+R}\theta\Big|^2(x) \\
&= \Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\} Q_t\hat{g}^{\rightarrow}(x, t).
\end{aligned}$$

From this,

$$\begin{aligned}
\hat{G}^{\rightarrow}\theta(x) &= \Big\{\int_0^\infty t\hat{g}^{\rightarrow}(x, t)\,dt\Big\}^{1/2} = \Big\{4\int_0^\infty t\hat{g}^{\rightarrow}(x, 2t)\,dt\Big\}^{1/2} \\
&\leq 2\Big\{\int_0^\infty t\Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\} Q_t\hat{g}^{\rightarrow}(x, t)\,dt\Big\}^{1/2} \\
&\leq 2\Big\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\Big\}^{1/2}\Big\{\int_0^\infty tQ_t\hat{g}^{\rightarrow}(x, t)\,dt\Big\}^{1/2}
\end{aligned}$$

$$= 2\left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}^{1/2} \hat{H}^{\rightarrow}\theta(x).$$

For the scalar function, it holds that $G^{\rightarrow}u \leq 2H^{\rightarrow}u$ since we have $|Q_t u(x)|^2 \leq Q_t|u|^2(x)$.

Let us next estimate $G^{\uparrow}u$.

$$\begin{aligned}
G^{\uparrow}u(x) &= \left\{\int_0^\infty t|\nabla Q_t^\lambda u(x)|^2\,dt\right\}^{1/2}\\
&= \left\{4\int_0^\infty t|\nabla Q_{2t}^\lambda u(x)|^2\,dt\right\}^{1/2}\\
&= 2\left\{\int_0^\infty t|\hat{Q}_t^{\lambda+R}\nabla Q_t^\lambda u(x)|^2\,dt\right\}^{1/2}\\
&\leq 2\left\{\int_0^\infty t\{Q_t^{\lambda+V}|\nabla Q_t^\lambda u(x)|\}^2\,dt\right\}^{1/2}\\
&\leq 2\left\{\int_0^\infty t\left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}Q_t|\nabla Q_t^\lambda u|^2(x)\,dt\right\}^{1/2}\\
&= 2\left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}^{1/2}\left\{\int_0^\infty tQ_t g^{\uparrow}(x,t)\,dt\right\}^{1/2}\\
&= 2\left\{\sup_{s\geq 0} T_s^{2(\lambda+V)}1(x)\right\}^{1/2} H^{\uparrow}u(x).
\end{aligned}$$

Thus we have (4.24). This completes the proof. □

In the next section, we use the diffusion process generated by $L+\partial_a^2$. So we will do some calculation on $L+\partial_a^2$.

Lemma 4.4. *For any θ, set $\hat{f}(x,a) = |\hat{Q}_a^{\lambda+R}\theta(x)|$ and for $\varepsilon > 0$, $\hat{f}_\varepsilon(x,a) = \sqrt{\hat{f}(x,a)^2+\varepsilon}$. Then we have*

$$\left(L+\partial_a^2\right)\hat{f}^2 \geq 2(\lambda+V)\hat{f}^2 + 2\hat{g}. \tag{4.25}$$

and for $1 < p \leq 2$, it holds that

$$(L+\partial_a^2)\hat{f}_\varepsilon^p \geq p(\lambda+V)\hat{f}^2\hat{f}_\varepsilon^{p-2} + p(p-1)\hat{f}_\varepsilon^{p-2}\hat{g} \tag{4.26}$$

where $\hat{g} = \hat{g}(x,a)$ was defined by

$$\hat{g}(x,a) = |\partial_a\hat{Q}_a^{\lambda+R}\theta(x)|^2 + \hat{\Gamma}(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta)(x).$$

For the scalar case, we define $f(x,a) = |Q_a^\lambda u(x)|$, $f_\varepsilon(x,a) = \sqrt{f(x,a)^2+\varepsilon}$. Then *we have*

$$\left(L+\partial_a^2\right) f^2 \geq 2\lambda f^2 + 2g \tag{4.27}$$

and for $1 < p \leq 2$,

$$\left(L+\partial_a^2\right) f_\varepsilon^p \geq p\lambda f^2 f_\varepsilon^{p-2} + p(p-1) f_\varepsilon^{p-2} g. \tag{4.28}$$

Proof. We first show (4.25). To show this, we note that $(\hat{L}-\lambda-R+\partial_a^2)\times \hat{Q}_a^{\lambda+R}\theta(x)=0$. Moreover, using the identity $2\hat{\Gamma}(\theta,\theta)=L|\theta|^2-2(\hat{L}\theta,\theta)$, it holds that

$$L|\hat{Q}_a^{\lambda+R}\theta|^2 = 2\left(\hat{L}\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta\right) + 2\hat{\Gamma}\left(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta\right).$$

Hence

$$\begin{aligned}
(L+\partial_a^2)\hat{f}^2 &= (L+\partial_a^2)|\hat{Q}_a^{\lambda+R}\theta|^2 \\
&= 2(\partial_a^2\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta) + 2(\partial_a\hat{Q}_a^{\lambda+R}\theta, \partial_a\hat{Q}_a^{\lambda+R}\theta) \\
&\quad + 2(\hat{L}\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta) + 2\hat{\Gamma}(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta) \\
&= -2\left((\hat{L}-\lambda-R)\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta\right) + 2|\partial_a\hat{Q}_a^{\lambda+R}\theta|^2 \\
&\quad + 2(\hat{L}\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta) + 2\hat{\Gamma}(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta) \\
&\geq 2(\lambda+V)|\hat{Q}_a^{\lambda+R}\theta|^2 + 2\hat{g}(x,a).
\end{aligned}$$

Secondly we show (4.26). To show this we recall the following fundamental relationship between L and ∇: for $F(\xi^1,\xi^2,\dots,\xi^n)\in C^\infty(\mathbb{R}^n)$ and $f^1,f^2,\dots,f^n\in \mathrm{Dom}(L)$,

$$LF(f^1,f^2,\dots,f^n) = \sum_{i=1}^n \frac{\partial F}{\partial \xi^i} Lf^i + \sum_{i,j=1}^n \frac{\partial^2 F}{\partial\xi^i\partial\xi^j}(\nabla f^i, \nabla f^j)$$

(see [5, Lemma 1]). Hence we have, for $1<p\leq 2$,

$$\begin{aligned}
(L+\partial_a^2)\hat{f}_\varepsilon^p &= (L+\partial_a^2)(\hat{f}_\varepsilon^2)^{p/2} \\
&= \frac{p}{2}(\hat{f}_\varepsilon^2)^{p/2-1}(L+\partial_a^2)\hat{f}_\varepsilon^2 \\
&\quad + \frac{p}{2}\left(\frac{p}{2}-1\right)(\hat{f}_\varepsilon^2)^{p/2-2}\{(\partial_a\hat{f}_\varepsilon^2)^2 + |\nabla\hat{f}_\varepsilon^2|^2\} \\
&= \frac{p}{2}\hat{f}_\varepsilon^{p-2}(L+\partial_a^2)\hat{f}_\varepsilon^2 + \frac{p}{4}(p-2)\hat{f}_\varepsilon^{p-4}\{(\partial_a\hat{f}^2)^2 + |\nabla\hat{f}^2|^2\}.
\end{aligned}$$

Let us recall that (see, e.g., [14, (3.11)])

$$|\nabla\hat{f}^2|^2 = |\nabla(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta)|^2 \leq 4\hat{\Gamma}(\hat{Q}_a^{\lambda+R}\theta, \hat{Q}_a^{\lambda+R}\theta)|\hat{Q}_a^{\lambda+R}\theta|^2.$$

Taking this into account, we have

$$\begin{aligned}(L+\partial_a^2)\hat{f}_\varepsilon^p &\geq \frac{p}{2}\hat{f}_\varepsilon^{p-2}\{2(\lambda+V)|\hat{Q}_a^{\lambda+R}\theta|^2+2\hat{g}\}\\ &\quad+\frac{p}{4}(p-2)\hat{f}_\varepsilon^{p-4}\{4(\partial_a\hat{Q}_a^{\lambda+R}\theta,\hat{Q}_a^{\lambda+R}\theta)^2+4\hat{\Gamma}(\hat{Q}_a^{\lambda+R}\theta,\hat{Q}_a^{\lambda+R}\theta)|\hat{Q}_a^{\lambda+R}\theta|^2\}\\ &\geq \frac{p}{2}\hat{f}_\varepsilon^{p-2}\{2(\lambda+V)|\hat{Q}_a^{\lambda+R}\theta|^2+2\hat{g}\}+p(p-2)\hat{f}_\varepsilon^{p-4}\hat{f}^2\hat{g}\\ &\geq p\hat{f}_\varepsilon^{p-2}(\lambda+V)|\hat{Q}_a^{\lambda+R}\theta|^2+p\hat{f}_\varepsilon^{p-2}\hat{g}+p(p-2)\hat{f}_\varepsilon^{p-2}\hat{g}\\ &\geq p(\lambda+V)\hat{f}_\varepsilon^{p-2}\hat{f}^2+p(p-1)\hat{f}_\varepsilon^{p-2}\hat{g}.\end{aligned}$$

The scalar case can be proved similarly. This completes the proof. □

5. Equivalence of L^p-norms

In this section, we give estimates of G and H functions by a probabilistic method and then show the domination of norms. The original idea is due to P.A. Meyer [9] but we mainly follow Bakry [4].

Let (X_t, P_x) be the diffusion process on M associated with $\mathcal{E}$ as before. We need an additional 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ and we regard M as a vertical space. We write $P_x^{\uparrow}$ in place of P_x. Let $(B_t, P_a^{\rightarrow})$ be a 1-dimensional Brownian motion starting at $a\in\mathbb{R}$ with the generator d^2/da^2. Note that this Brownian motion is different from the standard one up to constant. Let τ be the hitting time of (B_t) to 0, i.e.,

$$\tau=\inf\{t\geq 0\,;\ B_t=0\}.$$

We consider the following diffusion $(Y_t,\mathbf{P}_{(x,a)})$ on the state space $M\times\mathbb{R}$;

$$Y_t:=(X_t,B_t),\quad \mathbf{P}_{(x,a)}:=P_x^{\uparrow}\otimes P_a^{\rightarrow}. \tag{5.1}$$

So the generator of (Y_t) is $L+\partial_a^2$. We denote the integration with respect to $\mathbf{P}_{(x,a)}$ and $\int_M\mathbf{P}_{(x,a)}\mu(dx)$ by $\mathbf{E}_{(x,a)}$ and $\mathbf{E}_{\mu\times\delta_a}$, respectively.

We use the following identities (see Meyer [9] for the proof): Let $\eta\colon M\times\mathbb{R}_+\to[0,\infty)$ be measurable. Then, for $a>0$,

$$\mathbf{E}_{\mu\times\delta_a}\left[\int_0^\tau\eta(X_t,B_t)\,dt\right]=\int_M\mu(dx)\int_0^\infty(a\wedge t)\eta(x,t)\,dt \tag{5.2}$$

and

$$\mathbf{E}_{\mu\times\delta_a}\left[\int_0^\tau\eta(X_t,B_t)\,dt\ \middle|\ X_\tau=x\right]=\int_0^\infty(a\wedge t)Q_t\eta(x,t)\,dt. \tag{5.3}$$

We need an inequality for submartingales. Let (Z_t) be a *non-negative* continuous submartingale with the following Doob-Meyer decomposition;

$$Z_t = M_t + A_t$$

where (M_t) is a continuous martingale and (A_t) is a continuous increasing process with $A_0 = 0$. Then, for $p \geq 1$, it holds that

$$E[A_\infty^p] \leq C_p E[Z_\infty^p]. \tag{5.4}$$

For the proof, see Lenglart-Lépingle-Pratelli [8].

Before going to estimate G-function we prepare the following;

Proposition 5.1. *For any $p \geq 1$, we have*

$$\sup_{\substack{\alpha \geq 0 \\ N \geq 0}} \mathbf{E}_{\mu \times \delta_N}\left[\left\{\int_0^\tau \alpha e^{-\sqrt{\alpha} B_s}\, ds\right\}^p\right] < \infty. \tag{5.5}$$

Proof. By the Itô formula, we have

$$e^{-\sqrt{\alpha} B_t} = e^{-\sqrt{\alpha} B_0} - \sqrt{\alpha} \int_0^t e^{-\sqrt{\alpha} B_s}\, dB_s + \int_0^t \alpha e^{-\sqrt{\alpha} B_s}\, ds.$$

Hence

$$\int_0^{t \wedge \tau} \alpha e^{-\sqrt{\alpha} B_s}\, ds = e^{-\sqrt{\alpha} B_{t \wedge \tau}} - e^{-\sqrt{\alpha} B_0} + M_t$$

where (M_t) is a martingale defined by

$$M_t = \sqrt{\alpha} \int_0^{t \wedge \tau} e^{-\sqrt{\alpha} B_s}\, dB_s,$$

which satisfies

$$\langle M \rangle_t = 2\alpha \int_0^{t \wedge \tau} e^{-2\sqrt{\alpha} B_s}\, ds.$$

Now, by the Burkholder inequality

$$\begin{aligned}
\mathbf{E}_{\mu \times \delta_N}&\left[\left\{\int_0^\tau \alpha e^{-\sqrt{\alpha} B_s}\, ds\right\}^p\right] \\
&\leq C_p \mathbf{E}_{\mu \times \delta_N}\left[\left(e^{-\sqrt{\alpha} B_\tau} - e^{-\sqrt{\alpha} B_0}\right)^p\right] + C_p \mathbf{E}_{\mu \times \delta_N}[\langle M \rangle_\tau^{p/2}] \\
&\leq C_p + C_p \mathbf{E}_{\mu \times \delta_N}\left[\left\{\int_0^\tau 4\alpha e^{-\sqrt{4\alpha} B_s}\, ds\right\}^{p/2}\right].
\end{aligned}$$

Thus it is enough to show (5.5) when $p = 1$.

$$\mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau \alpha e^{-\sqrt{\alpha}B_s}\,ds\right] = \int_0^\infty (N\wedge a)\alpha e^{-\sqrt{\alpha}a}\,da \le \int_0^\infty a\alpha e^{-\sqrt{\alpha}a}\,da = 1.$$

This completes the proof. □

G-functions are now estimated as follows.

Proposition 5.2. *For any* $1 < p < q < 2$, *we have*

$$\|\hat{G}\theta\|_p \lesssim \|\theta\|_q \tag{5.6}$$

and

$$\|\theta\|_{q'} \lesssim \|\hat{G}^{\rightarrow}\theta\|_{p'} \tag{5.7}$$

where p' *and* q' *are the conjugate exponent of* p *and* q, *respectively.*

For scalar functions, we have

$$\|Gu\|_p \lesssim \|u\|_p. \tag{5.8}$$

Proof. Set $\hat{f}(x,a) = |\hat{Q}_a^{\lambda+R}\theta(x)|$ and for $\varepsilon > 0$, $\hat{f}_\varepsilon(x,a) = \sqrt{\hat{f}(x,a)^2 + \varepsilon}$. Define

$$Z_t^{(\varepsilon)} = \hat{f}_\varepsilon(X_{t\wedge\tau}, B_{t\wedge\tau})^p$$

and

$$Z_t = \hat{f}(X_{t\wedge\tau}, B_{t\wedge\tau})^p.$$

Then

$$M_t^{(\varepsilon)} = Z_t^{(\varepsilon)} - \int_0^{t\wedge\tau} (L + \partial_a^2)\hat{f}_\varepsilon(X_s, B_s)^p\,ds$$

is a martingale.

By Lemma 4.4, we have

$$\begin{aligned}(L + \partial_a^2)\hat{f}_\varepsilon^p &\ge p(\lambda + V)\hat{f}^2\hat{f}_\varepsilon^{p-2} + p(p-1)\hat{g}\hat{f}_\varepsilon^{p-2} \\ &\ge -p(\lambda + V)_-\hat{f}^2\hat{f}_\varepsilon^{p-2} + p(p-1)\hat{g}\hat{f}_\varepsilon^{p-2}.\end{aligned} \tag{5.9}$$

Hence

$$Z_t^{(\varepsilon)} + \int_0^\tau p(\lambda + V)_-(X_s)\hat{f}_\varepsilon(X_s, B_s)^p\,ds$$

$$= M_t^{(\varepsilon)} + \int_0^{t\wedge\tau} (L+\partial_a^2)\hat{f}_\varepsilon(X_s,B_s)^p\,ds + \int_0^\tau p(\lambda+V)_-(X_s)\hat{f}_\varepsilon(X_s,B_s)^p\,ds$$

is a non-negative submartingale. By letting $\varepsilon\to 0$ in (5.9), we have

$$\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p \geq -p(\lambda+V)_-\hat{f}^p + p(p-1)\hat{g}\hat{f}^{p-2}$$

which implies

$$\hat{g} \leq \frac{1}{p(p-1)}\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p\cdot\hat{f}^{2-p} + \frac{1}{p-1}V_-\hat{f}^2. \tag{5.10}$$

Now we can estimate $\hat{G}\theta$.

$$\begin{aligned}
\|\hat{G}\theta\|_p^p &= \left\|\left\{\int_0^\infty a\hat{g}(x,a)\,da\right\}^{p/2}\right\|_1\\
&\lesssim \left\|\left\{\int_0^\infty\left\{a\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p + ap(\lambda+V)_-\hat{f}^p\right\}\hat{f}^{2-p}\,da\right\}^{p/2}\right\|_1\\
&\lesssim \left\|\left\{\sup_{t\geq 0}T_t^{\lambda+V}|\theta|\right\}^{p(2-p)/2}\left\{\int_0^\infty a\left\{\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p + p(\lambda+V)_-\hat{f}^p\right\}da\right\}^{p/2}\right\|_1\\
&\leq \left\|\left\{\sup_{t\geq 0}T_t^{\lambda+V}|\theta|\right\}^p\right\|_1^{(2-p)/2}\times\left\|\int_0^\infty a\left\{\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p + p(\lambda+V)_-\hat{f}^p\right\}da\right\|_1^{p/2}.
\end{aligned}$$

The first factor of the right hand side can be estimated as follows. By Theorem 3.2, we have

$$\left\|\left\{\sup_{t\geq 0}T_t^{\lambda+V}|\theta|\right\}^p\right\|_1^{(2-p)/2} \leq \left\|\sup_{t\geq 0}T_t^{\lambda+V}|\theta|\right\|_p^{p(2-p)/2} \lesssim \|\theta\|_q^{p(2-p)/2}. \tag{5.11}$$

For the second factor, we have

$$\begin{aligned}
&\left\|\int_0^\infty a\left\{\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p + p(\lambda+V)_-\hat{f}^p\right\}da\right\|_1\\
&= \lim_{N\to\infty}\mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau\liminf_{\varepsilon\to 0}\left\{(L+\partial_a^2)\hat{f}_\varepsilon^p(X_t,B_t) + p(\lambda+V)_-(X_t)\hat{f}^p(X_t)\right\}dt\right]\\
&\leq \lim_{N\to\infty}\liminf_{\varepsilon\to 0}\mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau\left\{(L+\partial_a^2)\hat{f}_\varepsilon^p(X_t,B_t) + p(\lambda+V)_-(X_t)\hat{f}^p(X_t)\right\}dt\right]\\
&\qquad (\because\ \text{the Fatou lemma})\\
&= \lim_{N\to\infty}\liminf_{\varepsilon\to 0}\mathbf{E}_{\mu\times\delta_N}\left[Z_\infty^{(\varepsilon)} - Z_0^{(\varepsilon)} + \int_0^\tau p(\lambda+V)_-(X_t)\hat{f}^p(X_t)\,dt\right]\\
&= \lim_{N\to\infty}\mathbf{E}_{\mu\times\delta_N}\left[Z_\infty - Z_0 + \int_0^\tau p(\lambda+V)_-(X_t)\hat{f}^p(X_t)\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{N\to\infty} \mathbf{E}_{\mu\times\delta_N}[|\theta(X_\tau)|^p] + \lim_{N\to\infty} \mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau p(\lambda+V)_-(X_t)\hat{f}^p(X_t)\,dt\right] \\
&= \|\theta\|_p^p + \left\|\int_0^\infty a(\lambda+V)_-\hat{f}^p\,da\right\|_1.
\end{aligned}
$$

The second term can be estimated as follows:

$$
\begin{aligned}
\left\|\int_0^\infty a(\lambda+V)_-\hat{f}^p\,da\right\|_1 &\leq \int_0^\infty a\|V_-\hat{f}^p\|_1\,da \\
&\leq \int_0^\infty a\|V_-\|_r\|\hat{f}^p\|_{q/p}\,da \qquad \left(\frac{1}{q/p}+\frac{1}{r}=1\right) \\
&\leq \|V_-\|_r\int_0^\infty a\|\hat{Q}_a^{\lambda+R}\theta\|_q^p\,da \\
&\leq \|V_-\|_r\int_0^\infty a\|\hat{Q}_a^{\lambda+R}\|_{q\to q}^p\|\theta\|_q^p\,da \\
&\leq \|V_-\|_r\int_0^\infty ae^{-\sqrt{\lambda-\gamma_q}ap}\|\theta\|_q^p\,da \qquad (\because\ (4.5)) \\
&= \frac{\|V_-\|_r\|\theta\|_q^p}{(\lambda-\gamma_q)p^2}.
\end{aligned}
$$

Thus we have

$$
\|\hat{G}\theta\|_p^p \lesssim \|\theta\|_q^{p(2-p)/2}\left(\|\theta\|_p^p+\|\theta\|_q^p\right)^{p/2} \lesssim \|\theta\|_q^p
$$

which shows (5.6).

(5.7) is obtained by the duality argument. In fact, using Proposition 4.1, we have

$$
\begin{aligned}
\int_M \left(\theta(x),\eta(x)\right)_K\mu(dx) &= 4\int_M\mu(dx)\int_0^\infty a\left(\partial_a\hat{Q}_a^{\lambda+R}\theta(x),\partial_a\hat{Q}_a^{\lambda+R}\eta(x)\right)_K da \\
&\leq 4\int_M \hat{G}^{\to}\theta(x)\hat{G}^{\to}\eta(x)\mu(dx) \\
&\leq 4\|\hat{G}^{\to}\theta\|_p\|\hat{G}^{\to}\eta\|_{p'} \\
&\lesssim \|\theta\|_q\|\hat{G}^{\to}\eta\|_{p'}.
\end{aligned}
$$

Now (5.7) follows easily.

(5.8) for scalar functions can be shown much easily. □

When $p \geq 2$, we estimate $\hat{H}u$ and Hu.

Proposition 5.3. *For any $2 < p < r$, we have*

$$
\|\hat{H}\theta\|_p \lesssim \|\theta\|_r. \tag{5.12}
$$

For scalar functions, we have

$$\|Hu\|_p \lesssim \|u\|_p. \tag{5.13}$$

Proof. We set $\hat{f}(x,a) = |\hat{Q}_a^{\lambda+R}\theta(x)|$ and define

$$Z_t = \hat{f}(X_{t\wedge\tau}, B_{t\wedge\tau})^2.$$

Then,

$$M_t = Z_t - \int_0^{t\wedge\tau} (L + \partial_a^2)\hat{f}^2(X_s, B_s)\,ds$$

is a martingale. By Lemma 4.4, we have

$$2\hat{g} \le (L + \partial_a^2)\hat{f}^2 + 2V_-\hat{f}^2. \tag{5.14}$$

Then, setting

$$A_t = \int_0^{t\wedge\tau} \{(L + \partial_a^2)\hat{f}^2(X_s, B_s) + 2V_-\hat{f}^2(X_s, B_s)\}\,ds, \tag{5.15}$$

we can see that (A_t) is an increasing process and have that

$$Z_t + \int_0^{t\wedge\tau} 2V_-\hat{f}^2(X_s, B_s)\,ds = M_t + A_t. \tag{5.16}$$

Hence $Z_t + \int_0^{t\wedge\tau} 2V_- f^2\,ds$ is a non-negative submartingale and its increasing part (A_t) satisfies

$$A_t \ge \int_0^{t\wedge\tau} 2\hat{g}(X_s, B_s)\,ds. \tag{5.17}$$

Therefore, by (5.4), the following inequality hold.

$$\begin{aligned}
\mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau 2\hat{g}(X_s, B_s)\,ds\right\}^{p/2}\right] &\le \mathbf{E}_{\mu\times\delta_N}[A_\infty^{p/2}] \\
&\lesssim \mathbf{E}_{\mu\times\delta_N}\left[\left\{Z_\infty + \int_0^\tau 2V_-\hat{f}^2\,ds\right\}^{p/2}\right] \\
&\lesssim \mathbf{E}_{\mu\times\delta_N}[Z_\infty^{p/2}] + \mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau V_-\hat{f}^2\,ds\right\}^{p/2}\right] \\
&= \|\theta\|_p^p + \mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau V_-\hat{f}^2\,ds\right\}^{p/2}\right].
\end{aligned}$$

The second term can be estimated as follows. We take any $p < q < r$.

$$\begin{aligned}
&\mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau V_-\hat{f}^2\,ds\right\}^{p/2}\right]\\
&\quad= \mathbf{E}_{\mu\times\delta_n}\left[\left\{\int_0^\tau e^{-\eta B_s}e^{\eta B_s}V_-\hat{f}^2\,ds\right\}^{p/2}\right]\\
&\quad\le \mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau e^{-\eta B_s q/(q-2)}\,ds\right\}^{(q-2)p/2q}\left\{\int_0^\tau e^{\eta B_s q/2}V_-^{q/2}\hat{f}^q\,ds\right\}^{2p/2q}\right]\\
&\qquad(\because\ \text{the Hölder inequality for the exponents } q/(q-2) \text{ and } q/2)\\
&\quad\le \mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau e^{-\eta B_s q/(q-2)}\,ds\right\}^{u(q-2)p/2q}\right]^{1/u}\mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau e^{\eta B_s q/2}V_-^{q/2}\hat{f}^q\,ds\right]^{p/q}\\
&\qquad\left(\because\ \text{the Hölder inequality for the exponents } u \text{ and } q/p \text{ where } \frac{1}{u}+\frac{1}{q/p}=1\right)\\
&\quad\lesssim \mathbf{E}_{\mu\times\delta_N}\left[\int_0^\tau e^{\eta q B_s/2}V_-^{q/2}\hat{f}^q\,ds\right]^{p/q}\qquad(\because\ \text{Propostion 5.1})\\
&\quad= \left\{\int_M \mu(dx)\int_0^\infty (N\wedge a)e^{\eta q a/2}V_-^{q/2}(x)|\hat{Q}_a^{\lambda+R}\theta(x)|^q\,da\right\}^{p/q}.
\end{aligned}$$

To estimate the integral above, we recall that $\|\hat{Q}_t^{\lambda+R}\theta\|_r \le e^{-\sqrt{\lambda-\gamma_r}t}\|\theta\|_r$. Therefore,

$$\begin{aligned}
&\int_M \mu(dx)\int_0^\infty (N\wedge a)e^{\eta q a/2}V_-^{q/2}(x)|\hat{Q}_a^{\lambda+R}\theta(x)|^q\,da\\
&\qquad\le \int_0^\infty ae^{\eta q a/2}\,da\int_M V_-^{q/2}(x)|\hat{Q}_a^{\lambda+R}\theta(x)|^q\mu(dx)\\
&\qquad\le \int_0^\infty ae^{\eta q a/2}\,da\left\{\int_M V_-^{vq/2}(x)\mu(dx)\right\}^{1/v}\left\{\int_M |\hat{Q}_a^{\lambda+R}\theta(x)|^{qr/q}\mu(dx)\right\}^{q/r}\\
&\qquad\quad\left(\frac{1}{v}+\frac{1}{r/q}=1\right)\\
&\qquad\lesssim \int_0^\infty ae^{\eta q a/2}\|\hat{Q}_a^{\lambda+R}\theta(x)\|_r^q\,da\\
&\qquad\lesssim \int_0^\infty ae^{\eta q a/2}e^{-\sqrt{\lambda-\gamma_r}qa}\|\theta\|_r^q\,da\\
&\qquad\lesssim \|\theta\|_r^q\qquad(\because\ \sqrt{\lambda-\gamma_r}q > \eta q/2).
\end{aligned}$$

Thus we have obtained

$$\mathbf{E}_{\mu\times\delta_N}\left[\left\{\int_0^\tau \hat{g}(X_s,B_s)\,ds\right\}^{p/2}\right] \lesssim \|\theta\|_r^p. \tag{5.18}$$

Now we can estimate $\hat{H}\theta$.

$$
\begin{aligned}
\|\hat{H}\theta\|_p^p &= \left\| \left\{ \int_0^\infty a Q_a \hat{g}(x,a)\, da \right\}^{p/2} \right\|_1 \\
&= \lim_{N\to\infty} \int_M \mu(dx) \left\{ \int_0^\infty (a \wedge N) Q_a \hat{g}(x,a)\, da \right\}^{p/2} \\
&= \lim_{N\to\infty} \int_M \mu(dx) \mathbf{E}_{\mu\times\delta_N} \left[\int_0^\tau \hat{g}(X_s, B_s)\, ds \;\middle|\; X_\tau = x \right]^{p/2} \\
&\le \lim_{N\to\infty} \int_M \mu(dx) \mathbf{E}_{\mu\times\delta_N} \left[\left\{ \int_0^\tau \hat{g}(X_s, B_s)\, ds \right\}^{p/2} \;\middle|\; X_\tau = x \right] \\
&= \lim_{N\to\infty} \mathbf{E}_{\mu\times\delta_N} \left[\left\{ \int_0^\tau \hat{g}(X_s, B_s)\, ds \right\}^{p/2} \right] \\
&\lesssim \|\theta\|_r^p. \qquad (\because\ (5.18))
\end{aligned}
$$

The scalar case is easier. $\square$

Combining Propositions 4.3, 5.3, we can get

Proposition 5.4. *For any $2 \le p < q < \infty$, we have*

$$\|\hat{G}^{\to}\theta\|_p \lesssim \|\theta\|_q \tag{5.19}$$

and

$$\|\theta\|_{q'} \lesssim \|\hat{G}^{\to}\theta\|_{p'} \tag{5.20}$$

where p' and q' are the conjugate exponents of p and q, respectively.

For scalar functions, we have

$$\|Gu\|_p \lesssim \|u\|_p. \tag{5.21}$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We take λ to be large enough. Recall that $\{Q_t^\lambda\}$ is the subordination of $\{T_t^\lambda\}$. Then, by the intertwining property (2.7), we have

$$\nabla Q_t^\lambda = \hat{Q}_t^{\lambda+R} \nabla.$$

Now take any $1 < p < q < \infty$. Then we have

$$\|\nabla u\|_p \lesssim \|\hat{G}^{\to}\nabla u\|_q = \left\| \left\{ \int_0^\infty a \left| \partial_a \hat{Q}_a^{\lambda+R} \nabla u(x) \right|^2 da \right\}^{1/2} \right\|_q$$

$$
\begin{aligned}
&= \left\| \left\{ \int_0^\infty a \left| \nabla Q_a^\lambda \sqrt{\lambda - L}\, u \right|^2 da \right\}^{1/2} \right\|_q \\
&= \| G^\uparrow \sqrt{\lambda - L}\, u \|_q \\
&\lesssim \| \sqrt{\lambda - L}\, u \|_q
\end{aligned}
$$

which proves (2.10).

The reversed inequality (2.11) is obtained by the duality argument. This completes the proof. □

6. Riemannian manifold with boundary

In this section, we discuss the reflected Brownian motion on a Riemannian manifold with boundary. Let M be a compact Riemannian manifold with boundary ∂M. Let $(X_t, P_x)_{x\in M}$ be the Brownian motion on M with the Neumann boundary condition. We denote the Riemannian volume by m. In this section, the semigroup $\{T_t\}$ is generated by $L = \Delta$ with the Neumann boundary condition. $\{T_t\}$ is a symmetric and strongly continuous contraction semigroup in $L^2(m)$. Further $\{\hat{T}_t\}$ is the semigroup generated by the Hodge-Kodaira Laplacian $\hat{L} = -dd^* - d^*d$ with absolute boundary condition. The associated bilinear forms with L and $\hat{L}$ are denoted by $\mathcal{E}$ and $\hat{\mathcal{E}}$. We can see that the following intertwining property holds for $\{T_t\}$ and $\{\hat{T}_t\}$:

$$
\nabla T_t = \hat{T}_t \nabla. \tag{6.1}
$$

As in §5, we use an additional 1-dimensional Brownian motion $(B_t, P_a^{\rightarrow})$ generated by d^2/da^2. Let τ be the hitting time of (B_t) to 0, and $(Y_t, \mathbf{P}_{(x,a)})$ be the product diffusion process on the state space $M \times \mathbb{R}$.

$$
Y_t := (X_t, B_t), \quad \mathbf{P}_{(x,a)} := P_x \otimes P_a^{\rightarrow}. \tag{6.2}
$$

So the generator of (Y_t) is $L + \partial_a^2$.

We use the notation $\mathbf{E}_{m\times\delta_a} = \int_M \mathbf{P}_{(x,a)} m(dx)$ in the same way as in §5. For any $f \in C^\infty(M)$, we have

$$
f(X_t, B_t) - f(X_0, B_0) = \text{ a martingale } + \int_0^t (L + \partial_a^2) f(X_s, B_s)\, ds + \int_0^t \nabla_N f(X_s, B_s)\, dl_s.
$$

Here $\{l_t\}$ is an additive functional corresponding to the smooth measure σ (σ is the surface measure of ∂M), N is the inner normal vector and ∇ denotes the covariant differentiation. In particular we take 1-form θ with absolute boundary condition and set $f(x, a) = |Q_a \theta(x)|^2$. Then,

$$
f(X_{t\wedge\tau}, B_{t\wedge\tau}) - f(X_0, B_0)
$$

$$= M_t + \int_0^{t\wedge\tau} (L + \partial_a^2) f(X_s, B_s)\, ds + \int_0^{t\wedge\tau} \nabla_N f(X_s, B_s)\, dl_s$$
$$= M_t + \int_0^{t\wedge\tau} (L + \partial_a^2) f(X_s, B_s)\, ds + \int_0^{t\wedge\tau} \alpha\big(Q_{B_s}\theta(X_s), Q_{B_s}\theta(X_s)\big)\, dl_s.$$

Here α is the second fundamental form of ∂M (see [15] for this identity.) The quadratic variation of (M_t) is given by

$$\langle M\rangle_t = 2\int_0^{t\wedge\tau} |\nabla Q_{B_s}\theta(X_s)|^2 + |\partial_a Q_{B_s}\theta(X_s)|^2\, ds.$$

Hence we can do the same argument as in the previous section. But we have to tackle the additional term $\int_0^{t\wedge\tau} \alpha(Q_{B_s}\theta(X_s), Q_{B_s}\theta(X_s))\, dl_s$.

Next we see the semigroup domination. We note that for 1-forms θ, η and $f \in C^\infty(M)$,

$$\begin{aligned} &-\mathcal{E}\big((\theta,\eta), f\big) + \hat{\mathcal{E}}(f\theta,\eta) + \hat{\mathcal{E}}(\theta, f\eta) \\ &\quad = 2\int_M (\nabla\theta, \nabla\eta) f m(dx) + 2\int_M \mathrm{Ric}(\theta,\eta) f m(dx) + 2\int_{\partial M} \alpha(\theta,\eta) f\sigma(dx) \end{aligned} \tag{6.3}$$

where Ric is the Ricci curvature (refer to [15] for this identity.)

We take $\gamma \geq 0$ and $\beta \geq 0$ so that $\mathrm{Ric}(\theta,\theta) \geq -\gamma|\theta|^2$ and $\alpha(\theta,\theta) \geq -\beta|\theta|^2$. Then $\alpha(\theta,\theta)\sigma \geq -\beta|\theta|^2\sigma$ as measures. It is easy to see that σ is a smooth measure. We also note that in the interior of M, it holds that

$$L(\theta,\eta) - (\hat{L}\theta,\eta) - (\theta,\hat{L}\eta) = 2(\nabla\theta,\nabla\eta) - 2\,\mathrm{Ric}(\theta,\eta). \tag{6.4}$$

By (6.3) and (6.4), the semigroup domination theorem implies (see [14, 15])

$$|\hat{T}_t\theta| \leq T_t^{-\gamma-\beta\sigma}|\theta|. \tag{6.5}$$

Here $T_t^{-\gamma-\beta\sigma}$ is the semigroup which has $-\gamma-\beta\sigma$ as a potential. It can be represented as

$$T_t^{-\gamma-\beta\sigma}u(x) = E_x\big[u(X_t)e^{\gamma t+\beta l_t}\big] \tag{6.6}$$

We can also show that $(-l_t)$ satisfies the assumption of Theorem 3.1. To see this, take any function $h \in C^\infty(M)$ such that $\nabla_N h = 1$ on ∂M. Such a function can be constructed as follows. Take any local coordinate $(x_1,\dots,x_{n-1},r)$ such that $\partial M = \{r = 0\}$ and $r \mapsto (x_1,\dots,x_{n-1},r)$ is a geodesic with unit velocity perpendicular to ∂M. Then $h(x_1,\dots,x_{n-1},r) = r$ satisfies the property above. Global existence of h can be obtained by using the partition of unity. Then

$$h(X_t) - h(X_0) = M_t + \int_0^t \Delta h(X_s)\, ds + l_t$$

where (M_t) is a martingale with $d\langle M\rangle_t \le C\,dt$ for a constant $C > 0$. Hence

$$E_x[e^{ql_t}] = E_x\left[\exp\left\{qh(X_t) - qh(X_0) - qM_t - q\int_0^t \Delta h(X_s)\,ds\right\}\right].$$

The right hand side is bounded in x because h, Δh and $\langle M\rangle$ is bounded (this implies that σ is a Kato class potential; for Kato class potentials, see Albeverio-Ma [2]). Therefore, there exist constant $c_q > 0$ and $\beta_q > 0$ such that, for q.e.-x,

$$E_x[e^{ql_t}]^{1/q} \le c_q e^{\beta_q t}, \quad \forall t \ge 0. \tag{6.7}$$

Now we can apply Theorem 3.1 to $T_t^{\lambda-\gamma-\beta\sigma}$. For simplicity, we introduce the following notation:

$$M^{\lambda-\gamma-\beta\sigma}u(x) = \sup_{t\ge 0}|T_t^{\lambda-\gamma-\beta\sigma}u(x)|.$$

When $\lambda - \gamma = 0$ and $\beta = 0$, we simply denote Mu in place of $M^{\lambda-\gamma-\beta\sigma}u$. Then, if λ is large enough, we have for any $p > 1$,

$$\|M^{\lambda-\gamma-\beta\sigma}u\|_p \lesssim \|u\|_p. \tag{6.8}$$

We can also obtain an estimate for the subordination. Let $\{Q_t^{\lambda-\gamma-\beta\sigma}\}$ be the subordination of $\{T_t^{\lambda-\gamma-\beta\sigma}\}$. Then

$$\begin{aligned}
|Q_t^{\lambda-\gamma-\beta\sigma}u| &= \left|\int_0^\infty T_s^{\lambda-\gamma-\beta\sigma}u\mu_t(ds)\right| \\
&\le \int_0^\infty e^{-\alpha^2 s}\sup_{r\ge 0}|T_r^{\lambda-\alpha^2-\gamma-\beta\sigma}u|\mu_t(ds) \\
&= e^{-\alpha t}M^{\lambda-\alpha^2-\gamma-\beta\sigma}u.
\end{aligned}$$

Thus we have

$$\sup_{t\ge 0}\{e^{\alpha t}|Q_t^{\lambda-\gamma-\beta\sigma}u|\} \le M^{\lambda-\alpha^2-\gamma-\beta\sigma}u. \tag{6.9}$$

We also note that $\{\hat{T}_t\}$ is a bounded operator in L^p by virtue of (6.5) and there exist constants $c_p > 0$ and $\gamma_p > 0$ so that

$$\|\hat{T}_t\|_{p\to p} \le c_p e^{\gamma_p t}. \tag{6.10}$$

Let $\{\hat{Q}_t^\lambda\}$ be the subordination of $\{\hat{T}_t^\lambda = e^{-\lambda t}\hat{T}_t\}$. (6.5) implies $|\hat{Q}_t^\lambda\theta| \le Q_t^{\lambda-\gamma-\beta\sigma}|\theta|$. We define $\hat{G}$ and $\hat{H}$ in terms of $\{\hat{Q}_t^\lambda\}$. Now we can easily see that Proposition 4.3 holds in this case. We have more. In fact, by virtue of (6.7), we can and do take λ

large enough so that $\sup_{t\geq 0} T_t^{\lambda-\gamma-\beta\sigma}1(x)$ is bounded in q.e.-x and thereby we have

$$\hat{G}^{\rightarrow}\theta \lesssim \hat{H}^{\rightarrow}\theta. \tag{6.11}$$

Similar estimate holds for $G^{\uparrow}u$ and $H^{\uparrow}u$.

Lastly we note that, by combining the domination and (6.9),

$$\sup_{t\geq 0}\{e^{\alpha t}|\hat{Q}_t^{\lambda}\theta|\} \leq M^{\lambda-\alpha^2-\gamma-\beta\sigma}|\theta|. \tag{6.12}$$

Next we extend (5.2) to additive functionals. Take any smooth measure ρ and let A_t be the additive functional associated with ρ. Then we have the following identity.

Proposition 6.1. *For any non-negative function* f *on* $M \times [0,\infty)$ *and* k *on* M, *the following identity holds*:

$$\mathbf{E}_{m\times\delta_a}\left[\int_0^{\tau} f(X_t, B_t)\, dA_t\right] = \int_M \rho(dx)\int_0^{\infty}(a\wedge t)f(x,t)\, dt \tag{6.13}$$

$$\mathbf{E}_{m\times\delta_a}\left[k(X_{\tau})\int_0^{\tau} f(X_t, B_t)\, dA_t\right] = \int_M \rho(dx)\int_0^{\infty}(a\wedge t)Q_t k(x) f(x,t)\, dt \tag{6.14}$$

Proof. Let us first recall the resolvent kernel for the absorbing Brownian motion on $(0,\infty)$. Here, the generator is d^2/da^2. For $\alpha > 0$, set

$$g_{\alpha}(a,b) = \begin{cases} \dfrac{1}{2\sqrt{\alpha}}(e^{\sqrt{\alpha}a} - e^{-\sqrt{\alpha}a})e^{-\sqrt{\alpha}b}, & a \leq b, \\ \dfrac{1}{2\sqrt{\alpha}}e^{-\sqrt{\alpha}a}(e^{\sqrt{\alpha}b} - e^{-\sqrt{\alpha}b}), & a \geq b. \end{cases} \tag{6.15}$$

Then the resolvent $G_{\alpha} = (\alpha - d/(da^2))^{-1}$ is given by

$$G_{\alpha}h(a) = \int_0^{\infty} g_{\alpha}(a,b)h(b)\, db. \tag{6.16}$$

Moreover we note that $\lim_{\alpha\to 0} g_{\alpha}(a,b) = a\wedge b$.

By the Revuz correspondence, (see [6, the equation (5.1.14)]) we have

$$\begin{aligned} &\int_0^{\infty} h(a)\, da \mathbf{E}_{m\times\delta_a}\left[\int_0^{\tau} e^{-\alpha s} f(X_s, B_s)\, dA_s\right] \\ &\quad = \int_0^{\infty} G_{\alpha}h(a)\, da \int_M f(x,a)\rho(dx) \\ &\quad = \int_0^{\infty} h(a)da \int_M \rho(dx)\int_0^{\infty} g_{\alpha}(a,b)f(x,b)\, db \qquad (\because g_{\alpha} \text{ is symmetric}) \end{aligned}$$

Hence we have, for a.e.-a,

$$\mathbf{E}_{m\times\delta_a}\left[\int_0^\tau e^{-\alpha s} f(X_s, B_s)\, dA_s\right] = \int_M \rho(dx)\int_0^\infty g_\alpha(a,b) f(x,b)\, db.$$

But both hands are quasi-continuous in a and one point has positive capacity, the above identity holds for all $a \geq 0$. By letting $\alpha \to 0$, we can get (6.13).

To show (6.14), set

$$H_t = \int_0^{t\wedge\tau} f(X_s, B_s)\, dA_s.$$

Then H_t is a process of bounded variation. Hence, by the Itô formula,

$$\begin{aligned} Q_{B_{t\wedge\tau}} k(X_{t\wedge\tau}) H_t &= \text{ a martingale} + \int_0^{t\wedge\tau} (L+\partial_a^2) Q_{B_s} k(X_s) H_s\, ds + \int_0^{t\wedge\tau} Q_{B_s} k(X_s)\, dH_s \\ &= \text{ a martingale} + \int_0^{t\wedge\tau} Q_{B_s} k(X_s) f(X_s, B_s)\, dA_s. \end{aligned}$$

Here we used that $\nabla_N Q_a k = 0$ on ∂M because $Q_a k$ belongs to the domain of the Neumann Laplacian. By taking expectation and letting $t \to \infty$, we have

$$\begin{aligned} &\mathbf{E}_{m\times\delta_a}\left[Q_{B_\tau} k(X_\tau)\int_0^\tau f(X_s, B_s)\, dA_s\right] \\ &\quad = \mathbf{E}_{m\times\delta_a}\left[\int_0^\tau Q_{B_s} k(X_s) f(X_s, B_s)\, dA_s\right] \\ &\quad = \int_M \rho(dx)\int_0^\infty (a\wedge t) Q_t k(x) f(x,t)\, dt. \quad (\because\ (6.13)) \end{aligned}$$

This completes the proof. □

Recall that $\{Q_t^\lambda\}$, $\{\hat{Q}_t^\lambda\}$ are subordinations of $\{T_t^\lambda\}$, $\{\hat{T}_t^\lambda\}$, respectively and G and H functions are defined in terms of $\{\hat{Q}_t^\lambda\}$. Then we have the following estimate.

Proposition 6.2. *For $1 < p \leq 2$, we have*

(6.17) $$\|\hat{G}\theta\|_p \lesssim \|\theta\|_p$$

and

(6.18) $$\|\theta\|_{p'} \lesssim \|G^{\to}\theta\|_{p'}$$

where p' is the conjugate exponent of p.

For scalar functions, we have

(6.19) $$\|Gu\|_p \lesssim \|u\|_p.$$

Proof. We only show the 1-form case. We set $\hat{f}(x,a) = |\hat{Q}_a^\lambda \theta(x)|$ and $\hat{f}_\varepsilon = \sqrt{\hat{f}^2 + \varepsilon}$ $(\varepsilon > 0)$. Define

$$Z_t^{(\varepsilon)} = \hat{f}_\varepsilon(X_{t\wedge\tau}, B_{t\wedge\tau})^p$$

and

$$Z_t = \hat{f}(X_{t\wedge\tau}, B_{t\wedge\tau})^p.$$

Then,

$$M_t^{(\varepsilon)} = Z_t^{(\varepsilon)} - \int_0^{t\wedge\tau} (L + \partial_a^2)\hat{f}_\varepsilon(X_s, B_s)^p\, ds - \int_0^{t\wedge\tau} \nabla_N \hat{f}_\varepsilon(X_s, B_s)^p\, dl_s.$$

is a martingale. Note that

$$\nabla_N \hat{f}_\varepsilon^p = \nabla_N(\hat{f}^2 + \varepsilon)^{p/2} = \frac{p}{2}(\hat{f}^2 + \varepsilon)^{(p-2)/2}\alpha(\hat{Q}_a^\lambda\theta, \hat{Q}_a^\lambda\theta).$$

Therefore,

$$\begin{aligned}
&\mathbf{E}_{m\times\delta_N}\left[\int_0^\tau (L + \partial_a^2)\hat{f}_\varepsilon(X_s, B_s)^p\, ds\right] \\
&\quad = \mathbf{E}_{m\times\delta_N}[Z_\infty^{(\varepsilon)} - Z_0^{(\varepsilon)}] - \frac{p}{2}\mathbf{E}_{m\times\delta_N}\left[\int_0^\tau \hat{f}_\varepsilon^{p-2}\alpha(\hat{Q}_{B_s}^\lambda\theta, \hat{Q}_{B_s}^\lambda\theta)\, dl_s\right] \\
&\quad \le \mathbf{E}_{m\times\delta_N}[Z_\infty^{(\varepsilon)} - Z_0^{(\varepsilon)}] + \frac{p\beta}{2}\mathbf{E}_{m\times\delta_N}\left[\int_0^\tau \hat{f}_\varepsilon^{p-2}|\hat{Q}_{B_s}^\lambda\theta|^2\, dl_s\right].
\end{aligned}$$

By taking limit, we have

$$\begin{aligned}
&\mathbf{E}_{m\times\delta_N}\left[\liminf_{\varepsilon\to0}\int_0^\tau (L + \partial_a^2)\hat{f}_\varepsilon(X_s, B_s)^p\, ds\right] \\
&\quad \le \liminf_{\varepsilon\to0}\mathbf{E}_{m\times\delta_N}[Z_\infty^{(\varepsilon)} - Z_0^{(\varepsilon)}] + \frac{p\beta}{2}\liminf_{\varepsilon\to0}\mathbf{E}_{m\times\delta_N}\left[\int_0^\tau \hat{f}_\varepsilon^{p-2}|\hat{Q}_{B_s}^\lambda\theta(X_s)|^2\, dl_s\right] \\
&\quad \le \|u\|_p^p + \frac{p\beta}{2}\liminf_{\varepsilon\to0}\int_0^\infty (N\wedge a)\, da \int_M \hat{f}_\varepsilon^{p-2}|\hat{Q}_a^\lambda\theta|^2\sigma(dx) \\
&\quad \le \|u\|_p^p + \frac{p\beta}{2}\liminf_{\varepsilon\to0}\int_0^\infty (N\wedge a)\, da \int_M \hat{f}_\varepsilon^p\sigma(dx).
\end{aligned}$$

We estimate the second term. We use the interpolation space. Taking $\xi = 1-(1/p)$, we introduce the interpolation norm $\|\cdot\|_{\xi,p}$ of $\|\cdot\|_{0,p}$ and $\|\cdot\|_{1,p}$. Here $\|\cdot\|_{0,p}$ is the L^p norm in $L^p(M, dx)$ and $\|\cdot\|_{1,p}$ is the Sobolev norm:

$$\|u\|_{1,p}^p = \int_M |u|^p m(dx) + \int_M |\nabla u|^p m(dx).$$

Then the following inequality holds (see e.g., [1, Chapter VII]):

$$\int_{\partial M} |u|^p d\sigma(x) \lesssim \|u\|_{\xi,p}^p.$$

Moreover, the general theory of interpolation implies (see [1, LEMMA 7.16])

$$\|u\|_{\xi,p}^p \lesssim \|u\|_{0,p}^{(1-\xi)p} \|u\|_{1,p}^{\xi p}.$$

Thus we have

$$\int_{\partial M} |u|^p \sigma(dx) \lesssim \|u\|_{0,p}^{(1-\xi)p} \|u\|_{1,p}^{\xi p}. \tag{6.20}$$

On the other hand

$$|\nabla \hat{f}_\varepsilon| = \left|\nabla \sqrt{\hat{f}^2 + \varepsilon}\right| = \frac{1}{2}(\hat{f}^2 + \varepsilon)^{-1/2} |\nabla \hat{f}^2| \leq \frac{1}{2}(\hat{f}^2 + \varepsilon)^{-1/2} 2\hat{f} |\nabla \hat{Q}_a^\lambda \theta| \leq |\nabla \hat{Q}_a^\lambda \theta|.$$

Using these inequalities, we have

$$\begin{aligned}
\int_0^\infty (N \wedge a)\, da \int_M \hat{f}_\varepsilon^p \sigma(dx) &\lesssim \int_0^\infty (N \wedge a) \|\hat{f}_\varepsilon\|_{0,p}^{(1-\xi)p} \|\hat{f}_\varepsilon\|_{1,p}^{\xi p}\, da \\
&\lesssim \int_0^\infty (N \wedge a)\{\|\hat{f}_\varepsilon\|_p^p + \|\hat{f}_\varepsilon\|_p^{(1-\xi)p} \|\nabla \hat{f}_\varepsilon\|_p^{\xi p}\}\, da \\
&\lesssim \int_0^\infty (N \wedge a)\{\|\hat{f}_\varepsilon\|_p^p + \|\hat{f}_\varepsilon\|_p^{(1-\xi)p} \|\nabla \hat{Q}_a^\lambda \theta\|_p^{\xi p}\} da.
\end{aligned}$$

By taking limit, we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^\infty (N \wedge a)\, da \int_M \hat{f}_\varepsilon^p \sigma(dx) \\
&\quad \lesssim \int_0^\infty (N \wedge a)\{\|\hat{f}\|_p^p + \|\hat{f}\|_p^{(1-\xi)p} \|\nabla \hat{Q}_a^\lambda \theta\|_p^{\xi p}\}\, da \\
&\quad \lesssim \int_0^\infty a e^{-\sqrt{\lambda - \gamma_p} pa} \|\theta\|_p^p\, da + \int_0^\infty a e^{-\sqrt{\lambda-\gamma_p}(1-\xi)pa} \|\theta\|_p^{(1-\xi)p} \|\nabla \hat{Q}_a^\lambda \theta\|_p^{\xi p}\, da \\
&\quad \lesssim \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p} \int_0^\infty a e^{-\sqrt{\lambda-\gamma_p}(1-\xi)pa}\, da \left\{\int_M |\nabla \hat{Q}_a^\lambda \theta(x)|^p m(dx)\right\}^\xi \\
&\quad \leq \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p} \left\{\int_0^\infty a e^{-\sqrt{\lambda-\gamma_p}(1-\xi)pa} 1^{1/(1-\xi)} da\right\}^{1-\xi} \\
&\qquad \times \left\{\int_0^\infty a e^{-\sqrt{\lambda-\gamma_p}(1-\xi)pa}\, da \int_M |\nabla \hat{Q}_a^\lambda \theta(x)|^p m(dx)\right\}^\xi \\
&\qquad \left(\frac{1}{1/(1-\xi)} + \frac{1}{1/\xi} = 1\right)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p}\left\{\int_M m(dx)\int_0^\infty ae^{-\sqrt{\lambda-\gamma_p}(1-\xi)pa}|\nabla\hat{Q}_a^\lambda\theta(x)|^p\,da\right\}^\xi \\
&\lesssim \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p}\left[\int_M m(dx)\left\{\int_0^\infty ae^{-\sqrt{\lambda-\gamma_p}(1-\xi)p\nu a}da\right\}^{1/\nu}\right. \\
&\qquad \times \left.\left\{\int_0^\infty a|\nabla\hat{Q}_a^\lambda\theta(x)|^{p\cdot 2/p}\,da\right\}^{p/2}\right]^\xi \qquad \left(\frac{1}{\nu}+\frac{1}{2/p}=1\right) \\
&\lesssim \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p}\left[\int_M\left\{\int_0^\infty a|\nabla\hat{Q}_a^\lambda\theta(x)|^2\,da\right\}^{p/2} m(dx)\right]^\xi \\
&\lesssim \|\theta\|_p^p + \|\theta\|_p^{(1-\xi)p}\|\hat{G}^\uparrow\theta\|_p^{\xi p}.
\end{aligned}
$$

Further, as in the proof of Proposition 5.2, we can show that

$$
\|\hat{G}\theta\|_p^p \lesssim \|\theta\|_p^{p(2-p)/2}\left\|\int_0^\infty a\liminf_{\varepsilon\to 0}(L+\partial_a^2)\hat{f}_\varepsilon^p\,da\right\|^{p/2}.
$$

In fact, $(\lambda+V)_-$ in the proof of Proposition 5.2 vanishes in this case. Combining these inequalities, we have

$$
\begin{aligned}
\|\hat{G}\theta\|_p^p &\lesssim \|\theta\|_p^{p(2-p)/2}\{\|\theta\|_p^{p\cdot p/2} + \|\theta\|_p^{(1-\xi)p\cdot p/2}\|\hat{G}^\uparrow\theta\|_p^{\xi p\cdot p/2}\} \\
&\le \|\theta\|_p^p + \|\theta\|_p^{(2-\xi p)p/2}\|\hat{G}^\uparrow\theta\|_p^{\xi p^2/2} \\
&= \|\theta\|_p^p + \|\theta\|_p^{(3-p)p/2}\|\hat{G}^\uparrow\theta\|_p^{(p-1)p/2} \qquad \left(\xi = 1-\frac{1}{p}\right) \\
&\le \|\theta\|_p^p + \frac{3-p}{2}\delta^{-(p-1)/(3-p)}\|\theta\|_p^p + \frac{p-1}{2}\delta\|\hat{G}^\uparrow\theta\|_p^p \qquad \left(\frac{3-p}{2}+\frac{p-1}{2}=1\right) \\
&\lesssim \|\theta\|_p^p + \delta^{-(p-1)/(3-p)}\|\theta\|_p^p + \delta\|\hat{G}\theta\|_p^p.
\end{aligned}
$$

Since δ is arbitrary, we can get

$$
\|\hat{G}\theta\|_p^p \lesssim \|\theta\|_p^p
$$

which is (6.17). Now the rest is easy. □

When $p \ge 2$, we estimate $\hat{H}\theta$ and Hu.

Proposition 6.3. *We further assume that the second fundamental form α is non-negative definite. Then, for any $p \ge 2$, we have*

$$
\|\hat{H}\theta\|_p \lesssim \|\theta\|_p. \tag{6.21}
$$

For scalar functions, we have

$$
\|Hu\|_p \lesssim \|u\|_p. \tag{6.22}
$$

Proof. Set $\hat{f}(x,a) = |\hat{Q}_a^\lambda \theta(x)|$ for 1-form θ and define

$$Z_t = \hat{f}(X_{t\wedge\tau}, B_{t\wedge\tau})^2.$$

Then

$$Z_t = Z_0 + M_t - \int_0^{t\wedge\tau} (L+\partial_a^2)\hat{f}^2(X_s,B_s)\,ds - \int_0^{t\wedge\tau} \alpha\big(\hat{Q}_{B_s}^\lambda\theta(X_s), \hat{Q}_{B_s}^\lambda\theta(X_s)\big)\,dl_s$$

where (M_t) is a martingale with the quadratic variation

$$\langle M\rangle_t = 2\int_0^{t\wedge\tau} \{|\nabla \hat{f}^2(X_s,B_s)|^2 + |\partial_a \hat{f}^2(X_s,B_s)|^2\}\,ds.$$

By the assumption that α is non-negative definite, (Z_t) is a submartingale and the increasing part is given as

$$A_t := \int_0^{t\wedge\tau} (L+\partial_a^2)\hat{f}^2(X_s,B_s)\,ds + \int_0^{t\wedge\tau} \alpha\big(\hat{Q}_{B_s}^\lambda\theta(X_s), \hat{Q}_{B_s}^\lambda\theta(X_s)\big)\,dl_s.$$

Now, recalling that (see Lemma 4.4)

$$(L+\partial_a^2)\hat{f}^2 \geq 2\hat{g},$$

we have

$$A_t \geq \int_0^{t\wedge\tau} 2\hat{g}(X_s,B_s)\,ds.$$

By virtue of the submartingale inequality (5.4), we obtain

$$\begin{aligned}\mathbf{E}_{m\times\delta_N}\left[\left\{\int_0^\tau 2\hat{g}(X_s,B_s)\,ds\right\}^{p/2}\right] &\leq \mathbf{E}_{m\times\delta_N}[A_\infty^{p/2}]\\ &\lesssim \mathbf{E}_{m\times\delta_N}[Z_\infty^{p/2}]\\ &= \|\theta\|_p^p.\end{aligned}$$

Thus we have

$$\begin{aligned}\|\hat{H}\theta\|_p^p &= \left\|\left\{\int_0^\infty a Q_a \hat{g}(x,a)\,da\right\}^{p/2}\right\|_1\\ &= \lim_{N\to\infty}\int_M \mu(dx)\left\{\int_0^\infty (a\wedge N)Q_a\hat{g}(x,a)\,da\right\}^{p/2}\\ &= \lim_{N\to\infty}\int_M \mu(dx)\mathbf{E}_{m\times\delta_N}\left[\int_0^\tau \hat{g}(X_s,B_s)\,ds\;\middle|\; X_\tau = x\right]^{p/2}\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{N\to\infty}\int_M \mu(dx)\mathbf{E}_{m\times\delta_N}\left[\left\{\int_0^\tau \hat{g}(X_s, B_s)\,ds\right\}^{p/2} \,\middle|\, X_\tau = x\right] \\
&= \lim_{N\to\infty}\mathbf{E}_{m\times\delta_N}\left[\left\{\int_0^\tau \hat{g}(X_s, B_s)\,ds\right\}^{p/2}\right] \\
&\lesssim \|\theta\|_p^p.
\end{aligned}
$$

Scalar case is easier. □

By combining Propositions 4.3 and 6.3, we easily obtain the following estimates for G-functions:

Proposition 6.4. *Assume that α is non-negative definite. Then, for any $p \ge 2$, we have*

$$\|\hat{G}^{\to}\theta\|_p \lesssim \|\theta\|_p \tag{6.23}$$

and

$$\|\theta\|_{p'} \lesssim \|\hat{G}^{\to}\theta\|_{p'} \tag{6.24}$$

where p' is the conjugate exponent of p.

For scalar functions, we have

$$\|Gu\|_p \lesssim \|u\|_p. \tag{6.25}$$

Now the following theorem can be proved in the same way as Theorem 2.1.

Theorem 6.5. *For any $1 < p \le 2$, it holds that*

$$\|u\|_p + \|\nabla u\|_p \lesssim \|\sqrt{1-\Delta}u\|_p, \tag{6.26}$$

$$\|\sqrt{1-\Delta}u\|_{p'} \lesssim \|u\|_{p'} + \|\nabla u\|_{p'}. \tag{6.27}$$

where p' is the conjugate exponent of p.

If we further assume that the second fundamental form α is non-negative definite, then the inequalities above hold for $p \ge 2$.

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