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ON LIFTS OF IRREDUCIBLE 2-BRAUER CHARACTERS OF SOLVABLE GROUPS

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1. Introduction

Let G be a finite group. Write $|G| = p^a k$, where p is a prime number and $(k, p) = 1$, and let $\mathbf{Q}_k = \mathbf{Q}(e^{2\pi i/k})$, the field generated by $e^{2\pi i/k}$ over the field $\mathbf{Q}$ of rationals. Recall that an ordinary character χ of G is said to be p-rational if $\chi(x) \in \mathbf{Q}_k$, for every $x \in G$.

Now, let φ be an irreducible 2-Brauer character of G and denote by n (resp. m), the number of ordinary (resp. 2-rational) irreducible characters ξ of G such that the restriction $\xi_{G_{2'}}$ of ξ to the subset $G_{2'}$ of $2'$-elements of G, is equal to φ.

Let V be a simple FG-module affording φ, where F is an algebraically closed field of characteristic 2, and let Q be a vertex of V.

If Q is cyclic and $|Q| > 1$, the module V belongs to a 2-block B of G having Q as a defect group (see Theorem VII.15.1 in [2]). By the theory of blocks with cyclic defect groups (see, for instance, Theorem 68.1 in [1]), we have $n = |Q|$.

More generally, assume now that G has a normal subgroup N such that $Q \not\subseteq N$ and that the quotient group QN/N is cyclic. Then, in case G is solvable, the main result of this paper (Theorem 1) asserts that $n \geq |QN/N|$ and that $m \geq 2$.

It is worth mentioning that the statement concerning m above is really an exclusive feature of the prime 2. In fact, it has been shown by I.M. Isaacs that if ϕ is an irreducible p-Brauer character of a p-solvable group H, where p is odd, then there exists a unique irreducible p-rational character θ of H such that $\theta_{H_{p'}} = \phi$ (see Theorem X.2.3 in [2]).

2. Background

Although the main result of this paper (Theorem 1) concerns ordinary characters and 2-Brauer characters of solvable groups, its proof relies heavily on Isaacs' theory of partial characters developed in [6, 7]. In this section, we review few concepts of that theory needed for our purpose.

Let π be an arbitrary set of primes and assume throughout this section that G is a finite π-separable group. Recall that the π'-partial characters of G are just the restrictions χ^0 of ordinary characters χ of G to the set of π'-elements of G. Furthermore, χ^0 is said to be irreducible if it cannot be written as a sum of two π'-partial characters. The set of irreducible π'-partial characters of G is denoted by $\mathrm{I}_{\pi'}(G)$. For any $\xi \in \mathrm{Irr}(G)$, there are uniquely determined nonnegative integers $d_{\xi\psi}$, such that $\xi^0 = \sum_\psi d_{\xi\psi}\psi$, where ψ runs through $\mathrm{I}_{\pi'}(G)$.

In case $\pi = \{p\}$, it follows from the Fong-Swan theorem that the π'-partial characters of G are exactly the Brauer characters (at p) and consequently $\mathrm{I}_{\pi'}(G) = \mathrm{IBr}(G)$.

Next, assume that K is a subgroup of G and that ψ is a π'-partial character of G. Then, it is obvious that the restriction ψ_K is a π'-partial character of K. For $\varphi \in \mathrm{I}_{\pi'}(K)$, we denote by $\mathrm{I}_{\pi'}(G \mid \varphi)$, the set of all $\omega \in \mathrm{I}_{\pi'}(G)$ such that φ is a constituent of ω_K. Induction τ^G of a π'-partial character τ of K can also be defined by using the usual formula of induced characters and applying it only to π'-elements. It is easy to see that τ^G is a π'-partial character of G.

In [9], a vertex of $\psi \in \mathrm{I}_{\pi'}(G)$ is defined to be a Hall π-subgroup of some subgroup J of G for which there exists $\alpha \in \mathrm{I}_{\pi'}(J)$ such that $\alpha^G = \psi$ and $\alpha(1)$ is a π'-number. It turns out that the set of vertices of ψ is not empty and that it forms a single conjugacy class of π-subgroups of G (see Theorem B in [9]). If ψ is an irreducible p-Brauer character of a p-solvable group, then it is not hard to see that the vertices of ψ defined above (when $\pi = \{p\}$), are exactly the vertices of the simple module (in characteristic p) affording ψ.

It is clear from the definitions that for every $\psi \in \mathrm{I}_{\pi'}(G)$, there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \psi$. However, χ is not unique in general. Nevertheless, in [6], Isaacs has canonically defined a set $\mathrm{B}_{\pi'}(G)$ of irreducible characters of G such that the map $\chi \mapsto \chi^0$ is a bijection of $\mathrm{B}_{\pi'}(G)$ onto $\mathrm{I}_{\pi'}(G)$.

Let now $N \triangleleft G$ and $\mu \in \mathrm{B}_{\pi'}(N)$. Two characters $\chi_1, \chi_2 \in \mathrm{Irr}(G \mid \mu)$ are said to be linked if there exists $\psi \in \mathrm{I}_{\pi'}(G)$ such that $d_{\chi_1\psi} \neq 0$ and $d_{\chi_2\psi} \neq 0$. The equivalence classes defined by the transitive extension of this linking relation are called relative π-blocks of G with respect to (N, μ), and the set of all these relative π-blocks is denoted by $\mathrm{Bl}_\pi(G \mid \mu)$ (see Section 3 in [11]). In case $(N, \mu) = (\langle 1\rangle, 1_{\langle 1\rangle})$, where $1_{\langle 1\rangle}$ is the trivial character of $\langle 1\rangle$, the relative π-blocks of G with respect to (N, μ) are just the π-blocks defined by M. Slattery [12].

3. The main theorem

We start this section by stating the main theorem of this paper.

Theorem 1. *Let G be a finite solvable group and let F be an algebraically closed field of characteristic 2. Let V be a simple FG-module with vertex Q and let φ be the irreducible Brauer character afforded by V. Suppose that there exists a normal subgroup N of G such that $Q \nsubseteq N$ and QN/N is cyclic. Then*

(i) *G has at least $|QN/N|$ ordinary irreducible characters χ such that the restriction $\chi_{G_{2'}}$ of χ to the subset $G_{2'}$ of $2'$-elements of G, is equal to φ.*

(ii) *G has at least two 2-rational irreducible characters ξ such that $\xi_{G_{2'}} = \varphi$.*

In order to prove this theorem, we need few preliminary results. For the sake of generality, all but the last of these results are proved in the general setting of finite π-separable groups, where π is an arbitrary set of prime numbers. (Note that a solvable group is necessarily π-separable.)

Before stating our first preliminary result, recall that a character-triple is a triple (H, M, α), where M is a normal subgroup of the group H and α is an H-invariant irreducible character of M. By definition (see Definition 11.23 in [5]), if the triple (H, M, α) is isomorphic to (H', M', α'), then there exists an isomorphism $\tau\colon H/M \to H'/M'$. If $M \subseteq L \subseteq H$ and L' is the subgroup of H' containing M' such that $\tau(L/M) = L'/M'$, then also by the definition of character-triple isomorphism, we have a bijection $\sigma_L\colon \mathrm{Irr}(L \mid \alpha) \to \mathrm{Irr}(L' \mid \alpha')$. Let σ be the union of the maps σ_L. Then, the pair (τ, σ) is the corresponding isomorphism from (H, M, α) to (H', M', α').

Lemma 2. *Let π be a set of primes and let H be a π-separable group. Let $M \lhd H$ and let α be an H-invariant π'-special character of M. Then, there exist a central extension H' of $\overline{H} = H/M$ by a π'-subgroup M' of H', a linear character α' of M' and bijections Ψ of $\mathrm{Irr}(H \mid \alpha)$ onto $\mathrm{Irr}(H' \mid \alpha')$ and Ψ^0 of $\mathrm{I}_{\pi'}(H \mid \alpha^0)$ onto $\mathrm{I}_{\pi'}(H' \mid \alpha')$ such that*

(a) *For any $\theta \in I_{\pi'}(H \mid \alpha^0)$, if ξ is any character in $\mathrm{Irr}(H \mid \alpha)$ such that $\xi^0 = \theta$, we have $\Psi^0(\theta) = \Psi(\xi)^0$.*

(b) *For any $\chi \in \mathrm{Irr}(H \mid \alpha)$ and any $\theta \in \mathrm{I}_{\pi'}(H \mid \alpha^0)$, we have $d_{\chi\theta} = d_{\Psi(\chi)\Psi^0(\theta)}$.*

(c) *The correspondence $\mathcal{B} \mapsto \Psi(\mathcal{B})$ is a bijection of $\mathrm{Bl}_\pi(H \mid \alpha)$ onto the set of (Slattery) π-blocks of H' over α'.*

(d) *If $\theta \in \mathrm{I}_{\pi'}(H \mid \alpha^0)$, then θ has a vertex Q such that QM/M is isomorphic to some vertex Q' of $\Psi^0(\theta)$.*

Proof. This lemma without (d), is the invariant case of Theorem 3.1 in [10]. Recall, by the proof of that theorem, that the triple (H', M', α') is chosen to be isomorphic to (H, M, α). In other words, there exists a character-triple isomorphism $(\tau, \sigma)\colon (H, M, \alpha) \to (H', M', \alpha')$. The bijection Ψ is just the map σ_H introduced just before the lemma. All we need now is to show (d).

Let $\theta \in \mathrm{I}_{\pi'}(H \mid \alpha^0)$. Then, there exists $\xi \in \mathrm{B}_{\pi'}(H)$ such that $\theta = \xi^0$. Since the irreducible constituents of ξ_M are all in $\mathrm{B}_{\pi'}(M)$ (Corollary 7.5 in [6]) and θ lies over α^0, it follows that ξ lies over α. Now, as α is π'-special, Lemma 1.2 in [13] says that there exists a nucleus (K, ρ) of ξ such that $M \subseteq K$ and $\rho \in \mathrm{Irr}(K \mid \alpha)$ (see Section 4 of [6], for the definition of the nucleus). In particular, we have $\xi = \rho^H$ and hence $\theta = \xi^0 = (\rho^0)^H$. Moreover, since $\xi \in \mathrm{B}_{\pi'}(H)$, the character ρ is π'-special. Therefore, a Hall π-subgroup Q of K is a vertex for θ.

Next, by Lemma 11.35 in [5], we have

$$\Psi(\xi) = \Psi(\rho^H) = \sigma_H(\rho^H) = (\sigma_K(\rho))^{H'}.$$

As $\Psi^0(\theta) = \Psi(\xi)^0$ by (a), we get that $\Psi^0(\theta) = (\sigma_K(\rho)^0)^{H'}$. Let K' be the subgroup of H' containing M' such that $\tau(K/M) = K'/M'$. Now, since $\Psi^0(\theta) \in \mathrm{I}_{\pi'}(H')$, it follows that $\sigma_K(\rho)^0 \in \mathrm{I}_{\pi'}(K')$.

By Lemma 11.24 in [5], we have $\rho(1)\alpha'(1) = \sigma_K(\rho)(1)\alpha(1)$. Since $\rho(1)$, $\alpha(1)$ and $\alpha'(1)$ are all π'-numbers, we conclude that $\sigma_K(\rho)(1)$ is a π'-number. Hence, a Hall π-subgroup Q' of K' is a vertex of $\Psi^0(\theta)$.

Now, QM/M is a Hall π-subgroup of K/M and $Q'M'/M'$ is a Hall π-subgroup of K'/M'. As $K/M \cong K'/M'$, we obtain $QM/M \cong Q'M'/M' \cong Q'/Q' \cap M'$. Furthermore, since Q' is a π-group and M' is a π'-group, we get $Q' \cap M' = 1$ and it follows that $QM/M \cong Q'$. This proves (d) and completes the proof of the lemma. □

We can now improve Theorem 3.1 of [11].

Theorem 3. *Let N be a normal subgroup of a π-separable group G and let $\mu \in B_{\pi'}(N)$ with $T = \mathrm{I}_G(\mu)$. Then, there exist a central extension U of $\overline{T} = T/N$ by a π'-subgroup Z of U, a linear character ν of Z and bijections Γ of $\mathrm{Irr}(G \mid \mu)$ onto $\mathrm{Irr}(U \mid \nu)$ and Γ^0 of $I_{\pi'}(G \mid \mu^0)$ onto $I_{\pi'}(U \mid \nu)$ such that the following hold.*

(a) *For any $\chi \in \mathrm{Irr}(G \mid \mu)$ and any $\phi \in I_{\pi'}(G \mid \mu^0)$, we have $d_{\chi\phi} = d_{\Gamma(\chi)\Gamma^0(\phi)}$.*

(b) *The correspondence $\mathcal{B} \mapsto \Gamma(\mathcal{B})$ is a bijection of $\mathrm{Bl}_\pi(G \mid \mu)$ onto the set of (Slattery) π-blocks of U over ν.*

(c) *If $\phi \in I_{\pi'}(G \mid \mu^0)$, then ϕ has a vertex Q such that QN/N is isomorphic to some vertex P of $\Gamma^0(\phi)$.*

Proof. Let (W, γ) be a nucleus for μ and let $S = \mathrm{N}_T((W, \gamma))$, the stabilizer of (W, γ) in T. First, we note that this theorem without (c), is Theorem 3.1 in [11]. Recall, by its proof, that Theorem 3.1 of [11] is obtained by first applying Theorem 3.2 in [11] to the group G, the normal subgroup N and the character $\mu \in \mathrm{B}_{\pi'}(N)$, and then applying the invariant form of Theorem 3.1 in [10] (this is Lemma 2 above without (d)) to the group S, the normal subgroup W and the S-invariant π'-special character γ of W.

To complete the proof, we need to show (c). Let $\phi \in \mathrm{I}_{\pi'}(G \mid \mu^0)$. By Theorem 3.2 (b) in [11], there exists a partial character $\theta \in \mathrm{I}_{\pi'}(S \mid \gamma^0)$ such that $\phi = \theta^G$. Now, $\Gamma^0(\phi)$ is the element of $\mathrm{I}_{\pi'}(U \mid \nu)$ corresponding to θ via the bijection Ψ^0 of Lemma 2.

By Lemma 2 (d), θ has a vertex Q such that QW/W is isomorphic to some vertex P of $\Gamma^0(\phi)$ $(= \Psi^0(\theta))$.

Now, by Lemma 3.6 (a) of [11], we have $S \cap N = W$. Therefore, we get $Q \cap N = Q \cap S \cap N = Q \cap W$. Since $QN/N \cong Q/Q \cap N$ and $QW/W \cong Q/Q \cap W$, it follows that $QN/N \cong QW/W \cong P$. Finally, note that the subgroup Q is also a vertex of ϕ as $\phi = \theta^G$. This proves (c) and finishes the proof of the theorem. □

Let (H, M, α) be a character-triple, where H is a π-separable group and α is π'-special, and let $\mathcal{B}$ be a relative π-block of H with respect to (M, α).

Let R be the normal subgroup of H containing M such that $R/M = \mathrm{O}_{\pi'}(H/M)$. If $\zeta \in \mathrm{Irr}(H \mid \alpha)$, then by Lemma 2.3 in [6], there exists a π'-special character δ of R such that δ is a constituent of ζ_R. By Lemma 3.2 in [10], if $\beta \in \mathrm{B}_{\pi'}(H)$ satisfies $d_{\zeta\beta^0} \neq 0$, we have $\beta \in \mathrm{Irr}(H \mid \delta)$. Therefore, the constituents of β_R are precisely the constituents of ζ_R by Clifford's theorem (Theorem 6.2 in [5]). It follows that if $\zeta' \in \mathrm{Irr}(H \mid \alpha)$ also satisfies $d_{\zeta'\beta^0} \neq 0$, then ζ' also lies over the H-orbit of δ. This implies that the characters of $\mathcal{B}$ all lie over the H-orbit of some π'-special character η of R. So, there exists a relative π-block $\mathcal{B}_0$ of H with respect to (R, η) such that $\mathcal{B} \subseteq \mathcal{B}_0$. Assume now that $\xi \in \mathcal{B}$ and $\xi_0 \in \mathcal{B}_0$ satisfy $d_{\xi\omega} \neq 0$ and $d_{\xi_0\omega} \neq 0$ for some $\omega \in \mathrm{I}_{\pi'}(H)$. Then, as η lies over α, the character ξ_0 lies over α and it follows that $\xi_0 \in \mathcal{B}$. Consequently, $\mathcal{B} = \mathcal{B}_0$ and thus we may view $\mathcal{B}$ as a relative π-block of H with respect to (R, η).

We now have the following "Fong reduction" type result.

Lemma 4. *Let $N \lhd G$, where G is π-separable and let $\mu \in B_{\pi'}(N)$. If $\mathcal{B} \in \mathrm{Bl}_\pi(G \mid \mu)$, then there exist a subgroup A of G, a normal subgroup E of A satisfying $O_{\pi'}(A/E) = 1$ and an A-invariant π'-special character β of E such that induction defines a bijection of $\mathrm{Irr}(A \mid \beta)$ onto $\mathcal{B}$. Furthermore, if D is a Hall π-subgroup of A, then D is a defect group of $\mathcal{B}$ and DN/N is isomorphic to DE/E.*

Proof. Let (W, γ) be a nucleus for μ and let $S = \mathrm{N}_T((W, \gamma))$, where $T = \mathrm{I}_G(\mu)$. (Note that γ is π'-special as $\mu \in \mathrm{B}_{\pi'}(N)$.) By Theorem 3.2 of [11], there exists a relative π-block $\mathcal{B}_0 \in \mathrm{Bl}_\pi(S \mid \gamma)$ such that the induction map $\alpha \mapsto \alpha^G$ defines a bijection of $\mathcal{B}_0$ onto $\mathcal{B}$. Let now P be any defect group of $\mathcal{B}_0$. Then, P is also a defect group of $\mathcal{B}$ (see Section 4 in [11]). Since $P \subseteq S$, we have $P \cap N = P \cap W$, by Proposition 4.1 in [11] and it follows that $PN/N \cong PW/W$.

Now, to complete the proof, we may therefore assume that μ is a G-invariant π'-special character.

Let R be the normal subgroup of G containing N such that $R/N = \mathrm{O}_{\pi'}(G/N)$. By the discussion preceding the lemma, there exists a π'-special character η of R lying over μ such that $\mathcal{B} \in \mathrm{Bl}_\pi(G \mid \eta)$. Next, let $J = \mathrm{I}_G(\eta)$. Then, by Lemma 3.4 in [10], there is a unique relative π-block $\widehat{\mathcal{B}}$ of J with respect to (R, η) such that the induction map $\chi \mapsto \chi^G$ is a bijection of $\widehat{\mathcal{B}}$ onto $\mathcal{B}$.

CASE 1. Assume $J = G$. Then, the character η is G-invariant and so by Lemma 2, there exist a central extension G' of $\overline{G} = G/R$ by a π'-subgroup R' of G', a linear character η' of R' and a bijection Ψ of $\mathrm{Irr}(G \mid \eta)$ onto $\mathrm{Irr}(G' \mid \eta')$ such that the correspondence $b \mapsto \Psi(b)$ is a bijection of $\mathrm{Bl}_\pi(G \mid \eta)$ onto the set of (Slattery) π-blocks of G' over η'.

Since $G/R \cong G'/R'$ and $\mathrm{O}_{\pi'}(G/R) = 1$, we get that $\mathrm{O}_{\pi'}(G'/R') = 1$, and thence

$O_{\pi'}(G') = R'$. By Theorem 2.8 in [12], G' has a single π-block over η'. It follows that $\mathrm{Irr}(G \mid \eta)$ consists of the single relative π-block $\mathcal{B}$. We can then take $A = G$, $E = R$ and $\beta = \eta$.

By the definition of defect groups (see Section 4 in [10]), if D is a Hall π-subgroup of G, then D is a defect group for $\mathcal{B}$. Next, we show that $DN/N \cong DR/R$. Since $DN/N \cong D/D \cap N$ and $DR/R \cong D/D \cap R$, it suffices to show that $D \cap N = D \cap R$.

First, note that $D \cap N$ is a Hall π-subgroup of N. Since R/N is a π'-group, it follows that $D \cap N$ is also a Hall π-subgroup of R. Hence, $D \cap N = D \cap R$, as wanted.

CASE 2. Assume $J < G$. Then, working by induction on the group order, we can find a subgroup A of J, a normal subgroup E of A satisfying $O_{\pi'}(A/E) = 1$ and an A-invariant π'-special character β of E such that induction defines a bijection of $\mathrm{Irr}(A \mid \beta)$ onto $\widehat{\mathcal{B}}$. Moreover, if D is a Hall π-subgroup of A, then D is a defect group of $\widehat{\mathcal{B}}$ and $DR/R \cong DE/E$.

Now, since induction of characters defines a bijection of $\widehat{\mathcal{B}}$ onto $\mathcal{B}$, the induction map $\theta \mapsto \theta^G$ is a bijection of $\mathrm{Irr}(A \mid \beta)$ onto $\mathcal{B}$. Next, by the definition of defect groups (Section 4 of [10]), the subgroup D is a defect group of $\mathcal{B}$. Furthermore, by Lemma 4.1 in [10], $D \cap N$ is a Hall π-subgroup of N. Now, just as in case 1, it follows that $DN/N \cong DR/R$, and consequently $DN/N \cong DE/E$. This completes the proof of the lemma. □

Lemma 5. *Let $\theta \in B_{2'}(G)$, where G is solvable and let Q be a vertex of $\varphi = \theta^0$. Suppose that N is a normal subgroup of G such that $Q \nsubseteq N$ and QN/N is cyclic, and let μ be an irreducible constituent of θ_N. Then, $\mu \in B_{2'}(N)$ and if $\theta \in \mathcal{B} \in \mathrm{Bl}_2(G \mid \mu)$, we have $QN/N \cong DN/N$, for any defect group D of $\mathcal{B}$. Furthermore, $\mathcal{B} = \{\chi \in \mathrm{Irr}(G \mid \mu) : \chi^0 = \varphi\}$ and the number of elements of $\mathcal{B}$ is exactly $|QN/N|$.*

Proof. Since $\theta \in \mathrm{Irr}(G \mid \mu)$, the character μ lies in $B_{2'}(N)$ by Corollary 7.5 in [6]. Consequently, we have $\mu^0 \in I_{2'}(N)$ and $\varphi \in I_{2'}(G \mid \mu^0)$.

By Theorem 3, there exist a solvable group U, a $2'$-subgroup $Z \subseteq Z(U)$, a linear character ν of Z and bijections Γ of $\mathrm{Irr}(G \mid \mu)$ onto $\mathrm{Irr}(U \mid \nu)$ and Γ^0 of $I_{2'}(G \mid \mu^0)$ onto $I_{2'}(U \mid \nu)$. Furthermore, $\omega = \Gamma^0(\varphi)$ has a vertex P such that $P \cong QN/N$.

Let $b = \Gamma(\mathcal{B})$. By Theorem 3 (b), b is a 2-block of U over ν. Moreover, ω is an irreducible Brauer character associated with b. As QN/N is cyclic, the vertex P is cyclic and it follows by Theorem VII.15.1 in [2], that b has P as a defect group.

Let now D be any defect group for $\mathcal{B}$. Then, by Theorem 4.2 in [11], we have $D/D \cap N \cong P$. Since $QN/N \cong P$, it follows that $QN/N \cong DN/N$.

Next, by the theory of blocks with cyclic defect groups (Theorem 68.1 in [1]), ω is the unique irreducible Brauer character associated with b and there are exactly $|P|$ (= $|QN/N|$) ordinary irreducible characters λ in b. Furthermore, every character λ lies over ν and satisfies $\lambda^0 = \omega$. In particular, we have $b = \{\eta \in \mathrm{Irr}(U \mid \nu) : \eta^0 = \omega\}$.

It follows from Theorem 3 (a) that $\mathcal{B} = \Gamma^{-1}(b) = \{\chi \in \mathrm{Irr}(G \mid \mu) : \chi^0 = \varphi\}$, and consequently, the number of elements of $\mathcal{B}$ is equal to the number $|QN/N|$ of elements in b. The proof of the lemma is now complete. □

We are almost ready to start the proof of Theorem 1. All we need now are few general facts about p-rational characters.

Let G be any finite group and for an integer $h \geq 1$, denote by $\mathbf{Q}_h$ the field $\mathbf{Q}(e^{2\pi i/h})$ generated by $e^{2\pi i/h}$ over the field $\mathbf{Q}$ of rationals. Now, fix a prime p and write $|G| = l = p^a k$, where $(k, p) = 1$. Let $\mathcal{G}$ denote the Galois group $\mathrm{Gal}(\mathbf{Q}_l/\mathbf{Q}_k)$. If χ is a character (resp. irreducible character) of G and if $\sigma \in \mathcal{G}$, the function χ^σ defined by $\chi^\sigma(x) = \chi(x)^\sigma$ is also a character (resp. irreducible character) of G. It is clear from the definition of p-rational characters given in the introduction of this paper, that χ is p-rational if and only if $\chi^\sigma = \chi$ for all $\sigma \in \mathcal{G}$.

Next, let H be a subgroup of G. Then, $\mathbf{Q}_{|H|} \subseteq \mathbf{Q}_l$ and it follows that θ^σ is defined for every character θ of H and every $\sigma \in \mathcal{G}$.

We can now prove our main result.

Proof of Theorem 1. Let θ be the element of $\mathrm{B}_{2'}(G)$ such that $\theta^0 = \varphi$, and fix an irreducible constituent μ of θ_N. By Lemma 5, $\mu \in \mathrm{B}_{2'}(N)$ and if $\theta \in \mathcal{B} \in \mathrm{Bl}_2(G \mid \mu)$, we have, $\mathcal{B} = \{\chi \in \mathrm{Irr}(G \mid \mu) : \chi^0 = \varphi\}$, and the number of elements of $\mathcal{B}$ is $|QN/N|$. This suffices to prove (i). Next, we prove (ii).

By Lemma 4, there exist subgroups $E \lhd A \subseteq G$ satisfying $\mathrm{O}_{2'}(A/E) = 1$ and an A-invariant $2'$-special character β of E such that induction defines a bijection of $\mathrm{Irr}(A \mid \beta)$ onto $\mathcal{B}$. Furthermore, if D is a Sylow 2-subgroup of A, then D is a defect group of $\mathcal{B}$ and $DE/E \cong DN/N$. By Lemma 5, we have $DN/N \cong QN/N$. Therefore, $DE/E \cong QN/N$ and it follows that DE/E is a cyclic 2-group. Moreover, as $Q \nsubseteq N$, we have that $|DE/E| > 1$.

Next, write $|G| = 2^r k$ and $|A| = h = 2^s m$, where both k and m are odd integers. Assume that a character $\zeta \in \mathrm{Irr}(A \mid \beta)$ is 2-rational. Then, the values of ζ are in $\mathbf{Q}_m$. Since m divides k, the values of ζ lie in $\mathbf{Q}_k$ and it follows that the values of the character ζ^G all lie in $\mathbf{Q}_k$. In other words, ζ^G is 2-rational. Therefore, to show (ii), it suffices to find two 2-rational characters in $\mathrm{Irr}(A \mid \beta)$.

Recall that D is a Sylow 2-subgroup of A. Then, DE/E is a Sylow 2-subgroup of A/E. Since $\mathrm{O}_{2'}(A/E) = 1$ and DE/E is cyclic, it follows from Theorem 6.3.3 in [4], that $DE/E \lhd A/E$. Now, let R be the subgroup of A such that R/E is the unique subgroup of DE/E of index 2. As $DE/E \lhd A/E$, it is clear that $R/E \lhd A/E$, and so $R \lhd A$.

Since β is A-invariant, Corollary 4.8 in [3] implies that there exists a $2'$-special character $\zeta_0 \in \mathrm{Irr}(A \mid \beta)$. Set $(\zeta_0)^0 = \omega$. Next, fix an irreducible constituent η of $(\zeta_0)_R$ and write $S = \{\lambda \in \mathrm{Irr}(A \mid \eta) : \lambda^0 = \omega\}$. As ζ_0 is $2'$-special, D is a vertex of ω. Moreover, since DR/R is cyclic of order 2, Lemma 5 says that S contains exactly 2 elements, the character ζ_0, of course, and another character ζ_1.

As $\zeta_0 \in \mathrm{Irr}(A \mid \beta)$ and β is A-invariant, we have $\eta \in \mathrm{Irr}(R \mid \beta)$, and consequently $S \subseteq \mathrm{Irr}(A \mid \beta)$. So now, to complete the proof, it suffices to show that the characters ζ_0 and ζ_1 are 2-rational. First, note that ζ_0 is 2-rational by Lemma 3.1 in [8]. Next, we prove that ζ_1 is 2-rational.

Write $(\zeta_1)_R = \sum_{i=1}^{n} \eta_i$, where $\eta_i \in \mathrm{Irr}(R)$ and $\eta_1 = \eta$. Now, let $\sigma \in \mathrm{Gal}(\mathbf{Q}_h/\mathbf{Q}_m)$. Then, for each i, η_i^σ is well defined (see the remarks preceding the proof) and $(\zeta_1^\sigma)_R = \sum_{i=1}^{n} \eta_i^\sigma$. Since ζ_0 is $2'$-special, then so is η by Lemma 2.2 in [6]. Hence, η is 2-rational by Lemma 3.1 in [8]. In other words, the values of η are in $\mathbf{Q}_l$, where l is the order of a Hall $2'$-subgroup of R. As l divides m, we have $\mathbf{Q}_l \subseteq \mathbf{Q}_m$ and it follows that $\eta^\sigma = \eta$. This shows that $\zeta_1^\sigma \in \mathrm{Irr}(A \mid \eta)$. Next, we have $(\zeta_1)^0 = \omega$. So, clearly $(\zeta_1^\sigma)^0 = \omega$ and we conclude that $\zeta_1^\sigma \in S$. Now, as $S = \{\zeta_0, \zeta_1\}$ and ζ_0 is 2-rational, we have $\zeta_1^\sigma = \zeta_1$, necessarily. Hence, ζ_1 is 2-rational, as wanted. □

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