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STANDARD L-FUNCTIONS ATTACHED TO ALTERNATING TENSOR VALUED SIEGEL MODULAR FORMS

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1. Introduction

Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbb{C})$ on a finite-dimensional complex vector space V_ρ such that the signature of ρ is $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let f be a V_ρ-valued Siegel cuspform of type ρ with respect to $Sp(n, \mathbb{Z})$ (size $2n$). Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the standard L-function attached to f by

$$(1.1) \quad L(s, f, \underline{\mathrm{St}}) := \prod_p \left\{ (1 - p^{-s}) \prod_{j=1}^{n} (1 - \alpha_j(p) p^{-s})(1 - \alpha_j(p)^{-1} p^{-s}) \right\}^{-1},$$

where p runs over all prime numbers and $\alpha_j(p)$ $(1 \leq j \leq n)$ are the Satake p-parameters of f. The right-hand side of (1.1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n + 1$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_{\mathbb{R}}(s + \varepsilon) \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s + \lambda_j - j) L(s, f, \underline{\mathrm{St}})$$

with

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s),$$

and

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even}, \\ 1 & \text{for } n \text{ odd}. \end{cases}$$

Then by Takayanagi [15], we expect the following:

Conjecture. $\Lambda(s, f, \mathrm{St})$ *has a meromorphic continuation to the whole s-plane and satisfies a functional equation.*

For $\rho = \det^k$ (cf. Andrianov and Kalinin [1], Böcherer [2] and Mizumoto [12]), $\rho = \det^k \otimes \mathrm{sym}^l$ (cf. Takayanagi [15]), $\rho = \det^k \otimes \mathrm{alt}^{n-1}$ (cf. Takayanagi [16]), this conjecture holds. In this paper, for $\rho = \det^k \otimes \mathrm{alt}^l$ $(1 \le l \le n-1)$, we show the conjecture holds.

We note that the signature of $\det^k \otimes \mathrm{alt}^l$ is $(\underbrace{k+1, \ldots, k+1}_{l}, \underbrace{k, \ldots, k}_{n-l})$. Then the main result of this paper is the following (cf. Piatetski-Shapiro and Rallis [14], Weissauer [17]).

Theorem 1. *Let $n \in \mathbb{Z}_{>0}$, k, $l \in 2\mathbb{Z}$, and $2k \ge n > 2$. Let f be a cuspidal eigenform of type ρ. Then $\Lambda(s, f, \mathrm{St})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s, f, \mathrm{St}) = \Lambda(1-s, f, \mathrm{St}).$$

Moreover, $\Lambda(s, f, \mathrm{St})$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$. If n is odd, then $\Lambda(s, f, \mathrm{St})$ is entire.

2. Preliminaries

Let $n \in \mathbb{Z}_{>0}$. Let (ρ, V_ρ) be a finite-dimensional irreducible representation of $GL(n, \mathbb{C})$. We fix a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V_ρ such that

$$\langle \rho(g)v, w \rangle = \langle v, \rho({}^t\overline{g})w \rangle \quad \text{for } g \in GL(n, \mathbb{C}),\ v,\ w \in V_\rho.$$

Let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n, and $\mathfrak{H}_n$ the Siegel upper half space of degree n. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n$ and $Z \in \mathfrak{H}_n$, we put

$$M\langle Z \rangle := (AZ+B)(CZ+D)^{-1}, \quad j(M, Z) := \det(CZ+D),$$

and for $f : \mathfrak{H}_n \to V_\rho$,

$$(f|_\rho M)(Z) := \rho((CZ+D)^{-1}) f(M\langle Z \rangle).$$

We write $|_k$ for $\rho = \det^k$ and we omit subscripts ρ, k when there is no fear of confusion.

A C^∞-function $f : \mathfrak{H}_n \to V_\rho$ is called a V_ρ-valued C^∞-modular form of type ρ if it satisfies $f|_\rho M = f$ for all $M \in \Gamma^n$. The space of all such functions is denoted by $M^n(V_\rho)^\infty$. The space of V_ρ-valued Siegel modular forms of type ρ is defined by

$$M^n(V_\rho) := \{ f \in M^n(V_\rho)^\infty \mid f \text{ is holomorphic on } \mathfrak{H}_n \text{ (and its cusps)} \},$$

and the space of cuspforms by

$$S^n(V_\rho) := \left\{ f \in M^n(V_\rho) \;\middle|\; \lim_{\lambda\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}\right) = 0 \quad \text{for all } Z \in \mathfrak{H}_{n-1} \right\}.$$

If $\rho = \det^k$, we write $M_k^{n\infty}$, M_k^n, and S_k^n for $M^n(V_\rho)^\infty$, $M^n(V_\rho)$, and $S^n(V_\rho)$, respectively.

For $f, g \in M^n(V_\rho)^\infty$, the Petersson inner product of f and g is defined by

$$(f, g) := \int_{\Gamma^n\backslash\mathfrak{H}_n} \left\langle \rho\big(\sqrt{\mathrm{Im}(Z)}\big) f(Z), \rho\big(\sqrt{\mathrm{Im}(Z)}\big) g(Z) \right\rangle \det\big(\mathrm{Im}(Z)\big)^{-n-1} dZ$$

if the right-hand side is convergent.

If (ρ, V_ρ) is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $(\tilde{\rho}, V_{\tilde{\rho}})$ satisfying the following condition: There exists a unique one-dimensional vector subspace $\mathbb{C}\tilde{v}$ of $V_{\tilde{\rho}}$ such that for any upper triangular matrix of $GL(n, \mathbb{C})$,

$$\tilde{\rho}\left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix}\right) \tilde{v} = \left(\prod_{j=1}^n g_{jj}^{\lambda_j}\right) \tilde{v},$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then we call $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the signature of ρ.

Now, we put

$$G^+Sp(n, \mathbb{Q}) := \left\{ M \in GL(2n, \mathbb{Q}) \;\middle|\; {}^tM \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},\ \mu(M) > 0 \right\}.$$

For $g \in G^+Sp(n, \mathbb{Q})$, let $\Gamma^n g \Gamma^n = \bigcup_{j=1}^r \Gamma^n g_j$ be a decomposition of the double coset $\Gamma^n g \Gamma^n$ into left cosets. For $f \in M^n(V_\rho)$ (resp. $S^n(V_\rho)$, $M^n(V_\rho)^\infty$), we define the Hecke operator $(\Gamma^n g \Gamma^n)$ by

$$f|(\Gamma^n g \Gamma^n) := \sum_{j=1}^r f|g_j.$$

Let $f \in S^n(V_\rho)$ be an eigenform. We define the standard L-function attached to f by (1.1). We also define the following series:

$$D(s, f) := \sum_{T \in \mathbb{T}^{(n)}} \lambda(f, T) \det(T)^{-s}, \tag{2.1}$$

where

$$\mathbb{T}^{(n)} := \left\{ \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \;\middle|\; t_j \in \mathbb{Z}_{>0}(1 \le j \le n), \quad t_1 \mid \ldots \mid t_n \right\}.$$

and $\lambda(f, T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n\right)$, $T \in \mathbb{T}^{(n)}$. By Böcherer [3], we have:

$$\zeta(s) \prod_{j=1}^{n} \zeta(2s - 2j) D(s, f) = L(s - n, f, \mathrm{St}). \tag{2.2}$$

For $k \in 2\mathbb{Z}_{>0}$, $s \in \mathbb{C}$ and $Z \in \mathfrak{H}_n$, we define the Eisenstein series by

$$E_k^n(Z, s) := \det(\mathrm{Im}(Z))^s \sum_{M \in P_{n,0} \backslash \Gamma^n} j(M, Z)^{-k} \left| j(M, Z) \right|^{-2s},$$

where

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ 0^{(n+r,n-r)} & * \end{pmatrix} \in \Gamma^n \right\}.$$

Then $E_k^n(Z, s)$ converges absolutely and locally uniformly for $k + 2\,\mathrm{Re}(s) > n + 1$, and $E_k^n(Z, s) \in M_k^{n\,\infty}$. Moreover, we have the following:

Theorem 2 (Langlands [11], Kalinin [8] and Mizumoto [12, 13]). *Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$. For $Z \in \mathfrak{H}_n$, we put*

$$\mathbb{E}_k^n(Z, s) := \frac{\Gamma_n(s + k/2)}{\Gamma_n(s)} \xi(2s) \prod_{j=1}^{[n/2]} \xi(4s - 2j) E_k^n\left(Z, s - \frac{k}{2}\right),$$

where

$$\Gamma_n(s) := \prod_{j=1}^{n} \Gamma\left(s - \frac{j-1}{2}\right), \quad \xi(s) := \Gamma_{\mathbb{R}}(s)\zeta(s).$$

Then $\mathbb{E}_k^n(Z, s)$ is invariant under $s \to (n + 1)/2 - s$ and it is an entire function in s.

It is also known that every partial derivative (in the entries of Z) of the Eisenstein series $E_k^n(Z, s)$ is slowly increasing (locally uniformly in s).

Theorem 3 (Mizumoto [13]). *Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$.*
(i) *For each $s_0 \in \mathbb{C}$, there exist constants $\delta > 0$ and $d \in \mathbb{Z}_{\geq 0}$, depending only on n, k and s_0, such that*

$$(s - s_0)^d E_k^n(Z, s)$$

is holomorphic in s for $|s - s_0| < \delta$, and is C^∞ in $(\mathrm{Re}(Z), \mathrm{Im}(Z))$.
(ii) *Furthermore, for given $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geq 0}$, there exist constants $\alpha > 0$ and $\beta > 0$ depending only on n, k, d, s_0, ε, δ and N such that*

$$\left| (s - s_0)^d \frac{\partial^N}{\partial z_{\mu_1 \nu_1} \cdots \partial z_{\mu_N \nu_N}} E_k^n(Z, s) \right| \leq \alpha \det(\mathrm{Im}(Z))^\beta$$

for $\mathrm{Im}(Z) \geq \varepsilon 1_n$, $|s - s_0| < \delta$, *and* $1 \leq \mu_j, \nu_j \leq n$.

The assertion above for the case $N = 0$ has been proved by Langlands [11] and Kalinin [8].

3. Differential operator and the pullback formula

Let V be a finite-dimensional vector space. For a finite subset I of $\mathbb{Z}_{>0}$, we define V^I by

$$V^I := \underbrace{V \otimes \cdots \otimes V}_{\sharp I}.$$

Moreover for disjoint finite subsets I, J of $\mathbb{Z}_{>0}$, we identify $V^{I \cup J}$ with $V^I \otimes V^J$ by the following:

$$V^I \otimes V^J \ni (v_{i_1} \otimes \cdots \otimes v_{i_r}) \otimes (v_{j_1} \otimes \cdots \otimes v_{j_s}) \mapsto v_{k_1} \otimes \cdots \otimes v_{k_{r+s}} \in V^{I \cup J},$$

where $I = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$, $J = \{j_1, \ldots, j_s\}$ with $j_1 < \cdots < j_s$, and $I \cup J = \{k_1, \ldots, k_{r+s}\}$ with $k_1 < \cdots < k_{r+s}$.

For $\alpha \in \mathbb{Z}_{>0}$, we define the isomorphism $(\,\cdot\,)^\alpha : V \to V^{\{\alpha\}}$ by $(v)^\alpha := v$. We omit the tensor product $\otimes$ when there is no fear of confusion.

Now, we put

$$\begin{aligned} V_1 &:= \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n, & \boldsymbol{e}_1 &:= (e_1, \ldots, e_n), \\ V_2 &:= \mathbb{C}e_{n+1} \oplus \cdots \oplus \mathbb{C}e_{2n}, & \boldsymbol{e}_2 &:= (e_{n+1}, \ldots, e_{2n}). \end{aligned}$$

Let $\mathrm{alt}^l(V_1)$ (resp. $\mathrm{alt}^l(V_2)$) be the l-th alternating tensor product of V_1 (resp. V_2), i.e.,

$$\mathrm{alt}^l(V_j) := \mathrm{span}\left\{ \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma)(\boldsymbol{e}_j{}^t a_1)^{\sigma(1)} \cdots (\boldsymbol{e}_j{}^t a_l)^{\sigma(l)} \,\middle|\, a_1, \ldots, a_l \in \mathbb{C}^n \right\} \quad (j = 1, 2),$$

where $\mathfrak{S}_l$ is the l-th symmetric group. For each $g \in GL(n, \mathbb{C})$, $\rho_j(g) := \det^k \otimes \operatorname{alt}^l(g)$ acts on $\operatorname{alt}^l(V_j)$ $(j = 1, 2)$ by

$$\begin{aligned}\rho_j(g) \sum_{\sigma\in\mathfrak{S}_l} &\operatorname{sgn}(\sigma)(\boldsymbol{e}_j{}^t a_1)^{\sigma(1)} \cdots (\boldsymbol{e}_j{}^t a_l)^{\sigma(l)} \\ &:= \det(g)^k \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma)(\boldsymbol{e}_j g^t a_1)^{\sigma(1)} \cdots (\boldsymbol{e}_j g^t a_l)^{\sigma(l)}.\end{aligned}$$

Let ι be the isomorphism from V_1 to V_2 defined by $\iota(e_j) = e_{n+j}$ $(1 \le j \le n)$. It induces the isomorphism (also denoted by ι) from $\operatorname{alt}^l(V_1)$ to $\operatorname{alt}^l(V_2)$. For an $\operatorname{alt}^l(V_1)$-valued function f on $\mathfrak{H}_n$ and for $Z \in \mathfrak{H}_n$, we define $\iota(f)$ by

$$\big(\iota(f)\big)(Z) := \iota\big(f(Z)\big).$$

For a symmetric matrix A of size $2n$ and $\alpha, \beta \in \mathbb{Z}_{>0}$, we define

$$\begin{aligned}A^{\alpha\beta} &:= ((e_1)^\alpha, \dots, (e_n)^\alpha, 0, \dots, 0)A\,{}^t((e_1)^\beta, \dots, (e_n)^\beta, 0, \dots, 0),\\ A^\alpha_\beta &:= ((e_1)^\alpha, \dots, (e_n)^\alpha, 0, \dots, 0)A\,{}^t(0, \dots, 0, (e_{n+1})^\beta, \dots, (e_{2n})^\beta),\\ A_{\alpha\beta} &:= (0, \dots, 0, (e_{n+1})^\alpha, \dots, (e_{2n})^\alpha)A\,{}^t(0, \dots, 0, (e_{n+1})^\beta, \dots, (e_{2n})^\beta).\end{aligned}$$

Let $\mathfrak{Z} = (z_{\mu\nu})$ be a variable on $\mathfrak{H}_{2n}$. We put

$$\frac{\partial}{\partial\mathfrak{Z}} := \left(\frac{1 + \delta_{\mu\nu}}{2}\frac{\partial}{\partial z_{\mu\nu}}\right)_{1\le\mu,\nu\le 2n},$$

where $\delta_{\mu\nu}$ is the Kronecker's delta, and for C^∞-functions, we define the differential operator $\mathcal{D}$ by

$$\mathcal{D} := \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma)\left(\frac{\partial}{\partial\mathfrak{Z}}\right)^1_{\sigma(1)} \cdots \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^l_{\sigma(l)}.$$

Then we have:

Proposition 1. *Let $n, k \in \mathbb{Z}_{>0}$ and $2k \ge n$.*
(i) *Let F be any $\mathbb{C}$-valued C^∞-function on $\mathfrak{H}_{2n}$. Then for each $g_1, g_2 \in \Gamma^n$ and $\mathfrak{Z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)}\end{pmatrix} \in \mathfrak{H}_{2n}$, we get the following commutation relation*:

$$((\mathcal{D}F)|_{\rho_1}(g_1)_Z|_{\rho_2}(g_2)_W)(\mathfrak{Z}_0) = (\mathcal{D}(F|_k(g_1^\uparrow g_2^\downarrow)))(\mathfrak{Z}_0),$$

where $(\quad)_Z$ (resp. $(\quad)_W$) denotes the action on Z (resp. W) and for $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in$

Γ^n, *we put*

$$M^{\uparrow} := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_n & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix}, \quad M^{\downarrow} := \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_n & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

(ii) *The operator $\mathcal{D}$ sends modular forms to modular forms*:

$$\mathcal{D} : {M_k^{2n}}^{\infty} \longrightarrow M^n(\mathrm{alt}^l(V_1))^{\infty} \otimes M^n(\mathrm{alt}^l(V_2))^{\infty}.$$

Moreover, $\mathcal{D}$ is a holomorphic operator and it satisfies

$$\mathcal{D} : M_k^{2n} \longrightarrow M^n(\mathrm{alt}^l(V_1)) \otimes M^n(\mathrm{alt}^l(V_2)).$$

Proof. Let $X_j = (x_{\mu\nu}^{(j)})$ $(j = 1, 2)$ be variables on $\mathbb{C}^{(n,2k)}$. We put $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, and

$$Q(X^t X) := \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma)(X^t X)^1_{\sigma(1)} \cdots (X^t X)^l_{\sigma(l)}.$$

Then the polynomial Q is pluri-harmonic for X_1, X_2, i.e., for each $1 \leq \mu, \nu \leq n$,

$$\sum_{\kappa=1}^{2k} \frac{\partial}{\partial x_{\mu\kappa}^{(j)}} \frac{\partial}{\partial x_{\nu\kappa}^{(j)}} Q = 0 \quad (j = 1, 2).$$

Therefore, by Ibukiyama [7] (see also [16]), we get Proposition 1. □

Now we prove Theorem 1 according to Böcherer's method in [2]. For this, we prove the following:

Proposition 2. *Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$, $s \in \mathbb{C}$ and $k + 2\,\mathrm{Re}(s) > 2n + 1$. Suppose that $2k \geq n > 2$. For $\mathfrak{Z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we get*

$$(\mathcal{D}E_k^{2n})(\mathfrak{Z}_0, s) = \prod_{j=1}^{l} \left(-k - s + \frac{j-1}{2} \right) \sum_{r=0}^{n} \sum_{T \in \mathbb{T}^{(r)}} \mathcal{P}_r(Z, W, T, s),$$

where

$$\mathcal{P}_r(Z, W, T, s) := \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g_2' \in P_{n,r} \backslash \Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}_1' \in \Gamma^r(T) \backslash G_{n,r}} \cdot \Big\{ \det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s \left| \det(1_n - \tilde{T} W \tilde{T} Z) \right|^{-2s}$$

$$\cdot\rho_1((1_n - \tilde{T}W\tilde{T}Z)^{-1}) \sum_{\sigma\in\mathfrak{S}_l} \mathrm{sgn}(\sigma) \begin{pmatrix} * & \tilde{T} \\ \tilde{T} & * \end{pmatrix}^1_{\sigma(1)} \cdots \begin{pmatrix} * & \tilde{T} \\ \tilde{T} & * \end{pmatrix}^l_{\sigma(l)} \Bigg\}$$
$$|(\tilde{g}_1')_W|(\tilde{g}_1)_Z|(g_2')_W|(g_2)_Z,$$
$$G_{n,r} := \left\{ \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \right\},$$

and for $T \in \mathbb{T}^{(r)}$,

$$\Gamma^r(T) := \left\{ g \in \Gamma^r \;\middle|\; \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right\} \quad \textit{and} \quad \tilde{T}^{(n)} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}.$$

Proof. By Garrett [5], the left coset $P_{2n,0}\backslash\Gamma^{2n}$ has a complete system of representatives $g_{\tilde{T}}\tilde{g}_1^{\uparrow}g_2^{\uparrow}\tilde{g}_1'^{\downarrow}g_2'^{\downarrow}$ with

$$g_{\tilde{T}} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{T} & 1_n & 0 \\ \tilde{T} & 0 & 0 & 1_n \end{pmatrix}, \quad T \in \mathbb{T}^{(r)} \quad (0 \le r \le n),$$
$$\tilde{g}_1 \in G_{n,r}, \quad g_2 \in P_{n,r}\backslash\Gamma^n, \quad \tilde{g}_1' \in \Gamma^r(T)\backslash G_{n,r}, \quad g_2' \in P_{n,r}\backslash\Gamma^n.$$

Therefore, it follows from Proposition 1 that

$$(\mathcal{D}E_k^{2n})(\mathfrak{Z}_0, s) = \sum_{r=0}^{n} \sum_{T\in\mathbb{T}^{(r)}} \sum_{g_2\in P_{n,r}\backslash\Gamma^n} \sum_{g_2'\in P_{n,r}\backslash\Gamma^n} \sum_{\tilde{g}_1\in G_{n,r}} \sum_{\tilde{g}_1'\in\Gamma^r(T)\backslash G_{n,r}}$$
$$\{\mathcal{D}(\det(\mathrm{Im}(\mathfrak{Z}))^s|_k g_{\tilde{T}})|_{\mathfrak{Z}=\mathfrak{Z}_0}\}|(\tilde{g}_1')_W|(\tilde{g}_1)_Z|(g_2')_W|(g_2)_Z.$$

If for each $\tilde{T}$ we put $g_{\tilde{T}} = \begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix}$, we get

$$\mathcal{D}(\det(\mathrm{Im}(\mathfrak{Z}))^s|_k g_{\tilde{T}})|_{\mathfrak{Z}=\mathfrak{Z}_0} = \det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-s}\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}\det(\mathrm{Im}(\mathfrak{Z}))^s)|_{\mathfrak{Z}=\mathfrak{Z}_0},$$

by the form of $\mathcal{D}$ and $((1/2i)(\mathrm{Im}(\mathfrak{Z}_0))^{-1})^j_{\sigma(j)} = 0$,

$$= \det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-s}\det(\mathrm{Im}(\mathfrak{Z}_0))^s\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s})|_{\mathfrak{Z}=\mathfrak{Z}_0}.$$

To compute $\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s})$, we prove the following lemma.

Lemma 1. *We put* $\delta := \det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})$ *and* $\Delta := (\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}\mathfrak{C}$. *For* $\lambda \in \mathbb{C}$,

$$\sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^1_{\sigma(1)} \cdots \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^l_{\sigma(l)} \delta^\lambda = \delta^\lambda \prod_{j=1}^{l}\left(\lambda + \frac{j-1}{2}\right) \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma)\Delta^1_{\sigma(1)}\cdots\Delta^l_{\sigma(l)}.$$

Proof of Lemma 1. We use induction on l. Since

$$\frac{\partial}{\partial\mathfrak{Z}}\delta^\lambda = \delta^\lambda\lambda\Delta,$$

for $l=1$, the lemma holds. Let $l>1$.

$$\begin{aligned}
(3.1)\quad & \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^1_{\sigma(1)} \cdots \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^l_{\sigma(l)} \delta^\lambda \\
&= \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial\mathfrak{Z}}\right)^1_{\sigma(1)} \left\{\delta^\lambda \prod_{j=1}^{l-1}\left(\lambda+\frac{j-1}{2}\right)\Delta^2_{\sigma(2)}\cdots\Delta^l_{\sigma(l)}\right\} \\
&= \delta^\lambda \prod_{j=1}^{l-1}\left(\lambda+\frac{j-1}{2}\right)\left\{\lambda\sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)\Delta^1_{\sigma(1)}\cdots\Delta^l_{\sigma(l)}\right. \\
&\qquad \left. -\frac{1}{2}\sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)\sum_{\kappa=2}^{l}\Delta^2_{\sigma(2)}\cdots\widehat{\Delta^\kappa_{\sigma(\kappa)}}\cdots\Delta^l_{\sigma(l)}(\Delta^{1\kappa}\Delta_{\sigma(1)\sigma(\kappa)}+\Delta^1_{\sigma(\kappa)}\Delta^\kappa_{\sigma(1)})\right\}.
\end{aligned}$$

We note

$$(3.2)\qquad \sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)\Delta^2_{\sigma(2)}\cdots\widehat{\Delta^\kappa_{\sigma(\kappa)}}\cdots\Delta^l_{\sigma(l)}\Delta^{1\kappa}\Delta_{\sigma(1)\sigma(\kappa)} = 0$$

and

$$(3.3)\qquad \sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)\Delta^2_{\sigma(2)}\cdots\widehat{\Delta^\kappa_{\sigma(\kappa)}}\cdots\Delta^l_{\sigma(l)}\Delta^1_{\sigma(\kappa)}\Delta^\kappa_{\sigma(1)} = -\sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)\Delta^1_{\sigma(1)}\cdots\Delta^l_{\sigma(l)}.$$

Thus by (3.1), (3.2) and (3.3), the lemma holds. □

Using Lemma 1, we obtain

$$\begin{aligned}
\mathcal{D}(\det(\operatorname{Im}(\mathfrak{Z}))^s|_k g_{\tilde{T}})|_{\mathfrak{Z}=\mathfrak{Z}_0} &= \prod_{j=1}^{l}\left(-k-s+\frac{j-1}{2}\right)\det(\operatorname{Im}(\mathfrak{Z}_0))^s\,|\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})|^{-2s} \\
&\quad\cdot\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\sum_{\sigma\in\mathfrak{S}_l}\operatorname{sgn}(\sigma)((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})^1_{\sigma(1)}\cdots((\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C})^l_{\sigma(l)}.
\end{aligned}$$

Since $\det(\operatorname{Im}(\mathfrak{Z}_0)) = \det(\operatorname{Im}(Z))\det(\operatorname{Im}(W))$, $\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D}) = \det(1_n - \tilde{T}W\tilde{T}Z)$, and $(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & (1_n-\tilde{T}W\tilde{T}Z)^{-1}\tilde{T} \\ * & * \end{pmatrix}$, Proposition 2 is proved. □

4. Proof of Theorem 1

Theorem 4. *Let $n \in \mathbb{Z}_{>0}$, $k, l \in 2\mathbb{Z}_{>0}$, and $2k \geq n > 2$. If $f \in S^n(\mathrm{alt}^l(V_2))$ is an eigenform,*

$$\left(f, (\mathcal{D}\mathbb{E}_k^{2n})\left(\begin{pmatrix} -\overline{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\overline{s}+n}{2} \right) \right) = 2^{1-l}\pi^{(-n^2+\varepsilon)/2}(\pi i)^{nk+l}\gamma(s)\Lambda(s, f, \mathrm{St})(\iota^{-1}(f))(Z),$$

where

$$\gamma(s) := \frac{\Gamma_n((s+n)/2)}{\Gamma_{n-1}((s-1)/2)\Gamma((s+\varepsilon)/2)} = \gamma(1-s).$$

Proof of Theorem 4. It follows from Theorem 3 that

$$\left(f, (\mathcal{D}E_k^{2n})\left(\begin{pmatrix} -\overline{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \overline{s} \right) \right)$$

converges absolutely and locally uniformly for $k + 2\,\mathrm{Re}(s) > 2n + 1$. We consider that the Petersson inner product $(f, \mathcal{P}_r(-\overline{Z}, *, T, \overline{s}))$. For $r < n$, by the same reason as that Klingen [10, Satz 2],

$$(f, \mathcal{P}_r(-\overline{Z}, *, T, \overline{s})) = 0.$$

Therefore we only consider that $(f, \mathcal{P}_n(-\overline{Z}, *, T, \overline{s}))$.

Now, we have

$$\mathcal{P}_n(Z, W, T, s) = \det(T)^{-k-2s} \cdot \mathcal{P}(Z, W, s) \,\Big|\, \left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n \right)_W,$$

where

$$\mathcal{P}(Z, W, s) = \sum_{\tilde{g}_1 \in \Gamma^n} \Bigg\{ \det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s \left|\det(Z+W)\right|^{-2s} \cdot \rho_1((Z+W)^{-1}) \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma) \begin{pmatrix} * & 1_n \\ 1_n & * \end{pmatrix}^1_{\sigma(1)} \cdots \begin{pmatrix} * & 1_n \\ 1_n & * \end{pmatrix}^l_{\sigma(l)} \Bigg\} \Bigg|\, (\tilde{g}_1)_Z.$$

Since the Hecke operators are Hermitian operators and f is an eigenform, we have

$$(f, \mathcal{P}_n(-\overline{Z}, *, T, \overline{s})) = \lambda(f, T)\det(T)^{-k-2s}(f, \mathcal{P}(-\overline{Z}, *, \overline{s})).$$

If we compute the integral $(f, \mathcal{P}(-\overline{Z}, *, \overline{s}))$ according to Klingen [9, §1] (see also [2],

[4], [15]), we obtain

$$(f, \mathcal{P}(-\overline{Z}, *, \overline{s})) = 2^{n(n+1-2s)+1}(2^{-1}i)^{nk+l}c(s-n-1, \rho_1)(\iota^{-1}(f))(Z)$$

and

$$c(s-n-1, \rho_1)\,\mathrm{id} = \int_{S^n} \det(1_n - S\overline{S})^{s-n-1}\rho_1(1_n - S\overline{S})dS,$$

where $S^n := \{S \in \mathbb{C}^{(n)} \mid S = {}^tS,\ 1_n - S\overline{S} > 0\}$. Thus, by Proposition 2, (2.1) and (2.2),

(4.1) $$\left(f, (\mathcal{D}E_k^{2n})\left(\begin{pmatrix} -\overline{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\overline{s}+n-k}{2}\right)\right)$$
$$= 2^{n(1-s+k)+1}(2^{-1}i)^{nk+l}c\left(\frac{s+n-k}{2} - n - 1, \rho_1\right)(-2^{-1})^l \frac{\Gamma(s+n+k+1)}{\Gamma(s+n+k+1-l)}$$
$$\cdot\, \zeta(s+n)^{-1}\prod_{j=1}^{n}\zeta(2s+2n-2j)^{-1}L(s, f, \underline{\mathrm{St}})(\iota^{-1}(f))(Z).$$

To compute $c\big((s+n-k)/2 - n - 1, \rho_1\big)$, we prove the following lemma.

Lemma 2. *Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbb{C})$ whose signature is $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > -\lambda_n - 1$, we put*

$$\psi(s, \rho) := \int_{S^n} \det(1_n - S\overline{S})^s \rho(1_n - S\overline{S})dS.$$

Then there exists a constant $c(s, \rho)$ satisfying $\psi(s, \rho) = c(s, \rho)\,\mathrm{id}$, and

(4.2) $$c(s, \rho) = \frac{2^n\pi^{n(n+1)/2}}{\prod_{1\leq\mu\leq\nu\leq n}(\lambda_\mu + \lambda_\nu + 2s + 2n + 2 - \mu - \nu)}.$$

Proof of Lemma 2. The lemma is proved in the same way as that by Hua [6, §2.3] (see also [2], [4], [9], [15]).

For any unitary matrix $U \in U(n)$, we have $\psi(s, \rho) = \rho(U^{-1})\psi(s, \rho)\rho(U)$. Since ρ is an irreducible representation of $U(n)$, $\psi(s, \rho)$ is a homothety by Schur's lemma, i.e., there exists a constant $c(s, \rho)$ satisfying $\psi(s, \rho) = c(s, \rho)\,\mathrm{id}$.

We compute $c(s, \rho)$. Let $\mathfrak{v} \in V_\rho$ be the highest weight vector with $\langle \mathfrak{v}, \mathfrak{v}\rangle = 1$. Then,

$$\begin{aligned} c(s, \rho) &= \langle \psi(s, \rho)\mathfrak{v}, \mathfrak{v}\rangle \\ &= \int_{S^n} \det(1_n - S\overline{S})^s \langle \rho(1_n - S\overline{S})\mathfrak{v}, \mathfrak{v}\rangle dS. \end{aligned}$$

Let ρ_0 be an irreducible representation of $GL(n,\mathbb{C})$ such that $\rho = \det^{\lambda_n} \otimes \rho_0$ and the signature of ρ_0 is $(\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0)$. Then,

$$c(s,\rho) = \int_{S^n} \det(1_n - S\overline{S})^{s+\lambda_n} \langle \rho_0(1_n - S\overline{S})\mathfrak{v}, \mathfrak{v}\rangle dS.$$

We set $S = \begin{pmatrix} S_1^{(n-1)} & {}^t v \\ v & z \end{pmatrix}$ and $\rho_0'(g^{(n-1)}) = \rho_0\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)$, by Hua [6, Theorem 2.3.2],

$$\begin{aligned} c(s,\rho) &= \frac{\pi}{s+\lambda_n+1} \int_{1_{n-1} - S_1\overline{S_1} - {}^t v\overline{v} > 0} \frac{\det(1_{n-1} - S_1\overline{S_1} - {}^t v\overline{v})^{s+\lambda_n}}{(1 + \overline{v}(1_{n-1} - S_1\overline{S_1} - {}^t v\overline{v})^{-1}\,{}^t v)^{s+\lambda_n+2}} \\ &\quad \cdot \langle \rho_0'(1_{n-1} - S_1\overline{S_1} - {}^t v\overline{v})\mathfrak{v}, \mathfrak{v}\rangle dS_1 dv \\ &= \frac{\pi}{s+\lambda_n+1} \int_{1_{n-1} - S_1\overline{S_1} > 0} \det(1_{n-1} - S_1\overline{S_1})^{s+\lambda_n+1} \\ &\quad \cdot \int_{1 - \overline{u_0}\,{}^t u_0 > 0} (1 - \overline{u_0}\,{}^t u_0)^{2(s+\lambda_n+1)} \langle \rho_0'(\Gamma(1_{n-1} - {}^t u_0\overline{u_0})\,{}^t\overline{\Gamma})\mathfrak{v}, \mathfrak{v}\rangle du_0 dS_1, \end{aligned}$$

where $u_0 = (u_1, \ldots, u_{n-1})$ and $\Gamma\,{}^t\overline{\Gamma} = 1_{n-1} - S_1\overline{S_1}$. We put

$$\varphi(s,\rho) := \int_{1 - \overline{u}\,{}^t u > 0} (1 - \overline{u}\,{}^t u)^s \rho(1_n - {}^t u\overline{u}) du, \quad u = (u_1, \ldots, u_n).$$

Using Schur's lemma again, there exists a constant $d(s,\rho)$ satisfying $\varphi(s,\rho) = d(s,\rho)\,\mathrm{id}$. Therefore,

$$\begin{aligned} \text{(4.3)} \quad c(s,\rho) &= \frac{\pi}{s+\lambda_n+1} \int_{1_{n-1} - S_1\overline{S_1} > 0} \det(1_{n-1} - S_1\overline{S_1})^{s+\lambda_n+1} \\ &\quad \cdot \varphi(2(s+\lambda_n+1), \rho_0') \langle \rho_0'(1_{n-1} - S_1\overline{S_1})\mathfrak{v}, \mathfrak{v}\rangle dS_1 \\ &= \frac{\pi}{s+\lambda_n+1} c(s+\lambda_n+1, \rho_0') d(2(s+\lambda_n+1), \rho_0'). \end{aligned}$$

We compute $d(s,\rho)$.

$$\begin{aligned} d(s,\rho) &= \int_{1 - \overline{u}\,{}^t u > 0} (1 - \overline{u}\,{}^t u)^{s+\lambda_n} \langle \rho_0(1_n - {}^t u\overline{u})\mathfrak{v}, \mathfrak{v}\rangle du \\ &= \int_{\substack{1 - \overline{u_0}\,{}^t u_0 > 0 \\ |u_n| < 1 - \overline{u_0}\,{}^t u_0}} \left((1 - \overline{u}\,{}^t u)^{s+\lambda_n} du_n\right) \langle \rho_0'(1_{n-1} - {}^t u_0\overline{u_0})\mathfrak{v}, \mathfrak{v}\rangle du_0, \end{aligned}$$

where $u = (u_1, \ldots, u_n)$ and $u_0 = (u_1, \ldots, u_{n-1})$. Since

$$\int_{|u_n| < 1 - \overline{u_0}\,{}^t u_0} (1 - \overline{u}\,{}^t u)^{s+\lambda_n} du_n = \frac{\pi}{s+\lambda_n+1} (1 - \overline{u_0}\,{}^t u_0)^{s+\lambda_n+1},$$

$$d(s,\rho) = \frac{\pi}{s+\lambda_n+1} \int_{1 - \overline{u_0}\,{}^t u_0 > 0} (1 - \overline{u_0}\,{}^t u_0)^{s+\lambda_n+1} \langle \rho_0'(1_{n-1} - {}^t u_0\overline{u_0})\mathfrak{v}, \mathfrak{v}\rangle du_0$$

$$= \frac{\pi}{s+\lambda_n+1} d(s+\lambda_n+1, \rho_0').$$

Therefore

$$d(s,\rho) = \frac{\pi^n}{\prod_{j=1}^{n}(s+\lambda_{n-j+1}+j)}. \tag{4.4}$$

By (4.3) and (4.4), we get (4.2). □

Since the signature of ρ_1 is $(\underbrace{k+1,\ldots,k+1}_{l},\underbrace{k,\ldots,k}_{n-l})$, it follows from Lemma 2 that

$$\begin{aligned} c\left(\frac{s+n-k}{2}-n-1, \rho_1\right) &= \frac{2^n \pi^{n(n+1)/2}}{\prod_{1\le\mu\le\nu\le n}(\lambda_\mu+\lambda_\nu+s+n-k-\mu-\nu)} \\ &= 2^n \pi^{n(n+1)/2} \frac{\Gamma(s+n+k+1-l)}{\Gamma(s+n+k+1-2l)} \\ &\quad \cdot \prod_{j=1}^{l} \frac{\Gamma(s+k+1-j)}{\Gamma(s+n+k+3-2j)} \prod_{j=l+1}^{n} \frac{\Gamma(s+k-j)}{\Gamma(s+n+k+1-2j)}. \end{aligned}$$

Then, by (4.1), we obtain Theorem 4. □

It follows from Theorem 2 and Theorem 4 that the functional equation of $\Lambda(s, f, \mathrm{St})$. Moreover using the same way that by Mizumoto [12] (see also [15]), the holomorphy is proved. Hence Theorem 1 holds.

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