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ON THE GENERALIZED NOVIKOV FIRST EXT GROUP MODULO A PRIME

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1. Introduction

Let BP be the Brown-Peterson spectrum for a fixed prime p, whose homotopy is $BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$. In [6] §6.5, the second author has introduced the spectrum $T(m)$, whose BP-homology is

$$BP_*(T(m)) \cong BP_*[t_1, \dots, t_m].$$

This is homotopy equivalent to BP below dimension $2p^{m+1} - 3$.

The Adams-Novikov E_2-term converging to the homotopy groups of $T(m)$

$$E_2^{*,*}(T(m)) = \mathrm{Ext}_{BP_*(BP)}\big(BP_*, BP_*(T(m))\big)$$

is isomorphic by [6, Corollary 7.1.3] to

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) \cong BP_*[t_{m+1}, t_{m+2}, \dots].$$

In particular $\Gamma(1) = BP_*(BP)$ by definition. To get the structure of $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*)$, we will use the chromatic method introduced in [3].

Denote an ideal $(p, v_1, \dots, v_{n-1})$ of BP_* by I_n, and a comodule

$$v_{n+s}^{-1}BP_*/(p, v_1, \dots, v_{n-1}, v_n^\infty, \dots, v_{n+s-1}^\infty).$$

by M_n^s. Then we can consider the chromatic spectral sequence converging to

$$\mathrm{Ext}_{\Gamma(m+1)}\left(BP_*, BP_*/I_n\right)$$

with

$$E_1^{s,t} = \mathrm{Ext}^t_{\Gamma(m+1)}\left(BP_*, M_n^s\right).$$

Shimomura calls this Ext group the *general chromatic* E_1*-term*.

The limiting case as m approaches infinity is discussed by the second author in [7]. In this paper we will determine the module structure (over an appropriate generalization of $k(1)_*$) of

$$\mathrm{Ext}^0_{\Gamma(m+1)}\left(BP_*, M_1^1\right)$$

in Theorem 6.1, which is closely related to the group

$$\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, BP_*/(p)).$$

The structure of these two groups are described below in Theorems 6.1 and 7.1. Notice that our target $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, BP_*/(p))$ is different from the localized object, which is determined in Kamiya-Shimomura [2]. Hereafter we will often abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ by $\mathrm{Ext}_{\Gamma(m+1)}(M)$ for a $\Gamma(m+1)$-comodule M.

We begin by recalling the analogous result for $m = 0$, which was obtained long ago by Miller-Wilson in [4] (and reformulated in [6] as Theorems 5.2.13, Corollary 5.2.14, and Theorem 5.2.17). Recall that we have the 4-term exact sequence

$$0 \to BP_*/(p) \to M_1^0 \to M_1^1 \to N_1^2 \to 0 \tag{1.1}$$

obtained by splicing the two short exact sequences

$$0 \longrightarrow BP_*/(p) \longrightarrow M_1^0 \longrightarrow N_1^1 \longrightarrow 0,$$

and

$$0 \longrightarrow N_1^1 \longrightarrow M_1^1 \longrightarrow N_1^2 \longrightarrow 0.$$

From (1.1) we see that $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$ is a certain subquotient of

$$\mathrm{Ext}^1_{\Gamma(1)}(M_1^0) \oplus \mathrm{Ext}^0_{\Gamma(1)}(M_1^1). \tag{1.2}$$

For the first summand, we have (for p odd)

$$\mathrm{Ext}_{\Gamma(1)}(M_1^0) = \mathrm{Ext}_{\Gamma(1)}(v_1^{-1}BP_*/(p)) \cong K(1)_* \otimes E(h_{1,0}).$$

In particular we have

$$\mathrm{Ext}^1_{\Gamma(1)}(M_1^0) \cong K(1)_*\{h_{1,0}\}.$$

It turns out that the image of $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$ into this group is $k(1)_*\{h_{1,0}\}$, which is the v_1-torsion free component of $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$.

To describe $\mathrm{Ext}^0_{\Gamma(1)}(M_1^1)$, we recall the elements $x_k \in v_2^{-1}BP_*/(p)$ defined by

$$x_0 = v_2,$$

$$x_1 = v_2^p - v_1^p v_2^{-1} v_3,$$
$$x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3,$$

and
$$x_k = \begin{cases} x_{k-1}^2 & (p=2) \\ x_{k-1}^p - 2v_1^{(p+1)(p^{k-1}-1)} v_2^{(p-1)p^{k-1}+1} & (p>2) \end{cases} \quad \text{for } k \geq 3,$$

and integers $a(k)$ defined by

$$a(0) = 1,$$
$$a(1) = p,$$
$$a(k) = \begin{cases} 3 \cdot 2^{k-1} & (p=2) \\ p^k + p^{k-1} - 1 & (p>2) \end{cases} \quad \text{for } k \geq 2.$$

Then we have

Theorem 1.3 ([4]). *As a $k(1)_*$-module, $\mathrm{Ext}^0_{\Gamma(1)}\left(M_1^1\right)$ is the direct sum of*
(a) *the cyclic submodules generated by $x_k^s/v_1^{a(k)}$ for $k \geq 0$ and $p \nmid s \in \mathbf{Z}$; and*
(b) *$K(1)_*/k(1)_*$, generated by $1/v_1^j$ for $j \geq 1$.*

The odd prime case follows from the next proposition ([3, Proposition 5.4]). We refer the reader to the original sources for the case $p = 2$.

Proposition 1.4. *Let p be odd. Modulo $(p, v_1^{1+a(k)})$, the differential*

$$d = \eta_R - \eta_L : v_2^{-1} BP_*/(p) \to v_2^{-1} BP_*/(p) \otimes_{BP_*} BP_*(BP)$$

on x_k is

$$d(x_k) \equiv \begin{cases} v_1 t_1^p & \text{for } k = 0, \\ v_1^p v_2^{p-1} t_1 & \text{for } k = 1, \\ 2v_1^{a(k)} v_2^{(p-1)p^{i-1}} t_1 & \text{for } k \geq 2. \end{cases}$$

Before Theorem 1.3 was proved, the naive conjecture about $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$ would have had the exponents $a(k)$ being p^k for all $k \geq 0$. It was clear that

$$\frac{v_2^{sp^k}}{v_1^{p^k}} \in \mathrm{Ext}^0_{\Gamma(1)}(M_1^1),$$

but the existence of "deeper" elements such as

$$\frac{x_2}{v_1^{a(2)}} = \frac{v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2} v_2^{-p} v_3^p}{v_1^{p^2+p-1}}$$

and

$$\frac{x_3}{v_1^{a(3)}} = \frac{v_2^{p^3} - v_1^{p^3-p} v_2^{p^3-p^2+p} - v_1^{p^3} v_2^{-p^2} v_3^{p^2} - 2v_1^{p^3+p^2-p-1} v_2^{p^3-p^2+1}}{v_1^{p^3+p^2-1}}$$

(and that of $\beta_{sp^2/a(2)}$ and $\beta_{sp^3/a(3)}$ in $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$ for $s > 1$) came as a surprise, as did the fact that the limiting value (as $k \to \infty$) of $a(k)/p^k$ is $(p+1)/p$ (this limit is attained for $p = 2$ but not for odd primes) instead of 1.

Using these results one can deduce

Theorem 1.5. *For odd prime p, the group $\mathrm{Ext}^1_{\Gamma(1)}(BP_*/(p))$ is isomorphic to*

$$k(1)_* \left\{\beta_{sp^k/j} : s \geq 0,\ p \nmid s,\ k \geq 0 \text{ and } 0 < j \leq a_s(k)\right\} \oplus k(1)_*\{h_{1,0}\},$$

where $\beta_{sp^k/j}$ is the image of x_k^s/v_1^j under the connecting homomorphism

$$\delta\colon\ \mathrm{Ext}^0_{\Gamma(1)}(N_1^1) \to \mathrm{Ext}^1_{\Gamma(1)}(N_1^0).$$

and $a_s(k) = \begin{cases} p^k & (s = 1) \\ a(k) & (s > 1) \end{cases}.$

Our results (Theorems 6.1 and 7.1 below) have the same form as Theorems 1.3 and 1.5, but with x_k and $a(k)$ replaced by $\widehat{x}_k$ and $\widehat{a}(k)$ defined in (4.1) and (4.3), and with $k(1)_*$ replaced by a bigger ring $v_2^{-1}\widehat{k}(1)_*$ defined in (2.1). The $\widehat{a}(k)$ are the same for all $m > 0$ (except when $m = 1$ and $p = 2$) although the $\widehat{x}_k$ show a slight difference between the cases $m = 1$ and $m > 1$. The case $m = 1$ and $p = 2$ is different and has to be treated separately. For $m > 1$ there are no special conditions for the prime 2. The asymptotic behavior of the exponents is given by

$$\lim_{k\to\infty} \frac{\widehat{a}(k)}{p^k} = \frac{p^3 + p^2}{p^3 - 1},$$

a slightly larger value than for the case $m = 0$. However for $m > 0$ there are no deeper elements in $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$, i.e., no elements of the form $\widehat{\beta}_{sp^k/j}$ with $p \nmid s$ and $j > p^k$.

We found a new form of periodicity in our statement with no precedent in Theorem 1.3. For example, (except for $p = 2$ and $m = 1$) we have

$$\widehat{x}_k - \widehat{x}_{k-1}^p = -v_1^{p^{k-1}(p+1)} v_2^{p^{k-2}(p^{m+2}-p-1)} \widehat{x}_{k-3}^{p-1} \left(\widehat{x}_{k-3} - \widehat{x}_{k-4}^p\right) \qquad \text{for } k \geq 5,$$

and

$$\widehat{a}(k) = p^k + p^{k-1} + \widehat{a}(k-3) \qquad \text{for } k \geq 4.$$

A similar result for the chromatic module M_2^1 is obtained in a joint work with Ippei Ichigi [1]. There we get a similar periodicity with period 4 instead of 3 when $m \geq 5$.

We obtained our result in the summer of 1999. On the other hand, Kamiya-Shimomura [2] told us that they have determined all the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}\left(M_1^1\right)$ in the fall of 1999 independently.

We are grateful to the referee for suggesting some corrections to an earlier draft of this paper.

2. Prelimaries

For a $\Gamma(m+1)$-comodule M, consider the cobar complex

$$\left\{C^n_{\Gamma(m+1)}(M), d_n\right\}_{n\geq 0},$$

which is determined by

$$C^n_{\Gamma(m+1)}(M) = \underbrace{\Gamma(m+1)\otimes_{BP_*}\cdots\otimes_{BP_*}\Gamma(m+1)}_{n\text{-factors}}\otimes_{BP_*}M,$$

and

$$d_n\colon C^n_{\Gamma(m+1)}(M)\to C^{n+1}_{\Gamma(m+1)}(M).$$

Then $\mathrm{Ext}_{\Gamma(m+1)}(M)$ is the cohomology of this cobar complex. By the change-of-rings isomorphism (cf. [6, Theorem 6.1.1]), we have

$$\begin{aligned}\mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right) &\cong \mathrm{Ext}_{\Gamma(1)}\left(M_n^0\otimes_{BP_*} BP_*(T(m))\right)\\ &\cong \mathrm{Ext}_{\Sigma(n)}\Big(K(n)_*, K(n)_*(T(m))\Big),\end{aligned}$$

where $\Sigma(n) = K(n)_*\otimes_{BP_*} BP_*(BP)\otimes_{BP_*} K(n)_*$. This object is already known by [6, Corollary 6.5.6].

In order to avoid the excessive appearance of the index m, we will hereafter use the following notations.

$$(2.1)\qquad \begin{cases} \omega = p^m, \\ \widehat{v}_i = v_{m+i}, \\ \widehat{t}_i = t_{m+i}, \\ \widehat{h}_{i,j} = h_{m+i,j}, \\ \widehat{K}(n)_* = K(n)_*[v_{n+1},\ldots,v_{n+m}], \\ \text{and}\quad \widehat{k}(n)_* = k(n)_*[v_{n+1},\ldots,v_{n+m}], \end{cases}$$

where $h_{m+i,j}$ is the cocycle represented by $t_{m+i}^{p^j}$.

Theorem 2.2 ([6, Corollary 6.5.6]). *If $n < 2(p-1)(m+1)/p$ and $n < m+2$, then*

$$\mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right)\cong \widehat{K}(n)_*\otimes E\left(\widehat{h}_{i,j} : 1\leq i\leq n, 0\leq j\leq n-1\right).$$

In this paper we will need this result only for $n = 2$, for which it covers the cases $m > 0$ for odd p and $m > 1$ for $p = 2$. For the case $p = 2$ and $m = 1$, we need

Theorem 2.3 ([5]). *If $p = 2$ and $m = 1$, then*

$$\mathrm{Ext}_{\Gamma(2)}\left(M_2^0\right) \cong \widehat{K}(2)_* \otimes P\left(\widehat{h}_{1,0}, \widehat{h}_{1,1}\right) \Big/ \left(\widehat{h}_{1,1}^2 + v_2^2 \widehat{h}_{1,0}^2\right) \otimes E\left(\widehat{h}_{2,0}, \widehat{h}_{2,1}, \rho\right),$$

where $\rho = \widehat{h}_{3,1} + v_2^5 \widehat{h}_{3,0}$.

This information allow us to determine the structure of $\mathrm{Ext}_{\Gamma(m+1)}\left(M_1^1\right)$ using the Bockstein spectral sequence. In fact, we use the following convenient lemma.

Lemma 2.4 (cf. [3, Remark 3.11]). *Assume that there exists a $\widehat{k}(1)_*$-submodule B^t of $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right)$ for each $t < N$, such that the following sequence is exact:*

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(m+1)}\left(M_2^0\right) \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^0\right) \xrightarrow{1/v_1} B^t \xrightarrow{v_1} B^t \xrightarrow{\delta} \cdots$$

where δ is a restriction of the coboundary map

$$\delta\colon \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right) \to \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M_2^0\right).$$

Then the inclusion $i_t\colon B^t \to \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right)$ is an isomorphism between $\widehat{k}(1)_*$-modules for each $t < N$.

Proof. Because $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right)$ is a v_1-torsion module, we can filter B^t by

$$B^t(i) = \left\{x \in B^t : v_1^i x = 0\right\}$$

and $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right)$ by

$$E^t(i) = \left\{x \in \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_1^1\right) : v_1^i x = 0\right\}.$$

Assume that the inclusion i_k is an isomorphism for $k \le t - 1$ (the $t = 0$ case is obvious), and consider the following commutative ladder diagram where we abbreviate

$\mathrm{Ext}^s_{\Gamma(m+1)}(M^j_i)$ by $H^s(M^j_i)$.

$$\begin{array}{ccccccccc}
B^{t-1} & \xrightarrow{\delta} & H^t\left(M^0_2\right) & \xrightarrow{1/v_1} & B^t(i) & \xrightarrow{v_1} & B^t(i-1) & \xrightarrow{\delta} & H^t\left(M^0_2\right) \\
{\scriptstyle i_{t-1}}\downarrow\cong & & \| & & \downarrow{\scriptstyle i_t} & & \downarrow{\scriptstyle i_t} & & \| \\
H^{t-1}\left(M^1_1\right) & \xrightarrow{\delta} & H^t\left(M^0_2\right) & \xrightarrow{1/v_1} & E^t(i) & \xrightarrow{v_1} & E^t(i-1) & \xrightarrow{\delta} & H^t\left(M^0_2\right)
\end{array}$$

Using the Five Lemma, we obtain the desired isomorphism $B^t(i) \cong E^t(i)$ $(i \geq 1)$ by induction on i. □

In §3 and §4, we will define elements $\widehat{x}_k \in v_2^{-1}BP_*$ for $k \geq 0$ (see (4.1)) satisfying

$$\widehat{x}^s_k \equiv \widehat{v}_2^{sp^k} \quad \mod (p, v_1),$$

and integers $\widehat{a}(k)$ such that each $\widehat{x}^s_k/v^l_1$ is a cocycle of for all $1 \leq l \leq \widehat{a}(k)$.

Using these notations, we can describe the structure of B^0 fitting into the long exact sequence of Lemma 2.4. We have

Lemma 2.5. *For $m > 0$,*

$$B^0 = v_2^{-1}\widehat{k}(1)_* \left\{ \frac{\widehat{x}^s_k}{v_1^{\widehat{a}(k)}} : \ k \geq 0,\ s > 0 \text{ and } p \nmid s \right\} \oplus v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$$

is isomorphic as a $\widehat{k}(1)_*$-module to $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M^1_1\right)$, *if the set*

$$\left\{ \delta\left(\frac{\widehat{x}^s_k}{v_1^{\widehat{a}(k)}} \right) : \ k \geq 0,\ s > 0 \text{ and } p \nmid s \right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_2\right)$$

is linearly independent over

$$R = \mathbf{Z}/(p)[v_2, v_2^{-1}, v_3, \ldots, v_m, v_{m+1}],$$

where δ is the coboundary map in Lemma 2.4.

Proof. All exactness of the sequence

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(m+1)}\left(M^0_2\right) \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_2\right)$$

is obvious, except Ker $\delta \subset$ Im v_1. So we need to show only this inclusion. Separate the R-basis of B^0 into two parts,

$$A = \left\{ \frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} : \ k \geq 0 \text{ and } p \nmid s > 0 \right\}$$

and

$$B = \left\{ \frac{\widehat{x}_k^s}{v_1^l} : \ k \geq 0, \ p \nmid s > 0, \text{ and } 1 \leq l < \widehat{a}(k) \right\} \cup \left\{ \frac{1}{v_1^i} : i > 0 \right\}.$$

Then it is obvious that $\delta(\widehat{x}_\lambda) \neq 0 \in \operatorname{Ext}^1_{\Gamma(m+1)}\left(M_2^0\right)$ for $\widehat{x}_\lambda \in A$, but that $\delta(y_\mu) = 0 \in \operatorname{Ext}^1_{\Gamma(m+1)}\left(M_2^0\right)$ for $y_\mu \in B$. Thus for any element $z = \sum_\lambda a_\lambda \widehat{x}_\lambda + \sum_\mu b_\mu y_\mu$ of B^0 $(a_\lambda, b_\mu \in R)$, we have $\delta(z) = \sum_\lambda a_\lambda \delta(\widehat{x}_\lambda)$. The condition implies that all a_λ are zero when $\delta(z) = 0$, and so $v_1 \sum_\mu b_\mu y_\mu / v_1 = z$. This completes the proof. □

3. Definition of the elements $\widehat{w}_3$ and $\widehat{w}_4$

In this section we will introduce elements $\widehat{w}_3$ and $\widehat{w}_4$ in (3.2) to change the bases $\widehat{h}_{i,j}$ ($i = 1$, 2 and $j = 0$, 1) of $\operatorname{Ext}_{\Gamma(m+1)}(M_2^0)$ given in Theorems 2.2 and 2.3. First we recall the right unit η_R on $\widehat{v}_i$.

Lemma 3.1. *For any prime p and $m \geq 1$, the right unit*

$$\eta_R \colon BP_* \to \Gamma(m+1)/(p)$$

on the Hazewinkel generators are

$$\begin{cases} \eta_R\left(\widehat{v}_2\right) = \widehat{v}_2 + v_1 \widehat{t}_1^p - v_1^{p\omega} \widehat{t}_1, \\ \eta_R\left(\widehat{v}_3\right) = \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1 + v_1 \widehat{t}_2^p - v_1^{p^2\omega} \widehat{t}_2 \\ \qquad\qquad + v_1 w_1\left(\widehat{v}_2, v_1 \widehat{t}_1^p, -v_1^{p\omega} \widehat{t}_1\right) \\ \qquad\qquad (\text{add } v_1^{4\omega+1} \widehat{t}_1^2 \text{ for } p = 2) \\ \qquad\quad \equiv \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1 + v_1 \widehat{t}_2^p - v_1^2 \widehat{v}_2^{p-1} \widehat{t}_1^p \quad \bmod \left(v_1^3\right), \\ \eta_R\left(\widehat{v}_4\right) \equiv \widehat{v}_4 + v_3 \widehat{t}_1^{p^3} - v_3^{p\omega} \widehat{t}_1 + v_2 \widehat{t}_2^{p^2} - v_2^{p^2\omega} \widehat{t}_2 \quad \bmod \left(v_1\right). \end{cases}$$

where $w_1(-)$ is the first Witt polynomial satisfying

$$w_1\left(y_1, \ldots, y_t, \ldots\right) = \frac{\left(\sum_t y_t^p\right) - \left(\sum_t y_t\right)^p}{p}.$$

Now let

$$\begin{cases} \widehat{w}_3 = v_2^{-1} \widehat{v}_3, \\ \widehat{w}_4 = v_2^{-1}\left(\widehat{v}_4 - v_3 \widehat{w}_3^p\right). \end{cases} \tag{3.2}$$

Using Lemma 3.1, it is easily shown that

Lemma 3.3. *The differentials*

$$d = \eta_R - \eta_L \colon v_2^{-1}BP_*/(p) \to v_2^{-1}BP_*/(p) \otimes_{BP_*} \Gamma(m+1)$$

on the above $\widehat{w}_k$*'s are*

$$d(\widehat{w}_3) \equiv \widehat{t}_1^{p^2} - v_2^{p\omega-1}\widehat{t}_1 + v_1v_2^{-1}\widehat{t}_2^{p} - v_1^2v_2^{-1}\widehat{v}_2^{p-1}\widehat{t}_1^{p} \qquad \mathrm{mod}\ \left(v_1^3\right),$$

and $$d(\widehat{w}_4) \equiv \widehat{t}_2^{p^2} - v_2^{-1}v_3^{p\omega}\widehat{t}_1 + v_2^{p^2\omega-p-1}v_3\widehat{t}_1^{p} - v_2^{p^2\omega-1}\widehat{t}_2 \qquad \mathrm{mod}\ (v_1).$$

Then we can change the $\widehat{K}(n)_*$-module basis of Theorems 2.2 and 2.3 using Lemma 3.3. In particular, we have

Corollary 3.4.

$$\mathrm{Ext}^1_{\Gamma(m+1)}\left(M_2^0\right) \cong \begin{cases} \widehat{K}(2)_*\left\{\widehat{h}_{1,1}, \widehat{h}_{1,2}, \widehat{h}_{2,2}, \widehat{h}_{2,3}\right\} & \text{for } p > 2, \text{ or } p = 2 \text{ and } m > 1, \\ \widehat{K}(2)_*\left\{\widehat{h}_{1,1}, \widehat{h}_{1,2}, \widehat{h}_{2,2}, \widehat{h}_{2,3}, \rho\right\} & \text{for } p = 2 \text{ and } m = 1. \end{cases}$$

When we compute the connecting homomorphism δ of Lemma 2.5, this base-changing method actually works well to determine the structure of $\mathrm{Ext}^0_{\Gamma(m+1)}(M_n^1)$ for a general n. In fact, Kamiya-Shimomura [2] and Shimomura [9] recently determined the structure of $\mathrm{Ext}^0_{\Gamma(m+1)}(M_n^1)$ under some conditions on m and n in a similar way.

4. The elements $\widehat{x}_k$

In this section, we will define elements $\widehat{x}_k \in v_2^{-1}BP_*$ $(k \geq 0)$ to be used in Lemma 2.5 except for $p = 2$ and $m = 1$. The case $p = 2$ and $m = 1$ will be treated in the next section.

Define elements $\widehat{x}_k \in v_2^{-1}BP_*$ $(k \geq 0)$ inductively on k by

$$(4.1) \qquad \begin{cases} \widehat{x}_0 = \widehat{v}_2, \\ \widehat{x}_1 = \widehat{x}_0^{p}, \\ \widehat{x}_2 = \widehat{x}_1^{p} - v_1^{p^2-1}v_2^{\beta+1}\widehat{x}_0 - v_1^{p^2}\widehat{w}_3^{p}, \\ \widehat{x}_3 = \widehat{x}_2^{p}, \\ \widehat{x}_4 = \begin{cases} \widehat{x}_3^{p} + \widehat{y}_1 + \widehat{y}_2 & (m > 1) \\ \widehat{x}_3^{p} + \widehat{y}_1 + \dfrac{1}{2}\widehat{y}_3 & (m = 1 \text{ and } p > 2) \end{cases}, \\ \widehat{x}_k = \widehat{x}_{k-1}^{p} - v_1^{p^{k-1}\alpha}v_2^{p^{k-2}\beta}\widehat{x}_{k-3}^{p-1}(\widehat{x}_{k-3} - \widehat{x}_{k-4}^{p}) & \text{for } k \geq 5, \end{cases}$$

where $\alpha = p+1$ and $\beta = p^2\omega - p - 1$, and $\widehat{y}_i$ $(i = 1,\ 2,\ 3)$ are given by

$$
(4.2)\qquad \begin{cases}
\widehat{y}_1 = -v_1^{p^4+p^3-p^2-p} v_2^{p^2\beta+p}\widehat{x}_2 + v_1^{p^4+p^3-p} v_2^{-p^3-p^2} v_3^{p^3\omega}\widehat{x}_1 \\
\qquad -v_1^{p^4+p^3-1} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2}\widehat{x}_0 + v_1^{p^4+p^3} v_2^{-p^3}\widehat{w}_4^{p^2} \\
\qquad -v_1^{p^4+p^3} v_2^{(\beta-p)p^2} v_3^{p^2}\widehat{w}_3^{p}, \\
\widehat{y}_2 = -v_1^{p^4+p^3-p^2} v_2^{(\beta-p)p^2} v_3^{p^2}\widehat{x}_2, \\
\widehat{y}_3 = \widehat{y}_2 + v_1^{p^4+p^3-1} v_2^{(p^2+1)\beta-p^3+1}\widehat{x}_2\widehat{x}_0 + v_1^{p^4+p^3} v_2^{(\beta-p)p^2}\widehat{x}_2\widehat{w}_3^{p}.
\end{cases}
$$

Define integers $\widehat{a}(k)$ by

$$
(4.3)\qquad \widehat{a}(k) = \begin{cases}
p^k & \text{for } 0 \le k \le 1, \\
p^{k-1}\alpha & \text{for } 2 \le k \le 3, \\
p^{k-1}\alpha + \widehat{a}(k-3) & \text{for } k \ge 4.
\end{cases}
$$

Notice that the integers $\widehat{a}(k)$ are equivalently defined inductively on k by

$$
(4.4)\qquad \widehat{a}(k) = \begin{cases}
p\widehat{a}(k-1) & \text{for } 2 < k \equiv 0 \mod (3), \\
p\widehat{a}(k-1) + p & \text{for } 2 \le k \not\equiv 0 \mod (3).
\end{cases}
$$

Lemma 4.5. *Unless $p = 2$ and $m = 1$, the differentials*

$$
d = \eta_R - \eta_L \colon v_2^{-1}BP_*/(p) \to v_2^{-1}BP_*/(p) \otimes_{BP_*} \Gamma(m+1)
$$

on the above $\widehat{x}_k$'s are

$$
\begin{aligned}
d(\widehat{x}_0) &\equiv v_1\widehat{t}_1^{p} && \mod \left(v_1^2\right), \\
d(\widehat{x}_1) &\equiv v_1^{\widehat{a}(1)}\widehat{t}_1^{p^2} && \mod \left(v_1^{1+\widehat{a}(1)}\right), \\
d(\widehat{x}_2) &\equiv -v_1^{\widehat{a}(2)} v_2^{-p}\widehat{t}_2^{p^2} && \mod \left(v_1^{1+\widehat{a}(2)}\right), \\
d(\widehat{x}_3) &\equiv -v_1^{\widehat{a}(3)} v_2^{-p^2}\widehat{t}_2^{p^3} && \mod \left(v_1^{1+\widehat{a}(3)}\right), \\
d(\widehat{x}_k) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta}\widehat{v}_2^{(p-1)p^{k-3}} d(\widehat{x}_{k-3}) && \mod \left(v_1^{1+\widehat{a}(k)}\right) \qquad \textit{for } k \ge 4.
\end{aligned}
$$

Proof. By Lemma 3.1 we have

$$
(4.6)\qquad \begin{aligned}
d(\widehat{x}_0) &\equiv v_1\widehat{t}_1^{p} && \mod \left(v_1^{p\omega}\right), \\
d(\widehat{x}_1) &\equiv v_1^{p}\widehat{t}_1^{p^2} && \mod \left(v_1^{p^2\omega}\right).
\end{aligned}
$$

Moreover, we find that

$$
\begin{aligned}
d(\widehat{x}_1^{p}) &\equiv v_1^{p^2}\widehat{t}_1^{p^3} \qquad \mod \left(v_1^{p^3\omega}\right), \\
d(-v_1^{p^2}\widehat{w}_3^{p}) &\equiv -v_1^{p^2}(\widehat{t}_1^{p^3} - v_2^{\beta+1}\widehat{t}_1^{p} - v_1^{2p} v_2^{-p}\widehat{v}_2^{(p-1)p}\widehat{t}_1^{p^2} + v_1^{p} v_2^{-p}\widehat{t}_2^{p^2})
\end{aligned}
$$

$$\mod \left(v_1^{p^2+3p}\right),$$

and $$d(-v_1^{p^2-1}v_2^{\beta+1}\widehat{x}_0) \equiv -v_1^{p^2}v_2^{\beta+1}\widehat{t}_1^p \qquad \mod \left(v_1^{p\omega+p^2-1}\right).$$

Summing the above three congruences we obtain

$$\begin{aligned} d(\widehat{x}_2) &\equiv -v_1^{p^2+p}v_2^{-p}(\widehat{t}_2^{p^2} - v_1^p\widehat{v}_2^{(p-1)p}\widehat{t}_1^{p^2}) && \mod \left(v_1^{p^2+2p+2}\right) \\ &\equiv -v_1^{\widehat{a}(2)}v_2^{-p}\widehat{t}_2^{p^2} && \mod \left(v_1^{p^2+2p}\right), \end{aligned}$$

and $$d(\widehat{x}_3) \equiv -v_1^{\widehat{a}(3)}v_2^{-p^2}\widehat{t}_2^{p^3} \qquad \mod \left(v_1^{p^3+2p^2}\right).$$

(4.4) suggests that we should calculate $d(\widehat{x}_k)$ modulo $(v_1^{2+\widehat{a}(k)})$ rather than modulo $(v_1^{1+\widehat{a}(k)})$ when we apply induction on $k \geq 4$. For $k = 4$, we find that modulo $(v_1^{2+\widehat{a}(4)})$

(4.7) $$\begin{cases} d(v_1^{\widehat{a}(4)-p}v_2^{-p^3}\widehat{w}_4^{p^2}) \\ \qquad \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3}(\widehat{t}_2^{p^4} - v_2^{-p^2}v_3^{p^3\omega}\widehat{t}_1^{p^2} + v_2^{p^2\beta}v_3^{p^2}\widehat{t}_1^{p^3} - v_2^{(\beta+p)p^2}\widehat{t}_2^{p^2}) \\ d(v_1^{\widehat{a}(4)-2p}v_2^{-p^3-p^2}v_3^{p^3\omega}\widehat{x}_1) \\ \qquad \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3-p^2}v_3^{p^3\omega}\widehat{t}_1^{p^2} \\ d(-v_1^{\widehat{a}(4)-\widehat{a}(2)-p}v_2^{p^2\beta+p}\widehat{x}_2) \\ \qquad \equiv v_1^{\widehat{a}(4)-p}v_2^{p^2\beta}(\widehat{t}_2^{p^2} - v_1^p\widehat{v}_2^{p^2-p}\widehat{t}_1^{p^2}) \\ d(-v_1^{\widehat{a}(4)-p}v_2^{(\beta-p)p^2}v_3^{p^2}\widehat{w}_3^p) \\ \qquad \equiv -v_1^{\widehat{a}(4)-p}v_2^{(\beta-p)p^2}v_3^{p^2}(\widehat{t}_1^{p^3} - v_2^{\beta+1}\widehat{t}_1^p + v_1^pv_2^{-p}\widehat{t}_2^{p^2}) \\ d(-v_1^{\widehat{a}(4)-p-1}v_2^{(p^2+1)\beta-p^3+1}v_3^{p^2}\widehat{x}_0) \\ \qquad \equiv -v_1^{\widehat{a}(4)-p}v_2^{(p^2+1)\beta-p^3+1}v_3^{p^2}\widehat{t}_1^p. \end{cases}$$

Summing these congruences we obtain

$$d(\widehat{y}_1) \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3}\widehat{t}_2^{p^4} - v_1^{\widehat{a}(4)}v_2^{p^2\beta}\left(v_2^{-p^3-p}v_3^{p^2}\widehat{t}_2^{p^2} + \widehat{v}_2^{p^2-p}\widehat{t}_1^{p^2}\right) \\ \mod \left(v_1^{2+\widehat{a}(4)}\right).$$

On the other hand, we find that modulo $(v_1^{2+\widehat{a}(4)})$

$$d(\widehat{y}_2) \equiv \begin{cases} v_1^{\widehat{a}(4)}v_2^{p^2\beta-p^3-p}v_3^{p^2}\widehat{t}_2^{p^2} & (m \geq 2), \\ -v_1^{p^3\alpha}v_2^{(\beta-p)p^2}v_3^{p^2}(\widehat{t}_1^{p^3} - v_1^pv_2^{-p}\widehat{t}_2^{p^2}) & (m = 1). \end{cases}$$

In the $m \geq 2$ case, we see that

$$\begin{aligned} d(\widehat{x}_4) &\equiv -v_1^{\widehat{a}(4)}v_2^{p^2\beta}\widehat{v}_2^{(p-1)p}\widehat{t}_1^{p^2} \\ &\equiv -v_1^{p^3\alpha}v_2^{p^2\beta}\widehat{v}_2^{(p-1)p}d(\widehat{x}_1) \qquad \mod \left(v_1^{2+\widehat{a}(4)}\right). \end{aligned}$$

In the $m = 1$ case, we must modify the element $\widehat{y}_2$ into $\widehat{y}_3$ as defined in (4.2). We find that

$$\begin{cases} d\left(v_1^{\widehat{a}(4)-p} v_2^{(\beta-p)p^2} v_3^{p^2} \widehat{w}_3^p\right) \equiv v_1^{\widehat{a}(4)-p} v_2^{(\beta-p)p^2} v_3^{p^2} \left(\widehat{t}_1^{p^3} - v_2^{\beta+1} \widehat{t}_1^p + v_1^p v_2^{-p} \widehat{t}_2^{p^2}\right) \\ \qquad\qquad \mod \left(v_1^{\widehat{a}(4)+p}\right), \\ d\left(v_1^{\widehat{a}(4)-p-1} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2} \widehat{x}_0\right) \equiv v_1^{\widehat{a}(4)-p} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2} \widehat{t}_1^p \\ \qquad\qquad \mod \left(v_1^{\widehat{a}(4)+p^2-p-1}\right). \end{cases}$$

Summing the above congruences we obtain

$$d(\widehat{y}_3) \equiv 2 v_1^{\widehat{a}(4)} v_2^{p^2\beta-p^3-p} v_3^{p^2} \widehat{t}_2^{p^2} \qquad \mod \left(v_1^{2+\widehat{a}(4)}\right).$$

Consequently, we obtain the desired congruence of $d(\widehat{x}_4)$ in $m = 1$ case, too.

For $k \geq 5$, assume that

$$d(\widehat{x}_{k-1}) \equiv -v_1^{p^{k-2}\alpha} v_2^{p^{k-3}\beta} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) \qquad \mod \left(v_1^{2+\widehat{a}(k-1)}\right),$$

and denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by $\widehat{z}_k$. By definition (4.1), we note that $\widehat{z}_k = 0$ for $k \equiv 0 \mod 3$. In case that $k \not\equiv 0$ modulo 3, we have

$$\widehat{z}_k = -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3} \qquad \text{for } k \geq 5.$$

Notice that $\widehat{z}_{k-3}$ is divided by $v_1^{p^2-1}$ for $k = 5$, by $v_1^{p(p+1)(p^2-1)}$ for $k = 7$, and by $v_1^{p^{k-4}\alpha}$ for $k \geq 8$. On the other hand, by inductive hypothesis we see that $d(\widehat{x}_{k-3}^{p-1})$ is divisible by $v_1^{p^2+p}$ for $k = 5$ and by $v_1^{p^{k-4}\alpha}$ for $k \geq 7$. So we have

$$\begin{aligned} d(\widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3}) &= d(\widehat{x}_{k-3}^{p-1}) \eta_R(\widehat{z}_{k-3}) + \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \\ &\equiv \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \qquad \mod \left(v_1^{2+\widehat{a}(k-3)}\right). \end{aligned}$$

Therefore the differential on $\widehat{z}_k$ is

$$\begin{aligned} d(\widehat{z}_k) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} d(\widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3}) \\ &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \qquad \mod \left(v_1^{2+\widehat{a}(k)}\right). \end{aligned}$$

On the other hand, by inductive hypothesis we have

$$\begin{aligned} d(\widehat{x}_{k-1}^p) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-4}^{(p-1)p} d(\widehat{x}_{k-4}^p) \\ &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{x}_{k-4}^p) \qquad \mod \left(v_1^{2+\widehat{a}(k)}\right). \end{aligned}$$

Summing the above two congruences we obtain

$$d(\widehat{x}_k) \equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{x}_{k-3}) \qquad \mod \left(v_1^{2+\widehat{a}(k)}\right)$$

as desired. □

5. The case $p = 2$ and $m = 1$

In this section we recover some results of Shimomura [8] using the basis obtained in Corollary 3.4.

Define the elements $\widehat{x}_k \in v_2^{-1}BP_*$ in the same fashion as those in (4.1) for $0 \leq k \leq 3$, and

$$\begin{cases} \widehat{x}_4 = \widehat{x}_3^2 + \widehat{y}_1 + \widehat{y}_4, \\ \widehat{x}_k = \widehat{x}_{k-1}^2 + v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{x}_{k-2}(\widehat{x}_{k-2} + \widehat{x}_{k-3}^2) \qquad \text{for } k \geq 5, \end{cases} \tag{5.1}$$

where $\widehat{y}_4$ is

$$\widehat{y}_4 = v_1^{14}v_2^{14}\widehat{x}_3 + v_1^{23}v_2^{25}\widehat{x}_1 + v_1^{25}v_2^{8}v_3^{8}\widehat{x}_0 + v_1^{25}v_2^{25}\widehat{w}_3 + v_1^{26}v_2^{10}\widehat{w}_4^2.$$

Note that the construction of $\widehat{x}_k$ ($k \geq 4$) in this case is 2-periodic, although it is 3-periodic for the other cases. We are surprised at this difference.

Define integers $\widehat{a}(k)$ by

$$\widehat{a}(k) = \begin{cases} 2^k & \text{for } 0 \leq k \leq 1, \\ 3\cdot 2^{k-1} & \text{for } 2 \leq k \leq 3, \\ 5\cdot 2^{k-2} + \widehat{a}(k-2) & \text{for } k \geq 4. \end{cases} \tag{5.2}$$

This gives $\widehat{a}(0) = 1$, $\widehat{a}(1) = 2$, $\widehat{a}(2) = 6$, $\widehat{a}(3) = 12$, $\widehat{a}(4) = 26$, and so on. Notice that the integers $\widehat{a}(k)$ are equivalently defined inductively on k by

$$\widehat{a}(k) = \begin{cases} 2\widehat{a}(k-1) & \text{for odd } k, \\ 2\widehat{a}(k-1) + 2 & \text{for even } k. \end{cases} \tag{5.3}$$

Then we have

Lemma 5.4. *For $p = 2$ and $m = 1$, the differentials*

$$d = \eta_R - \eta_L : v_2^{-1}BP_*/(2) \to v_2^{-1}BP_*/(2) \otimes_{BP_*} \Gamma(m+1)$$

on the above $\widehat{x}_k$'s are

$$\begin{aligned} d(\widehat{x}_0) &\equiv v_1\widehat{t}_1^2 && \mod \left(v_1^2\right), \\ d(\widehat{x}_1) &\equiv v_1^{\widehat{a}(2)}\widehat{t}_1^4 && \mod \left(v_1^{1+\widehat{a}(1)}\right), \\ d(\widehat{x}_2) &\equiv v_1^{\widehat{a}(2)}v_2^{-2}\widehat{t}_2^4 && \mod \left(v_1^{1+\widehat{a}(2)}\right), \end{aligned}$$

$$d(\widehat{x}_3) \equiv v_1^{\widehat{a}(3)} v_2^{-4} \widehat{t}_2^8 \qquad \bmod \left(v_1^{1+\widehat{a}(3)}\right),$$

$$d(\widehat{x}_k) \equiv v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{v}_2^{2^{k-2}} d(\widehat{x}_{k-2}) \qquad \bmod \left(v_1^{1+\widehat{a}(k)}\right) \qquad \textit{for } k \geq 4.$$

Proof. The $k = 0$ and $k = 1$ cases follow directly from Lemma 3.1 (cf. (4.6)). For $k = 2$ case, we find that

$$\begin{cases} d\left(\widehat{x}_1^2\right) \equiv v_1^4 \widehat{t}_1^8 & \bmod \left(v_1^{16}\right), \\ d\left(v_1^4 \widehat{w}_3^2\right) \equiv v_1^4\left(\widehat{t}_1^8 + v_2^6 \widehat{t}_1^2 + v_1^2 v_2^{-2} \widehat{t}_2^4 + v_1^4 v_2^{-2} \widehat{v}_2^2 \widehat{t}_1^4\right) & \bmod \left(v_1^{10}\right), \\ d\left(v_1^3 v_2^6 \widehat{x}_0\right) \equiv v_1^4 v_2^6 \widehat{t}_1^2 + v_1^7 v_2^6 \widehat{t}_1 & \bmod \left(v_1^9\right). \end{cases}$$

Then we have

$$\begin{aligned} d\left(\widehat{x}_2\right) &\equiv v_1^6 v_2^{-2} \widehat{t}_2^4 + v_1^7 v_2^6 \widehat{t}_1 + v_1^8 v_2^{-2} v_3^2 \widehat{t}_1^4 && \bmod \left(v_1^9\right) \\ &\equiv v_1^6 v_2^{-2} \widehat{t}_2^4 && \bmod \left(v_1^7\right), \\ d\left(\widehat{x}_3\right) &\equiv v_1^{12} v_2^{-4} \widehat{t}_2^8 && \bmod \left(v_1^{14}\right). \end{aligned}$$

For $k = 4$ case, we obtain the same consequences as in (4.7), but with the third one replaced by

$$d\left(v_1^{18} v_2^{22} \widehat{x}_2\right) \equiv v_1^{24} v_2^{20} \widehat{t}_2^4 + v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \qquad \bmod \left(v_1^{27}\right),$$

and so

$$d\left(\widehat{y}_1\right) \equiv v_1^{24} v_2^{-8} \widehat{t}_2^{16} + v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{10} v_3^4 \widehat{t}_2^4 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \qquad \bmod \left(v_1^{27}\right).$$

On the other hand, we find that

$$\begin{cases} d\left(v_1^{25} v_2^{25} \widehat{w}_3\right) \equiv v_1^{25}\left(v_2^{25} \widehat{t}_1^4 + v_2^{28} \widehat{t}_1\right) + v_1^{26} v_2^{24} \widehat{t}_2^2, \\ d\left(v_1^{23} v_2^{25} \widehat{x}_1\right) \equiv v_1^{25} v_2^{25} \widehat{t}_1^4, \\ d\left(v_1^{26} v_2^{10} \widehat{w}_4^2\right) \equiv v_1^{26}\left(v_2^8 v_3^8 \widehat{t}_1^2 + v_2^{10} \widehat{t}_2^8 + v_2^{20} v_3^2 \widehat{t}_1^4 + v_2^{24} \widehat{t}_2^2\right), \\ d\left(v_1^{14} v_2^{14} \widehat{x}_3\right) \equiv v_1^{26} v_2^{10} \widehat{t}_2^8, \\ d\left(v_1^{25} v_2^8 v_3^8 \widehat{x}_0\right) \equiv v_1^{26} v_2^8 v_3^8 \widehat{t}_1^2 \end{cases}$$

modulo (v_1^{27}), so we have

$$d\left(\widehat{y}_4\right) \equiv v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \qquad \bmod \left(v_1^{27}\right).$$

Using the above congruences, we have

$$\begin{aligned} d(\widehat{x}_4) &\equiv v_1^{26} v_2^{10} v_3^4 \widehat{t}_2^4 \\ &\equiv v_1^{20} v_2^{12} v_3^4 d(\widehat{x}_2) && \bmod \left(v_1^{1+\widehat{a}(4)}\right). \end{aligned}$$

(5.3) suggests that we should calculate $d(\widehat{x}_k)$ modulo $(v_1^{2+\widehat{a}(k)})$ rather than modulo $(v_1^{1+\widehat{a}(k)})$ for $k \geq 5$ when we apply induction on k. Denote $\widehat{x}_k + \widehat{x}_{k-1}^2$ by $\widehat{z}_k$. By definition (5.1) we note that $\widehat{z}_k = 0$ for odd k. In case that k is even, we have

$$\widehat{z}_k = v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{x}_{k-2}\widehat{z}_{k-2} \qquad \text{for } k \geq 5.$$

Notice that $\widehat{z}_{k-2}$ is divisible by v_1^{14} for $k = 6$ and by $v_1^{5\cdot 2^{k-4}}$ for $k \geq 8$. On the other hand, by inductive hypothesis $d(\widehat{x}_{k-2})$ is divisible by $v_1^{\widehat{a}(k-2)}$. So we have

$$\begin{aligned} d(\widehat{x}_{k-2}\widehat{z}_{k-2}) &= d(\widehat{x}_{k-2})\eta_R(\widehat{z}_{k-2}) + \widehat{x}_{k-2}d(\widehat{z}_{k-2}) \\ &\equiv \widehat{x}_{k-2}d(\widehat{z}_{k-2}) \qquad \bmod \left(v_1^{2+\widehat{a}(k-2)}\right). \end{aligned}$$

Therefore the differential on $\widehat{z}_k$ is

$$\begin{aligned} d(\widehat{z}_k) &\equiv v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} d(\widehat{x}_{k-2}\widehat{z}_{k-2}) \\ &\equiv v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{x}_{k-2}d(\widehat{z}_{k-2}) \qquad \bmod \left(v_1^{2+\widehat{a}(k)}\right). \end{aligned}$$

On the other hand, by inductive hypothesis we have

$$d(\widehat{x}_{k-1}^2) \equiv v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{v}_2^{2^{k-2}} d(\widehat{x}_{k-3}^2) \qquad \bmod \left(v_1^{2+\widehat{a}(k)}\right)$$

because $2(1 + \widehat{a}(4)) = 2 + \widehat{a}(5)$ and $2(2 + \widehat{a}(k-1)) \geq 2 + \widehat{a}(k)$ for $k \geq 6$. Summing the above two congruences, we obtain

$$d(\widehat{x}_k) \equiv v_1^{5\cdot 2^{k-2}} v_2^{3\cdot 2^{k-2}} \widehat{x}_{k-2} d(\widehat{x}_{k-2}) \qquad \bmod \left(v_1^{2+\widehat{a}(k)}\right).$$

as desired. □

6. The structure of $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M_1^1\right)$

Theorem 6.1. *As a $v_2^{-1}\widehat{k}(1)_*$-module, $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M_1^1\right)$ for $m \geq 1$ is the direct sum of*
(a) *the cyclic submodules generated by $\widehat{x}_k^s/v_1^{\widehat{a}(k)}$ for $k \geq 0$, $s > 0$ and $p \nmid s$; and*
(b) $v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$, *generated by $1/v_1^j$ for $j \geq 1$,*
where $\widehat{x}_k$'s are the elements defined in (4.1) *and* (5.1)*.*

Proof. First we prove the theorem except for the $p = 2$ and $m = 1$ case. By Lemma 2.5 it suffices to show that the set

$$D = \left\{ \delta\left(\widehat{x}_k^s / v_1^{\widehat{a}(k)}\right) :\ k \geq 0,\ s > 0 \text{ and } p \nmid s \right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_2^0\right)$$

is linearly independent over

$$R = \mathbf{Z}/(p)[v_2, v_2^{-1}, v_3, \dots, v_m, v_{m+1}].$$

It follows from Corollary 3.4 that $\mathrm{Ext}^1_{\Gamma(m+1)}(M_2^0)$ is the free $\widehat{K}(2)_*$-module on the four classes represented by

$$\left\{\widehat{t}_1^p,\ \widehat{t}_1^{p^2},\ \widehat{t}_2^{p^2},\ \widehat{t}_2^{p^3}\right\},$$

so its basis over R is

$$\left\{\widehat{v}_2^t\widehat{t}_1^p,\ \widehat{v}_2^t\widehat{t}_1^{p^2},\ \widehat{v}_2^t\widehat{t}_2^{p^2},\ \widehat{v}_2^t\widehat{t}_2^{p^3} : t \geq 0\right\}.$$

Now define integers $\widehat{b}(k)$ and $\widehat{c}(k)$ for $k \geq 0$ by

$$\widehat{b}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 1, \\ -p^{k-1} & \text{for } 2 \leq k \leq 3, \\ p^{k-2}\beta + \widehat{b}(k-3) & \text{for } k \geq 4, \end{cases}$$

where $\beta = p^2\omega - p - 1$ as before, and

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ (p-1)p^{k-3} + \widehat{c}(k-3) & \text{for } k \geq 4. \end{cases}$$

Then Lemma 4.5 implies that

$$d(\widehat{x}_k) \equiv \pm v_1^{\widehat{a}(k)} v_2^{\widehat{b}(k)} \widehat{v}_2^{\widehat{c}(k)} \begin{cases} \widehat{t}_1^p & \text{for } k = 0, \\ \widehat{t}_1^{p^2} & \text{for } k > 0 \text{ and } k \equiv 1 \bmod 3, \\ \widehat{t}_2^{p^2} & \text{for } k > 0 \text{ and } k \equiv 2 \bmod 3, \\ \widehat{t}_2^{p^3} & \text{for } k > 0 \text{ and } k \equiv 3 \bmod 3 \end{cases}$$

modulo $\left(v_1^{1+\widehat{a}(k)}\right)$, where $\widehat{a}(k)$ is defined in (4.3). Since

$$d(\widehat{x}_k^s) \equiv s\widehat{x}_k^{s-1} d(\widehat{x}_k) \equiv s\widehat{v}_2^{(s-1)p^k} d(\widehat{x}_k) \bmod \left(v_1^{1+\widehat{a}(k)}\right),$$

it follows that

$$\delta\left(\frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}}\right) = \pm s v_2^{\widehat{b}(k)} \widehat{v}_2^{(s-1)p^k + \widehat{c}(k)} \begin{cases} \widehat{t}_1^p & \text{for } k = 0, \\ \widehat{t}_1^{p^2} & \text{for } k > 0 \quad \text{and } k \equiv 1 \bmod 3, \\ \widehat{t}_2^{p^2} & \text{for } k > 0 \quad \text{and } k \equiv 2 \bmod 3, \\ \widehat{t}_2^{p^3} & \text{for } k > 0 \quad \text{and } k \equiv 3 \bmod 3. \end{cases} \tag{6.2}$$

In order to show that these elements $\delta\left(\widehat{x}_k^s/v_1^{\widehat{a}(k)}\right)$ (with $k \geq 0$ and $s > 0$ not divisible by p) are linearly independent over R, it suffices to observe the exponents of $\widehat{v}_2$ in the right hand side of (6.2).

So we consider the sets $D_0 = \{\widehat{v}_2^{s-1} : s > 0 \text{ and } p \nmid s\}$ for $k = 0$, and $D_{k_0} = \{\widehat{v}_2^{(s-1)p^k+\widehat{c}(k)} : k = k_0 + 3k_1,\ s > 0 \text{ and } p \nmid s\}$ for a fixed k_0 $(1 \leq k_0 \leq 3)$. Since the integer $\widehat{c}(k)$ is

$$\widehat{c}(k) = (p-1)p^{k_0}(1 + p^3 + \cdots + p^{3k_1-3})$$

for $k = k_0 + 3k_1 \geq 4$ with $1 \leq k_0 \leq 3$, we see

$$(s-1)p^k + \widehat{c}(k) \equiv sp^k - \frac{p^{k_0}}{1+p+p^2} \qquad \text{mod } (p^{k+1}).$$

If $(s-1)p^k + \widehat{c}(k) = (t-1)p^l + \widehat{c}(l)$ with $k \equiv l \equiv k_0$ modulo 3, then it follows that $k = l$ and hence $s = t$. Thus all the entries in the sets D_0 and D_{k_0} $(1 \leq k_0 \leq 3)$ are disparate, respectively.

In the $p = 2$ and $m = 1$ case our argument is the same subject to the following changes. The integers $\widehat{b}(k)$ and $\widehat{c}(k)$ are defined by

$$\widehat{b}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 1, \\ -2^{k-1} & \text{for } 2 \leq k \leq 3, \\ 3 \cdot 2^{k-2} + \widehat{b}(k-2) & \text{for } k \geq 4, \end{cases}$$

and

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ 2^{k-2} + \widehat{c}(k-2) & \text{for } k \geq 4, \end{cases}$$

which is

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ \dfrac{4}{3}(2^{k-2}-1) & \text{for even } k \geq 4, \\ \dfrac{8}{3}(2^{k-3}-1) & \text{for odd } k \geq 5. \end{cases}$$

Then (6.2) gets replaced by

$$\delta\left(\frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}}\right) = v_2^{\widehat{b}(k)}\widehat{v}_2^{(s-1)p^k+\widehat{c}(k)} \begin{cases} \widehat{t}_1^2 & \text{for } k = 0, \\ \widehat{t}_1^4 & \text{for } k = 1, \\ \widehat{t}_2^4 & \text{for } k > 0 \text{ and } k \equiv 0 \text{ mod } 2, \\ \widehat{t}_2^8 & \text{for } k > 1 \text{ and } k \equiv 1 \text{ mod } 2, \end{cases}$$

and we can argue for linear independence as before. □

7. The group $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$

In this section we will use the structure of $\mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1)$ given in Theorem 6.1 to determine the group $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$. As in the case $m = 0$, this group is the direct sum of subquotients of $\mathrm{Ext}^1_{\Gamma(m+1)}(M_1^0)$ and $\mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1)$.

In Lemma 7.2 we will show that the former subquotient has the same form as in the case $m = 0$, i.e., it is $\widehat{k}(1)_*\{\widehat{h}_{1,0}\}$. We will also see that unlike in the classical case, the element $v_1^{-1}\widehat{h}_{1,0}$ supports a nontrivial d_2 in the chromatic spectral sequence.

The summand $v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$ of $\mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1)$ is the image of

$$d_1\colon E_1^{0,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^0) \longrightarrow E_1^{1,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1),$$

so it maps trivially to $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$. The kernel of the map

$$d_1\colon E_1^{1,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1) \longrightarrow E_1^{2,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^2),$$

consists of all elements, each of which does not have any monomial with negative v_2-exponent. We will see in Corollary 7.7 that these are the elements

$$\frac{\widehat{x}_k^s}{v_1^j} \in \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1) \qquad \text{with } k \geq 0,\ s > 0,\ p \nmid s, \text{ and } 0 < j \leq p^k.$$

Combining these results we get

Theorem 7.1. *For any prime p and $m \geq 1$, the group*

$$\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$$

is isomorphic to

$$\widehat{k}(1)_*\left\{\widehat{\beta}_{sp^k/j} : s \geq 0,\ p \nmid s,\ k \geq 0 \text{ and } 0 < j \leq p^k\right\} \bigoplus \widehat{k}(1)_*\{\widehat{h}_{1,0}\},$$

where $\widehat{\beta}_{sp^k/j}$ is the image of $\widehat{x}_k^s/v_1^j$ under the connecting homomorphism

$$\delta\colon \mathrm{Ext}^0_{\Gamma(m+1)}(N_1^1) \longrightarrow \mathrm{Ext}^1_{\Gamma(m+1)}(N_1^0).$$

First we consider the subquotient of $\mathrm{Ext}^1_{\Gamma(m+1)}(M_1^0)$.

Lemma 7.2. *For any prime p and $m \geq 1$, the group $E_\infty^{0,1}$ in the chromatic spectral sequence is $\widehat{k}(1)_*\{\widehat{h}_{1,0}\}$. Moreover there is a nontrivial differential in the chromatic spectral sequence,*

$$d_2\left(v_1^{-1}\widehat{h}_{1,0}\right) = \frac{z}{v_1^{p+1}v_2^{p\omega-1}},$$

where $z = \widehat{v}_2^p - v_1^p v_2^{-1}\widehat{v}_3$.

Proof. We use the chromatic cobar complex

$$\left\{CC^n_{\Gamma(m+1)}(BP_*/(p)), d_c\right\}_{n\geq 0}$$

given by

$$\begin{aligned} CC^n_{\Gamma(m+1)}(BP_*/(p)) &= \bigoplus_{s+t=n} C^s(M_1^t), \\ d_c = d_e + (-1)^t d_i &: C^s(M_1^t) \to C^s(M_1^{t+1}) \oplus C^{s+1}(M_1^t), \end{aligned}$$

where $d_e\colon C^s(M_1^t) \to C^s(M_1^{t+1})$ is induced by the composite map $M_1^t \to N_1^{t+1} \to M_1^{t+1}$ and $d_i\colon C^s(M_1^t) \to C^{s+1}(M_1^t)$ is the differential in the cobar complex (see [6, Definition 5.1.10]).

By Theorem 2.2, we have

$$E_1^{0,1} = \mathrm{Ext}^1_{\Gamma(m+1)}(M_1^0) \cong \widehat{K}(1)_* \left\{\widehat{h}_{1,0}\right\}.$$

The element $\widehat{h}_{1,0}$ is represented by $\widehat{t}_1$ in the cobar complex and is clearly a permanent cycle in the chromatic spectral sequence. We need to show that $v_1^{-1}\widehat{h}_{1,0}$ does not survive to $E_\infty^{0,1}$. If it does, then the element $\widehat{h}_{1,0} \in \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$ is divisible by v_1 and therefore has trivial image under the composite

$$\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p)) \to \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/I_2) \to \mathrm{Ext}^1_{\Gamma(m+1)}(v_2^{-1}BP_*/I_2).$$

The target group was computed in [5], and the element in question is one of its generators.

For the chromatic differential d_2, we have

$$d(z) \equiv v_1^p v_2^{p\omega-1}\widehat{t}_1 \qquad \mathrm{mod}\ \left(v_1^{p+1}\right).$$

It follows that in the chromatic cobar complex $CC_{\Gamma(m+1)}(BP_*/(p))$ the differential

$$d_c\colon C^1(M_1^0) \oplus C^0(M_1^1) \to C^2(M_1^0) \oplus C^1(M_1^1) \oplus C^0(M_1^2)$$

satisfies

$$\begin{aligned} d_c\left(v_1^{-1}\widehat{t}_1\right) &= \frac{\widehat{t}_1}{v_1} \quad \in C^1(M_1^1), \\ d_c\left(\frac{v_2^{1-p\omega}z}{v_1^{p+1}}\right) &= -\frac{\widehat{t}_1}{v_1} + \frac{z}{v_1^{p+1}v_2^{p\omega-1}} \in C^1(M_1^1) \oplus C^0(M_1^2), \end{aligned}$$

so $$d_c\left(v_1^{-1}\widehat{t}_1 + \frac{v_2^{1-p\omega}z}{v_1^{p+1}}\right) = \frac{z}{v_1^{p+1}v_2^{p\omega-1}}.$$

In terms of the double complex associated with the chromatic resolution, we have the following picture:

$$\begin{array}{llll}
s = 1: & v_1^{-1}\widehat{t}_1 \xrightarrow{\ d_e\ } & \dfrac{\widehat{t}_1}{v_1} & \\
 & & \uparrow d_i & \\
s = 0: & & \dfrac{v_2^{1-p\omega}z}{v_1^{p+1}} \xrightarrow{\ d_e\ } & \dfrac{z}{v_1^{p+1}v_2^{p\omega-1}} \\
 & t = 0 & t = 1 & t = 2
\end{array}$$

This means that in the chromatic spectral sequence we have the indicated d_2. Its target must be nontrivial in E_2, i.e., it is not in the image under

$$d_1\colon E_1^{1,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1) \longrightarrow E_1^{2,0} = \mathrm{Ext}^0_{\Gamma(m+1)}(M_1^2).$$

because otherwise $v_1^{-1}\widehat{h}_{1,0}$ would survive to $E_\infty^{0,1}$, contradicting the nondivisibilty result above. □

Now we turn to the v_1-torsion in $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*/(p))$. Let $\widehat{d}(k)$ be the maximum exponent of v_1 satisfying

$$\widehat{x}_k \equiv \widehat{x}_{k-1}^p \qquad \mathrm{mod}\ \left(p, v_1^{\widehat{d}(k)}\right).$$

(if $\widehat{x}_k = \widehat{x}_{k-1}^p$, then we set $\widehat{d}(k) = \infty$.) Thus the integers $\widehat{d}(k)$ ($k \geq 5$) are given inductively by

(7.3) $$\widehat{d}(k) = p^{k-1}\alpha + \widehat{d}(k-3)$$

with $\widehat{d}(2) = p^2 - 1$, $\widehat{d}(3) = \infty$, $\widehat{d}(4) = p^4 + p^3 - p^2 - p$ unless $p = 2$ and $m = 1$, but

(7.4) $$\widehat{d}(k) = 5 \cdot 2^{k-2} + \widehat{d}(k-2)$$

with $\widehat{d}(3) = \infty$, $\widehat{d}(4) = 14$ in the case $p = 2$ and $m = 1$.

Lemma 7.5. *For any prime p and $m \geq 1$,*

$$\widehat{x}_k \equiv \widehat{x}_2^{p^{k-2}}\ \mathrm{mod}\ \left(p, v_1^{p^{k-4}\widehat{d}(4)}\right).$$

Furthermore, $\widehat{x}_k \equiv \widehat{x}_4^{2^{k-4}}$ *modulo* $(2, v_1^{2^{k-6}\widehat{d}(6)})$ *in the case* $p = 2$ *and* $m = 1$.

Proof. From (7.3) and (7.4) it follows that $\widehat{d}(k) > p^{k-4}\widehat{d}(4)$ for $k \geq 5$ unless $p = 2$ and $m = 1$, and that $\widehat{d}(k) > 2^{k-6}\widehat{d}(6)$ for $k \geq 7$ in the case $p = 2$ and $m = 1$. Therefore it is obvious that

$$\min\left\{\widehat{d}(k), p\widehat{d}(k-1), \dots, p^{k-4}\widehat{d}(4), p^{k-3}\widehat{d}(3)\right\} = p^{k-4}\widehat{d}(4) = p^k + p^{k-1} - p^{k-2} - p^{k-3}$$

unless $p = 2$ and $m = 1$, and

$$\min\left\{\widehat{d}(k), 2\widehat{d}(k-1), \dots, 2^{k-6}\widehat{d}(6), 2^{k-5}\widehat{d}(5)\right\} = 2^{k-6}\widehat{d}(6) = 94 \cdot 2^{k-6}$$

when $p = 2$ and $m = 1$. This completes the proof. □

Lemma 7.6. *Let* $\widehat{x}_k^s/v_1^j$ $(j \leq \widehat{a}(k))$ *be one of the generators of* $\mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1)$. *Then the image of this element by the map*

$$\mathrm{Ext}^0_{\Gamma(m+1)}(M_1^1) \to \mathrm{Ext}^0_{\Gamma(m+1)}(N_1^2)$$

is non-trivial if and only if $k \geq 2$ *and* $p^k < j \leq \widehat{a}(k)$.

Proof. We may assume that $k \geq 2$. From definition of $\widehat{x}_2$, it follows that

$$\widehat{x}_2^{p^{k-2}} \equiv \widehat{v}_2^{p^k} - v_1^{p^k - p^{k-2}} v_2^{\beta p^{k-2}} \widehat{v}_3^{p^{k-2}} + v_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_3^{p^{k-1}} \qquad \mathrm{mod}\ (p).$$

Then, using the fact that

$$\begin{aligned} 2(p^k - p^{k-2}) &\geq \widehat{a}(k) && \text{for } k = 2 \text{ or } 3 \\ 2(p^k - p^{k-2}) &> p^k\widehat{d}(4) && \text{for } k \geq 4 \end{aligned}$$

and Lemma 7.5 we have

$$\widehat{x}_k^s \equiv \widehat{v}_2^{sp^k} - s\widehat{v}_2^{(s-1)p^k}\left(v_1^{p^k - p^{k-2}} v_2^{\beta p^{k-2}} \widehat{v}_3^{p^{k-2}} - v_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_3^{p^{k-1}}\right)$$

modulo (p, v_1^j) for $k = 2$ and 3, and modulo $(p, v_1^{p^{k-4}\widehat{d}(4)})$ for $k \geq 4$.

In the right hand side the first and the second terms do not have a negative v_2-exponent, but the third term in $\widehat{x}_k^s/v_1^j$ is

$$\frac{s v_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_2^{(s-1)p^k} \widehat{v}_3^{p^{k-1}}}{v_1^j}.$$

which may be mapped non-trivially to N_1^2. Unless $p = 2$ and $m = 1$, we notice that $p^{k-4}\widehat{d}(4) > p^k$. Then we observe that $\widehat{x}_k^s/v_1^j$ is mapped non-trivially to N_1^2 if and only if $j > p^k$ except when $p = 2$, $m = 1$ and $k \geq 4$.

On the other hand, in the $p = 2$ and $m = 1$ case we find that $\widehat{x}_k \equiv \widehat{x}_4^{2^{k-4}}$ modulo $(v_1^{2^{k-6}\widehat{d}(6)})$ $(k \geq 6)$ and

$$\begin{aligned}\widehat{x}_4 &\equiv \widehat{x}_3^2 + v_1^{14}v_2^{14}\widehat{x}_3\\ &\equiv \widehat{v}_2^{16} + v_1^{12}v_2^{24}\widehat{v}_2^4 + v_1^{14}v_2^{14}\widehat{v}_2^8 + v_1^{16}v_2^{-8}\widehat{v}_3^8 \qquad \mathrm{mod}\ (2, v_1^{18}),\end{aligned}$$

so that

$$\widehat{x}_4^{2^{k-4}} \equiv \widehat{v}_2^{2^k} + v_1^{2^k}v_2^{-2^{k-1}}\widehat{v}_3^{2^{k-1}} + v_1^{3\cdot 2^{k-2}}v_2^{3\cdot 2^{k-1}}\widehat{v}_2^{2^{k-2}} + v_1^{7\cdot 2^{k-3}}v_2^{7\cdot 2^{k-3}}\widehat{v}_2^{2^{k-1}}$$

modulo $(2, v_1^{9\cdot 2^{k-3}})$. Notice that $2^{k-6}\widehat{d}(6) > 9 \cdot 2^{k-3} > 2^k$ and that we may ignore the terms except the second one, because the other terms don't have a negative v_2-exponent. Then we can complete the proof in similar way as the above. □

Corollary 7.7. *The only elements of $E_1^{1,0}$ which survive to $E_\infty^{1,0}$ are*

$$\frac{\widehat{x}_k^s}{v_1^j} \quad \textit{for } s \geq 0,\ p \nmid s,\ k \geq 0 \textit{ and } 0 < j \leq p^k.$$

Proof. The summand $v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$ of $E_1^{1,0}$ is killed by the chromatic differential

$$d_1\colon \operatorname{Ext}^0_{\Gamma(m+1)}(M_1^0) \to \operatorname{Ext}^0_{\Gamma(m+1)}(M_1^1).$$

Joining this result with Lemma 7.6, we have the desired result. □

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