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ON SMOOTH Sp(p, q)-ACTIONS ON $S^{4p+4q-1}$

Dedicated to the memory of Professor Katsuo Kawakubo

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0. Introduction

Consider the standard $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ action on the $(4p+4q-1)$-sphere $S^{4p+4q-1}$. This action has codimension-one principal orbits with $\mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$ as the principal isotropy subgroup. Furthermore, the fixed point set of the restricted $\mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$ action is diffeomorphic to the seven-sphere S^7.

In the previous papers [4, 5], we have studied smooth $\mathbf{SO}_0(p, q)$-actions on S^{p+q-1}, each of which is an extension of the standard $\mathbf{SO}(p) \times \mathbf{SO}(q)$ action on S^{p+q-1}. In this paper, we shall study smooth $\mathbf{Sp}(p, q)$-actions on $S^{4p+4q-1}$, each of which is an extension of the standard $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ action on $S^{4p+4q-1}$, and we shall show such an action is characterized by a pair (ϕ, f) satisfying certain conditions, where ϕ is a smooth $\mathbf{Sp}(1, 1)$-action on S^7, and $f : S^7 \to \mathbf{P}_1(\mathbf{H})$ is a smooth mapping.

The pair (ϕ, f) was introduced by Asoh [1] to consider smooth $\mathbf{SL}(2, \mathbf{C})$-actions on the 3-sphere, and was improved by our previous papers [4, 5]. The pair was used also by Mukōyama [2] to consider smooth $\mathbf{Sp}(2, \mathbf{R})$-actions on the 4-sphere. He studies also smooth $\mathbf{SU}(p, q)$-actions on $S^{2p+2q-1}$ [3]. Here, we notice that the Lie groups $\mathbf{SL}(2, \mathbf{C})$ and $\mathbf{Sp}(2, \mathbf{R})$ are locally isomorphic to $\mathbf{SO}_0(3, 1)$ and $\mathbf{SO}_0(3, 2)$, respectively.

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1. Standard representation of Sp(p, q)

Let $\mathbf{Sp}(p, q)$ denote the group of complex matrices of degree $2p + 2q$ defined by the equations

$$ {}^tAJ_{p+q}A = J_{p+q}, \quad {}^tAK_{p,q}\overline{A} = K_{p,q}. $$

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Here,

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad K_{p,q} = \begin{bmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{bmatrix}.$$

Consider the linear mapping $\mathbf{J} = cKJ : \mathbf{C}^{2p+2q} \to \mathbf{C}^{2p+2q}$. Here, $K = K_{p,q}$, $J = J_{p+q}$ and c is the complex conjugation. Since $K^2 = I$, $J^2 = -I$ and $KJ = JK$, we obtain $\mathbf{J}^2 = -I$. Furthermore, we see $\mathbf{J}(zX) = \bar{z}\mathbf{J}(X)$ for each $X \in \mathbf{C}^{2p+2q}$ and $z \in \mathbf{C}$. Hence, the linear mapping $\mathbf{J}$ defines a quaternion structure on $\mathbf{C}^{2p+2q}$. We see $\mathbf{J}(AX) = A\mathbf{J}(X)$ for each $A \in \mathbf{Sp}(p,q)$ and $X \in \mathbf{C}^{2p+2q}$, by the definition of $\mathbf{Sp}(p,q)$. Therefore, the quaternion structure $\mathbf{J}$ is $\mathbf{Sp}(p,q)$-equivariant.

Now we decompose an element X of $\mathbf{C}^{2p+2q}$ into $X = {}^t[U_1, V_1, U_2, V_2]$, where $U_1, U_2 \in \mathbf{C}^p$ and $V_1, V_2 \in \mathbf{C}^q$. Then we see

$$\mathbf{J}^t[U_1, V_1, U_2, V_2] = {}^t[-\bar{U}_2, \bar{V}_2, \bar{U}_1, -\bar{V}_1].$$

Hence we obtain the following equation for each $\alpha, \beta \in \mathbf{C}$:

$$(\alpha I + \beta \mathbf{J}) \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} \alpha U_1 - \beta \bar{U}_2 \\ \alpha V_1 + \beta \bar{V}_2 \\ \alpha U_2 + \beta \bar{U}_1 \\ \alpha V_2 - \beta \bar{V}_1 \end{bmatrix}.$$

Therefore, we can identify naturally $\mathbf{C}^{2p+2q}$ having the quaternion structure $\mathbf{J}$ with the quaternion vector space $\mathbf{H}^{p+q}$ having the right scalar multiplication by the following correspondence:

$${}^t[U_1, V_1, U_2, V_2] \to {}^t[U_1 + jU_2, V_1 - jV_2].$$

Denote by $\mathbf{I}(a,b,c,d)$ the isotropy group at

$$a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}$$

with respect to the standard representation of $\mathbf{Sp}(p,q)$ on $\mathbf{C}^{2p+2q}$, where $\mathbf{e}_1$, $\mathbf{e}_2, \ldots$, $\mathbf{e}_{2p+2q}$ are the standard basis of $\mathbf{C}^{2p+2q}$ and a,b,c,d are complex numbers with $(a,b,c,d) \neq (0,0,0,0)$. Then, we see the followings:

$$\begin{aligned} &\dim \frac{\mathbf{Sp}(p,q)}{\mathbf{I}(a,b,c,d)} = 4p + 4q - 1, \\ &\mathbf{I}(1,0,0,0) = \mathbf{I}(0,0,1,0) = \mathbf{Sp}(p-1,q), \\ &\mathbf{I}(0,1,0,0) = \mathbf{I}(0,0,0,1) = \mathbf{Sp}(p,q-1), \end{aligned}$$

$$\bigcap_{(a,b,c,d)\neq(0,0,0,0)} \mathbf{I}(a,b,c,d) = \mathbf{Sp}(p-1,q-1).$$

For $(a,b,c,d) \neq (0,0,0,0)$ and $(a',b',c',d') \neq (0,0,0,0)$, we define an equivalence relation:

$$(a+jc, b-jd) \sim (a'+jc', b'-jd') \Longleftrightarrow \begin{cases} a'+jc' = (a+jc)(\alpha+j\beta), \\ b'-jd' = (b-jd)(\alpha+j\beta) \end{cases}$$

for some quaternion $\alpha + j\beta \neq 0$. The set of equivalence classes is naturally identified with the 1-dimensional quaternion projective space $\mathbf{P}_1(\mathbf{H})$. Then, we see the following:

$$(a+jc, b-jd) \sim (a'+jc', b'-jd') \Longleftrightarrow \mathbf{I}(a,b,c,d) = \mathbf{I}(a',b',c',d').$$

2. Certain closed subgroups of Sp(p, q)

Put

$$\begin{aligned} &\mathbf{Sp}(p)\times\mathbf{Sp}(q) = \mathbf{Sp}(p,q)\cap\mathbf{U}(2p+2q), \\ &\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1) = \mathbf{I}(1,0,0,0)\cap\mathbf{I}(0,1,0,0)\cap\mathbf{U}(2p+2q). \end{aligned}$$

Then, $\mathbf{Sp}(p)\times\mathbf{Sp}(q)$ is the maximal compact subgroup of $\mathbf{Sp}(p,q)$, and $\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1)$ is the principal isotropy subgroup of the standard $\mathbf{Sp}(p)\times\mathbf{Sp}(q)$ action on $\mathbf{C}^{2p+2q}$ which is the restriction of the standard representation of $\mathbf{Sp}(p,q)$.

Now we shall search all subalgebras $\mathcal{G}$ of Lie $\mathbf{Sp}(p,q)$ satisfying the following conditions:

$$\begin{aligned} &\mathcal{G} \supset \mathrm{Lie}(\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1)),\ \ \mathcal{G}\neq \mathrm{Lie}\,\mathbf{Sp}(p,q), \\ &\dim \mathrm{Lie}\,\mathbf{Sp}(p,q) - \dim\mathcal{G} \leq 4p+4q-1. \end{aligned}$$

Here, Lie $\mathbf{Sp}(p,q)$ denotes the Lie algebra of $\mathbf{Sp}(p,q)$ which is a Lie subalgebra of $M_{2p+2q}(\mathbf{C})$ with the bracket operation $[A,B] = AB - BA$, and so on.

Let Ad: $\mathbf{Sp}(p,q) \to \mathrm{Aut}(\mathrm{Lie}\,\mathbf{Sp}(p,q))$ be the adjoint representation defined by AMA^{-1}; $A \in \mathbf{Sp}(p,q)$, $M \in \mathrm{Lie}\,\mathbf{Sp}(p,q)$. Then we can decompose Lie $\mathbf{Sp}(p,q)$ into

$$\mathrm{Lie}\,\mathbf{Sp}(p,q) = \mathcal{K}\oplus\mathcal{S}\oplus\mathcal{U}\oplus\mathcal{V}\oplus\mathcal{T}$$

as a direct sum of $\mathrm{Ad}|_{(\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1))}$-invariant vector spaces. Here,

$$\begin{aligned} \mathcal{K} &= \mathrm{Lie}(\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1)), \\ \mathcal{S} &= \nu_{p-1}\otimes\nu^*_{q-1}, \\ \mathcal{U} &= \nu_{p-1}\oplus\nu_{p-1}, \\ \mathcal{V} &= \nu_{q-1}\oplus\nu_{q-1}, \end{aligned}$$

$$\mathcal{T} = \mathbf{R}^{10}.$$

Then the desired algebra $\mathcal{G}$ can be decomposed into

$$\mathcal{G} = \mathcal{K} \oplus (\mathcal{G} \cap \mathcal{S}) \oplus (\mathcal{G} \cap \mathcal{U}) \oplus (\mathcal{G} \cap \mathcal{V}) \oplus (\mathcal{G} \cap \mathcal{T}).$$

Under the bracket operation, we obtain the following data.

$$\begin{array}{llll} [\mathcal{K}, \mathcal{S}] = \mathcal{S}, & [\mathcal{K}, \mathcal{U}] = \mathcal{U}, & [\mathcal{K}, \mathcal{V}] = \mathcal{V}, & [\mathcal{K}, \mathcal{T}] = \mathbf{0}, \\ [\mathcal{T}, \mathcal{S}] = \mathbf{0}, & [\mathcal{T}, \mathcal{U}] = \mathcal{U}, & [\mathcal{T}, \mathcal{V}] = \mathcal{V}, & [\mathcal{T}, \mathcal{T}] = \mathcal{T}, \\ [\mathcal{S}, \mathcal{U}] = \mathcal{V}, & [\mathcal{S}, \mathcal{V}] = \mathcal{U}, & [\mathcal{U}, \mathcal{V}] = \mathcal{S}, & \\ [\mathcal{U}, \mathcal{U}] \subset \mathcal{K} \oplus \mathcal{T}, & [\mathcal{V}, \mathcal{V}] \subset \mathcal{K} \oplus \mathcal{T}. & & \end{array}$$

Moreover we obtain the following.

$$\begin{array}{ll} \dim \mathcal{S} = 4(p-1)(q-1), & \dim \mathcal{U} = 8p - 8, \\ \dim \mathcal{V} = 8q - 8, & \dim \mathcal{T} = 10. \end{array}$$

By a routine work, we obtain the following result.

Lemma 2.1. *Suppose $p \geq 2$ and $q \geq 2$. Let $\mathcal{G}$ be a proper Lie subalgebra of* Lie $\mathbf{Sp}(p, q)$ *satisfying the following conditions*:

$$\mathcal{G} \supset \mathrm{Lie}(\mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)), \quad \mathcal{G} \neq \mathrm{Lie}\, \mathbf{Sp}(p, q),$$
$$\dim \mathrm{Lie}\, \mathbf{Sp}(p, q) - \dim \mathcal{G} \leq 4p + 4q - 1.$$

Then, $\mathcal{G}$ is one of the following:
(1) $\mathcal{G} \supset \mathrm{Lie}\, \mathbf{I}(a, b, c, d)$ *for some* $(a, b, c, d) \neq (0, 0, 0, 0)$ *such that* $\mathcal{G} \cap (\mathcal{U} \oplus \mathcal{V}) = (\mathrm{Lie}\, \mathbf{I}(a, b, c, d)) \cap (\mathcal{U} \oplus \mathcal{V})$.
(2) $\mathcal{G} = \mathrm{Lie}(\mathbf{Sp}(p, 1) \times \mathbf{Sp}(q-1))$ *for* $q = 2$.
(3) $\mathcal{G} = \mathrm{Lie}(\mathbf{Sp}(p-1) \times \mathbf{Sp}(1, q))$ *for* $p = 2$.
(4) $p = q = 2$, $\dim \mathcal{G} = 21$ *and $\mathcal{G}$ satisfies the following condition*: $\mathcal{G} \cap \mathrm{Lie}(\mathbf{Sp}(2) \times \mathbf{Sp}(2)) = A^{-1} \mathrm{Lie}(\Delta\, \mathbf{Sp}(1) \times (\mathbf{Sp}(1) \times \mathbf{Sp}(1)))A$, *for some* $A \in \mathbf{Sp}(2) \times \mathbf{Sp}(2)$.

3. Smooth Sp(p, q) actions on $S^{4p+4q-1}$

Consider the standard action of $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ on $S^{4p+4q-1}$ defined by

$$\psi \colon (\mathbf{Sp}(p) \times \mathbf{Sp}(q)) \times S^{4p+4q-1} \longrightarrow S^{4p+4q-1},$$
$$\psi(A, X) = AX; \;\; A \in \mathbf{Sp}(p) \times \mathbf{Sp}(q), \;\; X \in S^{4p+4q-1}.$$

The action ψ has $\mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$ as the principal isotropy type and $\mathbf{Sp}(p) \times \mathbf{Sp}(q-1)$ and $\mathbf{Sp}(p-1) \times \mathbf{Sp}(q)$ as singular isotropy types. Moreover the codimension of principal orbits is one.

Put $G = \mathbf{Sp}(p,q)$, $K = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$ and $H = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$.

Here, we consider $S^{4p+4q-1}$ as the unit sphere of C^{2p+2q}. Then the fixed point set $F(H)$ of restricted H-action is the 7-sphere as follows:

$$F(H) = \{a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}\},$$

where a, b, c, d are complex numbers satisfying $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

Let us consider a smooth G-action Φ on $S^{4p+4q-1}$ such that the restricted K-action of Φ coincides with the standard action ψ.

Then we obtain a mapping $f : F(H) \to \mathbf{P}_1(\mathbf{H})$ defined by the condition

$$f(Y) = (a + jc : b - jd) \Longleftrightarrow G_Y \supset \mathbf{I}(a, b, c, d).$$

Since the isotropy subgroup G_Y at $Y \in F(H)$ contains H, G_Y contains a unique subgroup of the form $\mathbf{I}(a, b, c, d)$ by Lemma 2.1.

Lemma 3.1. *For any smooth G-action Φ on $S^{4p+4q-1}$ such that the restricted K-action of Φ coincides with the standard action ψ, the relations $G_{\mathbf{e}_1} = \mathbf{Sp}(p-1, q)$ and $G_{\mathbf{e}_{p+1}} = \mathbf{Sp}(p, q-1)$ are hold. In particular, the orbits through $\mathbf{e}_1$ and $\mathbf{e}_{p+1}$ are open in $S^{4p+4q-1}$.*

Proof. First we obtain $G_{\mathbf{e}_1} \supset \mathbf{Sp}(p-1, q)$ and $G_{\mathbf{e}_{p+1}} \supset \mathbf{Sp}(p, q-1)$ by the following facts:

$$\mathbf{Sp}(p-1) \times \mathbf{Sp}(q) \subset \mathbf{I}(a, b, c, d) \Longleftrightarrow b = d = 0,$$
$$\mathbf{Sp}(p) \times \mathbf{Sp}(q-1) \subset \mathbf{I}(a, b, c, d) \Longleftrightarrow a = c = 0,$$
$$\mathbf{I}(1, 0, 0, 0) = \mathbf{Sp}(p-1, q), \quad \mathbf{I}(0, 1, 0, 0) = \mathbf{Sp}(p, q-1).$$

On the other hand, by Lemma 2.1 we obtain $G_{\mathbf{e}_1} \subset \mathbf{Sp}(1) \times \mathbf{Sp}(p-1, q)$ and $G_{\mathbf{e}_{p+1}} \subset \mathbf{Sp}(p, q-1) \times \mathbf{Sp}(1)$. By considering the restricted K-action ψ, we obtain $G_{\mathbf{e}_1} = \mathbf{Sp}(p-1, q)$ and $G_{\mathbf{e}_{p+1}} = \mathbf{Sp}(p, q-1)$. In particular, since $\dim G/\mathbf{Sp}(p-1, q) = \dim G/\mathbf{Sp}(p, q-1) = 4p + 4q - 1$, the orbits through $\mathbf{e}_1$ and $\mathbf{e}_{p+1}$ are open in $S^{4p+4q-1}$. □

Lemma 3.2. *For any smooth G-action Φ on $S^{4p+4q-1}$ such that the restricted K-action of Φ coincides with the standard action ψ, the mapping $f : F(H) \to \mathbf{P}_1(\mathbf{H})$ defined by the condition*

$$f(Y) = (a + jc : b - jd) \Longleftrightarrow G_Y \supset \mathbf{I}(a, b, c, d)$$

is smooth.

Proof. First we define 10 elements of Lie G as follows:

$$\begin{aligned}
A_1 &= E_{1,p} - E_{p,1} + E_{p+q+1,2p+q} - E_{2p+q,p+q+1},\\
A_2 &= -i(E_{1,p} + E_{p,1} - E_{p+q+1,2p+q} - E_{2p+q,p+q+1}),\\
A_3 &= E_{2p+q,1} - E_{1,2p+q} + E_{p+q+1,p} - E_{p,p+q+1},\\
A_4 &= i(E_{2p+q,1} + E_{1,2p+q} + E_{p+q+1,p} + E_{p,p+q+1}),\\
C &= E_{p,p+1} + E_{p+1,p} - E_{2p+q,2p+q+1} - E_{2p+q+1,2p+q},\\
B_1 &= E_{p+1,p+q} - E_{p+q,p+1} + E_{2p+q+1,2p+2q} - E_{2p+2q,2p+q+1},\\
B_2 &= -i(E_{p+1,p+q} + E_{p+q,p+1} - E_{2p+q+1,2p+2q} - E_{2p+2q,2p+q+1}),\\
B_3 &= E_{2p+2q,p+1} - E_{p+1,2p+2q} + E_{2p+q+1,p+q} - E_{p+q,2p+q+1},\\
B_4 &= i(E_{2p+2q,p+1} + E_{p+1,2p+2q} + E_{2p+q+1,p+q} + E_{p+q,2p+q+1}),\\
D &= E_{p+q,1} + E_{1,p+q} - E_{2p+2q,p+q+1} - E_{p+q+1,2p+2q}.
\end{aligned}$$

Then we see the following relations:

$$\begin{aligned}
&b_1A_1 + b_2A_2 + d_1A_3 + d_2A_4 + C \in \operatorname{Lie} \mathbf{I}(1, b, 0, d),\\
&a_1B_1 + a_2B_2 + c_1B_3 + c_2B_4 + D \in \operatorname{Lie} \mathbf{I}(a, 1, c, 0),
\end{aligned}$$

where each coefficients are real numbers defined by $a = a_1+ia_2$, $b = b_1+ib_2$, $c = c_1+ic_2$ and $d = d_1 + id_2$. Moreover, we see that each of A_1, A_2, A_3, A_4, B_1, B_2, B_3 and B_4 is an element of Lie K.

Now we define a Lie algebra homomorphism $\Phi^+\colon \operatorname{Lie} G \longrightarrow \Gamma(S^{4p+4q-1})$ by

$$\Phi^+(M)_Y(h) = \lim_{t\to 0} \frac{h(\Phi(\exp(-tM), Y)) - h(Y)}{t},$$

where $\Gamma(-)$ denotes the Lie algebra consisting of smooth vector fields on a given manifold, $M \in \operatorname{Lie} G$ and h is a smooth function defined on an open neighborhood of Y. For $M \in \operatorname{Lie} G$, we see $M \in \operatorname{Lie} G_Y \Longleftrightarrow \Phi^+(M)_Y = 0$.

Now we see that the tangent vector fields $\Phi^+(A_1)$, $\Phi^+(A_2)$, $\Phi^+(A_3)$, $\Phi^+(A_4)$, $\Phi^+(B_1)$, $\Phi^+(B_2)$, $\Phi^+(B_3)$ and $\Phi^+(B_4)$ are linearly independent at each point Y of $F(H)$. Because, if they are linearly dependent at $Y \in F(H)$, a non-trivial linear combination of A_1, A_2, A_3, A_4, B_1, B_2, B_3 and B_4 is contained in Lie G_Y and it is a contradiction to the isotropy types of the standard K-action ψ.

Let us denote by $(M, M')_Y$ the inner product of two tangent vector fields $\Phi^+(M)$, $\Phi^+(M')$ at Y with respect to the standard Riemannian metric on $S^{4p+4q-1}$. Denote by $A[Y]$, $B[Y]$ the Gram matrices as follows:

$$\begin{aligned}
(A_s, A_t)_Y &: \quad (s, t)\text{-component of } A[Y],\\
(B_s, B_t)_Y &: \quad (s, t)\text{-component of } B[Y].
\end{aligned}$$

Then $A[Y]$, $B[Y]$ are non-singular at each point $Y \in F(H)$. Moreover, we see the following:

$$f(Y) = (1 : b - jd) \implies A[Y]\begin{bmatrix} b_1 \\ b_2 \\ d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} (A_1, C)_Y \\ (A_2, C)_Y \\ (A_3, C)_Y \\ (A_4, C)_Y \end{bmatrix},$$

$$f(Y) = (a + jc : 1) \implies B[Y]\begin{bmatrix} a_1 \\ a_2 \\ c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} (B_1, D)_Y \\ (B_2, D)_Y \\ (B_3, D)_Y \\ (B_4, D)_Y \end{bmatrix}.$$

Hence we see that each of $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 is a smooth function of Y on an open set of $F(H)$. In fact, b_i, d_j are smooth on the open set of $F(H)$ defined by $(a, c) \neq (0, 0)$ and a_i, c_j are smooth on the open set of $F(H)$ defined by $(b, d) \neq (0, 0)$.

Therefore, the mapping $f : F(H) \to \mathbf{P}_1(\mathbf{H})$ is smooth. □

Denote by $N(p, q)$ the centralizer of $\mathbf{Sp}(p-1, q-1)$ in $\mathbf{Sp}(p, q)$. Then the group $N(p, q)$ acts naturally on

$$\mathbf{C}^4 = \{a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}\}$$

as the restriction of the standard action of $\mathbf{Sp}(p, q)$ on $\mathbf{C}^{2p+2q}$. By the correspondence

$$\mathbf{C}^4 \ni a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1} \longleftrightarrow \begin{bmatrix} a + jc \\ b - jd \end{bmatrix} \in \mathbf{H}^2,$$

the group $N(p, q)$ acts naturally on $\mathbf{P}_1(\mathbf{H})$. In fact, for $n \in N(p, q)$

$$n(a + jc : b - jd) = (a' + jc' : b' - jd')$$

if and only if

$$\begin{aligned} &n(a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}) \\ &= a'\mathbf{e}_1 + b'\mathbf{e}_{p+1} + c'\mathbf{e}_{p+q+1} + d'\mathbf{e}_{2p+q+1}. \end{aligned}$$

Notice that $N(p, q)$ is naturally isomorphic to $\mathbf{Sp}(1, 1)$. On the other hand, the group $N(p, q)$ acts naturally on $F(H)$ as the restriction of the given action Φ.

Lemma 3.3. *For any smooth G-action Φ on $S^{4p+4q-1}$ such that the restricted K-action of Φ coincides with the standard action ψ, the mapping $f : F(H) \to \mathbf{P}_1(\mathbf{H})$*

defined in Lemma 3.2 *is* $N(p,q)$*-equivariant. In particular,*

$$f(Y) = (a + jc : b - jd) \Longrightarrow N(p,q)_Y \supset N(p,q) \cap \mathbf{I}(a,b,c,d).$$

Proof. Suppose $f(Y) = (a + jc : b - jd)$ for $Y \in F(H)$. Then G_Y contains $\mathbf{I}(a,b,c,d)$. Let $n \in N(p,q)$. Then $G_{\Phi(n,Y)} = nG_Y n^{-1}$ contains $n\mathbf{I}(a,b,c,d)n^{-1}$. On the other hand, we see that $n(a + jc : b - jd) = (a' + jc' : b' - jd')$ if and only if $n\mathbf{I}(a,b,c,d)n^{-1} = \mathbf{I}(a',b',c',d')$. By these fact, we obtain $f(\Phi(n,Y)) = nf(Y)$. Hence the mapping $f: F(H) \to \mathbf{P}_1(\mathbf{H})$ is $N(p,q)$-equivariant. Moreover, $G_Y \supset \mathbf{I}(a,b,c,d)$ implies

$$N(p,q)_Y \supset N(p,q) \cap \mathbf{I}(a,b,c,d). \qquad \square$$

4. Construction of Sp(p, q)-actions

Under the natural isomorphism of $N(p,q)$ to $\mathbf{Sp}(1,1)$, we define $M(\theta) \in N(p,q)$ as the matrix corresponding to the following

$$\left[\begin{array}{cc|cc} \cosh\theta & \sinh\theta & & \\ \sinh\theta & \cosh\theta & & \\ \hline & & \cosh\theta & -\sinh\theta \\ & & -\sinh\theta & \cosh\theta \end{array}\right].$$

Now we prepare the following result.

Lemma 4.1. *The equation*

$$\mathbf{Sp}(p,q) = (\mathbf{Sp}(p) \times \mathbf{Sp}(q))N(p,q)\mathbf{I}(a,b,c,d)$$

holds for each $(a,b,c,d) \neq (0,0,0,0)$.

Proof. Consider the standard action of $\mathbf{Sp}(p,q)$ on $\mathbf{C}^{2p+2q}$. Put

$$Y = a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}.$$

For any $g \in \mathbf{Sp}(p,q)$, we decompose $gY = {}^t[U_1, V_1, U_2, V_2]$, where $U_1, U_2 \in \mathbf{C}^p$ and $V_1, V_2 \in \mathbf{C}^q$. Then we see

$$-\|U_1\|^2 + \|V_1\|^2 - \|U_2\|^2 + \|V_2\|^2 = -|a|^2 + |b|^2 - |c|^2 + |d|^2.$$

Hence, we can choose $k \in K = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$ as follows:

$$k^{-1}gY = s\mathbf{e}_1 + t\mathbf{e}_{p+1} \ : \ s = \sqrt{\|U_1\|^2 + \|U_2\|^2}, \ t = \sqrt{\|V_1\|^2 + \|V_2\|^2}.$$

Next, we can choose $M(\theta) \in N(p,q)$ as follows:

$$M(-\theta)k^{-1}gY = \sqrt{|a|^2 + |c|^2}\mathbf{e}_1 + \sqrt{|b|^2 + |d|^2}\mathbf{e}_{p+1}.$$

Finally, we can choose $n \in N(p,q) \cap K$ such that $n^{-1}M(-\theta)k^{-1}gY = Y$. In particular, we obtain $n^{-1}M(-\theta)k^{-1}g \in \mathbf{I}(a,b,c,d)$. □

As in the previous section, we use the notations $G = \mathbf{Sp}(p,q)$, $K = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$ and $H = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$.

Moreover, we use the notations $\mathbf{I}(a,b,c,d)$, $F(H)$ and $N(p,q)$.

In this section, we suppose the following situation:

1. a smooth action ϕ: $N(p,q) \times F(H) \longrightarrow F(H)$ is given.
2. an $N(p,q)$-equivariant smooth mapping $f : F(H) \longrightarrow \mathbf{P}_1(\mathbf{H})$ is given.
3. the following conditions are satisfied:
   (a) $n \in N(p,q) \cap K,\ Y \in F(H) \Longrightarrow \phi(n,Y) = \psi(n,Y)$.
   (b) $f(Y) = (a + jc : b - jd) \Longrightarrow N(p,q)_Y \supset N(p,q) \cap \mathbf{I}(a,b,c,d)$.

Notice that such a situation is realized if there is a smooth G-action on $S^{4p+4q-1}$ which is an extension of the standard K-action ψ on $S^{4p+4q-1}$. These facts are proved in lemmas 3.2, 3.3.

We shall show how to construct a smooth $G = \mathbf{Sp}(p,q)$-action on $S^{4p+4q-1}$ from the pair (ϕ, f). First, we prepare several lemmas.

Lemma 4.2. *The following relations hold.*

$$f(Y) = (1:0) \Longleftrightarrow K_Y = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q),$$
$$f(Y) = (0:1) \Longleftrightarrow K_Y = \mathbf{Sp}(p) \times \mathbf{Sp}(q-1).$$

Proof. Notice that the isotropy subgroup K_Y for $Y \in F(H)$ is one of the following:

$$\mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1),\ \ \mathbf{Sp}(p-1) \times \mathbf{Sp}(q),\ \ \mathbf{Sp}(p) \times \mathbf{Sp}(q-1).$$

Under the natural isomorphism of $N(p,q)$ to $\mathbf{Sp}(1,1)$, the group $K \cap N(p,q)$ can be identified with $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$. Here we denote

$$K \cap N(p,q) = \mathbf{Sp}(1) \times \mathbf{Sp}(1).$$

Under this identification, we see $(\mathbf{Sp}(1) \times \mathbf{Sp}(1))_{(\alpha:\beta)} = 1 \times 1$ for each $(\alpha : \beta) \in \mathbf{P}_1(\mathbf{H})$ satisfying $\alpha\beta \neq 0$. Hence we see that $K_Y = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1)$, if $f(Y) = (\alpha : \beta)$ satisfying $\alpha\beta \neq 0$. On the other hand, if $f(Y) = (a + jc : b - jd)$, then we see

$$K_Y \supset K \cap N(p,q)_Y \supset (\mathbf{Sp}(1) \times \mathbf{Sp}(1)) \cap \mathbf{I}(a,b,c,d).$$

In particular, we see

$$(\mathbf{Sp}(1) \times \mathbf{Sp}(1)) \cap \mathbf{I}(1,0,0,0) = 1 \times \mathbf{Sp}(1),$$
$$(\mathbf{Sp}(1) \times \mathbf{Sp}(1)) \cap \mathbf{I}(0,1,0,0) = \mathbf{Sp}(1) \times 1.$$

By these facts, we obtain the desired result. □

Lemma 4.3. *$Y \in F(H)$, $f(Y) = (a + jc : b - jd)$ be given. Then*

$$g = k_1 n_1 h_1 = k_2 n_2 h_2 \Longrightarrow \psi(k_1, \phi(n_1, Y)) = \psi(k_2, \phi(n_2, Y))$$

for any k_1, $k_2 \in K$; n_1, $n_2 \in N(p,q)$; h_1, $h_2 \in \mathbf{I}(a,b,c,d)$.

Proof. Put

$$X = X(a,b,c,d) = a\mathbf{e}_1 + b\mathbf{e}_{p+1} + c\mathbf{e}_{p+q+1} + d\mathbf{e}_{2p+q+1}.$$

First, we consider the standard representation of $G = \mathbf{Sp}(p,q)$ on $\mathbf{C}^{2p+2q}$. We can describe by the above notation

$$n_t X(a,b,c,d) = X_t = X(a_t, b_t, c_t, d_t), \quad (t = 1, 2).$$

By the assumption $g = k_1 n_1 h_1 = k_2 n_2 h_2$, we obtain

$$gX(a,b,c,d) = k_1 X(a_1, b_1, c_1, d_1) = k_2 X(a_2, b_2, c_2, d_2).$$

Hence we obtain $gX = k_1 X_1 = k_2 X_2$. Put $k = k_1^{-1} k_2$. Then we obtain $K_{X_1} = K_{kX_2} = kK_{X_2}k^{-1}$. By the form of isotropy subgroups, we obtain

(a) $$K_{X_1} = K_{X_2}, \quad k \in N(K_{X_t}) \quad (t = 1, 2)$$

By Lemma 4.2, we obtain the following:

(b) $$\begin{aligned} (a_t, c_t) \neq (0,0) \neq (b_t, d_t) &\Longleftrightarrow K_{X_t} = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q-1) \\ (a_t, c_t) \neq (0,0) = (b_t, d_t) &\Longleftrightarrow K_{X_t} = \mathbf{Sp}(p-1) \times \mathbf{Sp}(q) \\ (a_t, c_t) = (0,0) \neq (b_t, d_t) &\Longleftrightarrow K_{X_t} = \mathbf{Sp}(p) \times \mathbf{Sp}(q-1) \end{aligned}$$

Moreover, we obtain

(c) $$k_1^{-1} k_2 n_2 n_1^{-1} \in \mathbf{I}(a_1, b_1, c_1, d_1)$$

because the element $k_1^{-1} k_2 n_2 n_1^{-1}$ leaves the point X_1 fixed.

Now we consider case by case.

[1] The case $(b_1,d_1)=(0,0)$. By (a), (b), we see $(b_2,d_2)=(0,0)$. By $n_1X=X_1$,

$$f(\phi(n_1,Y))=n_1f(Y)=(a_1+jc_1:0)=(1:0).$$

Then, by (b), we see $K_{\phi(n_1,Y)}=\mathbf{Sp}(p-1)\times\mathbf{Sp}(q)$. On the other hand,

$$k_1^{-1}k_2n_2n_1^{-1}\in\mathbf{I}(a_1,0,c_1,0)=\mathbf{I}(1,0,0,0)=\mathbf{Sp}(p-1,q)$$

by (c). By the second half of (a), we obtain $k_1^{-1}k_2\in(\mathbf{Sp}(1)\times\mathbf{Sp}(p-1))\times\mathbf{Sp}(q)$ and hence we can decompose

$$k_1^{-1}k_2=k'k'':\ \ k'\in\mathbf{Sp}(p-1)\times\mathbf{Sp}(q),\ \ k''\in\mathbf{Sp}(1)\times 1.$$

Then $k''n_2n_1^{-1}\in N(p,q)\cap\mathbf{Sp}(p-1,q)=1\times\mathbf{Sp}(1)$ and hence we obtain

$$k_1^{-1}k_2n_2n_1^{-1}\in K\cap\mathbf{Sp}(p-1,q)=\mathbf{Sp}(p-1)\times\mathbf{Sp}(q).$$

Under these preparation, we obtain

$$\begin{aligned}\psi(k_2,\phi(n_2,Y)) &= \psi(k_2,\phi(n_2n_1^{-1}n_1,Y))\\ &= \psi(k_2,\phi(n_2n_1^{-1},\phi(n_1,Y)))\\ &= \psi(k_2,\psi(n_2n_1^{-1},\phi(n_1,Y)))\\ &= \psi(k_2n_2n_1^{-1},\phi(n_1,Y))\\ &= \psi(k_1,\psi(k_1^{-1}k_2n_2n_1^{-1},\phi(n_1,Y)))\\ &= \psi(k_1,\phi(n_1,Y)).\end{aligned}$$

[2] The case $(a_1,c_1)=(0,0)$ is similarly proved.
[3] The case $(a_1,c_1)\neq(0,0)\neq(b_1,d_1)$. In this case, we see $(a_2,c_2)\neq(0,0)\neq(b_2,d_2)$ by (a), (b). Now we can decompose

$$k_1^{-1}k_2=k'k'':\ \ k'\in\mathbf{Sp}(p-1)\times\mathbf{Sp}(q-1),\ \ k''\in\mathbf{Sp}(1)\times\mathbf{Sp}(1)$$

by the second half of (a). Then, $k''n_2n_1^{-1}\in\mathbf{I}(a_1,b_1,c_1,d_1)$ by (c). Since $\mathbf{I}(a_1,b_1,c_1,d_1)=n_1\mathbf{I}(a,b,c,d)n_1^{-1}$, we obtain $k''n_2=n_1h;\ h\in\mathbf{I}(a,b,c,d)$, where $h\in N(p,q)\cap\mathbf{I}(a,b,c,d)\subset N(p,q)_Y$. Under these preparation, we obtain

$$\begin{aligned}\psi(k_2,\phi(n_2,Y)) &= \psi(k_1k'k'',\phi(n_2,Y))\\ &= \psi(k_1k'',\phi(n_2,Y))\\ &= \psi(k_1,\phi(k'',\phi(n_2,Y)))\\ &= \psi(k_1,\phi(k''n_2,Y))\\ &= \psi(k_1,\phi(n_1h,Y))\end{aligned}$$

$$= \psi(k_1, \phi(n_1, \phi(h, Y)))$$
$$= \psi(k_1, \phi(n_1, Y)).$$

This completes the proof. □

Now we define $\Phi(g, Y) \in S^{4p+4q-1}$ for each $g \in G, Y \in F(H)$ by

$$\Phi(g, Y) = \psi(k, \phi(n, Y)).$$

Here we decompose $g = knh : k \in K,\ n \in N(p, q)$ and $h \in \mathbf{I}(a, b, c, d)$, for $f(Y) = (a + jc : b - jd)$. Lemma 4.3 assures the well-definedness of $\Phi(g, Y)$.

Lemma 4.4. *Suppose*

$$\psi(k_1, Y_1) = \psi(k_2, Y_2)\ ;\ Y_1, Y_2 \in F(H),\ k_1, k_2 \in K.$$

Then the relation $\Phi(gk_1, Y_1) = \Phi(gk_2, Y_2)$ *holds for any* $g \in G = \mathbf{Sp}(p, q)$.

Proof. By the assumption, $K_{Y_1} = K_{Y_2}$ and there is a decomposition

$$k_1^{-1}k_2 = k''k' : \quad k' \in K_{Y_2},\ k'' \in \mathbf{Sp}(1) \times \mathbf{Sp}(1).$$

Now we give a decomposition

$$gk_1 = knh : \quad k \in K,\ n \in N(p, q),\ h \in \mathbf{I}(a_1, b_1, c_1, d_1).$$

Here we assume $f(Y_t) = (a_t + jc_t : b_t - jd_t)$, $(t = 1, 2)$. Then

$$gk_2 = gk_1k''k' = knhk''k'.$$

On the other hand, we obtain

$$\mathbf{I}(a_1, b_1, c_1, d_1) = k''\mathbf{I}(a_2, b_2, c_2, d_2)(k'')^{-1}$$

from $Y_1 = \psi(k'', Y_2) = \phi(k'', Y_2)$. Hence we see

$$h \in \mathbf{I}(a_1, b_1, c_1, d_1) \Longrightarrow h' = (k'')^{-1}hk'' \in \mathbf{I}(a_2, b_2, c_2, d_2).$$

Put $n' = nk''$. Then, $n' \in N(p, q)$ and $gk_2 = kn'h'k'$. In this decomposition, we can show $k' \in \mathbf{I}(a_2, b_2, c_2, d_2)$ by considering the isotropy subgroup at Y_2 case by case. Hence we see

$$\Phi(gk_2, Y_2) = \psi(k, \phi(n', Y_2))$$

$$
\begin{aligned}
&= \psi(k, \phi(n, \psi(k'', Y_2))) \\
&= \psi(k, \phi(n, Y_1)) \\
&= \Phi(gk_1, Y_1). \qquad \square
\end{aligned}
$$

By this lemma, we may define a mapping $\Phi\colon G \times S^{4p+4q-1} \longrightarrow S^{4p+4q-1}$ by $\Phi(g, \psi(k, Y)) = \Phi(gk, Y) : g \in G, k \in K, Y \in F(H)$. The right-hand side is already defined.

It is easy to see that the mapping Φ is an abstract action of G on $S^{4p+4q-1}$ which is an extension of the standard K-action ψ and an extension of the given $N(p,q)$-action ϕ. It remains to show Φ is smooth.

First we state the following result which is an accurate form of Lemma 4.1. The proof is quite similar, so we omit it.

Lemma 4.5. *There is a decomposition*

$$g = kM(\theta)h \;:\quad k \in K,\;\; \theta \in \mathbf{R},\;\; h \in \mathbf{I}(1, \beta, 0, 0)$$

for any $\beta > 0$ *and any* $g \in G$.

Put

$$\mathbf{P}_1(\mathbf{R}) = \{(a:b) \in \mathbf{P}_1(\mathbf{H}) \mid a, b \in \mathbf{R}\}.$$

Then, $\mathbf{P}_1(\mathbf{R})$ is a 1-dimensional submanifold of $\mathbf{P}_1(\mathbf{H})$. Define

$$S = f^{-1}(\mathbf{P}_1(\mathbf{R})).$$

Because the isotropy subgroups at two points $(1:0)$, $(0:1)$ are both $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$ with respect to the standard $N(p,q)$-action on $\mathbf{P}_1(\mathbf{H})$, we see that the orbits through these points are open and hence the given $N(p,q)$-equivariant smooth mapping $f : F(H) \to \mathbf{P}_1(\mathbf{H})$ is transversal on $\mathbf{P}_1(\mathbf{R})$. Hence S is a 4-dimensional submanifold of $F(H)$. Put

$$S_+ = \{Y \in S \mid f(Y) = (1:\beta),\;\; \beta > 0\}.$$

Then S_+ is an open submanifold of S.

Hereafter, we denote $\beta = \beta(Y)$ for $Y \in S_+$ such that $f(Y) = (1:\beta)$.

Now we see the following:

$$f(\phi(M(\theta), Y)) = (\cosh\theta + \beta\sinh\theta : \sinh\theta + \beta\cosh\theta)$$

for $Y \in S_+$ and θ, where $\beta = \beta(Y)$. Hence $\phi(M(\theta), Y) \in S$ in general. Therefore,

$\phi(M(\theta), Y) \in S_+$ if and only if

$$(\cosh\theta + \beta\sinh\theta)(\sinh\theta + \beta\cosh\theta) > 0.$$

In this case, we obtain the following:

$$\beta(\phi(M(\theta), Y)) = \beta + \frac{(1-\beta^2)\tanh\theta}{1+\beta\tanh\theta}.$$

Here we define a matrix $P(Y)$ of degree $2p+2q$ as follows:

$$P(Y) = \frac{1}{1+\beta^2}(E_{1,1} + \beta E_{1,p+1} + \beta E_{p+1,1} + \beta^2 E_{p+1,p+1}).$$

We see trace $P(Y) = 1$. Notice that

$$\mathrm{trace}(gP(Y)g^*) = \cosh 2\theta + \frac{2\beta}{1+\beta^2}\sinh 2\theta$$

for the decomposition $g = kM(\theta)h : k \in K,\ h \in \mathbf{I}(1, \beta, 0, 0)$, where $Y \in S_+, \beta = \beta(Y)$.

Now we define

$$\mathbf{D}_+ = \{(\theta, Y) \in \mathbf{R} \times S_+ \mid \phi(M(\theta), Y) \in S_+\},$$
$$W_+ = \left\{(g, Y) \in G \times S_+ \;\middle|\; \pm\,\mathrm{trace}(gP(Y)g^*) \neq \frac{1-\beta^2}{1+\beta^2},\ \ \beta = \beta(Y)\right\}.$$

Clearly $\mathbf{D}_+$ is an open set of $\mathbf{R} \times S_+$ and W_+ is an open set of $G \times S_+$.

Now we have the following results, whose proof is quite similar to that of [4, Lemma 4.7]. So we omit the proof.

Lemma 4.6. *For $(g, Y) \in G \times S_+$, $(g, Y) \in W_+$ if and only if there is a decomposition*

$$g = kM(\theta)h :\ \ k \in K,\ \ h \in \mathbf{I}(1, \beta, 0, 0), \quad \phi(M(\theta), Y) \in S_+$$

where $\beta = \beta(Y)$.

Lemma 4.7. *There is a smooth mapping $\Delta\colon W_+ \to K/H \times \mathbf{D}_+$ defined by $\Delta(g, Y) = (kH, (\theta, Y))$, where $g = kM(\theta)h$; $k \in K, \theta \in \mathbf{R}$, and $h \in \mathbf{I}(1, \beta, 0, 0)$ for $\beta = \beta(Y)$.*

Put $W(\Phi) = (1 \times \psi)(\mu \times 1)^{-1}(W_+)$, where ψ is the K-action and μ is the multiplication on G. Then $W(\Phi)$ is an open set of $G \times S^{4p+4q-1}$ and we obtain the following commutative diagram:

$$\begin{array}{ccc}
G\times K\times S_+ & \xrightarrow{1\times\psi} & G\times S^{4p+4q-1} \\
\Big\downarrow{\scriptstyle \mu\times 1} & & \Big\uparrow{\scriptstyle \cup} \\
G\times S_+\supset W_+ & \longrightarrow & W(\Phi) \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Phi} \\
K/H\times \mathbf{D}_+ & \xrightarrow{\phi'} & S^{4p+4q-1},
\end{array}$$

where $\phi'(kH,(\theta,Y)) = \psi(k,\phi(M(\theta),Y))$. Since $1\times\psi$ is a smooth submersion, we see that the restriction $\Phi|_{W(\Phi)}$ is a smooth mapping.

Define $S_1(\Phi)=\{\Phi(g,\mathbf{e}_1)\mid g\in G\}$ and $S_2(\Phi)=\{\Phi(g,\mathbf{e}_{p+1})\mid g\in G\}$.

We shall show that these two sets are open in $S^{4p+4q-1}$ and the G-action Φ is smooth on these sets.

Here we define the standard G-action Ψ_0 on $S^{4p+4q-1}$ by

$$\Psi_0(g,X)=\|gX\|^{-1}gX;\ \ g\in G,\ \ X\in S^{4p+4q-1}.$$

Define $S_1(\Psi_0)=\{\Psi_0(g,\mathbf{e}_1)\mid g\in G\}$, and $S_2(\Psi_0)=\{\Psi_0(g,\mathbf{e}_{p+1})\mid g\in G\}$. By the natural correspondence

$$\Phi(g,\mathbf{e}_1)\mapsto\Psi_0(g,\mathbf{e}_1),\quad \Phi(g,\mathbf{e}_{p+1})\mapsto\Psi_0(g,\mathbf{e}_{p+1}),$$

we obtain G-equivariant mappings $F_\varepsilon\colon S_\varepsilon(\Phi)\to S_\varepsilon(\Psi_0)$ for $\varepsilon=1,\ 2$.

We can denote $\Phi(M(\theta),\mathbf{e}_1)=\phi(M(\theta),\mathbf{e}_1)=X(a(\theta),b(\theta),c(\theta),d(\theta))$. Since $f(X(*,0,*,0))=(1:0)$ and $f(X(0,*,0,*))=(0:1)$, we see

$$\text{(a)}\qquad \begin{array}{l}(b(\theta),d(\theta))\neq(0,0)\quad(\forall\theta\neq 0),\\ (a(\theta),c(\theta))\neq(0,0)\quad(\forall\theta).\end{array}$$

Next, using

$$-K_{p,q}\in K\cap\mathbf{I}(1,0,0,0),\quad(-K_{p,q})M(\theta)=M(-\theta)(-K_{p,q}),$$

we obtain

$$\begin{aligned}
\Phi((-K_{p,q})M(\theta),\mathbf{e}_1) &= \psi(-K_{p,q},X(a(\theta),b(\theta),c(\theta),d(\theta)))\\
&= X(a(\theta),-b(\theta),c(\theta),-d(\theta)),\\
\Phi(M(-\theta)(-K_{p,q}),\mathbf{e}_1) &= X(a(-\theta),b(-\theta),c(-\theta),d(-\theta)).
\end{aligned}$$

Hence we see that $a(\theta)$ and $c(\theta)$ are even functions, and $b(\theta)$ and $d(\theta)$ are odd functions. In particular, there exist smooth even functions $b_0(\theta)$, $d_0(\theta)$ such that $b(\theta)=b_0(\theta)\theta$ and $d(\theta)=d_0(\theta)\theta$.

Now we define $\Delta\,\mathbf{Sp}(1)$ as the subgroup of $K \cap N(p,q) = \mathbf{Sp}(1)\times\mathbf{Sp}(1)$ consisting of matrices in the form

$$\left[\begin{array}{c|c|c|c} a & & -\bar{c} & \\ \hline & a & & \bar{c} \\ \hline c & & \bar{a} & \\ \hline & -c & & \bar{a} \end{array}\right].$$

By direct calculation, we see

(b) $\qquad M(\theta)$ is commutative with each element of $\Delta\,\mathbf{Sp}(1)$.

Moreover, we obtain

$$\left[\begin{array}{c|c|c|c} a & & -\bar{c} & \\ \hline & a & & \bar{c} \\ \hline c & & \bar{a} & \\ \hline & -c & & \bar{a} \end{array}\right] X(x,y,x',y') \longleftrightarrow (a+jc)\begin{bmatrix} x+jx' \\ y-jy' \end{bmatrix}$$

under the natural correspondence

$$X(x,y,x',y') \longleftrightarrow \begin{bmatrix} x+jx' \\ y-jy' \end{bmatrix}.$$

This means the action of $\Delta\,\mathbf{Sp}(1)$ on $F(H)$ correspondents to the left scalar multiplication. In particular, we obtain

(c) $\qquad$ The $\Delta\,\mathbf{Sp}(1)$-action on $F(H)$ is free.

Moreover, we see the set $S = f^{-1}(\mathbf{P}_1(\mathbf{R}))$ is $\Delta\,\mathbf{Sp}(1)$-invariant.

Since $f(\phi(M(\theta),\mathbf{e}_1)) = (1 : \tanh\theta)$, we see the curve $\phi(M(\theta),\mathbf{e}_1)$ is transverse to each orbit of the $\Delta\,\mathbf{Sp}(1)$-action, by the facts (b), (c). Hence we obtain

(d) $$\frac{d}{d\theta}(|b(\theta)|^2 + |d(\theta)|^2) \neq 0 \quad (\forall\theta \neq 0)$$

Here we obtain $(a'(\theta), b'(\theta), c'(\theta), d'(\theta)) \neq (0,0,0,0)$ $(\forall\theta)$ by making use of the equation $f(\phi(M(\theta),\mathbf{e}_1)) = (1 : \tanh\theta)$. Since $a(\theta), c(\theta)$ are even functions, we see $a'(0) = c'(0) = 0$, and hence $(b_0(0), d_0(0)) = (b'(0), d'(0)) \neq (0,0)$. Combining this result with (a), we obtain

(e) $$(a(\theta), c(\theta)) \neq (0,0) \neq (b_0(\theta), d_0(\theta)) \quad (\forall\theta)$$

Here we define new smooth functions by

$$\sigma(\theta) = \sqrt{|a(\theta)|^2 + |c(\theta)|^2}, \qquad \tau_0(\theta) = \sqrt{|b_0(\theta)|^2 + |d_0(\theta)|^2}$$

$$\alpha(\theta) = \frac{\overline{a(\theta) + jc(\theta)}}{\sigma(\theta)}, \qquad \beta(\theta) = \frac{\overline{b_0(\theta) - jd_0(\theta)}}{\tau_0(\theta)}$$

Moreover we define $\tau(\theta) = \tau_0(\theta)\theta$. Then, $\tau(\theta)$ is an odd function and $\alpha(\theta)$, $\beta(\theta)$ are even function with values in quaternions of modulus one. Moreover,

$$\begin{bmatrix} (a(\theta) + jc(\theta))\alpha(\theta) \\ (b(\theta) - jd(\theta)\beta(\theta) \end{bmatrix} = \begin{bmatrix} \sigma(\theta) \\ \tau(\theta) \end{bmatrix}.$$

By (d), we obtain

$$\frac{d}{d\theta}\tau(\theta) = \frac{(d/d\theta)(|b(\theta)|^2 + |d(\theta)|^2)}{2\sqrt{|b(\theta)|^2 + |d(\theta)|^2}} \neq 0 \quad (\forall\theta \neq 0).$$

Then $\tau'(0) = \tau_0(0) > 0$ by (e). Hence we see $\tau'(\theta) > 0$ $(\forall\theta)$. Therefore, $\tau\colon \mathbf{R} \longrightarrow (-r, r)$ $(0 < r \leq 1)$ is a smooth diffeomorphism. The existence of such r is assured by the equation $|a(\theta)|^2 + |b(\theta)|^2 + |c(\theta)|^2 + |d(\theta)|^2 = 1$ $(\forall\theta)$.

Here we use the following identification again

$$\mathbf{C}^{2p+2q} \ni U_1 \oplus V_1 \oplus U_2 \oplus V_2 \longleftrightarrow (U_1 + jU_2) \oplus (V_1 - jV_2) \in \mathbf{H}^{p+q}.$$

By the diffeomorphism $\tau\colon \mathbf{R} \longrightarrow (-r, r)$, we can describe

$$S_1(\Phi) = \{U \oplus V \in \mathbf{H}^{p+q} \mid \|V\| < r, \|U\|^2 + \|V\|^2 = 1\}.$$

First we define $h_1\colon S_1(\Phi) \longrightarrow S_1(\Phi)$ by

$$h_1(U \oplus V) = U\alpha(\tau^{-1}(\|V\|)) \oplus V\beta(\tau^{-1}(\|V\|)).$$

Then h_1 is a K-equivariant deffeomorphism by definition. Moreover, we obtain the following:

(f) $$h_1(\Phi(M(\theta), \mathbf{e}_1)) = \sigma(\theta)\mathbf{e}_1 \oplus \tau(\theta)\mathbf{e}_{p+1} \quad (\forall\theta)$$

Since the function $\tanh\theta/\sqrt{1 + (\tanh\theta)^2}$ is a diffeomorphism and odd function from $\mathbf{R}$ onto the open interval $(-1/\sqrt{2}, 1/\sqrt{2})$, we can define $\gamma\colon (-r, r) \longrightarrow (-1/\sqrt{2}, 1/\sqrt{2})$ by the equation

$$\gamma(\tau(\theta)) = \frac{\tanh\theta}{\sqrt{1 + (\tanh\theta)^2}} \quad (\forall\theta).$$

Then the mapping γ is a diffeomorphism and odd function. So we define an even function $\gamma_0\colon (-r, r) \to \mathbf{R}$ by $\gamma(\theta) = \gamma_0(\theta)\theta$ $(\forall\theta)$.

Next we define $h_2: S_1(\Phi) \longrightarrow S_1(\Psi_0)$ by $U \oplus V \mapsto U\gamma_1 \oplus V\gamma_0(\|V\|)$, where $\gamma_1 = \|U\|^{-1}\sqrt{1-\gamma(\|V\|)^2}$. Then h_2 is also a K-equivariant deffeomorphism by definition. Moreover, we obtain the following:

$$\text{(g)} \qquad h_2(\sigma(\theta)\mathbf{e}_1 \oplus \tau(\theta)\mathbf{e}_{p+1}) = \Psi_0(M(\theta), \mathbf{e}_1)$$

The composition $h_2 \circ h_1$ is also a K-equivariant diffeomorphism and

$$(h_2 \circ h_1)(\Phi(M(\theta), \mathbf{e}_1)) = \Psi_0(M(\theta), \mathbf{e}_1)$$

by (f), (g). By making use of Lemma 4.5, we see $(h_2 \circ h_1)(\Phi(g, \mathbf{e}_1)) = \Psi_0(g, \mathbf{e}_1)$ for each $g \in G$.

Consequently, we see $F_1 = h_2 \circ h_1$ and hence $F_1: S_1(\Phi) \longrightarrow S_1(\Psi_0)$ is a smooth diffeomorphism. By the quite similar argument, we see that the G-equivariant mapping $F_2: S_2(\Phi) \longrightarrow S_2(\Psi_0)$ is also a smooth diffeomorphism.

Since the family of three open sets $W(\Phi)$, $G \times S_1(\Phi)$ and $G \times S_2(\Phi)$ is an open covering of $G \times S^{4p+4q-1}$ and the restriction of $\Phi: G \times S^{4p+4q-1} \longrightarrow S^{4p+4q-1}$ is smooth on these three open sets, we see that the action Φ of G on $S^{4p+4q-1}$ is smooth.

Consequently, we obtain the following result.

Theorem 4.8. *Let a smooth action $\phi: N(p,q) \times F(H) \longrightarrow F(H)$ and an $N(p,q)$-equivariant smooth mapping $f: F(H) \longrightarrow \mathbf{P}_1(\mathbf{H})$ be given. Suppose that the following conditions are satisfied*:

1. $n \in N(p,q) \cap K,\ Y \in F(H) \Longrightarrow \phi(n,Y) = \psi(n,Y)$.
2. $f(Y) = (a + jc : b - jd) \Longrightarrow N(p,q)_Y \supset N(p,q) \cap \mathbf{I}(a,b,c,d)$.

Then there exists a smooth G-action Φ on $S^{4p+4q-1}$ uniquely, which is an extension of the standard K-action ψ and an extension of the given $N(p,q)$-action ϕ. Moreover, the isotropy subgroup at $Y \in F(H)$ contains $\mathbf{I}(a,b,c,d)$, if $f(Y) = (a + jc : b - jd)$.

5. Construction of (ϕ, f)

In the previous section, we show how to construct a smooth action of $\mathbf{Sp}(p,q)$ on $S^{4p+4q-1}$ from a pair (ϕ, f), where ϕ is a smooth $N(p,q)$-action on $S^7 = F(H)$ whose restriction on $K \cap N(p,q)$ coincides with the restriction of the standard action of $K = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$ and $f: F(H) \to \mathbf{P}_1(\mathbf{H})$ is a smooth $N(p,q)$-equivariant mapping satisfying the conditions in Theorem 4.8.

Now we consider how to construct such a pair (ϕ, f). Define the circle S_0 in $S^{4p+4q-1}$ and involutions $J_\pm$ on S_0 by

$$S_0 = \{s\mathbf{e}_1 + t\mathbf{e}_{p+1} \mid s^2 + t^2 = 1;\ s, t \in \mathbf{R}\},$$

$$J_\varepsilon(s\mathbf{e}_1 + t\mathbf{e}_{p+1}) = \begin{cases} -s\mathbf{e}_1 + t\mathbf{e}_{p+1} & (\varepsilon = +), \\ s\mathbf{e}_1 - t\mathbf{e}_{p+1} & (\varepsilon = -). \end{cases}$$

Now we give a pair (ϕ_0, f_0) of a smooth one-parameter group $\phi_0 : \mathbf{R} \times S_0 \to S_0$ and a smooth function $f_0 : S_0 \to \mathbf{P}_1(\mathbf{R})$ satisfying the conditions

(a) $J_\varepsilon \phi_0(\theta, Y) = \phi_0(-\theta, J_\varepsilon(Y)) \quad (\varepsilon = \pm)$

(b) $f_0(Y) = (a : b) \Longrightarrow f_0(J_\varepsilon(Y)) = (-a : b) \quad (\varepsilon = \pm)$

(c) $f_0(Y) = (a : b) \Longrightarrow$
$f_0(\phi_0(\theta, Y)) = (a \cosh\theta + b\sinh\theta : a\sinh\theta + b\cosh\theta)$

(d) $f_0(Y) = (1 : 0) \Longleftrightarrow Y = \pm\mathbf{e}_1$

(e) $f_0(Y) = (0 : 1) \Longleftrightarrow Y = \pm\mathbf{e}_{p+1}$

From the pair (ϕ_0, f_0), we can construct a desired pair (ϕ, f). The method is quite similar as one in the previous section and as one in [5, §5], so we omit the description. Notice that each open orbit of $N(p,q)$-action ϕ corresponds to an equivalence class of open orbits of the one-parameter group ϕ_0, where two open orbits of the one-parameter group ϕ_0 are equivalent if the one is mapped onto the other by the involutions $J_\pm$.

The next problem is how to construct a pair (ϕ_0, f_0) satisfying the conditions (a)–(e). First we prepare the following lemma [1, Lemma 10.1].

Lemma 5.1. *There exist smooth functions A, B defined on $\mathbf{R}$ satisfying the conditions*

(1) $A(x)$: *odd function,* $B(x)$: *even function,*
(2) $|A(x)| < 1 (|x| < 1)$, $A(x) = 1\ (x \geq 1)$, $A(x) = -1\ (x \leq -1)$,
(3) $B(x) = 0\ (|x| \geq 1)$,
(4) $A'(x) > 0\ (|x| < 1)$,
(5) $B(x)A'(x) = A(x)^2 - 1\ (\forall x)$.

For each positive integer m, define new smooth functions A_m, B_m, C_m by

$$\begin{aligned} A_m(\tau) &= A(\omega_0)^{-1}A(\omega_{2m-1})A(\omega_{4m-2})^{-1} \quad (0 < \tau < \pi), \\ B_m(\tau) &= s\sum_{j=0}^{4m-2}(-1)^j B(\omega_j) \quad (0 \leq \tau \leq \pi), \\ C_m(\tau) &= -A_m\left(\tau + \frac{\pi}{2}\right) \quad \left(-\frac{\pi}{2} < \tau < \frac{\pi}{2}\right). \end{aligned}$$

Here $s = \pi/(8m-4)$ and $\omega_j = (\tau - 2js)/s$. Then the following conditions are satisfied by (1)–(5):

(6) $B_m(\tau)A'_m(\tau) = A_m(\tau)^2 - 1$,

(7) $A_m(\pi - \tau) = -A_m(\tau),\ B_m(\pi - \tau) = B_m(\tau)$,
(8) $A_m(\tau)C_m(\tau) = 1\ (0 < \tau < \pi/2)$.

Put

$$L_Y = -t\left(\frac{\partial}{\partial s}\right)_Y + s\left(\frac{\partial}{\partial t}\right)_Y, \quad Y = s\mathbf{e}_1 + t\mathbf{e}_{p+1},$$

which is the unit tangent vector field on S_0. We see $L(\xi J_\pm) = -L(\xi) \circ J_\pm$ for any smooth function ξ on S_0. Denote by $Y = Y(\tau) \in S_0$ as follows:

$$Y(\tau) = (\cos\tau)\mathbf{e}_1 + (\sin\tau)\mathbf{e}_{p+1}.$$

Now we define smooth functions on an open set of S_0 by

$$g(Y) = \begin{cases} B_m(\tau) & 0 \le \tau \le \pi, \\ B_m(-\tau) & -\pi \le \tau \le 0, \end{cases}$$

$$h(Y) = \begin{cases} -A_m(\tau) & 0 < \tau < \pi, \\ A_m(-\tau) & -\pi < \tau < 0, \end{cases}$$

$$k(Y) = \begin{cases} -C_m(\tau) & -\dfrac{\pi}{2} < \tau < \dfrac{\pi}{2}, \\ C_m(\pi - \tau) & \dfrac{\pi}{2} < \tau < \dfrac{3\pi}{2}. \end{cases}$$

Moreover we define

$$f_0(Y) = \begin{cases} (h(Y) : 1) & Y \ne \pm\mathbf{e}_1, \\ (1 : k(Y)) & Y \ne \pm\mathbf{e}_{p+1}. \end{cases}$$

Then we obtain a smooth function $f_0 \colon S_0 \to \mathbf{P}_1(\mathbf{R})$ by (7), (8).

Since $J_+Y(\tau) = Y(\pi - \tau)$ and $J_-Y(\tau) = Y(-\tau)$, we obtain

$$g(J_\pm(Y)) = g(Y),$$
$$f_0(Y) = (a : b) \Longrightarrow f_0(J_\pm(Y)) = (-a : b).$$

Then we see that the function f_0 satisfies the conditions (b), (d), (e).

Now we define a one-parameter group ϕ_0 on S_0 as the one corresponding to the tangent vector field gL, that is, ϕ_0 is defined by the following:

$$g(Y)L_Y(\xi) = \lim_{\theta \to 0} \frac{\xi(\phi_0(\theta, Y)) - \xi(Y)}{\theta}$$

for $Y \in S_0$ and any smooth function ξ on S_0. On the other hand, we see

$$g(Y)L_Y(h) = 1 - h(Y)^2 \quad \text{for} \quad Y \ne \pm\mathbf{e}_1,$$
$$g(Y)L_Y(k) = 1 - k(Y)^2 \quad \text{for} \quad Y \ne \pm\mathbf{e}_{p+1}$$

by (6)–(8). Hence we obtain $(d\xi/d\theta)(\phi_0(\theta, Y)) = 1 - \xi(\phi_0(\theta, Y))^2$ for $\xi = h, k$. Therefore we obtain $\xi(\phi_0(\theta, Y)) = (\xi(Y) + \tanh\theta)/(1 + \xi(Y)\tanh\theta)$ for $\xi = h, k$. Then we see the pair (ϕ_0, f_0) satisfies the condition (c). Moreover, we obtain $J_{\pm}\phi_0(\theta, J_{\pm}Y) = \phi_0(-\theta, Y)$. So the condition (a) holds for ϕ_0.

Consequently, the pair (ϕ_0, f_0) satisfies all conditions (a)–(e). Put Φ_m the corresponding smooth action of $\mathbf{Sp}(p,q)$ on $S^{4p+4q-1}$. Then we see the action Φ_m has just $2m$ open orbits on $S^{4p+4q-1}$.

Now we can state the following result.

Theorem 5.2. *For any positive integer m, there exists a smooth action of $\mathbf{Sp}(p,q)$ on $S^{4p+4q-1}$, which has just $2m$ open orbits.*

6. Concluding remark

For any real number c, a smooth action Ψ_c of $\mathbf{Sp}(p,q)$ on $S^{4p+4q-1}$ is defined by $\Psi_c(A, X) = AX\|AX\|^{-1}\exp(ic\log\|AX\|)$, where $i = \sqrt{-1}$. We call Ψ_c the twisted linear action [6]. For $c = 0$, the action Ψ_0 is described by $\Psi_0(A, X) = AX\|AX\|^{-1}$. This is the standard action considerd in the second half of the section 4.

The restricted $\mathbf{Sp}(p)\times\mathbf{Sp}(q)$-action of the twisted linear action Ψ_c is the standard action and we see that the twisted linear action Ψ_c has just three orbits and two of them are open orbits and one of them is compact orbit of codimension 1. Moreover we see that a matrix M is contained in the isotropy algebra at a point X of the compact orbit, if and only if $MX = (1 - ic)mX$ for some real number m.

By a routine work, we obtain the following result.

Theorem 6.1. *Between two twisted linear actions Ψ_c and $\Psi_{c'}$, there exists an equivariant homeomorphism if and only if $|c| = |c'|$.*

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