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STABLE EQUIVALENCE INDUCED BY A SOCLE EQUIVALENCE

YOSUKE OHNUKI

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0. Introduction

Throughout the paper K will denote a fixed field. By an algebra, we mean a basic connected finite dimensional non-simple associative K-algebra with an identity. For an algebra Λ, we shall denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules, and by $\underline{\operatorname{mod}}\,\Lambda$ the stable module category of $\operatorname{mod}\Lambda$. We denote by $\overline{a}$ the residue class of $a \in \Lambda$ in $\Lambda/\operatorname{soc}\Lambda$.

Let Λ and Λ' be selfinjective algebras. In this paper, we are interested in the question of when a socle equivalence between Λ and Λ' induces a stable equivalence $\underline{\operatorname{mod}}\,\Lambda \simeq \underline{\operatorname{mod}}\,\Lambda'$, where Λ and Λ' are said to be socle equivalent if there is an algebra isomorphism between $\Lambda/\operatorname{soc}\Lambda$ and $\Lambda'/\operatorname{soc}\Lambda'$. It is proved in [6] that Λ and Λ' are socle equivalent and stably equivalent provided that Λ and Λ' are Hochschild extension algebras of an algebra, say A, without oriented cycles in its quiver, by $\operatorname{Hom}_K(A, K)$. It is, however, that a socle equivalence does not imply a stable equivalence, in general (see [3], [2]). The aim of this paper is to show a sufficient condition in terms of a K-regular map for socle equivalent selfinjective algebras to be stable equivalence, where K-linear map $\lambda : \Lambda \to K$ is called a regular map of Λ if $\lambda = \varphi(1_\Lambda) : \Lambda \to K$ for an isomorphism $\varphi : \Lambda \to \operatorname{Hom}_K(\Lambda, K)$ in $\operatorname{mod}\Lambda$. Our main theorem is stated as follows.

Theorem. *Let Λ and Λ' be socle equivalent selfinjective algebras, say $p : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$. Assume that there are regular maps λ of Λ and λ' of Λ' such that $\lambda(ab) = \lambda'(a'b')$ for all $a, b \in \operatorname{rad}\Lambda$ and $a', b' \in \operatorname{rad}\Lambda'$ with $\overline{a}' = p(\overline{a})$ and $\overline{b}' = p(\overline{b})$. Then the stable categories $\underline{\operatorname{mod}}\,\Lambda$ and $\underline{\operatorname{mod}}\,\Lambda'$ are equivalent.*

In the last section, we shall see that Hochschild extension algebras, of an algebra without oriented cycles in its quiver, satisfy the assumption for regular maps in the theorem.

For basic background and notations, we refer to [1] and [7].

1. Preliminaries

Let Λ be a selfinjective algebra. We denote by $\mathcal{P}_\Lambda$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of finite dimensional projective left Λ-modules, and by $\mathrm{mor}\,\mathcal{P}_\Lambda$ the category of morphisms in $\mathcal{P}_\Lambda$ whose objects are the morphisms $f : P_1 \to P_0$ such that $\mathrm{soc}\, P_0 \subseteq \mathrm{Im}\, f \subseteq \mathrm{rad}\, P_0$. A morphism $\varphi : f \to f'$ between two objects $f : P_1 \to P_0$ and $f' : P_1' \to P_0'$ is a pair $\varphi = (\varphi_1, \varphi_0)$ formed by two homomorphisms $\varphi_1 : P_1 \to P_1'$ and $\varphi_0 : P_0 \to P_0'$ with $\varphi_0 f = f'\varphi_1$.

On the set of morphisms $\varphi = (\varphi_1, \varphi_0)$ from $f : P_1 \to P_0$ to $f' : P_1' \to P_0'$, we have an equivalence relation $\sim$ such that $\varphi = (\varphi_1, \varphi_0) \sim \psi = (\psi_1, \psi_0)$ if $\varphi_0 - \psi_0 = f'\omega$ for some $\omega : P_0 \to P_1'$. We denote by $\mathrm{hom}\,\mathcal{P}_\Lambda$ the homotopy category of $\mathrm{mor}\,\mathcal{P}_\Lambda$ whose objects are the objects of $\mathrm{mor}\,\mathcal{P}_\Lambda$ and the morphisms $[\varphi]$ in $\mathrm{hom}\,\mathcal{P}_\Lambda$ are the equivalence classes of morphisms $\varphi = (\varphi_1, \varphi_0)$ in $\mathrm{mor}\,\mathcal{P}_\Lambda$ with respect to $\sim$. Then we have the fully faithful and dense K-linear functor $\mathrm{Cok} : \mathrm{hom}\,\mathcal{P}_\Lambda \to \mathrm{mod}(\Lambda/\mathrm{soc}\,\Lambda)$ which assigns to an object $f : P_1 \to P_0$ of $\mathrm{hom}\,\mathcal{P}_\Lambda$ its cokernel $\mathrm{Cok}\, f$ and to a morphism $[\varphi] : f \to f'$ the induced morphism $\mathrm{Cok}\,\varphi : \mathrm{Cok}\, f \to \mathrm{Cok}\, f'$. The categories $\mathrm{hom}\,\mathcal{P}_\Lambda$ and $\mathrm{mod}(\Lambda/\mathrm{soc}\,\Lambda)$ are equivalent. See [6] or [4] in detail.

The following lemma is proved in [4].

Lemma 1.1. *Let $[\varphi] : f \to f'$ be a morphism in $\mathrm{hom}\,\mathcal{P}_\Lambda$. Then $\mathrm{Cok}\,\varphi$ factors through a projective module in $\mathrm{mod}\,\Lambda$ if and only if there is a morphism $\psi = (\psi_1, \psi_0) : f \to f'$ in $\mathrm{mor}\,\mathcal{P}_\Lambda$ such that $[\varphi] = [\psi]$ and $\psi_0 f = 0$.*

By Lemma 1.1, we have an equivalence relation $\approx$ such that $[\varphi] = [(\varphi_1, \varphi_0)] \approx [\psi] = [(\psi_1, \psi_0)]$ in $\mathrm{hom}\,\mathcal{P}_\Lambda$ if and only if $\mathrm{Cok}(\varphi - \psi)$ factors through a projective module in $\mathrm{mod}\,\Lambda$. We denote by $\underline{\mathrm{hom}}\,\mathcal{P}_\Lambda$ the stable homotopy category whose objects are objects in $\mathrm{hom}\,\mathcal{P}_\Lambda$ and morphisms $[[\varphi]]$ are equivalence classes of morphisms $[\varphi]$ in $\mathrm{hom}\,\mathcal{P}_\Lambda$ with respect to $\approx$. Then two categories $\underline{\mathrm{hom}}\,\mathcal{P}_\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda$ are naturally equivalent.

For an automorphism ν of Λ and a Λ-module M, ${}_\nu M$ denotes the Λ-module obtained from M by changing the operation of Λ as follows: $a \cdot m = \nu(a)m$ for each $a \in \Lambda$ and $m \in M$. Similarly, N_ν is defined for a right Λ-module N. We recall a regular map attached to a selfinjective algebra. A map λ is called a regular map of Λ if λ is a K-linear map from Λ to K and satisfies $\lambda(\Lambda a) \neq 0$ for any nonzero element a of Λ. A regular map λ of Λ is called ν-commutative if $\lambda(ba) = \lambda(\nu(a)b)$ for all $a, b \in \Lambda$. Let $\varphi : \Lambda \to \mathrm{Hom}_K(\Lambda, K)$ be a left Λ-module isomorphism. The right multiplication map $R_a : \Lambda \to \Lambda, b \mapsto ba$ induces an algebra automorphism ν of Λ such that $\varphi : \Lambda \to \mathrm{Hom}_K(\Lambda, K)_\nu$ is a Λ-bimodule isomorphism, and then $\varphi(1_\Lambda)$ is a ν-commutative regular map of Λ, because $\{\varphi(1_\Lambda)\}(\nu(a)b) = \{\varphi(1_\Lambda)a\}(b) = \{\varphi(a)\}(b) = \{a\varphi(1_\Lambda)\}(b) = \{\varphi(1_\Lambda)\}(ba)$. The automorphism ν is uniquely determined by Λ up to inner automorphism, and is called the Nakayama automorphism of Λ. Note that the Nakayama automorphism always exists for a selfinjective algebra.

Conversely, given a regular map λ of Λ, we have the left Λ-module isomorphism $\varphi_\lambda : \Lambda \to \mathrm{Hom}_K(\Lambda, K)$, $a \mapsto (b \mapsto \lambda(ba))$. Note that a regular map λ of Λ is a ν-commutative, where ν is the automorphism defined by φ_λ. We denote by $\overline{\nu}$ the algebra automorphism of $\Lambda/\operatorname{soc}\Lambda$ given by $\overline{\nu}(\overline{a}) = \overline{\nu(a)}$ for $a \in \Lambda$, because $\nu(\operatorname{soc}\Lambda) = \operatorname{soc}\Lambda$. Also, note that $\mathrm{Hom}_K(\Lambda/\operatorname{rad}\Lambda, K) \simeq {}_{\overline{\nu}}\operatorname{soc}\Lambda \simeq \operatorname{soc}\mathrm{Hom}_K(\Lambda/\operatorname{soc}\Lambda, K)$, because Λ is non-simple selfinjective. We also denote by $\overline{x}$ the residue class in $\mathrm{Hom}_K(\Lambda/\operatorname{soc}\Lambda, K)/\mathrm{Hom}_K(\Lambda/\operatorname{rad}\Lambda, K)$ of $x \in \mathrm{Hom}_K(\Lambda/\operatorname{soc}\Lambda, K)$. We need the following isomorphisms.

Lemma 1.2. *The Λ-bimodule isomorphism $\varphi : \Lambda \to \mathrm{Hom}_K(\Lambda, K)_\nu$ induces the following isomorphisms*:

$$\begin{aligned} \varphi_1 &: \Lambda \to {}_{\nu^{-1}}\mathrm{Hom}_K(\Lambda, K) \\ \varphi_2 &: \operatorname{rad}\Lambda \to {}_{\overline{\nu}^{-1}}\mathrm{Hom}_K(\Lambda/\operatorname{soc}\Lambda, K) \\ \varphi_3 &: \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \to {}_{\overline{\nu}^{-1}}\mathrm{Hom}_K(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda, K), \end{aligned}$$

where φ_1 is a Λ-bimodule isomorphism and φ_2, φ_3 are $\Lambda/\operatorname{soc}\Lambda$-bimodule isomorphisms.

Proof. We set $\lambda = \varphi(1_\Lambda)$ which is ν-commutative. Then $\lambda(ab) = \lambda(\nu^{-1}(ab)) = \lambda(b\nu^{-1}(a))$ for all $a, b \in \Lambda$. The φ_1, φ_2 or φ_3, respectively, is defined by $\{\varphi_1(a)\}(b) = \lambda(ab)$, $\{\varphi_2(c)\}(\overline{a}) = \lambda(ca)$ or $\{\varphi_3(\overline{c})\}(\overline{d}) = \lambda(cd)$, respectively, for all $a, b \in \Lambda$ and $c, d \in \operatorname{rad}\Lambda$. □

2. Stable equivalence

In this section we shall prove the theorem stated in Introduction. The idea of the proof owes to the previous works [6] and [4].

Let Λ and Λ' be two selfinjective algebras which are socle equivalent, say $p : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$. Let $\{e_i\}_{i=1}^n$ and $\{e'_i\}_{i=1}^n$ be complete sets of orthogonal primitive idempotents of Λ and Λ', respectively, such that $p(\overline{e_i}) = \overline{e_i}'$ for each i. For each $a \in \Lambda$, we choose a representative, say $a' \in \Lambda'$, of the residue class $p(\overline{a})$, and define a map by the correspondence induced by algebra isomorphism p

$$\widetilde{p} : \Lambda \to \Lambda', a \mapsto a'.$$

For $a \in e_i\Lambda e_j$, the right multiplication map R_a from Λe_i to Λe_j is also denoted by a simply.

Let $\varphi : \Lambda \to \mathrm{Hom}_K(\Lambda, K)_\nu$ and $\varphi' : \Lambda' \to \mathrm{Hom}_K(\Lambda', K)_{\nu'}$ be bimodule isomorphisms. It is immediate that

$$\mathrm{Hom}_K(p^{-1}, K) : {}_{\overline{\nu}^{-1}}\mathrm{Hom}_K(\Lambda/\operatorname{soc}\Lambda, K) \to {}_{p\overline{\nu}^{-1}}\mathrm{Hom}_K(\Lambda'/\operatorname{soc}\Lambda', K)_p$$

is a $\Lambda/\operatorname{soc}\Lambda$-bimodule isomorphism. From this, the composite

$$\psi := {\varphi'_2}^{-1}\operatorname{Hom}(p^{-1}, K)\varphi_2 : \operatorname{rad}\Lambda \to {}_{\bar{\nu}' p \bar{\nu}^{-1}}(\operatorname{rad}\Lambda')_p$$

is a $\Lambda/\operatorname{soc}\Lambda$-bimodule isomorphism

$$\begin{array}{ccc} \operatorname{rad}\Lambda & \xrightarrow{\psi} & {}_{\bar{\nu}' p \bar{\nu}^{-1}}(\operatorname{rad}\Lambda')_p \\ \varphi_2 \big\downarrow & & \big\downarrow \varphi'_2 \\ {}_{\bar{\nu}^{-1}}\operatorname{Hom}(\Lambda/\operatorname{soc}\Lambda, K) & \xrightarrow{\operatorname{Hom}(p^{-1},K)} & {}_{p\bar{\nu}^{-1}}\operatorname{Hom}(\Lambda'/\operatorname{soc}\Lambda', K)_p \end{array} \tag{2.1}$$

where $\varphi_2 : \operatorname{rad}\Lambda \to {}_{\bar{\nu}^{-1}}\operatorname{Hom}(\Lambda/\operatorname{soc}\Lambda, K)$ and $\varphi'_2 : {}_{\bar{\nu}'}\operatorname{rad}\Lambda' \to \operatorname{Hom}(\Lambda'/\operatorname{soc}\Lambda', K)$ are the isomorphisms in Lemma 1.2. We set $q = \bar{\nu}' p \bar{\nu}^{-1} : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$ and $q(\bar{e_i}) = \bar{e_i}''$ for each i. Then, $\psi(arb) = \tilde{q}(a)\psi(r)\tilde{p}(b)$ for any $a, b \in \Lambda$ and $r \in \operatorname{rad}\Lambda$, and we can lift $\{\bar{e_i}''\}_{i=1}^n$ to $\{e''_i\}_{i=1}^n$ a complete set of orthogonal primitive idempotents of Λ'. There uniquely exists j for each i such that $\Lambda' e'_i \simeq \Lambda' e''_j$ as Λ'-modules, because Λ' is assumed to be basic.

Assuming $\overline{\psi(a)} = p(\bar{a})$ for all $a \in \operatorname{rad}\Lambda$, let us define a functor $G = (G_0, G_1) : \hom\mathcal{P}_\Lambda \to \hom\mathcal{P}_{\Lambda'}$, which plays an important role for the proof of the theorem. The object correspondence is

$$G_0(r) = (\psi(r_{ij}))_{ij} : \oplus_i \Lambda' e''_i \to \oplus_j \Lambda' e'_j$$

for an object $r = (r_{ij})_{ij} : \oplus_i \Lambda e_i \to \oplus_j \Lambda e_j$ of $\hom\mathcal{P}_\Lambda$, where $r_{ij} \in \operatorname{rad}\Lambda$ by the definition of morphisms in $\hom\mathcal{P}_\Lambda$. Let $f = (b, a) : r \to t$ be a morphism of $\operatorname{mor}\mathcal{P}_\Lambda$ between two objects $r = (r_{ij})_{ij} : \oplus_i \Lambda e_i \to \oplus_j \Lambda e_j$ and $t = (t_{kl})_{kl} : \oplus_k \Lambda e_k \to \oplus_l \Lambda e_l$, and let $b = (b_{ik})_{ik}$, $a = (a_{jl})_{jl}$. The morphism correspondence is then given by

$$G_1([f]) = [\tilde{f}] : G_0(r) \to G_0(t),$$

where $\tilde{f} = (\tilde{q}(b), \tilde{p}(a)) \in \operatorname{mor}\mathcal{P}_{\Lambda'}$.

Lemma 2.1. *G is a well-defined functor.*

Proof. We put $\tilde{p}(a) = (\tilde{p}(a_{jl}))_{jl}$, $\tilde{q}(b) = (\tilde{q}(b_{ik}))_{ik}$, and $\tilde{f} = (\tilde{q}(b), \tilde{p}(a))$. We shall show that $G_1([f])$ is a morphism in $\hom\mathcal{P}_{\Lambda'}$. For this, we first claim that $\tilde{f} = (\tilde{q}(b), \tilde{p}(a)) : G_0(r) \to G_0(t)$ is a morphism of $\operatorname{mor}\mathcal{P}_{\Lambda'}$, that is, the following diagram is commutative:

$$\begin{array}{ccc} \oplus_i \Lambda' e''_i & \xrightarrow{G_0(r)} & \oplus_j \Lambda' e'_j \\ \tilde{q}(b) \big\downarrow & & \big\downarrow \tilde{p}(a) \\ \oplus_k \Lambda' e''_k & \xrightarrow{G_0(t)} & \oplus_l \Lambda' e'_l. \end{array}$$

In fact, it follows that

$$\begin{aligned} R_{G_0(t)}R_{\widetilde{q}(b)} &= (\widetilde{q}(b_{ik}))_{ik}(\psi(t_{kl}))_{kl} = \left(\psi\left(\sum_k b_{ik}t_{kl}\right)\right)_{il} \\ &= \left(\psi\left(\sum_j r_{ij}a_{jl}\right)\right)_{il} = (\psi(r_{ij}))_{ij}(\widetilde{p}(a_{jl}))_{jl} \\ &= R_{\widetilde{p}(a)}R_{G_0(r)}. \end{aligned}$$

Next, we have to show that G preserves the equivalence relation $\sim$ in $\operatorname{mor}\mathcal{P}_\Lambda$. Assume that there is a map $c = (c_{jk})_{jk} : \oplus_j \Lambda e_j \to \oplus_k \Lambda e_k$ such that $R_a = R_t R_c = R_{ct}$. Then $a_{jl} = \sum_k c_{jk}t_{kl} \in \operatorname{rad}\Lambda$ because $t_{kl} \in \operatorname{rad}\Lambda$, and

$$(\psi(a_{jl}))_{jl} = \left(\psi\left(\sum_k c_{jk}t_{kl}\right)\right)_{jl} = \left(\sum_k \widetilde{q}(c_{jk})\psi(t_{kl})\right)_{jl} = (\widetilde{q}(c_{jk}))_{jk}(\psi(t_{kl}))_{kl}.$$

On the other hand, $\psi(a_{jl}) = \widetilde{p}(a_{jl}) + s_{jl}$ for some $s_{jl} \in \operatorname{soc}\Lambda'$ by our assumption. Moreover, we have $(s_{jl})_{jl} = (\iota_{jk})_{jk}(\psi(t_{kl}))_{kl}$ for some map $(\iota_{jk})_{jk} : \oplus_j \Lambda' e'_j \to \oplus_k \Lambda' e''_k$, because $\operatorname{Im}(s_{jl})_{jl} \subseteq \oplus_l \operatorname{soc}\Lambda' e'_l \subseteq \operatorname{Im}(\psi(t_{kl}))_{kl}$. It holds that

$$(\widetilde{p}(a_{jl}))_{jl} = (\psi(a_{jl}) - s_{jl})_{jl} = (\widetilde{q}(c_{jk}) - \iota_{jk})_{jk}(\psi(t_{kl}))_{kl}.$$

Thus, G_1 is well-defined. Now it is easy to check that $G = (G_0, G_1)$ is a functor (cf. [6] or [4]). □

Proposition 2.2. *Assume that $\overline{\psi(a)} = p(\overline{a})$ for all $a \in \operatorname{rad}\Lambda$. Then stable homotopy categories $\underline{\hom}\,\mathcal{P}_\Lambda$ and $\underline{\hom}\,\mathcal{P}_{\Lambda'}$ are equivalent.*

Proof. We can construct the inverse functor $G' : \hom\mathcal{P}_{\Lambda'} \to \hom\mathcal{P}_\Lambda$ whose object correspondence is $G'_0(r') = (\psi^{-1}(r'_{ij}))_{ij} : \oplus_i \Lambda e_i \to \oplus_j \Lambda e_j$ for all objects $r' = (r'_{ij})_{ij} : \oplus_i \Lambda' e''_i \to \oplus_j \Lambda' e'_j$ of $\hom\mathcal{P}_{\Lambda'}$. The morphism is given by $G'_1([(b', a')]) = [((\widetilde{q^{-1}}(b'_{ik}))_{ik}, (\widetilde{p^{-1}}(a'_{jl}))_{jl})]$ for any morphism $[(b', a')] = [((b'_{ik})_{ik}, (a'_{jl})_{jl})] : r' \to t'$ of $\hom\mathcal{P}_{\Lambda'}$, where $r' = (r'_{ij})_{ij} : \oplus_i \Lambda' e''_i \to \oplus_j \Lambda' e'_j$ and $t' = (t'_{kl})_{kl} : \oplus_k \Lambda' e''_k \to \oplus_l \Lambda' e'_l$ are objects of $\hom\mathcal{P}_{\Lambda'}$. Therefore G is an equivalent functor.

We shall show that the functor G is stable (in the sense of Lemma 1.1). If a morphism $[(b, a)] : r \to t$ in $\hom\mathcal{P}_\Lambda$ satisfies $R_a R_r = 0$, then

$$R_{\widetilde{p}(a)}R_{\psi(r)} = \psi(r)\widetilde{p}(a) = \psi(ra) = 0,$$

thus the proof is completed. □

Proof of Theorem. Let Λ and Λ' be socle equivalent selfinjective algebras, say $p : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$. Note that we may identify $\underline{\operatorname{mod}}\,\Lambda$ (or $\underline{\operatorname{mod}}\,\Lambda'$) with

$\underline{\mathrm{hom}}\,\mathcal{P}_\Lambda$ (or $\underline{\mathrm{hom}}\,\mathcal{P}_{\Lambda'}$, respectively). Assume that there are regular maps λ of Λ and λ' of Λ' such that $\lambda(ab) = \lambda'(\tilde{p}(a)\tilde{p}(b))$ for all $a, b \in \mathrm{rad}\,\Lambda$. There are bimodule isomorphisms $\varphi : \Lambda \to \mathrm{Hom}(\Lambda, K)_\nu$ and $\varphi' : \Lambda' \to \mathrm{Hom}(\Lambda', K)_{\nu'}$ such that $\lambda = \varphi(1_\Lambda)$ and $\lambda' = \varphi'(1_{\Lambda'})$. Let $a, b \in \mathrm{rad}\,\Lambda$. It holds that

$$\begin{aligned}\{\mathrm{Hom}(p^{-1}, K)\varphi_3(\overline{b})\}(p(\overline{a})) &= \varphi_3(\overline{b})(\overline{a}) \\ &= \lambda(ab) = \lambda'(\tilde{p}(a)\tilde{p}(b)) \\ &= \{\varphi_3'(p(\overline{b}))\}(p(\overline{a})).\end{aligned}$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{rad}\,\Lambda/\,\mathrm{soc}\,\Lambda & \xrightarrow{\quad p \quad} & \mathrm{rad}\,\Lambda'/\,\mathrm{soc}\,\Lambda' \\ {\scriptstyle\varphi_3}\downarrow & & \downarrow{\scriptstyle\varphi_3'} \\ \mathrm{Hom}(\mathrm{rad}\,\Lambda/\,\mathrm{soc}\,\Lambda, K) & \xrightarrow{\mathrm{Hom}(p^{-1},K)} & \mathrm{Hom}(\mathrm{rad}\,\Lambda'/\,\mathrm{soc}\,\Lambda', K). \end{array}$$

By (2.1), we have $\overline{\psi(a)} = \overline{\varphi_2'^{-1}\,\mathrm{Hom}_K(p^{-1}, K)\varphi_2(a)} = \varphi_3'^{-1}\,\mathrm{Hom}_K(p^{-1}, K)\varphi_3(\overline{a}) = p(\overline{a})$ for all $a \in \mathrm{rad}\,\Lambda$. Consequently, the stable homotopy categories $\underline{\mathrm{hom}}\,\mathcal{P}_\Lambda$ and $\underline{\mathrm{hom}}\,\mathcal{P}_{\Lambda'}$ are equivalent by Proposition 2.2. □

3. Example: Hochschild extension algebras

Let A be a K-algebra without oriented cycles in its ordinary quiver and $DA = \mathrm{Hom}_K(A, K)$. A K-bilinear map $\alpha : A \times A \to DA$ is called a 2-cocycle if $\alpha(ab, c) + \alpha(a, b)c = \alpha(a, bc) + a\alpha(b, c)$ for all $a, b, c \in A$. For any 2-cocycle $\alpha : A \times A \to DA$, we denote by $A \ltimes_\alpha DA$ the Hochschild extension algebra of A by DA corresponding to α, that is, $A \ltimes_\alpha DA$ is equal to $A \oplus DA$ as a K-vectorspace, and its multiplication is given by $(a, u)(b, v) = (ab, av + ub + \alpha(a, b))$ for all $a, b \in A$ and $u, v \in DA$. In case $\alpha = 0$, $A \ltimes_0 DA$ is called a trivial extension algebra of A by DA and denoted simply by $A \ltimes DA$. An algebra $A \ltimes_\alpha DA$ is selfinjective. In [6], Yamagata proved that $A \ltimes_\alpha DA$ and $A \ltimes DA$ are socle equivalence algebras which naturally induce a stable equivalence. In this section, we shall show that these algebras $A \ltimes_\alpha DA$ and $A \ltimes DA$ satisfy the assumption of the main theorem, namely, we shall construct regular maps satisfying the required condition.

We consider a fixed 2-cocycle $\alpha : A \times A \to DA$. Let $\{e_i\}_{i=1}^n$, $\{\varepsilon_i\}_{i=1}^n$ and $\{\varepsilon_i'\}_{i=1}^n$ be complete sets of orthogonal primitive idempotents of A, $A \ltimes DA$ and $A \ltimes_\alpha DA$, respectively, such that $\varepsilon_i = (e_i, 0)$ and $\varepsilon_i' = (e_i, f_i)$ for some $f_i \in DA$ as elements of $A \oplus DA$. For any elements $x, y \in A \oplus DA$, we denote by xy or $x \cdot y$ the multiplication in $A \ltimes DA$ or $A \ltimes_\alpha DA$, respectively. We define a K-linear map $p : (A \ltimes DA)/\,\mathrm{soc}(A \ltimes DA) \to (A \ltimes_\alpha DA)/\,\mathrm{soc}(A \ltimes_\alpha DA)$ by $p(\overline{x}) = \sum_{ij} \overline{\varepsilon_i' \cdot \varepsilon_i x \varepsilon_j \cdot \varepsilon_j'}$ for all $x \in A \ltimes DA$.

Lemma 3.1. *p is an algebra isomorphism.*

Proof. We set $x' = \sum_{ij} \varepsilon_i' \cdot \varepsilon_i x \varepsilon_j \cdot \varepsilon_j'$ for all $x = \sum_{ij} \varepsilon_i x \varepsilon_j \in A \ltimes DA$. Note that $x' \in \operatorname{soc}(A \ltimes_\alpha DA)$ for all $x \in \operatorname{soc}(A \ltimes DA)$ because $\operatorname{soc}(A \ltimes DA) = \operatorname{soc} DA = \operatorname{soc}(A \ltimes_\alpha DA)$. Then p is well-defined and bijective because $\varepsilon_i(\varepsilon_i' \cdot \varepsilon_i) = \varepsilon_i = (\varepsilon_i \cdot \varepsilon_i')\varepsilon_i$ for each i, and its inverse is defined by $x = \sum_{ij} \varepsilon_i(\varepsilon_i' \cdot x \cdot \varepsilon_j')\varepsilon_j$ for $x \in A \ltimes_\alpha DA$.

Let $r \in \operatorname{rad}(A \ltimes DA)$ and $x, y \in A \ltimes DA$. Note that $r' \cdot x' = (rx)'$ ([6, Lemma 3.2]), and hence $r' \cdot \{(xy)' - x' \cdot y'\} = 0$. Since $\operatorname{rad}(A \ltimes_\alpha DA) = \{r' \mid r \in \operatorname{rad}(A \ltimes DA)\}$, we have $(xy)' - x' \cdot y' \in \operatorname{soc}(A \ltimes_\alpha DA)$. Consequently, p is an algebra isomorphism. $\square$

Let λ_α and λ_α' be K-linear maps from $A \ltimes_\alpha DA$ to K defined by $\lambda_\alpha(a, u) = u(1_A)$ and $\lambda_\alpha'(a, u) = u(1_A) - \sum_{ij}\{f_i(e_i a e_j) + f_j(e_i a e_j) + \alpha(e_i a e_j, e_j)(e_i) + \alpha(e_i, e_i a e_j)(1_A)\}$ for all $a \in A$ and $u \in DA$. Then both λ_α and λ_α' are regular maps of $A \ltimes_\alpha DA$.

Lemma 3.2. $\lambda_0((a, u)(b, v)) = \lambda_\alpha'(\tilde{p}(a, u) \cdot \tilde{p}(b, v))$ *for all* (a, u) *and* $(b, v) \in \operatorname{rad}(A \ltimes DA)$.

Proof. Let $a, b \in \operatorname{rad} A$ and $u, v \in DA$. It holds that $\lambda_0((a, u)(b, v)) = v(a) + u(b)$.

On the other hand, we may assume $\widetilde{p}(x) = \sum_{ij} \varepsilon_i' \cdot \varepsilon_i x \varepsilon_j \cdot \varepsilon_j'$ for all $x \in A \ltimes DA$. It holds that

$$
\begin{aligned}
\tilde{p}(a, u) \cdot \tilde{p}(b, v) &= \sum_{ijk} \varepsilon_i' \cdot \varepsilon_i(a, u)\varepsilon_j \cdot \varepsilon_j' \cdot \varepsilon_j(b, v)\varepsilon_k \cdot \varepsilon_k' \\
&= \sum_{ik} \varepsilon_i' \cdot \varepsilon_i(a, u)(b, v)\varepsilon_k \cdot \varepsilon_k' \\
&= \sum_{ik} \Big(e_i a b e_k, e_i a v e_k + e_i u b e_k + f_i e_i a b e_k \\
&\qquad\qquad + e_i a b e_k f_k + e_i \alpha(e_i a b e_k, e_k) + \alpha(e_i, e_i a b e_k) \Big),
\end{aligned}
$$

because $r' \cdot x' = (rx)'$ for all $x \in A \ltimes DA$ and $r \in \operatorname{rad}(A \ltimes DA)$ ([6, Lemma 3.2]). Therefore, we have $\lambda_\alpha'(\tilde{p}(a, u) \cdot \tilde{p}(b, v)) = \sum_{ik}\{v(e_k e_i a) + u(b e_k e_i)\} = v(a) + u(b)$. $\square$

If a base field K is an algebraically closed and a K-algebra A contains no oriented cycles in its quiver, then any Hochschild extension algebra $A \ltimes_\alpha DA$ is isomorphic to the trivial extension algebra $A \ltimes DA$. However, if K is not an algebraically closed, there exists a Hochschild extension algebra which is not isomorphic to $A \ltimes DA$ (see [5], [2]).

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Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai, Tsukuba
Ibaraki 305-8571, Japan
e-mail: ohnuki@cc.tuat.ac.jp