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ON THE *n*-COMPLETENESS OF COVERINGS OF PROPER FAMILIES OF ANALYTIC SPACES

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0. Introduction

In this paper we investigate complex analytic completeness of certain unramified covers of proper families of analytic spaces with n-dimensional fibers. When $n = 1$, T. Ohsawa has studied the stability of unramified covering spaces of complex analytic families of Riemann surfaces, and proved the following ([13], [14]):

Theorem O. (1) *Let X be a connected complex manifold of dimension* 2 *and T the unit disk of* **C**. *Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic submersion. Then every unramified covering space of X is holomorphically convex.* (2) *Let T be any contractible complex space, and X a complex space. Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map with one-dimensional fibers, and $\sigma : \widetilde{X} \longrightarrow X$ an unramified cover. Then a point $z \in T$ has an open neighborhood U such that $(\pi \circ \sigma)^{-1}(U)$ is holomorphically convex if and only if $(\pi \circ \sigma)^{-1}(z)$ is holomorphically convex.*

In connection with Theorem O, the author ([10]) and M. Coltoiu and V. Vâjâitu ([4]) have investigated completeness of the covering spaces of proper families with higher dimensional fibers. Here we shall prove a new result in this direction.

Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map of connected complex manifolds, and $n = \dim X - \dim T$ the relative dimension. Let $\sigma : \widetilde{X} \longrightarrow X$ be an unramified cover. We remark that when A is an analytic subset, $\pi^{-1}(A)$ and $(\pi\circ\sigma)^{-1}(A)$ have possibly non-reduced structures. Then we prove the following.

Theorem. *Let z be a point of T satisfying the following two conditions*: (i) $\pi^{-1}(z)$ *is a reduced connected complex space of dimension n*, (ii) $(\pi \circ \sigma)^{-1}(z)$ *has no compact irreducible component of dimension n, where $n = \dim X - \dim T$ is the relative dimension. Then there exists an open neighborhood U of z such that $(\pi\circ\sigma)^{-1}(U)$ is n-complete.*

It is well known that every n-dimensional reduced paracompact complex space is n-complete if it has no compact irreducible component of dimension n ([12], [6]). Our theorem is a relative version of this fact. We also remark that Coltoiu and Vâjâitu have shown in [4] the result in the case where π is a holomorphic submersion in our theorem.

In §1, we prepare notation and terminology. In §2, we study convexities of certain subdomains in X. In §3, we recall the construction of n-convex functions by using the argument of J.P. Demailly([6]) and prove the existence of special n-convex functions on $(\pi \circ \sigma)^{-1}(z)$. In §4, we use the argument of Coltoiu and Vâjâitu in [4] and prove the above theorem.

In Appendix, we explain the following two facts. In Appendix A, we describe constructions of 'holomorphic motions' of complex analytic families of relatively compact complex manifolds by using the argument of M. Kuranishi([8]). In Appendix B, we prove the existence of certain exhaustion functions on the unramified covering spaces by using the argument of T. Napier([11], [2]).

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1. Preliminaries

Let X be a complex space and T_pX the Zariski tangent space of X at $p \in X$. We put $TX = \cup_{p\in X} T_pX$.

A real-valued C^∞-function φ on X is said to be *q-convex* if there exists an open covering $\{A_\lambda\}_{\lambda\in\Lambda}$ of X such that each A_λ is isomorphic to a closed analytic set in an open set $\Omega_\lambda \subset \mathbf{C}^{N_\lambda}$ and each $\varphi|_{A_\lambda}$ has an extension $\widetilde{\varphi}_\lambda$ to Ω_λ such that the Levi form of $\widetilde{\varphi}_\lambda$ has at most $(q-1)$ non positive eigenvalues at each point of Ω_λ. This property does not depend on the covering nor on the local embeddings.

A real-valued function φ on a topological space Y is said to be an *exhaustion function* if the sublevel set $X_c := \{p \in X \mid \varphi(p) < c\}$ is a relatively compact set of Y for any $c \in \mathbf{R}$.

A complex space X is said to be *q-complete* if there exists an exhaustion function φ, which is q-convex on X.

A set $\mathcal{M} \subset TX$ is said to be *a linear set over* X if, for every point $p \in X$, $\mathcal{M}(p) := \mathcal{M} \cap T_pX$ is a complex vector space. We put $\operatorname{codim}\mathcal{M} := \sup_{p\in X}\operatorname{codim}\mathcal{M}(p)$ and $\mathcal{M}|_A := \mathcal{M} \cap \left(\cup_{p\in A} T_pX\right)$ for $A \subset X$.

DEFINITION 1.1. Let X be a complex space and $\mathcal{M}$ be a linear set over X.
(1) Let p be a point of X. A real-valued C^∞-function φ is said to be weakly 1-convex with respect to $\mathcal{M}(p)$ if there exists a local embedding $\iota : A \longrightarrow \Omega$, where A denotes an open neighborhood of p in X and Ω denotes an open set in $\mathbf{C}^N$, and an C^∞-extension $\widetilde{\varphi}$ to Ω of $\varphi|_A$ such that $i\partial\overline{\partial}\widetilde{\varphi}(\iota(p))(\iota_*(\xi), \iota_*(\xi)) \geq 0$ for every

$\xi \in \mathcal{M}(p)$.

The function φ is said to be weakly 1-convex with respect to $\mathcal{M}$ if φ is weakly 1-convex with respect to $\mathcal{M}(p)$ for every $p \in X$.

(2) The function φ is said to be 1-convex with respect to $\mathcal{M}$ if every point of X has an open neighborhood $U \subset X$ and a 1-convex function ψ on U such that $\varphi - \psi$ is weakly 1-convex with respect to $\mathcal{M}|_U$.

Then the following hold.

Proposition 1.2 ([15]). *Let X be a complex space and φ a q-convex function on X. Then there exists a linear set $\mathcal{M}$ over X with* $\operatorname{codim} \mathcal{M} \leq q - 1$ *such that φ is* 1*-convex with respect to $\mathcal{M}$.*

Lemma 1.3 ([15], Lemma 1.2). *Let X be a complex manifold with a hermitian metric g. Let $\mathcal{M}$ be a linear set over X. Then a real-valued C^∞-function φ is* 1*-convex with respect to $\mathcal{M}$ if and only if, for every compact set $K \subset X$, there exists a constant $\delta > 0$ such that $i\partial\overline{\partial}\varphi(p)(\xi, \xi) \geq \delta \cdot \|\xi\|_g^2$ holds for every $p \in K$ and $\xi \in \mathcal{M}$, where $\|\cdot\|_g$ denotes the norm induced by g.*

We introduce the following class which consists of continuous functions.

Definition 1.4. Let X be a complex space and $\mathcal{M}$ a linear set over X. A real-valued continuous function f on X is said to be $\mathcal{M}$-convex if every point of X has an open neighborhood U and finitely many functions $f_1, \dots, f_k : U \longrightarrow \mathbf{R}$ which are 1-convex with respect to $\mathcal{M}|_U$ satisfying $f|_U = \max\{f_1, \dots, f_k\}$.

We denote by $\mathcal{B}(X, \mathcal{M})$ the set of all $\mathcal{M}$-convex functions on X.

From the argument of [17], we approximate an $\mathcal{M}$-convex function up to second order derivative. Then we have the following result.

Proposition 1.5 (cf. [17], Theorem 1). *Let Y be a complex manifold with a hermitian metric ω and $\mathcal{L}$ a linear set over Y. Let $\eta : Y \longrightarrow (0, \infty)$ and $\kappa : Y \longrightarrow (0, \infty)$ be continuous functions. Let $w : Y \longrightarrow \mathbf{R}$ be an $\mathcal{L}$-convex function such that every point of Y has an open neighborhood $O = O(p)$ and finitely many functions $w'_1, \dots, w'_k : O \longrightarrow \mathbf{R}$ with*

$$w|_O = \max\{w'_1, \dots, w'_k\},$$

$$i\partial\overline{\partial}w'_k(p)(\xi, \xi) \geq \kappa\|\xi\|_\omega^2$$

for $p \in O$ and $\xi \in \mathcal{L}(p)$. Then there exists a C^∞-function $\widetilde{w} : Y \longrightarrow \mathbf{R}$ which is

1-*convex with respect to* $\mathcal{L}$ *such that*

$$w \leq \widetilde{w} < w + \eta,$$

$$i\partial\overline{\partial}\widetilde{w}(p)(\xi, \xi) \geq \kappa\|\xi\|_{\omega}^{2}$$

for every $p \in Y$ *and* $\xi \in \mathcal{L}(p)$, *where* $\|\cdot\|_{\omega}$ *denotes the norm induced by* ω.

On the other hand, we can approximate q-convex functions by q-convex Morse functions as follows.

Proposition 1.6 ([3], [17]). *Let* (Y, ω) *be a hermitian manifold and* φ *be a* q-*convex function. Then, for any continuous function* $\varepsilon : Y \longrightarrow (0, \infty)$, *there exists a* q-*convex Morse function* ψ *on* Y *with distinct critical values such that, for every* $p \in Y$, (i) $|\varphi(p) - \psi(p)| < \varepsilon(p)$, (ii) $\|d\varphi(p) - d\psi(p)\|_{\omega} < \varepsilon(p)$, (iii) $\|\partial\overline{\partial}\varphi(p) - \partial\overline{\partial}\psi(p)\|_{\omega} < \varepsilon(p)$.

Let Y be an n-dimensional complex manifold with a hermitian metric g. Let $\{(z_1, \ldots, z_n), U\}$ be a local coordinate neighborhood of $p \in Y$ and $(g_{i\overline{j}})_{1\leq i,j\leq n}$ the matrix representation of g with respect to $\{(z_1, \ldots, z_n), U\}$. For a real-valued C^{∞}-function v on Y, we introduce the trace of the Levi form with respect to g defined by

$$\triangle_g v(p) = \mathrm{Trace}_g\, i\partial\overline{\partial}v(p) := \sum_{1\leq i,j\leq n} g^{i\overline{j}}(p)\frac{\partial^2 v}{\partial z_i \partial\overline{z}_j}(p),$$

where $(g^{i\overline{j}})$ is the conjugate of the inverse matrix of $(g_{i\overline{j}})$ (cf. [6]). Then $\triangle_g v$ is a C^{∞}-function on Y. We will say that v is *strongly* g-*subharmonic* if $\triangle_g v(p) > 0$ for every point $p \in Y$. The C^{∞}-function v is n-convex if v is strongly g-subharmonic. Let Z be a complex submanifold of Y and $v : Z \longrightarrow \mathbf{R}$ be a C^{∞}-function. We define

$$\triangle_g v|_Z(p) := \sum g^{i\overline{j}}\Big|_Z(p) \cdot \frac{\partial^2 v}{\partial z_i \partial\overline{z}_j}\Big|_Z (p)$$

for $p \in Z$ in the similar way. We will say that v is *strongly* g-*subharmonic on* Z if $\triangle_g v|_Z(p) > 0$ for every point $p \in Z$.

Let X be a reduced complex space of dimension n. It is known that X is n-complete if X has no compact irreducible component of dimension n ([12], [6]). Indeed, we can prove it as follows. We find that there exists a sufficiently small n-complete neighborhood of $\mathrm{Sing}(X)$. Then we prove the n-completeness by showing the following proposition. We need this to show our claim.

Proposition 1.7 ([6], p. 290). *Let* X *be a reduced complex space of dimension* n *with no compact irreducible component of dimension* n. *Let* M *be a proper open* *subset of* X, *which is* n*-complete neighborhood of* $\mathrm{Sing}(X)$. *Let* $\varphi : M \longrightarrow [0,\infty)$ *be an* n*-convex exhaustion function on* M. *Let* $d \in (0,\infty)$ *be a constant with* $\{\varphi < d\} \supset \mathrm{Sing}(X)$. *Then there exist a hermitian metric* g *on* $\mathrm{Reg}(X)$ *and an* n*-convex exhaustion function* ψ *on* X *such that* (i) $\psi = \varphi$ *on* $\{p \in M \mid \varphi < d\}$, *and* $\{\varphi < d\} = \{\psi < d\}$, (ii) ψ *is strongly* g*-subharmonic on* $\mathrm{Reg}(X)$.

Moreover, we use the following theorem in [6] to examine neighborhoods of singularities.

Theorem 1.8 ([6], Theorem 1). *Let* N_1 *be an analytic subset in a complex space* N_2. *If* N_1 *is* q*-complete, then* N_1 *has a fundamental family of* q*-complete neighborhoods* N' *in* N_2.

To construct special n-convex functions, we also need to notice the following claims whose proofs are more or less immediate. For the detailed proofs, the reader is referred to [4].

Lemma 1.9 (cf. [4], Lemma 3). *Let* Z *be a complex space and* Z^* *an analytic subset containing the singular part* $\mathrm{Sing}(Z)$ *of* Z. *Let* $\{Z_\lambda\}_{\lambda\in\Lambda}$ *be connected components of* $Z \setminus Z^*$. *Let* N *be an open neighborhood of* Z^* *with* $Z \setminus \overline{N} \neq \phi$. *For* $\lambda \in \Lambda$, *let* $R_\lambda := \{a_1, \dots, a_m\}$ *be finite points of* $Z_\lambda \setminus \overline{N}$ *with* $a_i \neq a_j$ *if* $i \neq j$, *and let* $S_\lambda := \{b_1, \dots, b_m\}$ *be finite points of* $Z_\lambda \setminus \overline{N}$ *with* $b_i \neq b_j$ *if* $i \neq j$. *Then there exist a diffeomorphism* $F : Z \longrightarrow Z$ *and a compact subset* K_λ *of* Z_λ *with* $K_\lambda \cap \overline{N} = \phi$ *such that* (i) $F(R_\lambda) = S_\lambda$, (ii) F *is biholomorphic near* R_λ, (iii) F *is the identity map on* $Z \setminus K_\lambda$.

Lemma 1.10 (cf. [4], Lemma 4, [10], Proposition 3.2). *Let* Z *be a complex space and* Z^* *an analytic subset containing the singular part* $\mathrm{Sing}(Z)$ *of* Z. *Let* $\{Z_\lambda\}_{\lambda\in\Lambda}$ *be connected components of* $Z \setminus Z^*$. *Let* N *be an open neighborhood of* Z^* *with* $Z_\lambda \setminus \overline{N} \neq \phi$ *for each* $\lambda \in \Lambda$. *Let* $\{L_\nu\}_{\nu\in\mathbf{N}}$ *be a family of open sets of* Z, *and* $\{M_{\lambda,\nu} \subset Z_\lambda \mid \nu \in \mathbf{N}$ *with* $(Z_\lambda \cap L_\nu) \setminus \overline{N} \neq \phi\}$ *be a family of open sets of* Z_λ *for each* $\lambda \in \Lambda$ *such that*

(1) $\{L_\nu\}$ *is a locally finite open covering of* Z *with relatively compact connected sets,*

(2) $M_{\lambda,\nu}$ *is a non empty relatively compact set of* $(Z_\lambda \cap L_\nu) \setminus \overline{N}$ *and* $M_{\lambda,\nu} \cap L_\mu = \phi$ *if* $\nu \neq \mu$.

For each $\lambda \in \Lambda$, *let* R_λ *be a discrete set of* $Z_\lambda \setminus \overline{N}$ *and we put* $R := \cup_\lambda R_\lambda$. *Then there exists a diffeomorphism* $F : Z \longrightarrow Z$ *such that* (i) $F(R_\lambda) \subset \cup_{\nu\in\mathbf{N}} M_{\lambda,\nu}$ *for* $\lambda \in \Lambda$, *and* $F(R) \subset \cup_{\lambda,\nu} M_{\lambda,\nu}$, (ii) F *is biholomorphic near* R_λ *for* $\lambda \in \Lambda$, (iii) F *is the identity map on* $\overline{N}$.

Proof. We may assume that $Z \setminus Z^*$ has only one connected component Z_1, and $R = R_1$ is non empty set. We put $L_\nu := L_{1,\nu}$ and $M_\nu := M_{1,\nu}$ for $\nu \in \mathbf{N}$.

We put $O_l = \cup_{\nu=1}^{l} L_\nu$ for $l \in \mathbf{N}$. By using induction, we will construct a sequence of diffeomorphisms $\{F_l : Z \longrightarrow Z\}_{l \in \mathbf{N}}$ satisfying the following:
$(A)_l$ there exists a compact set K_l of O_l with $K_l \cap \overline{N} = \phi$ and F_l is the identity map on $Z \setminus K_l$,
$(B)_l$ $F_l(R_1 \cap O_l) \subset \cup_{\nu \leq l} M_\nu$,
$(C)_l$ F_l is biholomorphic near $R_1 \cap O_l$,
$(D)_l$ $F_l = F_{l-1}$ on $Z \setminus L_l$.

For $l = 1$, we put $R_1 \cap L_1 = \{a_1, \ldots, a_m\}$ and choose finitely distinct many points $\{b_1, \ldots, b_m\} \subset M_1$ with $\{a_1, \ldots, a_m\} \cap \{b_1, \ldots, b_m\} = \phi$. Let K_1 be a compact set of O_1 with $K_1 \cap \overline{N} = \phi$. From Lemma 1.9, there exists a diffeomorphism $F_1 : Z \longrightarrow Z$ such that $F_1(a_i) = b_i$ for $i = 1, \ldots, m$, and F_1 is biholomorphic near $R_1 \cap O_1$, and F_1 is the identity map on $Z \setminus K_1$.

Suppose that there exist diffeomorphisms $F_1, \ldots, F_l$ satisfying $(A)_j$,$(B)_j$,$(C)_j$,$(D)_j$ for $j = 1, \ldots, l$. From Lemma 1.9, there exists a diffeomorphism $f_{l+1} : Z \longrightarrow Z$ and a compact set K'_{l+1} with $K'_{l+1} \cap \overline{N} = \phi$ such that $f_{l+1}(R_1 \cap L_{l+1}) \subset M_{l+1}$, and f_{l+1} is biholomorphic near $R_1 \cap L_{l+1}$ and f_{l+1} is the identity map on $Z \setminus K_{l+1}$. We put $F_{l+1} := F_l \circ f_{l+1}$ and $K_{l+1} := K_l \cup K'_{l+1}$. Then the map F_{l+1} satisfies $(A)_{l+1} - (D)_{l+1}$.

From $(D)_l$, there exists the limit $F := \lim F_l$. Then F is a diffeomorphism satisfying (i)–(iii). □

Let Y be an $(n+m)$-dimensional complex manifold and T be a domain of $\mathbf{C}^m$ which contains $\overline{U}$, where U denotes the unit ball in $\mathbf{C}^m$. Let $\varpi : Y \longrightarrow T$ be a surjective holomorphic map with maximal rank. We put $Y_A := \varpi^{-1}(A)$ for $A \subset T$. Suppose that there exists a C^∞-map $S : Y_0 \times T \longrightarrow Y$ satisfying the following: (i) S is a diffeomorphism, (ii) $T \ni z \longmapsto S(y,z) \in Y$ is a holomorphic retraction over U for every $y \in Y_0$, and Y is the disjoint union of $\{S(y,T) \mid y \in Y_0\}$, (iii) the map $r : Y \longrightarrow Y_0$, defined by $S(r(p), \varpi(p)) = p$, is a C^∞-retraction onto Y_0, (iv) there exist an n-convex Morse function $h : Y_0 \longrightarrow [0, \infty)$ with distinct critical values and an open neighborhood V_0 of all critical points of h such that $r|_{r^{-1}(V_0)}$ is holomorphic.

We put $\Sigma_p := \{S(r(p), z) \mid z \in T\}$ and $F_p := \varpi^{-1} \circ \varpi(p)$ for $p \in Y$. Then the following holds.

Lemma 1.11 (cf. [4], Lemma 7). *There exists a hermitian metric G on Y such that $T_p\Sigma_p$ and T_pF_p are orthogonal with respect to G.*

We fix an open neighborhood V'_0 of all critical points of h satisfying $\overline{V'_0} \subset V_0$. We put $V_1 := r^{-1}(V_0)$ and $V_2 := Y_0 \setminus r^{-1}(\overline{V'_0})$. Let $\mathcal{N}$ be a linear set over V_0 with $\operatorname{codim} \mathcal{N} \leq n-1$ such that h is 1-convex with respect to $\mathcal{N}$ over V_0. We put $\mathcal{M}_1 := r^*(\mathcal{N})$ over V_1, which is a linear set over V_1 with $\operatorname{codim} \mathcal{M}_1 \leq n-1$. Let Γ'_p denotes the holomorphic tangent space at p to the real smooth hypersurface $\{h \circ r = h \circ r(p)\}$ and Γ''_p denotes its orthogonal complement in T_pY with respect to a hermitian metric G on Y in Lemma 1.11. We put $\mathcal{M}_2(p) := T_p\Sigma_p \oplus \Gamma''_p$ for $p \in V_2$, which is a linear set over V_2 with $\operatorname{codim} \mathcal{M}_2 = n - 1$.

We also suppose the following for the function h: (v) There exists a C^∞-function $c : [0, \infty) \longrightarrow (0, \infty)$ such that

$$|\langle \partial(h \circ r)(p), \xi \rangle| \geq c(t) \|\xi\|_G^2$$

holds for $t \in [0, \infty)$ and $p \in \{h \circ r = t\} \cap \varpi^{-1}(U) \cap V_2$ and $\xi \in \Gamma''_p$, where $\|\cdot\|_G$ denotes the norm induced by G. Then the following holds.

Proposition 1.12 (cf. [4], Lemma 9). *There exists a strictly increasing convex function $\lambda : \mathbf{R} \longrightarrow \mathbf{R}$ with $\lambda(0) = 0$ such that* (i) $i\partial\overline{\partial}(\lambda \circ h \circ r)(p)(\xi, \xi) \geq \|\xi\|_G^2$ *for $p \in \varpi^{-1}(U) \cap V_1$ and $\xi \in \mathcal{M}_1(p) \cap T_pF_p$,* (ii) $i\partial\overline{\partial}(\lambda \circ h \circ r)(p)(\xi, \xi) \geq \|\xi\|_G^2$ *for $p \in \varpi^{-1}(U) \cap V_2$ and $\xi \in \Gamma''_p$.*

We put $h^* := \lambda \circ h \circ r$, where λ denotes the C^∞-function in Proposition 1.12. We put $\varphi_{t,A} := -\log(t - h^*) + A\|\varpi\|^2$, where A denotes a positive constant. Then we have the following.

Theorem 1.13 (cf. [4], Lemma 10, Lemma 11). (1) *The function $\varphi_{t,A}$ is* 1-*convex with respect to $\mathcal{M}_1$ on $V_1 \cap \{h < t\}$ for every constant $A > 0$ and $t > 0$. Moreover, for any positive constant $s > 0$ and any relatively compact open set K of Y_0, there exists a constant $A_s > 0$ such that $i\partial\overline{\partial}\varphi_{t,A}(p)(\xi, \xi) \geq \|\xi\|_G^2/s$ holds for $0 < t \leq s$ and $p \in V_1 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_1(p)$ and $A \geq A_s$.*
(2) *Let $s > 0$ be a positive constant and K be any relatively compact open set of Y_0. Then there exists a sufficiently large constant $A_s > 0$ such that $i\partial\overline{\partial}\varphi_{t,A}(p)(\xi, \xi) \geq \|\xi\|_G^2/s$ holds for $0 < t \leq s$ and $p \in V_2 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_2(p)$ and $A \geq A_s$. Especially $\varphi_{t,A}$ is* 1-*convex with respect to $\mathcal{M}_2$ on $V_2 \cap \{h < t\} \cap r^{-1}(K)$ for $A \geq A_s$.*

Proof. (1) The first part follows from Proposition 1.12 (i) and the fact that r is holomorphic on V_1.

We show the second half as follows. We put $\Lambda(p) := \mathcal{M}_1(p) \cap T_pF_p$ for $p \in V_1$. Then $\Lambda(p)$ and $T_p\Sigma_p$ is orthogonal with respect to the hermitian metric G. We put $\xi = (\xi', \xi'') \in \mathcal{M}_1(p)$ for the orthogonal decomposition $\mathcal{M}_1(p) = T_p\Sigma_p \oplus \Lambda(p)$. Then there exists a constant $M_0 = M_0(s) > 0$ satisfying the following:

$$i\partial\overline{\partial}\|\varpi\|^2(p)(\xi, \xi) \geq M_0\|\xi'\|_G^2$$

for $0 < t \leq s$ and every $p \in V_1 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_1(p)$. By using

Proposition 1.12 (i), we have

$$i\partial\overline{\partial}\varphi_{t,A}(p)(\xi,\xi) \geq AM_0\|\xi'\|_G^2 + \frac{\|\xi''\|_G^2}{t-h^*(p)}$$

for $0 < t \leq s$ and every $p \in V_1 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_1(p)$. We put $A_s = (sM_0)^{-1}$. Then we have

$$i\partial\overline{\partial}\varphi_{t,A}(p)(\xi,\xi) \geq \frac{\|\xi'\|_G^2 + \|\xi''\|_G^2}{s} = \frac{\|\xi\|_G^2}{s}$$

for $0 \leq t \leq s$ and every $p \in V_1 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_1(p)$.

(2) We put $\xi = (\xi', \xi'') \in \mathcal{M}_2(p)$ for the orthogonal decomposition $\mathcal{M}_2(p) = T_p\Sigma_p \oplus \Gamma''_p$. Since $\overline{K}$ is a compact set of Y_0, there exist constants $M_i = M_i(s) > 0$ for $i = 1, 2, 3$ satisfying the following:

$$|\langle \partial h^*(p), \xi\rangle| \geq M_1\|\xi''\|_G^2,$$
$$i\partial\overline{\partial}h^*(p)(\xi,\xi) \geq -2M_2\|\xi'\|_G\|\xi''\|_G + \|\xi''\|_G^2,$$
$$i\partial\overline{\partial}\|\varpi\|^2(p)(\xi,\xi) \geq M_3\|\xi\|_G^2$$

for $0 < t \leq s$ and every $p \in V_2 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_2(p)$. Then we have

$$\begin{aligned} i\partial\overline{\partial}\varphi_{t,A}(p)(\xi,\xi) &\geq \frac{M_1}{(t-h^*(p))^2}\|\xi''\|_G^2 + \frac{\left(-2M_2\|\xi'\|_G \cdot \|\xi''\|_G + \|\xi''\|_G^2\right)}{t-h^*(p)} + AM_3\|\xi'\|_G^2 \\ &\geq \left(-\alpha M_2^2 + AM_3\right)\|\xi'\|_G^2 + \left(\frac{M_1}{(t-h^*(p))^2} - \frac{1}{\alpha\cdot(t-h^*(p))} + \frac{1}{t-h^*(p)}\right)\|\xi''\|_G^2 \end{aligned}$$

for $0 < t \leq s$ and every $p \in V_2 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_2(p)$ and $\alpha > 0$. We put $\alpha = 1/M_1$ and $A_s = M_3(1/s + M_2^2/M_1)$. Then we have

$$i\partial\overline{\partial}\varphi_{t,A}(p)(\xi,\xi) \geq \frac{\|\xi'\|_G^2 + \|\xi''\|_G^2}{s} = \frac{\|\xi\|_G^2}{s}$$

for $0 < t \leq s$ and every $p \in V_2 \cap \{h < t\} \cap r^{-1}(K)$ and $\xi \in \mathcal{M}_2(p)$. □

2. Convexity properties of certain subdomains of X

Let X be a connected complex manifold of dimension $N = n+m$ and T be the unit ball in $\mathbf{C}^m$. Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map and $\sigma : \widetilde{X} \longrightarrow X$ an unramified cover. We put $X_A := \pi^{-1}(A)$ and $\widetilde{X_A} := (\pi\circ\sigma)^{-1}(A)$ for every subset $A \subset T$. We denote by $\mathrm{Sing}(\pi)$ the set of all points $p \in X$ such that the differential of π does not have maximal rank at p. We denote by $\mathrm{Reg}(\pi)$ the complement of

$\mathrm{Sing}(\pi)$ in X. We put $\mathrm{Reg}(\varpi) := \sigma^{-1}(\mathrm{Reg}(\pi))$. To show our theorem, we may prove the following claim from the result of Coltoiu and Vâjâitu in [4].

CLAIM. Suppose that (i) X_0 is a reduced connected complex space of dimension n, (ii) $\dim \mathrm{Sing}(\pi) \cap X_0 \le n-1$, (iii) $\widetilde{X_0}$ is connected, (iv) $\mathrm{Sing}(\varpi) \cap \widetilde{X_0}$ and $\mathrm{Reg}(\varpi) \cap \widetilde{X_0}$ are non empty, and $\widetilde{X_0}$ has no compact irreducible component of dimension n. Then there exists an open neighborhood U of 0 such that $\widetilde{X_U}$ is n-complete.

Here we note that the assumption (i) in our Claim contains (ii) and the latter part of (iv). Indeed, π is a flat morphism on each point of X_0 from the assumption (i). On the other hand, X_0 contains a smooth Zariski open subset W such that π is a holomorphic submersion on W since X_0 is a reduced complex space. Hence $\dim \mathrm{Sing}(\pi) \cap X_0 \le \dim(X_0 \setminus W) \le n-1$ holds. Moreover, since $\mathrm{Reg}(\pi) \cap X_0$ contains W and σ is surjective, $\mathrm{Reg}(\varpi) \cap \widetilde{X_0} = \sigma^{-1}(\mathrm{Reg}(\pi) \cap X_0)$ is non empty. However we formulate such assumptions in our Claim for the plainness of this paper.

First of all, we have the following by using the argument of Kuranishi. It is a revising version for complex analytic families of relatively compact complex manifolds of so-called ‘holomorphic motions’ (cf. [7]).

Theorem 2.1 (cf. [8], [10], [4]). *For every relatively compact open set* $W \Subset \mathrm{Reg}(\pi) \cap X_0$ *and every set of finitely many points* $\mathcal{P} := \{p_1, \dots, p_s\} \subset W$, *there exist an open neighborhood* U *of* $0 \in T$ *and a* C^∞*-map* $S : \overline{W} \times U \ni (x, z) \longmapsto S(x, z) \in X_U$, *where* $\overline{W}$ *denotes the topological closure of* W *in* X_0, *satisfying the following*: (i) $S : \overline{W} \times U \longrightarrow S(\overline{W}, U)$ *is a diffeomorphism*, (ii) $U \ni z \longmapsto S(x, z) \in X_U$ *is a holomorphic section over* U *for every* $x \in \overline{W}$, *and* $S(\overline{W}, U)$ *is the disjoint union of* $\{S(x, U) \mid x \in \overline{W}\}$, (iii) *The map* $r : S(\overline{W}, U) \ni p \longmapsto r(p) \in \overline{W}$, *defined by* $S(r(p), \pi(p)) = p$, *is a* C^∞*-retraction such that there exists a relatively compact open neighborhood* $Q \Subset W$ *of* $\mathcal{P}$ *in* W *such that* $r|_{r^{-1}(Q)}$ *is holomorphic.*

Proof. See Appendix A. □

From now on, U denotes a Stein neighborhood of 0 in T and u denotes a strictly plurisubharmonic exhaustion function on U. We will replace U with a sufficiently small one if necessary.

For any subset A of a topological space Y, we denote by ∂A the topological boundary of A in Y. Let U be a neighborhood of $0 \in \mathbf{C}^m$ and r be a C^∞-retraction in Theorem 2.1 for any relatively compact open set $W \Subset \mathrm{Reg}(\pi) \cap X_0$ and any finitely many points $\mathcal{P}$ in W. Then we define the following from Theorem 2.1.

DEFINITION 2.2 (cf. [14]). For any $A \subset X_0$ such that $\overline{W}$ contains its boundary ∂A, we define $A^* \subset X_U$ with the boundary $\partial A^* = r^{-1}(\partial A)$ in X_U, and $A^* \cap X_0 = A$.

Here we consider the geometry of $\mathrm{Sing}(\pi)\cap X_0$ and its neighborhood.

Lemma 2.3. *There exist an open neighborhood* U *of* $0\in T$, *and a sufficiently small open neighborhoods* N' *of* $\mathrm{Sing}(\pi)\cap X_0$ *in* X, *and an* n*-convex exhaustion function* $\varphi : N' \longrightarrow [0,\infty)$ *satisfying the following*: (i) $N'\cap X_0$ *and* $\widetilde{X_0}\setminus\sigma^{-1}(N'\cap X_0)$ *have no compact irreducible component of dimension* n, (ii) N' *contains* $\mathrm{Sing}(\pi)\cap X_U$.

Proof. Let N_1 be an open neighborhood of $\mathrm{Sing}(\pi)\cap X_0$ in X_0 such that N_1 and $\widetilde{X_0}\setminus\sigma^{-1}(N_1)$ have no compact irreducible component of dimension n. Such a neighborhood N_1 exists from assumptions (ii) and (iv) of our claim and the fact that $\widetilde{X}$ is locally isomorphic to X and X_0 is compact.

Then there exists an n-convex exhaustion function $\varphi_1 : N_1 \longrightarrow [0,\infty)$ from result of Ohsawa in [12] (cf. [6]). Let N_2 be an open set of X such that N_1 is an analytic subset in N_2 with $N_2\cap X_0 = N_1$. From Theorem 1.8, there exist an open neighborhood N' of N_1 in N_2 and an n-convex exhaustion function $\varphi : N' \longrightarrow [0,\infty)$ with $N'\cap X_0 = N_1$. Moreover there exists an open neighborhood U of $0\in T$ such that N' contains $\mathrm{Sing}(\pi)\cap X_U$, since π is a proper holomorphic map. Hence we have Lemma 2.3. □

Let W be a relatively compact open set of $\mathrm{Reg}(\pi)\cap X_0$ satisfying

$$(N'\cap X_0)\cup W = X_0.$$

Let V' be an open neighborhood of $\overline{N'}$ in X_0 such that V' and $\widetilde{X_0}\setminus\sigma^{-1}(V')$ have no irreducible compact component of dimension n. Such a neighborhood V' exists from Lemma 2.3 (i). Let $d^*>0$ be a constant such that

$$N := \{p\in X_0 \mid \varphi(p) < d^*\} \supset X_0\setminus W \supset \mathrm{Sing}(\pi)\cap X_0,$$

where $\varphi|_N$ is an n-convex exhaustion function on N with $N\cup W = X_0$. We put

$$\mathcal{N} := \{p\in X \mid \varphi(p) < d^*\}.$$

Proposition 2.4. *For a sufficiently small positive constant* ε, *there exist a hermitian metric* g_0 *on* $\mathrm{Reg}(\pi)\cap V'$ *and an* n*-convex exhaustion Morse function* $h_0 : V' \longrightarrow [0,\infty)$ *satisfying* (i) $|h_0(p)-\varphi(p)| < \varepsilon$ *for* $p\in N$ *and* $N\subset\{h_0 < d^*+\varepsilon\}$, (ii) h_0 *is strongly* g_0*-subharmonic on* $\mathrm{Reg}(\pi)\cap V'$ *and* φ *is strongly* g_0*-subharmonic on* $\mathrm{Reg}(\pi)\cap N$.

Proof. From Proposition 1.7 There exists an n-convex exhaustion function $h_0' : V' \longrightarrow [0,\infty)$ with $h_0' = \varphi$ on N and $\{h_0' < d^*\} = N$.

From Lemma 6 in [6], there exists a hermitian metric g_0 on $\mathrm{Reg}(\pi)\cap V'$ such that h_0' is strongly g_0-subharmonic on $\mathrm{Reg}(\pi)\cap V'$. We apply Proposition 1.6 and approximate h_0' by an n-convex Morse function up to second order derivative. Then there exists an n-convex Morse function h_0 such that $|h_0 - h_0'| < \varepsilon$ and $|\triangle_{g_0} h_0(p) - \triangle_{g_0} h_0'(p)| < \varepsilon$ on V'. This function h_0 satisfies (i) and (ii), if ε is sufficiently small. □

Let U be a neighborhood of $0 \in T$ and r a C^∞-retraction satisfying properties of Theorem 2.1 for the relatively compact open set W of $\mathrm{Reg}(\pi) \cap X_0$ and finitely many points $\mathcal{P} \subset W$, which will be chosen later. By replacing U with a sufficiently small one, we may suppose that

$$N^* \cup W^* = X_U$$

holds. Then the following holds.

Proposition 2.5. *For a sufficiently small U of $0 \in T$, there exist an open set V of V' with $\overline{N} \subset V$, and a hermitian metric g on $\overline{X_U}$ and positive constants d_1 and d_0 with $d_1 > d_0$ and a bounded C^∞-function $h_0^* : V^* \longrightarrow [0, d_1)$ such that*
(i) *V and $\widetilde{X_0} \setminus \sigma^{-1}(V(t))$ have no compact irreducible component of dimension n, where we put $V(t) := \{p \in V \mid h_0^* < t\}$ for $t \in [d_0, d_1]$,*
(ii) *$h_0^* \circ \sigma \equiv d_1$ on ∂V_i and $[d_0, d_1) \subset (h_0^* \circ \sigma)(V_i)$ for every $i \in I$, where $\{V_i\}_{i \in I}$ denotes connected components of $\sigma^{-1}(V)$,*
(iii) *W contains $V \setminus V(d_0)$ and $V^* \cup W^* = X_U$,*
(vi) *h_0^* has no critical point on $V \setminus V(d_0)$, and $h_0^* = h_0^* \circ r$ on $V^* \setminus V(d_0)^*$,*
(v) *h_0^* is n-convex on $V^* \cap X_z$ for every $z \in U$, and $\{p \in X_z \mid h_0^* < t\}$ is a relatively compact open set of $V^* \cap X_z$ for every $z \in U$ and $t \in [0, d_1)$,*
(vi) *h_0^* is strongly g-subharmonic on each subset $V^* \cap \mathrm{Reg}(\pi) \cap X_z$ of X_z for every $z \in U$,*
(vii) *h_0^* is n-convex on the subset $V^* \setminus W^*$.*

Proof. Let d_1 be a regular value of h_0 satisfying the following:
(2.5.1-a) $\overline{N} = \overline{\{p \in X_0 \mid \varphi(p) < d^*\}} \subset \{h_0 < d_1\}$,
(2.5.1-b) $h_0 \circ \sigma \equiv d_1$ on ∂V_i for every $i \in I$, where $\{V_i\}_{i \in I}$ denotes connected components of $\sigma^{-1}(\{h_0 < d_1\})$.
Such a constant d_1 exists from Proposition 2.4 (i) and the fact that $\widetilde{X_0}$ is locally isomorphic to X_0, and X_0 is compact.

Let $d_0 < d_1$ be a constant satisfying the following:
(2.5.2-a) $\overline{N} \subset \{h_0 < d_0\}$,
(2.5.2-b) $[d_0, d_1]$ has no critical value of h_0,
(2.5.2-c) $[d_0, d_1) \subset (h_0 \circ \sigma)(V_i)$ for every $i \in I$.
Such a constant d_1 exists if $d_1 - d_0$ is sufficiently small. We put

$$V := \{h_0 < d_1\}.$$

Then W contains $V \setminus V(d_0)$ since $N \cup W = X_0$, and $V^* \cup W^* = X_U$ for a sufficiently small U. Hence (ii) and (iii) hold.

The set $V' \setminus V(t)$ has no compact component in V, and the boundary of each connected component of $V' \setminus V(t)$ intersects ∂V in X_0. Indeed, if one of the two claims does not hold, there does not exist an n-convex exhaustion function $h_0 : V' \longrightarrow [0, \infty)$ in Proposition 2.4. Then $\sigma^{-1}(V' \setminus V(t))$ has no compact component in $\sigma^{-1}(V)$ and the boundary of each connected component of $\sigma^{-1}(V' \setminus V(t))$ intersects $\sigma^{-1}(\partial V)$ in $\widetilde{X_0}$, because $\widetilde{X_0}$ is locally isomorphic to X_0. On the other hand, $\widetilde{X_0} \setminus \sigma^{-1}(V')$ has no compact irreducible component of dimension n. Hence (i) holds because $\widetilde{X_0} \setminus \sigma^{-1}(V(t))$ is the union of $\widetilde{X_0} \setminus \sigma^{-1}(V')$ and $\sigma^{-1}(V' \setminus V(t))$.

From Proposition 2.4, there exists a hermitian metric g_0 on $\mathrm{Reg}(\pi) \cap V$ such that
(2.5.3-a) h_0 is strongly g_0-subharmonic on $\mathrm{Reg}(\pi) \cap V$,
(2.5.3-b) φ is strongly g_0-subharmonic on $\mathrm{Reg}(\pi) \cap N$.

Then there exist a sufficiently small neighborhood U of $0 \in T$ and a hermitian metric g on $\overline{X_U}$ such that
(2.5.4-a) $h_0 \circ r$ is strongly g-subharmonic on $(W^* \cap V^*) \cap X_z$,
(2.5.4-b) φ is strongly g-subharmonic on $\mathrm{Reg}(\pi) \cap \mathcal{N} \cap X_z$ for every $z \in U$,
since h_0 is bounded up to second order derivative on V and r is a C^∞-map (cf. [14], Theorem 1).

Let $a_1 \in (0, d^*)$ be a constant with

$$V^* \setminus W^* \subset \{p \in X_U \mid h_0 \circ r(p) < a_1\} \Subset \mathcal{N}$$

for a sufficiently small neighborhood U of $0 \in T$. Let $a_2 \in (a_1, d_1)$ be a constant and $\rho_1 : V \longrightarrow [0, 1]$ be a C^∞-function with $\rho_1 \equiv 1$ on $\{h_0 \geq a_2\}$ and $\operatorname{supp} \rho_1 \subset \{h_0 \geq a_1\}$. Let L_1 be an open set with $\{h_0 \leq a_2\}^* \subset L_1 \subset \mathcal{N}$, and L_2 an open set with $\overline{L_1} \Subset L_2 \subset \mathcal{N}$. Let $\rho_2 : V^* \longrightarrow [0, 1]$ be a C^∞-function with $\rho_2 \equiv 1$ on $\overline{L_1}$ and $\operatorname{supp} \rho_2 \subset \overline{L_2}$. Then we have $\rho_2 \equiv 0$ on $\{d_0 \leq h_0 \leq d_1\}^*$.

We put $h_0^* := \lambda_1 \circ (\rho_1 \cdot h_0) \circ r + C \rho_2 \cdot \varphi$, where C denotes a positive constant and $\lambda_1 : \mathbf{R} \longrightarrow \mathbf{R}$ denotes a C^∞-function such that $\lambda_1 \equiv 1$ on $\{t \leq a_2\}$ and λ_1 is strictly increasing convex on $\{t > a_2\}$. Then the function h_0^* is bounded on V^*, and equal to $(\rho_1 \cdot h_0) \circ r + C \rho_2 \cdot \varphi$ on L_1. Hence there exists a sufficiently large constant $C > 0$ such that h_0^* is n-convex on L_1 and strongly g-subharmonic on $L_1 \cap \mathrm{Reg}(\pi) \cap X_z$ for every $z \in U$. We fix a positive constant δ such that L_1 contains $\{h_0 \leq a_2 + \delta\}^*$. Then h_0^* is strongly g-subharmonic on $V^* \cap \mathrm{Reg}(\pi) \cap X_z$ for every $z \in U$ if λ_1' and λ_1'' are sufficiently large on $\{t \geq a_2 + \delta\}$. For such λ_1 and $C > 0$, we replace $\lambda_1(d_i)$ with d_i for $i = 0, 1$. Then h_0^* satisfies (iv), (vi). Since $\lambda_1' \geq 0$ and $\lambda_1'' \geq 0$ hold, h_0^* also satisfies (v), (vii). □

REMARK 2.6. Since X_0 is compact, there exist a finite open covering $\{O_\mu \subset X_0\}_{\mu=1,\dots,s}$ of X_0 and finitely many points $\mathcal{P} := \{p_\mu\}$ satisfying the following:
(i) each O_μ is connected,

(ii) σ is biholomorphic from each connected component of $\widetilde{O_\mu}$ to O_μ,
(iii) if $O_\mu \setminus \overline{V} \neq \phi$, $\cup_{\iota \neq \mu} O_\iota \setminus \overline{V}$ does not contain $O_\mu \setminus \overline{V}$,
(vi) if $O_\mu \setminus \overline{V} \neq \phi$, there exists a point $p_\mu \in O_\mu \setminus \overline{V}$ with $p_\mu \notin \overline{O_\iota}$ if $\mu \neq \iota$.

From now on, let U and r be the same as those in Theorem 2.1 for W and

$$\mathcal{P} = \{p_\mu \in O_\mu \setminus \overline{V} \mid \mu = 1, \ldots, s \text{ with } O_\mu \setminus \overline{V} \neq \phi\}$$

in Remark 2.6. Obviously, this modifications of U and r do not affect Proposition 2.5.

For every $p \in W^*$, we put $\Sigma_p := \{S(r(p), z) \mid z \in U\}$ and $F_p := \pi^{-1} \circ \pi(p)$. Then Σ_p and F_p are closed complex manifolds. From Lemma 1.11, we have the following.

Lemma 2.7. *There exists a hermitian metric G on W^* such that, for any $p \in W^*$, the complex vector subspaces $T_p\Sigma_p$ and T_pF_p of T_pW^* are orthogonal with respect to G.*

We denote by $\|\cdot\|_G$ the norm induced by G. We may assume that g and G are quasi-isometrically equivalent on W^*, by replacing a relatively compact open set W' of W satisfying $N \cup W' = X_0$ (resp. $G|_{W'}$) with W (resp. G), if necessary.

We fix constants d_3 and d_2 with $d_0 < d_3 < d_2 < d_1$. Then the following holds.

Lemma 2.8 (cf. [10], Proposition 3.3 (II)). *There exist constants $c_1^* > 0$, $A_1 > 0$ and a linear set $\mathcal{M}_1$ over $V(d_3)^*$ with* codim $\mathcal{M}_1 \leq n - 1$ *such that*

$$i\partial\overline{\partial}\left(-\log(t - h_0^*) + A\|\pi\|^2\right)(p)(\xi, \xi) \geq c_1^*\|\xi\|_g^2$$

holds for $t \in [d_2, d_1]$, $p \in V(d_3)^*$ and $\xi \in \mathcal{M}_1(p)$ and any $A \geq A_1$.

Proof. From Proposition 2.5 (vi), there exists a 1-dimensional complex subspace $l(p)$ of T_pF_p such that $i\partial\overline{\partial}(-\log(t - h_0^*))(p)(\xi', \xi') > 0$ for $t \in [d_2, d_1]$, $p \in V(t)^* \cap \operatorname{Reg}(\pi)$ and $\xi' \in l(p)$.

Let d_3' be a constant with $d_3 < d_3' < d_2$. For $p \in V(d_3')^* \cap \operatorname{Reg}(\pi)$, let $Z(p)$ be an m-dimensional complex subspace of T_pX_U with $\dim \pi_* Z(p) = m$.

We put $\mathcal{M}_1'(p) := Z(p) \oplus l(p)$ for $p \in V(d_3')^* \cap \operatorname{Reg}(\pi)$, which is an $(m+1)$-dimensional complex subspace of T_pX_U. Let $\theta(p) = \{\theta_i(p)\}$ be a basis of $\mathcal{M}_1'(p)$ satisfying $\operatorname{span}\langle\theta_1(p), \ldots, \theta_m(p)\rangle_{\mathbf{C}} = Z(p)$ and $\operatorname{span}\langle\theta_{m+1}(p)\rangle_{\mathbf{C}} = l(p)$ for any $p \in V(d_3')^* \cap \operatorname{Reg}(\pi)$ with $\|\theta_i(p)\|_g = 1$. We may assume that θ_i's are C^∞-sections, by taking each $Z(p)$ and $l(p)$ adequately.

Then we have

$$i\partial\overline{\partial}\|\pi\|^2 = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$$

as the matrix representation with respect to θ, where C_1 is an $m \times m$-matrix valued function, since π is constant on X_z for $z \in U$. The matrix C_1 is positive definite on $V(d_3')^* \cap \mathrm{Reg}(\pi)$.

Let K_1 be an open neighborhood of $\mathrm{Sing}(\pi)$ with $\overline{K_1} \subset V^* \setminus W^*$. Then there exists a constant $c_1 > 0$ such that $\det C_1 > c_1$ holds on $V(d_3')^* \setminus K_1$. We put $i\partial\overline{\partial}(-\log(t - h_0^*)) = (b_{ij})$ with respect to θ. The functions b_{ij} are bounded on $V(d_3')^* \cap \mathrm{Reg}(\pi)$. Since b_{m+1m+1} is the matrix representation of $i\partial\overline{\partial}(-\log(t - h_0^*))|_l$ with respect to θ, there exists a constant $c_2 > 0$ such that $b_{m+1m+1} > c_2$ on $V(d_3')^* \setminus K_1$.

We put $C_2 = (b_{ij})_{1 \le i,j \le m}$ and $C_3 = (d_{ij})_{1 \le i,j \le m}$. Then there exists a positive constant $c_3 > 0$ such that $AC_1 + C_2 + C_3 > c_3 I_m$ on $V(d_3')^* \setminus K_1$ if $A \ge A_1'$ for a large constant $A_1' > 0$, where I_m denotes the $m \times m$-identity matrix. We denote by M the matrix representation of the hermitian form $i\partial\overline{\partial}(-\log(t - h_0^*)) + A \cdot \|\pi\|^2)$ with respect to θ. By a calculation we have

$$\begin{aligned} \det M &= (b_{m+1m+1})A^m \det C_1 + Q_{m-1}(A) \\ &\ge c_1 c_2 A^m + Q_{m-1}(A), \end{aligned}$$

where $Q_{m-1}(A)$ stands for a polynomial of A of degree $m-1$ whose coefficients are bounded functions on $V(d_3')^* \setminus K_1$. Hence there exists a positive constant $c_4 > 0$ such that $\det M > c_4$ on $V(d_3')^* \setminus K_1$ if $A \ge A_1''$ for a large constant $A_1'' > 0$. Hence, from Theorem 13.3.2 in [9],

$$i\partial\overline{\partial}\left(-\log(t - h_0^*) + A\|\pi\|^2\right)(p)(\xi, \xi) > 0$$

holds for $t \in [d_2, d_1]$, $p \in V(d_3')^* \setminus K_1$ and $\xi \in \mathcal{M}_1'(p)$ and $A \ge A_1 := \max\{A_1', A_1''\}$. Since the function $-\log(t - h_0^*) + A\|\pi\|^2$ is a C^∞-function on $V(d_3')^*$, there exists a constant $c_5 > 0$ such that

$$i\partial\overline{\partial}\left(-\log(t - h_0^*) + A\|\pi\|^2\right)(p)(\xi, \xi) \ge c_5\|\xi\|_g^2$$

holds for $t \in [d_2, d_1]$, $p \in V(d_3)^* \setminus K_1$ and $\xi \in \mathcal{M}_1'(p)$ and $A \ge A_1$.

On the other hand, from Proposition 2.5 (vii), the function $-\log(t - h_0^*)$ is n-convex on $V^* \setminus W^*$. Let K_2 be an open neighborhood of $\mathrm{Sing}(\pi)$ with $K_1 \Subset K_2$ and $\overline{K_2} \subset V^* \setminus W^*$. Then we have $\left(V(d_3)^* \setminus K_1\right) \cup K_2 = V(d_3)^*$. From Proposition 1.2 and Lemma 1.3, there exist a constant $c_6 > 0$ and a linear set $\mathcal{M}_1''(p)$ over K_2 with $\mathrm{codim}\,\mathcal{M}_1'' \le n - 1$ such that

$$i\partial\overline{\partial}\left(-\log(t - h_0^*) + A\|\pi\|^2\right)(p)(\xi, \xi) \ge c_6\|\xi\|_g^2$$

holds for $t \in [d_2, d_1]$, $p \in K_2$ and $\xi \in \mathcal{M}_1''(p)$ for every $A > 0$, by replacing U with a sufficiently small one.

We denote by $\mathcal{M}_1'''$ be the restriction of $\mathcal{M}_1'$ on $V(d_3)^* \setminus K_2$. We put $\mathcal{M}_1 := \mathcal{M}_1'' \cup \mathcal{M}_1'''$, which is a linear set over $V(d_3)^*$ with $\mathrm{codim}\,\mathcal{M}_1 \le n - 1$. We put

$c_1^* := \min\{c_5, c_6\}$. Then we have

$$i\partial\overline{\partial}\left(-\log(t-h_0^*)+A\|\pi\|^2\right)(p)(\xi,\xi)\geq c_1^*\|\xi\|_g^2$$

for $t\in[d_2,d_1]$, $p\in V(d_3)^*$ and $\xi\in\mathcal{M}_1(p)$ and $A\geq A_1$. $\square$

Let $\lambda_2:\mathbf{R}\longrightarrow\mathbf{R}$ be a C^∞-function in Proposition 1.12 with respect to the function h_0^* and the hermitian metric G on W^*, and we replace $\lambda_2(h_0^*)$ and $\lambda_2(d_i)$ for $i=0,1,2,3$ with h_0^* and d_i for $i=0,1,2,3$, respectively. Then the following holds.

Proposition 2.9. *There exist a linear set $\mathcal{M}_2$ over V^* with* $\operatorname{codim}\mathcal{M}_2\leq n-1$ *and positive constants A_2 and c_2^* such that $i\partial\overline{\partial}(-\log(t-h_0^*)+A\|\pi\|^2)(p)(\xi,\xi)\geq c_2^*\cdot\|\xi\|_g^2$ holds for $t\in[d_2,d_1)$, $p\in V^*(t)$ and $\xi\in\mathcal{M}_2(p)$ and any $A\geq A_2$.*

Proof. Let $d_4\in(d_0,d_3)$ be a constant. From Theorem 1.13 (2), there exist a linear set $\mathcal{M}_2'$ over $V^*\setminus V(d_4)^*$ and a positive constant A_2' such that

$$i\partial\overline{\partial}\left(-\log(t-h_0^*)+A\|\pi\|^2\right)(p)(\xi,\xi)\geq\frac{1}{d_1}\|\xi\|_G^2$$

holds for $p\in V^*(t)\setminus V(d_4)^*$ and $\xi\in\mathcal{M}_2'(p)$ and any $A\geq A_2'$. Since g and G is quasi-isometrically equivalent on W^*, there exists a constant $c_1>0$ such that

$$\|\xi\|_G^2\geq c_1\|\xi\|_g^2$$

holds for $p\in V^*(t)\setminus V(d_4)^*$ and $\xi\in\mathcal{M}_2'(p)$.

On the other hand, from Lemma 2.8, there exist positive constants c_1^* and A_1 and a linear set $\mathcal{M}_1$ over $V(d_3)^*$ such that

$$i\partial\overline{\partial}\left(-\log(t-h_0^*)+A\|\pi\|^2\right)(p)(\xi,\xi)\geq c_1^*\|\xi\|_g^2$$

holds for $p\in V(d_3)^*$ and $\xi\in\mathcal{M}_1(p)$ and any $A\geq A_1$.

We denote by $\mathcal{M}_2''$ the restriction of $\mathcal{M}_2'$ on $V^*\setminus V(d_3)^*$ and put $\mathcal{M}_2:=\mathcal{M}''\cup\mathcal{M}_1$, which is a linear set over V^* with $\operatorname{codim}\mathcal{M}_2\leq n-1$. We put $c_2^*:=\min\{c_1/d_1,c_1^*\}$. Then we have

$$i\partial\overline{\partial}\left(-\log(t-h_0^*)+A\|\pi\|^2\right)(p)(\xi,\xi)\geq c_2^*\cdot\|\xi\|_g^2$$

for $t\in[d_2,d_1]$, $p\in V^*$, $\xi\in\mathcal{M}_2(p)$ and $A\geq A_2:=\max\{A_2',A_1\}$. $\square$

3. Constructions of special n-convex functions on the fiber $\widetilde{X_z}$

We put $\varpi:=\pi\circ\sigma$, $\operatorname{Sing}(\varpi):=\sigma^{-1}(\operatorname{Sing}(\pi))$ and $\operatorname{Reg}(\varpi):=\sigma^{-1}(\operatorname{Reg}(\pi))$. We denote by $\widetilde{A}$ the set $\sigma^{-1}(A)$ for $A\subset X$. For $B\subset\widetilde{X_U}$, we denote by $Cl(B)$ or $\overline{B}$ the topological closure of B in $\widetilde{X_U}$.

Let S and r be the same as those in Theorem 2.1 for W and $\mathcal{P}$, which are fixed in §2. Let $\widetilde{S}$ (resp. $\widetilde{r}$) be the lift of S (resp. r) over $Cl(\widetilde{W^*})$. We put $\widetilde{\mathcal{M}_2} := \sigma^*\mathcal{M}_2$ over $\widetilde{V^*}$, $\widetilde{g} := \sigma^*g$ on $Cl(\widetilde{X_U})$, and $\widetilde{G} := \sigma^*G$ on $\widetilde{W^*}$. Then $\widetilde{g}$ and $\widetilde{G}$ are quasi-isometrically equivalent on $\widetilde{W^*}$ since g and G are quasi-isometrically equivalent on W^*.

Here, we can define the following by lifting results of Theorem 2.1.

DEFINITION 3.1 (cf. [14]). For any $A \subset \widetilde{X_0}$ with $Cl(\widetilde{W})$ contains ∂A, we define $A^* \subset \widetilde{X_U}$ with the boundary $\partial A^* = \widetilde{r}^{-1}(\partial A)$ in $\widetilde{X_U}$ and $A^* \cap \widetilde{X_0} = A$.

Let $\{V_i \subset \widetilde{X_0}\}_{i\in I}$ be connected components of $\widetilde{V} = \sigma^{-1}(V)$. From Proposition 2.5, $h_0^* \circ \sigma \equiv d_1$ on ∂V_i and $[d_0, d_1) \subset (h_0^* \circ \sigma)(V_i)$ hold for every $i \in I$. We put $V_i(t) = \{p \in V_i \mid h_0^* \circ \sigma < t\}$ for $d_0 \le t \le d_1$ and $i \in I$. Then we have $\widetilde{V} = \cup_{i\in I} V_i$ and $\widetilde{V(t)} = \cup_{i\in I} V_i(t)$.

In this section, we will construct a special n-convex function h_2 on $\widetilde{X_0}$ by using the argument of Demailly (cf. [6]), and show the existence of a C^∞-function h_3 on $\widetilde{X_U}$ such that h_3 is n-convex on $\widetilde{X_z}$ for every $z \in U$ as follows.

Lemma 3.2. *There exists an open subset* Y_1 *of* $\widetilde{X_0}$ *satisfying the following*: (i) $\widetilde{X_0} \setminus Y_1$ *has no compact component,* (ii) $\cup_{i\in I} V_i(d_2) \subset Y_1 \subset \widetilde{V}$, *and* $\partial Y_1 \cap \partial\widetilde{V} = \phi$ *in* $\widetilde{X_0}$, (iii) *for every* $j \in J := \{j \in I \mid V_j \Subset \widetilde{X_0}\}$, $Y_1 \cap V_j = V_j(d_2)$, (iv) *for every* $i \in I \setminus J$, *there exists a sequence of compact sets* $\{K_i(t)\}_{t\in[d_2,d_1)}$ *of* V_i *such that* $(Y_1 \cap V_i(t)) \setminus K_i(t) = V_i(t) \setminus K_i(t)$ *for* $t \in [d_2, d_1)$.

Proof. Let $i \in I \setminus J$. Then each V_i is noncompact and connected, and $[d_0, d_1]$ has no critical value of $h_0^* \circ \sigma|_{\widetilde{V}}$ from Proposition 2.5. Hence there exist an open subset $Y_{1,i}$ of V_i and a sequence of compact subsets $\{K_i(t)\}_{t\in[d_2,d_1)}$ of V_i satisfying the following:
(3.2.1-a) $V_i \setminus Y_{1,i}$ has no compact component of V_i, and the boundary of each connected component of $V_i \setminus Y_{1,i}$ in $\widetilde{X_0}$ intersects ∂V_i,
(3.2.1-b) $V_i(d_2) \subset Y_{1,i}$ and $\partial Y_{1,i} \cap \partial V_i = \phi$ in V_i,
(3.2.1-c) $(Y_{1,i} \cap V_i(t)) \setminus K_i(t) = V_i(t) \setminus K_i(t)$ for any $t \in [d_2, d_1)$.

We put

$$Y_1 := \big(\cup_{i\in I\setminus J} Y_{1,i}\big) \cup \big(\cup_{j\in J} V_j(d_2)\big).$$

Then $\widetilde{X_0} \setminus Y_1$ has no compact component from Proposition 2.5 (i) and (3.2.1-a) and the fact that $V_j \setminus V_j(d_2)$ has no compact component of V_j and the boundary of each connected component of $V_j \setminus V_j(d_2)$ in $\widetilde{X_0}$ intersects ∂V_j for every for $j \in J$. Property (ii) follows from (3.2.1-b) and the definition of Y_1. Property (iii) follows from the definition of Y_1, and (iv) follows from (3.2.1-c). □

Let $\{\Omega_j\}_{j\in\mathbf{N}}$ be connected components of $\mathrm{Reg}(\varpi)$. From the assumption (iv) of our claim, $\widetilde{X}_0$ has no compact irreducible component of dimension n. Hence $\overline{\Omega_j}$ is noncompact in $\widetilde{X_0}$. Then we have the following.

Lemma 3.3 ([6], Lemma 10). *For each* $j \in \mathbf{N}$, *there exists a family of open sets* $\{U_{j,k} \Subset \Omega_j\}_{k\in\mathbf{N}}$ *such that* (i) $\Omega_j \setminus \widetilde{V} \subset \cup_{k\in\mathbf{N}} U_{j,k} \subset \Omega_j \setminus Y_1$, (ii) *for every connected component* $U_{j,k,s}$ *of* $U_{j,k}$, *there exists a connected component* $U_{j,k+1,t(s)}$ *of* $U_{j,k+1}$, *such that* $U_{j,k+1,t(s)} \cap U_{j,k,s} \neq \phi$ *and* $U_{j,k+1,t(s)} \setminus \overline{U}_{j,k,s} \neq \phi$.

We put

$$\Psi := -\log(d_1 - h_0^* \circ \sigma) + C_i \quad \text{on} \quad V_i^* \quad \text{for} \quad i \in I, \tag{3A}$$

where $\{C_i\}_{i\in I}$ are positive constants satisfying
(3A-a) $C_1 = 0$,
(3A-b) $\Psi(V_i(d_2)^*)$ and $\Psi(V_j(d_2)^*)$ do not intersect if $i \neq j$,
(3A-c) Ψ is exhaustive on $\cup_{j\in J} V_j(d_2)$.
Such constants $\{C_i\}$ exist since V_i^*(resp. $V_i(d_2)^*$) and V_j^*(resp. $V_j(d_2)^*$) do not intersect if $i \neq j$. We put

$$Y_2 := \widetilde{X_0} \setminus \overline{\cup_{j,k} U_{j,k}},$$

which satisfies $\widetilde{V(d_2)} \subset Y_1 \subset Y_2 \subset \widetilde{V}$ from Lemma 3.3 (i). Then we have the following.

Proposition 3.4 (cf. [6], p. 290). *There exists an n-convex function* $h_1 : \widetilde{X_0} \longrightarrow [0, \infty)$ *such that* (i) $h_1 = \Psi$ *on* Y_2, (ii) *for any* $c \in [0, \infty)$, $\{p \in \widetilde{X_0} \mid h_1 < c\} \setminus Y_2$ *is relatively compact in* $\widetilde{X_0}$.

Proof. There exists a C^∞-function $v : \widetilde{X_0} \longrightarrow [0, \infty)$ such that $v = \Psi$ on Y_2 and v is exhaustive on $\widetilde{X_0} \setminus Y_2$. From Proposition 2.5 (v) and (3A), the function v is n-convex on Y_2. Hence, from Lemma 6 in [6], there exists a hermitian metric g_1 on $\widetilde{X_0} \setminus \mathrm{Sing}(\pi)$ such that v is strongly g_1-subharmonic on Y_2. Lemma 7 in [6] implies that, for any j and k, there exists a C^∞-function $v_{j,k} : U_{j,k} \longrightarrow [0, \infty)$ with support in $U_{j,k} \cup U_{j,k+1}$ which is strongly g_1-subharmonic on $U_{j,k}$, where $\{U_{j,k}\}$ denotes a family of open sets in Lemma 3.3. We put $h_1 := v + \sum_{j,k} C_{j,k} v_{j,k}$, for large constants $C_{j,k}$. By induction, we have a sequence of positive constants $\{C_{j,k}\}$ such that h_1 is strongly g_1-subharmonic on $\mathrm{Reg}(\varpi) \cap \widetilde{X_0}$ and $h_1 = \Psi$ on Y_2. Since v is exhaustive on $\widetilde{X_0} \setminus Y_2$, h_1 is exhaustive on $\widetilde{X_0} \setminus Y_2$. Hence h_1 satisfies properties (i), (ii). $\square$

Let r and Q be the same as those in Theorem 2.1 for W and $\mathcal{P}$, which are fixed in §2. Then the following holds.

Proposition 3.5. *There exist a n-convex function $h_2 : \widetilde{X_0} \longrightarrow [0,\infty)$ and a strictly increasing convex function $\lambda_3 : \mathbf{R} \longrightarrow \mathbf{R}$ such that* (i) *$h_2 = \lambda_3(\Psi)$ on Y_1,* (ii) *for any $c \in [0,\infty)$, $\{p \in \widetilde{X_0} \mid h_2 < c\} \setminus Y_1$ is relatively compact in $\widetilde{X_0}$,* (iii) *h_2 is a Morse function with distinct critical values on $\widetilde{X_0} \setminus \overline{Y_1}$,* (iv) *all critical points of h_2 in $\widetilde{X_0} \setminus \overline{Y_1}$ is contained by $\widetilde{Q} = \sigma^{-1}(Q)$.*

Proof. From Proposition 2.5 (iv) and Proposition 3.4 (i), the function $h_1 = \Psi$ has no critical point on $Y_2 \cap \sigma^{-1}(V \setminus V(d_2))$. Let $\varepsilon : \widetilde{X_0} \longrightarrow (0,\infty)$ be a continuous function. From Proposition 1.6, there exists an n-convex Morse function with distinct critical values $h_2' : \widetilde{X_0} \setminus \overline{Y_1} \longrightarrow [0,\infty)$ satisfying $|h_2'(p) - h_1(p)| < \varepsilon(p)$ and $\|dh_2'(p) - dh_1(p)\|_{\widetilde{g}} < \varepsilon(p)$ for any $p \in \widetilde{X_0} \setminus \overline{Y_1}$.

Let Y_3 be an open set of $\widetilde{X_0}$ with $\overline{Y_1} \subset Y_3$ and $\overline{Y_3} \subset Y_2$. Let Y_4 be an open set of $\widetilde{X_0}$ with $\overline{Y_3} \subset Y_4$ and $\overline{Y_4} \subset Y_2$. Let $\rho : \widetilde{X_0} \longrightarrow [0,1]$ be a C^∞-function with $\rho \equiv 1$ on Y_3 and $\mathrm{supp}(\rho) \subset Y_4$. We put

$$h_2'' := \rho h_1 + (1-\rho)h_2'.$$

Then h_2'' is a C^∞-function on $\widetilde{X_0}$ with $h_2'' = h_1$ on Y_1. By a calculation, we have

$$\|dh_2''(p) - dh_1(p)\|_{\widetilde{g}} < \varepsilon(p)(\|d\rho(p)\|_{\widetilde{g}} + \rho(p)) \quad \text{for} \quad p \in \widetilde{X_0}.$$

Hence $h_2'' = \Psi$ holds on Y_3, and h_2'' is an n-convex Morse function with distinct critical values on $\widetilde{X_0} \setminus \overline{Y_1}$ if ε is sufficiently small. Moreover, for any $c \in [0,\infty)$, $\{p \in \widetilde{X_0} \mid h_1 < c\} \setminus Y_1$ is relatively compact in $\widetilde{X_0}$.

Let $\{O_\mu\}$ and $\mathcal{P}$ be the same as those in Remark 2.6. Then each connected component of $\widetilde{X_0} \setminus \overline{Y_1}$ intersects $\sigma^{-1}(\mathcal{P})$. For $\mu = 1, \ldots, s$ with $O_\mu \setminus \overline{V} \neq \phi$, let Q_μ be a relatively compact open neighborhood of p_μ with $Q_\mu \subset Q \cap (O_\mu \setminus \overline{V})$.

Let $\{L_{\mu,\lambda}\}$ be connected components of $\widetilde{O_\mu}$ and $\{M_{\mu,\lambda}\}$ connected components of $\widetilde{Q_\mu}$. Then we have $M_{\mu,\lambda} \Subset \widetilde{X_0}$ and $\phi \neq M_{\mu,\lambda} \subset L_{\mu,\lambda}$. By replacing the set of indices $\{(\mu,\lambda) \mid \mu = 1,\ldots,s,\ \lambda \in \mathbf{N}\}$ with $\{\nu = \nu(\mu,\lambda) \in \mathbf{N}\}$, we put $L_\nu := L_{\mu,\lambda}$ and $M_\nu := M_{\mu,\lambda}$. Let R be all critical points of h_2'' in $\widetilde{X_0} \setminus \overline{Y_1}$. Then R is a discrete set in $\widetilde{X_0} \setminus \overline{Y_1}$.

We apply Lemma 1.10 for $Z = \widetilde{X_0}$, $Z^* = \mathrm{Sing}(\varpi) \cap \widetilde{X_0}$ and $N = \overline{Y_1}$. Then there exists a diffeomorphism $F : \widetilde{X_0} \longrightarrow \widetilde{X_0}$ with $F(R) \subset \cup_{\nu \in \mathbf{N}} M_\nu$ and F is holomorphic near R and F is the identity map on $\overline{Y_1}$. Then the function $h_2'' \circ F^{-1}$ is a C^∞-function on $\widetilde{X_0}$ such that

(3.5.2-a) $h_2'' \circ F^{-1} = \Psi$ on $\overline{Y_1}$,
(3.5.2-b) $h_2'' \circ F^{-1}$ is a Morse function with distinct critical values on $\widetilde{X_0} \setminus \overline{Y_1}$,
(3.5.2-c) all critical points on $\widetilde{X_0} \setminus \overline{Y_1}$ is contained by $\widetilde{Q}$ and $h_2'' \circ F^{-1}$ is n-convex in an open neighborhood of all critical points of $h_2'' \circ F^{-1}$.

We put $h_2 := \lambda_3 \circ h_2'' \circ F^{-1}$, where λ_3 denotes a strictly increasing convex function on $\mathbf{R}$. Then h_2 is n-convex if λ_3''/λ_3' is sufficiently large, and satisfies properties (i)–(iv). □

We put

$$h_3 := \begin{cases} \lambda_3(\Psi) & \text{on } \sigma^{-1}(V(d_2)^*), \\ h_2 \circ \widetilde{r} & \text{on } \widetilde{X_U} \setminus \sigma^{-1}(V(d_2)^*). \end{cases} \tag{3B}$$

From Proposition 2.5 (iv) and (3A), we have $\Psi = \Psi \circ \widetilde{r}$ on $\widetilde{V} \setminus \widetilde{V(d_0)}$, where d_0 denotes a constant in Proposition 2.5 with $d_0 < d_2$. Hence h_3 is a C^∞-function on $\widetilde{X_U}$ from Proposition 3.5 (i).

We put $\widetilde{\Sigma}_p := \{\widetilde{S}(\widetilde{r}(p), z) \mid z \in T\}$ for $p \in \widetilde{W^*}$ and $\widetilde{F}_p := \varpi^{-1} \circ \varpi(p)$ for $p \in \widetilde{X}$. From Lemma 2.7, $T_p\widetilde{\Sigma}_p$ and $T_p\widetilde{F}_p$ are orthogonal with respect to $\widetilde{G} = \sigma^* G$.

We fix an open set Q' containing $\mathcal{P}$ with $\overline{Q'} \subset Q$. We put $\widetilde{W}_2 := \widetilde{W}^* \setminus (r \circ \sigma)^{-1}(\overline{Q'})$. Let $\mathcal{L}$ be a linear set over $\widetilde{Q}$ with $\operatorname{codim} \mathcal{L} \leq n-1$ such that $h_3|_{\widetilde{X_0}}$ is 1-convex with respect to $\mathcal{L}$ over $\widetilde{Q}$. We put $\mathcal{L}^* := r^*(\mathcal{L})$ over $(r \circ \sigma)^{-1}(Q)$. Let $\widetilde{\Gamma}'_p$ be the holomorphic tangent space at $p \in \widetilde{W}_2$ to the real smooth hypersurface $\{h_3 \circ \widetilde{r} = h_3(\widetilde{r}(p))\}$ and $\widetilde{\Gamma}''_p$ be its orthogonal complement.

For any $c \in [0, \infty)$, there exists a compact set $K_c \subset \widetilde{X_0}$ such that $h_3 = \lambda_3(-\log(d_1 - h_0^* \circ \sigma))$ on $\{h_3 = c\} \setminus K_c$ holds from Proposition 3.5 and (3A), (3B). Hence there exists a C^∞-function $c : [0, \infty) \longrightarrow (0, \infty)$ such that

$$|\langle \partial h_3(p), \xi \rangle| \geq c(t) \|\xi\|^2_{\widetilde{G}}$$

holds for $t \in [0, \infty)$ and $p \in \{h_3 = t\} \cap \varpi^{-1}(U) \cap \widetilde{W}_2$ and $\xi \in \widetilde{\Gamma''}_p$. Then, by using Proposition 1.12, we have the following lemma.

Lemma 3.6. *There exists a strictly increasing convex C^∞-function $\lambda_4 : \mathbf{R} \longrightarrow \mathbf{R}$ with $\lambda_4(0) = 0$ such that* (i) *$\lambda_4(h_3)$ is n-convex on $\widetilde{X}_z$ for every $z \in U$,* (ii) *$i\partial\overline{\partial}\lambda_4(h_3)(p)(\xi, \xi) \geq \|\xi\|^2_{\widetilde{G}}$ for $p \in \varpi^{-1}(U) \cap (r \circ \sigma)^{-1}(Q)$ and $\xi \in \mathcal{L}(p) \cap T_p\widetilde{F_p}$,* (iii) *$i\partial\overline{\partial}\lambda_4(h_3)(p)(\xi, \xi) \geq \|\xi\|^2_{\widetilde{G}}$ for every $p \in \varpi^{-1}(U) \cap \widetilde{W}_2$, $\xi \in \widetilde{\Gamma}''_p$.*

4. Constructions of *n*-convex exhaustion functions on $\widetilde{X_U}$

We put $h^* := \lambda_4(h_3)$, where λ_4 denotes a C^∞-function in Lemma 3.6. Then h^* is n-convex on X_z for every $z \in U$. From (3B), we have $h^* = h^* \circ \widetilde{r}$ on $\widetilde{X_U} \setminus \sigma^{-1}(V(d_2)^*)$.

We put $Z_i := V_i(d_2)^*$ and $\alpha_i := \inf\{h^*(p) \mid p \in Z_i\}$ and $\beta_i := \sup\{h^*(p) \mid p \in Z_i\}$ for $i \in I$. Since $\Psi(Z_i)$ and $\Psi(Z_j)$ do not intersect from (3A-b), we may assume that $\beta_i < \alpha_{i+1}$ for $i \in I$, by replacing the index I with an adequate one if necessary. We put $D(t) := \{p \in \widetilde{X_0} \mid h^*(p) < t\} \setminus V_i(d_2)$ for $t \in [\beta_{i-1}, \beta_i)$. Let $D(t)^*$ be the open set of $\widetilde{X_U}$ in view of Definition 3.1, which is well-defined since $Cl(\widetilde{W})$ contains $\partial D(t)$.

Proposition 4.1. *Let* $\kappa > 0$ *be a constant. Then there exist a linear set* $\mathcal{M}$ *over* $\widetilde{X_U}$ *over* $\widetilde{X_U}$ *with* $\operatorname{codim}\mathcal{M} \le n-1$ *and sequences* $\{A_l \in (0,\infty)\}_{l\in\mathbf{N}}$ *and* $\{B_l \in (0,\infty)\}_{l\in\mathbf{N}}$ *such that* $i\partial\overline{\partial}B\cdot\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge 2\kappa\cdot\|\xi\|^2_{\tilde{g}}$ *holds for* $t\in(0,l)$, $p\in D(t)^*$, $\xi\in\mathcal{M}(p)$ *and* $A\ge A_l$ *and* $B\ge B_l$ *for* $l\in\mathbf{N}$,

Proof. Let $\mathcal{M}_2$ be the linear set over V^* with $\operatorname{codim}\mathcal{M}_2 \le n-1$ in Proposition 2.9. From (3A) and (3B), we have

$$h^* = \lambda_4\circ\lambda_3\circ\big(-\log(d_1-h_0^*\circ\sigma)\big) \quad \text{on} \quad Y_1^*, \tag{4.1.1}$$

where λ_3 (resp. λ_4) is a strictly increasing convex function on $\mathbf{R}$ in Proposition 3.5 (resp. Lemma 3.6). Hence, by using Proposition 2.9, there exist positive constants A^* and c^* such that

$$i\partial\overline{\partial}\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge c^*\cdot\|\xi\|^2_{\tilde{g}} \tag{4.1.2}$$

for $0\le t<l$ and $p\in Y_1^*\cap D(t)^*$ and $\xi\in\widetilde{\mathcal{M}}_2(p)$ and $A\ge A^*$, where $\widetilde{\mathcal{M}}_2$ denotes the lift of $\mathcal{M}_2$.

On the other hand, from Proposition 3.5, $D^*(t)\setminus Y_1^*$ is relatively compact in $\widetilde{X_U}$. From Proposition 3.5 and Lemma 3.6 and Theorem 1.13, there exist a constant $A_l^*>0$ and a linear set $\mathcal{M}_3$ over $\widetilde{X_U}\setminus Y_1^*$ with $\operatorname{codim}\mathcal{M}_3\le n-1$ such that

$$i\partial\overline{\partial}\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge \frac{1}{l}\cdot\|\xi\|^2_{\widetilde{G}}$$

holds for $0\le t<l$, $p\in D^*(t)\setminus Y_1^*$, $\xi\in\mathcal{M}_3(p)$ and $A\ge A_l^*$. Hence there exists a constant $c_l^*>0$ such that

$$i\partial\overline{\partial}\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge c_l^*\cdot\|\xi\|^2_{\tilde{g}} \tag{4.1.3}$$

holds for $0\le t<l$, $p\in D^*(t)\setminus Y_1^*$, $\xi\in\mathcal{M}_3(p)$ and $A\ge A_l^*$, since $\tilde{g}$ and $\widetilde{G}$ is quasi-isometrically equivalent on $\widetilde{W^*}$.

We denote by $\widetilde{\mathcal{M}}_2'$ the restriction of $\widetilde{\mathcal{M}}_2$ on Y_1^*, and $\widetilde{\mathcal{M}}_3'$ the restriction of $\widetilde{\mathcal{M}}_3$ on $\widetilde{X_U}\setminus Y_1^*$. We put $\mathcal{M} := \widetilde{\mathcal{M}}_2'\cup\mathcal{M}_3'$, which is a linear set over $\widetilde{X_U}$ with $\operatorname{codim}\mathcal{M}\le n-1$. We put $A_l := \max\{A^*, A_l^*\}$, and $c_l := \min\{c^*, c_l^*\}$. From (4.1.2) and (4.1.3), we have

$$i\partial\overline{\partial}\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge c_l\cdot\|\xi\|^2_{\tilde{g}}$$

for $0\le t<l$, $p\in D^*(t)$, $\xi\in\mathcal{M}(p)$ and $A\ge A_l$. We put $B_l := 2\kappa/c_l$. Then we have

$$i\partial\overline{\partial}B\cdot\big(-\log(t-h^*)+A\cdot\|\varpi\|^2\big)(p)(\xi,\xi) \ge 2\kappa\cdot\|\xi\|^2_{\tilde{g}}$$

for $0\le t<l$ and $p\in D(t)^*$, $\xi\in\mathcal{M}(p)$ and $A\ge A_l$, $B\ge B_l$. □

Let $\kappa > 0$ be a positive constant. Let $\{A_l\}$ and $\{B_l\}$ be strictly increasing sequences in Proposition 4.1 for the constant κ. We put

$$\Phi_t := B_l \cdot \big(-\log(t - h^*) + A_l \cdot \|\varpi\|^2 \big) + u \circ \varpi$$

for $l \le t < l+1$, where u is a strictly plurisubharmonic exhaustion function on U.

Lemma 4.2 (cf. [4], Lemma 6, [10]). *There exist strictly increasing sequences* $\{\gamma_i \in (0,\infty)\}_{i\in\mathbf{N}}$ *with* $\lim \gamma_i = \infty$, *and* $\{\delta_i \in (0,\infty)\}_{i\in\mathbf{N}}$ *with* $\lim \delta_i = \infty$ *such that, if we set* $u_i := \Phi_{\delta_i}$ *and* $D_i := D(\delta_i)^*$, *the following hold*: (i) $\{p \in D_{i+1} \mid u_{i+1} < \gamma_i\} \subset D_i$ *for every* $i \in \mathbf{N}$, (ii) *for every set* $K \subset \widetilde{X_U}$ *such that there exists a compact set* $\Omega = \Omega(K) \subset \widetilde{X_U}$ *with* $\sigma(K \setminus \Omega)$ *is relatively compact in* V^*, *there exists an index* $j = j(K) \in \mathbf{N}$ *such that* $K \subset \{p \in D_{i+1} \mid u_{i+1} < \gamma_i\}$ *holds for every* $i \ge j$.

Proof. Let $\{H_l \in (0,\infty)\}_{l\in\mathbf{N}}$ be a strictly increasing sequence such that $0 \le A_l \cdot \|\varpi\| \le H_{l+1}$ holds for every $l \in \mathbf{N}$.

Let a and b be real numbers with $l \le a < b < l+1$. If

$$b - a < \exp(-(B_{l+1}+1) \cdot H_{l+1})$$

holds, we have

$$\{p \in D(b) \mid \Phi_b < (B_{l+1}+1) \cdot H_{l+1}\} \subset D(a)^*. \tag{4.2.1}$$

On the other hand, we have

$$\begin{aligned} &\left\{p \in \widetilde{X_U} \,\middle|\, u \circ \varpi < H_{l+1}\right\} \cap D(b-1)^* \\ &\quad \subset \left\{p \in \widetilde{X_U} \,\middle|\, u \circ \varpi < H_{l+1}\right\} \\ &\qquad \cap \left\{p \in \widetilde{X_U} \,\middle|\, B_{l+1} \cdot \big(-\log(b - h^*) + A_{l+1}\|\varpi\|^2\big) < B_{l+1} \cdot H_{l+1}\right\} \\ &\quad \subset \{p \in D(b)^* \mid \Phi_b < (B_{l+1}+1) \cdot H_{l+1}\}. \end{aligned} \tag{4.2.2}$$

For every $l \in \mathbf{N}$ with $0 \le l$, we take a sequence $\{l = x_{l,0} < x_{l,1} < \cdots x_{l,t(l)} = l+1\}$ with $x_{l,k+1} - x_{l,k} < \exp(-(B_{l+2}+1) \cdot H_{l+2})$. We put $\delta_i := x_{l,k}$, where we set $i = i(l,k) = \sum_{a=1}^{l-1} t(a) + k + 1$ and $t(0) = 0$. For every $i \in \mathbf{N}$, we put $\gamma_i = (B_{l(i)+1}+1) \cdot H_{l(i)+1}$, where $l(i)$ denotes the integer part of δ_i. Then the sequences γ_i and δ_i satisfy (i) from (4.2.1), and satisfy (ii) from (4.1.1) and (4.2.2). □

Proposition 4.3 (cf. [4], Theorem 3). *There exists a* C^∞*-function* $\widetilde{w} : \widetilde{X_U} \longrightarrow [0,\infty)$ *such that* (i) $i\partial\overline{\partial}\widetilde{w}(p)(\xi,\xi) \ge 2\kappa\|\xi\|^2_{\widetilde{g}}$ *holds for* $p \in \widetilde{X_U}$ *and* $\xi \in \widetilde{\mathcal{M}}(p)$, (ii) $\widetilde{w}$ *is exhaustive on* $\widetilde{X_U} \setminus Y_1^*$.

Proof. Let $\{\gamma_i\}_{i\in\mathbf{N}}$ and $\{\delta_i\}_{i\in\mathbf{N}}$ be sequences in Lemma 4.2. By modifying the sequences, we may assume that there exist sequences $\{\gamma_i\}$, $\{u_i := \Phi_{\delta_i}\}$, $\{D_i := D(\delta_i)^*\}$ and $\{\varepsilon_i > 0\}$ such that
(i)′ $\{p \in \widetilde{X_U} \mid u_{i+1} \le \gamma_i + \varepsilon_i\} \subset D_i$ for every $i \in \mathbf{N}$,
(ii)′ for every subset $K \in \widetilde{X_U}$ such that there exists a subset $\Omega = \Omega(K) \subset \widetilde{X_U}$ with $\varpi(K \setminus \Omega)$ is a relatively compact subset of $\widetilde{V^*}$, there exists a number $j = j(K) \in \mathbf{N}$ such that $K \subset \{p \in D_{i+1} \mid u_{i+1}(p) < \gamma_i - \varepsilon_i\}$ holds for every $i \ge j$.

By using induction, we will construct the following sequence $\{w_i\}_{i\in\mathbf{N}}$ such that
$(a)_i$ $w_i \in \mathcal{B}(D_i, \mathcal{M})$ for every $i \in \mathbf{N}$.
$(b)_i$ $w_i|_{D_k \setminus D_{i-1}} \ge k$ holds for every $i > k$.
$(c)_i$ $w_i = w_{i-1}$ holds on $\{p \in D_i \mid u_i \ge \gamma_{i-1} - \varepsilon_{i-1}\}$ for $i = 2, 3, \ldots$.

Indeed, we put $w_1 = u_1$. Suppose that there exist functions $w_1, \ldots, w_i$ satisfying $(a)_k, (b)_k, (c)_k$ for $k = 1, 2, \ldots, i$. We define a continuous function

$$w_{i+1} := \begin{cases} w_i & \text{on } \{u_{i+1} \le \gamma_i - \varepsilon_i\}, \\ \max\{w_i, \chi_i(u_{i+1})\} & \text{on } \{\gamma_i - \varepsilon_i \le u_{i+1} \le \gamma_i + \varepsilon_i\}, \\ \chi_i(u_{i+1}) & \text{on } \{u_{i+1} \ge \gamma_i + \varepsilon_i\}\ , \end{cases}$$

where we put $\chi_i(t) = a_i t - b_i$ for constant a_i and b_i with

$$a_i \ge 1,\ b_i > 0,\ a_i(\gamma_i - \varepsilon_i) - b_i < 0, \tag{4.3.1}$$

$$a_i(\gamma_i + \varepsilon_i) - b_i > \max\{w_i(p) \mid u_{i+1}(p) = \gamma_i + \varepsilon_i\}, \tag{4.3.2}$$

$$a_i(\gamma_i + \varepsilon_i) - b_i > i + 1. \tag{4.3.3}$$

From (4.3.1) and (4.3.2), w_{i+1} is continuous and $(a)_{i+1}$ holds. From (4.3.3), we have

$$w_{i+1} \ge i + 1 \quad \text{on} \quad \{u_{i+1} \ge \gamma_i + \varepsilon_i\} \cap D_i.$$

On the other hand, by the condition $(b)_i$, we have

$$w_{i+1} \ge w_i \ge i \quad \text{on} \quad \{u_{i+1} \ge \gamma_i + \varepsilon_i\} \cap (D_{i+1} \setminus D_i).$$

Hence $(b)_{i+1}$ holds. Moreover, $(c)_{i+1}$ holds from the condition $(b)_*$ and the definition of w_{i+1}. From $(c)_*$, the sequence $\{w_i\}$ has the limit $w := \lim w_i$. Then the function w is continuous and $\mathcal{M}$-convex, and exhaustive on $\widetilde{X_U} \setminus Y_1^*$. Every point of $\widetilde{X_U}$ has an open neighborhood O and at most two functions $\{w'_j\}$ on O with

$$w|_O = \max\{w'_j\},$$
$$i\partial\overline{\partial} w'_k(p)(\xi, \xi) \ge 2\kappa \|\xi\|_g^2$$

for $p \in O$ and $\xi \in \mathcal{M}(p)$ from Proposition 4.1. Let η be a sufficiently small positive constant. We apply Proposition 1.5 for $Y = \widetilde{X_U}$, $\mathcal{L} = \mathcal{M}$ and $\omega = \widetilde{g}$. Then there exists a C^∞-function $\widetilde{w} : \widetilde{X_U} \longrightarrow [0, \infty)$ which is 1-convex with respect to $\mathcal{M}$ such that

$$w \le \widetilde{w} < w + \eta, \qquad (4.3.4)$$
$$i\partial\overline{\partial}\widetilde{w}(p)(\xi, \xi) \ge 2\kappa\|\xi\|^2_{\widetilde{g}}$$

hold for $p \in \widetilde{X_U}$ and $\xi \in \mathcal{M}(p)$. Property (ii) follows from (4.3.4) and the fact that w is exhaustive on $\widetilde{X_U} \setminus Y_1^*$. □

Here we observe general geometric properties of unramified covering spaces of relatively compact open subsets. For $p, q \in \widetilde{X}$, let $d(p, q)$ be the distance between p and q with respect to the metric $\widetilde{g}$. Fix a point $o \in \widetilde{X_U}$ and, for each point $p \in \widetilde{X_U}$, we put $r(p) := d(o, p)$. Then we have the following by using the argument of Lemma 3.2 of [11].

Lemma 4.4. *There exist a C^∞-function $\tau : \widetilde{X_U} \longrightarrow [0, \infty)$ and a positive constant C such that* (i) $C\cdot r \le \tau \le C\cdot(r+1)$, (ii) $\|d\tau\|_{\widetilde{g}} \le C$, (iii) $-C\cdot\widetilde{g} \le i\partial\overline{\partial}\tau \le C\cdot\widetilde{g}$.

Proof. See Appendix B. □

A function τ on $(\widetilde{X}, \widetilde{g})$ is said to be *the Napier's function on $\widetilde{X}$ with respect to $\widetilde{g}$* if τ satisfies properties of Lemma 4.4.

Proof of Theorem. Let τ be the Napier's function on $\widetilde{X_U}$ with respect to $\widetilde{g}$ such that there exists a constant $C > 0$ with (i) $C\cdot r \le \tau \le C\cdot(r+1)$, (ii) $\|d\tau\|_{\widetilde{g}} \le C$, (iii) $-C\cdot\widetilde{g} \le i\partial\overline{\partial}\tau \le C\cdot\widetilde{g}$.

Let $\widetilde{w}$ be a function in Proposition 4.3 for $\kappa = 2C$. We put

$$\Theta := \widetilde{w} + \tau$$

on $\widetilde{X_U}$. Then the function Θ is an n-convex exhaustion function on $\widetilde{X_U}$ from Proposition 4.3 and Lemma 4.4. □

Appendix

A. Constructions of partial holomorphic motions on X (cf. [8], [10])

We consider C^∞-families of relatively compact manifolds in the complex manifold X and examine their properties. Then we show Theorem 2.1.

Let $\mathcal{X}$ be a real $2n$-dimensional C^∞-manifold and T a domain of $\mathbf{C}^m$. *A C^∞-family of complex local coordinates of $\mathcal{X}$ over T* denotes a C^∞-map $w : V \times U \longrightarrow \mathbf{C}^n$, where V (resp. U) is an open set of $\mathcal{X}$ (resp. T), such that the map $w^z : V \longrightarrow \mathbf{C}^n$ defined by $w^z(x) := w(x, z)$ is a local coordinate of $\mathcal{X}$ for any $z \in U$.

The map $\widetilde{w} : V \times U \longrightarrow \mathbf{C}^n \times U$, which is defined by $\widetilde{w}(x, z) := (w(x, z), z)$, is bijective from $V \times U$ to the range. Let $v : V' \times U' \longrightarrow \mathbf{C}^n$ be another one of $\mathcal{X}$ over T. The change of local coordinates from w to v denotes the map $\widetilde{v} \circ (\widetilde{w})^{-1}$.

A holomorphic family of complex structures on $\mathcal{X}$ *over* T denotes a collection Λ of C^∞-families of complex local coordinates of $\mathcal{X}$ over T satisfying (A) for any $v, w \in \Lambda$, the change of local coordinates from w to v is holomorphic if it is defined, (B) for any $x \in \mathcal{X}$, $z \in T$, there exists a $w \in \Lambda$ with the domain $V \times U$ such that $(x, z) \in V \times U$, (C) if w and w^* are C^∞-families of complex local coordinates of $\mathcal{X}$ over T, and the change of local coordinates from w to w^* is holomorphic for any $w \in \Lambda$, w^* is in Λ.

Let Λ be a holomorphic family of complex structures on $\mathcal{X}$ over T. Then there exists a unique complex structure $\mathcal{X}(\Lambda)$ on $\mathcal{X} \times T$ such that, for any $w \in \Lambda$, $\widetilde{w}$ is holomorphic from $V \times U$, which is considered as an open set of $\mathcal{X}(\Lambda)$, to the product complex manifold $\mathbf{C}^n \times U$. We set $\varphi(x, z) := z$ for $(x, z) \in \mathcal{X} \times T \cong \mathcal{X}(\Lambda)$. Then $\varphi : \mathcal{X}(\Lambda) \longrightarrow T$ is a smooth surjective holomorphic map.

From now on, let X be a complex manifold of dimension $N = n + m$ and T a domain of $\mathbf{C}^m$ which contains $0 \in \mathbf{C}^m$. Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map. We put $X_A := \pi^{-1}(A)$ for $A \subset T$. By $\mathrm{Sing}(\pi)$ we denote the set of all $p \in X_0$ such that the differential of π at p does not have maximal rank. We put $\mathrm{Reg}(\pi) := X \setminus \mathrm{Sing}(\pi)$. We suppose that $\dim X_0 = n$ and $\mathrm{Reg}(\pi) \cap X_0 \neq \phi$, $\mathrm{Sing}(\pi) \cap X_0 \neq \phi$.

Let K be any relatively compact connected open set in $\mathrm{Reg}(\pi) \cap X_0$. Let $\mathcal{P} = \{p_1, \dots, p_k\}$ be finitely many points in K. Let K^* be an open neighborhood of K in $\mathrm{Reg}(\pi)$ with $K^* \cap X_0 = K$ satisfying the following:
(A) there exists a local coordinate system of $K^* : \{h_\mu : \mathcal{U}_\mu \longrightarrow V_\mu \times U\}_{\mu=1,\dots,k}$, where $\{\mathcal{U}_\mu\}_{\mu=1,\dots,k}$ are open subsets of $\mathrm{Reg}(\pi)$ and each V_μ is biholomorphic to a bounded open neighborhood of $0 \in \mathbf{C}^n$, and U is biholomorphic to the unit ball of $\mathbf{C}^m$,
(B) for every $\mu = 1, \dots, k$, the point $p_\mu \in \mathcal{P}$ is contained by V_μ and p_μ is not contained by V_ν for any $\nu \neq \mu$.

We denote by $\mathcal{K}$ the underlying C^∞-manifold of K. Then the following hold.

Proposition A.1 (cf. [10], Proposition A.1). *There exist an open neighborhood* U *of* $0 \in T$, *an open covering* $\{V'_\mu\}_{\mu=1,\dots,k}$ *of* $\mathcal{K}$, *a diffeomorphism* $G : K^* \longrightarrow \mathcal{K} \times U$, *a holomorphic family* $\Lambda = \{w_\mu : V'_\mu \times U \longrightarrow \mathbf{C}^n\}$ *of complex structures on* $\mathcal{K}$ *over* U, *an open neighborhood* $Q'_\mu \Subset V_\mu$ *of* p_μ *for any* μ *satisfying* (i) $G(K^* \cap X_z) = \mathcal{K} \times \{z\}$ *for any* $z \in U$, *and* $G : K^* \longrightarrow \mathcal{K} \times U \cong \mathcal{K}(\Lambda)$ *is a biholomorphic map,* (ii) $G \circ h_\mu^{-1}$ *is the identity map from* $Q'_\mu \times U$ *to* $Q'_\mu \times U \subset \mathcal{X}_0 \times U$ *for every* μ.

Proof. We can use the argument of Proposition A.1 in [10] since K^* has the finite open covering $\{\mathcal{U}_\mu\}$. We apply the argument, by replacing X_U, $\mathcal{X}_0$, $\{W'_\nu\}$ with K^*, $\mathcal{K}$, $\{Q'_\mu\}$ respectively. Then the desired conclusion follows. ☐

Lemma A.2 (cf. [8] p. 26, [10], Lemma A.2). *Let* $\{V'_\mu\}_{\mu=1,\dots,k}$ *be the open covering of* $\mathcal{K}$ *and* Λ *the holomorphic family of complex structures of* $\mathcal{K}$ *over* U *in* Proposition A.1. *Then there exist an open neighborhood* U^* *of* $0 \in T$*, a* C^∞*-map* $F : \mathcal{K} \times \mathcal{K} \times U^* \longrightarrow T^{1,0}K \times U^*$ *satisfying* (i) $F(x,q,z) \in T^{1,0}_x K \times \{z\}$ *for any* $(x,q,z) \in \mathcal{K} \times \mathcal{K} \times U^*$, (ii) $F(x,x,0) = (0,0) \in T^{1,0}_x K \times U^*$ *for any* $x \in \mathcal{K}$, (iii) *for an open neighborhood* $\mathcal{K}' \subset \mathcal{K} \times \mathcal{K}$ *of the diagonal set* $\{(x,x) \mid x \in \mathcal{K}\}$, $F_x : \mathcal{K}(\Lambda) \supset \mathcal{K}'_x \times U^* \longrightarrow F_x(\mathcal{K}'_x \times U^*) \subset T^{1,0}K \times U^*$ *is biholomorphic, where we put* $\{x\} \times \mathcal{K}'_x := (\{x\} \times \mathcal{K}) \cap \mathcal{K}'$ *and* $F_x(q,z) := F(x,q,z)$ *for* $(x,q,z) \in \mathcal{K} \times \mathcal{K} \times U^*$.

Proof. We apply the argument of Lemma A.2 in [10], by replacing X_U, X_0, $\mathcal{X}_0$ and $\mathcal{W}$ with K^*, K, $\mathcal{K}$ and $\mathcal{K}'$ respectively. Then the desired conclusion follows. $\square$

Then we have the following by applying the argument in Appendix in [10] in the similar way to the previous. For the completeness of this paper, we explain the detail of its proof.

Theorem A.3 (cf. [8], [10]). *For every relatively compact connected open set* $K \Subset \mathrm{Reg}(\pi) \cap X_0$ *and every set of finitely many points* $\mathcal{P} := \{p_1, \dots, p_s\} \subset K$*, there exist an open neighborhood* U *of* $0 \in T$ *and a* C^∞*-map* $S : K \times U \ni (x,z) \longmapsto S(x,z) \in X_U$ *satisfying the following*: (i) $S : K \times U \longrightarrow S(K,U)$ *is a diffeomorphism,* (ii) $U \ni z \longmapsto S(x,z) \in X_U$ *is a holomorphic section over* U *for every* $x \in K$ *and* $S(K,U)$ *is the disjoint union of* $\{S(x,U) \mid x \in K\}$, (iii) *The map* $r : S(K,U) \ni p \longmapsto r(p) \in K$ *defined by* $S(r(p), \pi(p)) = p$ *is a* C^∞*-retraction such that there exists a relatively compact open neighborhood* $Q \Subset K$ *of* $\mathcal{P}$ *in* K *such that* $r|_{r^{-1}}(Q)$ *is holomorphic.*

Proof. Let $G : K^* \longrightarrow \mathcal{K} \times U, \{V'_\mu\}, \Lambda = \{w_\mu\}$ be the same as those in Proposition A.1. Let $\{V''_\mu\}_{\mu=1,\dots,k}$ be an open covering of $\mathcal{K}$ satisfying $\overline{V''_\mu} \subset V'_\mu$. We define a C^∞-map $f^*_\mu : V''_\mu \ni (x,z) \longmapsto f^*_\mu(x,z) \in V'_\mu$ such that $w_\mu(f^*_\mu(x,z),z) = w_\mu(x,0)$ for any $x \in V''_\mu$, $z \in U$. Then the map $U \ni z \longmapsto (f^*_\mu(x,z),z) \in V'_\mu \times U \subset \mathcal{K} \times U$ is holomorphic for any fixed $x \in V''_\mu$. Indeed we have $(w_\mu(f^*_\mu(x,z),z),z) = (w_\mu(x,0),z) \subset \mathbf{C}^n \times U$ for $z \in U$.

Then there exist a C^∞-map $F : \mathcal{K} \times \mathcal{K} \times U \longrightarrow T^{1,0}K \times U$ and a neighborhood $\mathcal{K}' \subset \mathcal{K} \times \mathcal{K}$ of $\{(x,x) | x \in \mathcal{K}\}$ satisfying conditions of Lemma A.2 for Λ, by replacing U^* with a sufficiently small U if necessary. We may assume that $(x, f^*_\mu(x,z)) \in \mathcal{K}'$ for any $z \in U$. Let $\{\widetilde{\rho}_\mu\}_{\mu=1,\dots,k}$ be a partition of unity subordinated to $\{V''_\mu\}$. We set $\gamma_x : U \ni z \longmapsto \sum_\mu \widetilde{\rho}_\mu(x) F(x, f^*_\mu(x,z), z) \in T^{1,0}K \times U$. The map γ_x is holomorphic on U for any fixed $x \in \mathcal{K}$. Moreover $F_x : (\mathcal{K}(\Lambda) \supset) \mathcal{K}'_x \times U \ni (f^*_\mu(x,z),z) \longmapsto F(x, f^*_\mu(x,z), z) \in T^{1,0}K \times U$ is holomorphic for any fixed $x \in \mathcal{K}$ from Lemma A.2 (iii).

We may assume that $F_x^{-1}\circ\gamma_x : U \longrightarrow \mathcal{K}(\Lambda)$ is well-defined. We consider a C^∞-map $S : \mathcal{K}\times U \ni (x,z) \longmapsto S(x,z) := F_x^{-1}\circ\gamma_x(z) \in K^* \cong \mathcal{K}(\Lambda)$. Then the Jacobian of S has the maximal rank on $K\times\{0\}$. Hence S is a diffeomorphism from $\mathcal{K}\times U$ to K^* and $S(\mathcal{K},z) = K^*\cap X_z$ for $z\in U$ if U is sufficiently small. The open set K^* is the disjoint union of $\{S(x,U)\mid x\in K\}$ and $r : K^* \longrightarrow \mathcal{K}\times U$, which is defined by $S(r(p),\pi(p)) = p$ for $p\in K^*$, is a C^∞-retraction. Moreover, $S(x,\cdot) : U\ni z\longmapsto S(x,z)\in K^*$ is a holomorphic section over U for any fixed $x\in K$ since γ_x is holomorphic on U for any fixed $x\in\mathcal{K}$.

Let $Q_\mu \subset Q'_\mu$ be an open neighborhood of p_μ satisfying $Q_\mu\cap\operatorname{supp}\widetilde{\rho}_\nu$ is empty for any $\nu\neq\mu$, where Q'_μ is the neighborhood of p_μ in Proposition A.1 and we put $Q := \cup_\mu Q_\mu$. Let (x,z) be any point of $Q_\mu\times U\subset\mathcal{K}\times U$. Then $w_\mu(x,z) = \mathrm{Proj}_\mu\circ h_\mu\circ G^{-1}(x,z) = x = w_\mu(x,0)$ holds since $G\circ h_\mu^{-1}$ is the identity map on $Q_\mu\times U$ from Proposition A.1.

Hence we have $f_\mu^*(x,z) = x$ for any $(x,z)\in Q_\mu\times U\subset\mathcal{K}\times U$. Then we have $S(x,z) = F_x^{-1}\circ\gamma_x(z) = (x,z)\in Q_\mu\times U\subset K^*$. Hence $S\circ h_\mu^{-1}$ is the identity map on $Q_\mu\times U$ and the C^∞-retraction $r(p)$ is the natural projection from $h_\mu^{-1}(Q_\mu\times U)$ to Q_μ. Therefore $r|_{Q_\mu\times U} = r|_{S(Q_\mu,U)}$ is a holomorphic retraction. ☐

By using Theorem A.3, we have Theorem 2.1 as follows.

Proof of Theorem 2.1. It is suffice to show Theorem 2.1 for the case where W is connected. Let K be an open set with $W\Subset K\Subset \mathrm{Reg}(\pi)\cap X_0$ and $\mathcal{P} := \{p_1,\dots,p_s\}$ a finitely many point in W. From Theorem A.3, there exist an open neighborhood U of $0\in T$ and a C^∞-map $S : K\times U\longrightarrow X_U$. We replace $S|_{\overline{W}\times U}$, $r|_{S(\overline{W},U)}$, $Q\cap W$ with $S : \overline{W}\times U\longrightarrow X_U$, $r : S(\overline{W},U)\longrightarrow \overline{W}$,$Q$, respectively. Then they satisfy properties of Theorem 2.1. ☐

B. Existences of Napier's functions on unramified covering spaces([11], [2])

We use the argument of Lemma 3.2 in [11] and will show Lemma 4.4. We remark that E. Ballico [2] and Napier [11] have stated existences of functions satisfying conditions which are similar to those of Lemma 4.4.

To prove Lemma 4.4, it is suffices to show the following lemma.

Let (Y,g) be a hermitian manifold and $\sigma : \widetilde{Y}\longrightarrow Y$ an unramified covering map. Let D be a relatively compact subdomain of Y such that $\widetilde{D} := \sigma^{-1}(D)$ is connected. We put the hermitian metric $\widetilde{g} := \sigma^* g$ on $\widetilde{D}$.

For $p,q\in\widetilde{Y}$, let $d(p,q)$ be the distance between p and q with respect to the metric $\widetilde{g}$. Fix a point $o\in\widetilde{D}$ and, for each point $p\in\widetilde{Y}$, we put $r(p) := d(o,p)$. For any $A\subset\widetilde{Y}$, we put $r(A) := \inf_{q\in A} r(q)$. For $A\subset\widetilde{Y}$, we denote by $Cl(A)$ or $\overline{A}$ the topological closure of A in $\widetilde{Y}$. Then we have the following.

Its proof is similar to Lemma 3.2 of [11]. For the completeness of this paper, we describe the detail of its proof.

Lemma 4.4$'$. *Suppose that there exists a sequence of points* $\{y_\nu\}_{\nu\in\mathbf{N}}$ *with* $\lim_{\nu\to\infty} r(y_\nu)=\infty$. *Then there exist a* C^∞*-function* $\tau:\widetilde{D}\longrightarrow[0,\infty)$ *and a positive constant* C *such that* (i) $C\cdot r\le\tau\le C\cdot(r+1)$, (ii) $\|d\tau\|_{\widetilde{g}}\le C$, (iii) $-C\cdot\widetilde{g}\le i\partial\overline{\partial}\tau\le C\cdot\widetilde{g}$.

Proof. The manifold $(Cl(\widetilde{D}),\widetilde{g})$ has 'bounded geometry'. Namely, for every $p\in Cl(\widetilde{D})$, there exist an open neighborhood U_p of p in $\widetilde{Y}$ and positive constants R and C and a surjective biholomorphic map $\Psi_p:U_p\longrightarrow E(0,R)$ with
(1-a) $\Psi_p(p)=0$ holds,
(1-b) $\Psi_p^* g_e/C\le\widetilde{g}\le C\cdot\Psi_p^* g_e$ holds,
where R and C are independent of $p\in Cl(\widetilde{D})$ and g_e denotes the Euclidian metric in $\mathbf{C}^n$, and we put $E(0,R):=\{z\in\mathbf{C}^N\,|\,\|z\|\le R\}$.

Hence there exist constants r_i for $i=0,1,2,3$ and R_j for $j=0,1,2$ such that
(2-a) $2r_2\le r_2<r_1<3r_1\le r_0$, $R_2<R_1<R_0$,
(2-b) for every $p\in Cl(\widetilde{D})$, there exist an open neighborhood U_p of p in $\widetilde{Y}$ and a surjective biholomorhic map $\Psi_p:U_p\longrightarrow E(0,R_0)$ with
(2-b.1) $\Psi_p(p)=0$,
(2-b.2) $\Psi_p^* g_e/C_1\le\widetilde{g}\le C_1\cdot\Psi_p^* g_e$ for a positive constant C_1,
(2-b.3) $B(p,r_3)\Subset B(p,r_2)\subset U(p,R_2)\Subset U(p,R_1)\subset B(p,r_1)\Subset U(p,R_0)$, where we put $B(p,r):=\{q\in\widetilde{Y}\mid d(p,q)<r\}$ and $U(p,r):=\Psi_p^{-1}(E(0,r))$ for $0<r<R$,
(2-c) $\mathrm{vol}(B(p,r_3))/C_2<\mathrm{vol}(B(p,r_0))<C_2$ for a positive constant C_2.

Then there exists a sequence of points $\{p_\nu\in Cl(\widetilde{D})\}_{\nu\in\mathbf{N}}$ such that
(3-a) $B(p_\nu,r_3)\cap B(p_\mu,r_3)\neq\phi$ if $\nu\neq\mu$,
(3-b) $\{B(p_\nu,r_1)\}$ is uniformly locally finite. Namely, there exists a constant $N\in\mathbf{N}$ such that each point has an open neighborhood U which intersects at most N elements of $\{B(p_\nu,r_1)\}_{\nu\in\mathbf{N}}$,
(3-c) $\{B(p_\nu,r_2)\}_{\nu\in\mathbf{N}}$ is an open covering of $Cl(\widetilde{D})$.

Indeed, we put $p_1:=o$ and points $p_1,\ldots,p_{\nu-1}$ are given. Let r_i and R_j be positive constants satisfying (1-a) and (1-b). Let $p_\nu\in\partial((\cup_{k=1}^{\nu-1}B(p_k,r_2))\cap Cl(\widetilde{D}))$ satisfying $r(p_\nu)=r(\cup_{k=1}^{\nu-1}B(p_k,r_2))\cap Cl(\widetilde{D}))$. We have defined a sequence $\{p_\nu\}$ inductively in this way. Then we have (3-a)–(3-c) as follows.

Proof of (3-a). Let ν and μ be natural numbers with $\mu<\nu$. Then $p_\nu\in\partial(B(p_1,r_2)\cup\cdots\cup B(p_\mu,r_2)\cup\cdots\cup B(p_{\nu-1},r_2))$ holds. Since $p_\nu\notin B(p_\mu,r_2)$, we have $d(p_\nu,p_\mu)\ge r_2\ge 2r_3$. Hence $B(p_\nu,r_3)\cap B(p_\mu,r_3)\neq\phi$ holds. □

Proof of (3-b). Let $p\in Cl(\widetilde{D})$ with $B(p_\nu,r_1)\cap B(p,r_1)\neq\phi$. Since $\max\{d(p,q)\mid q\in B(p_\nu,r_1)\}\le 3r_1\le r_0$, we have $B(p_\nu,r_3)\subset B(p_\nu,r_1)\subset B(p,r_0)$. Let $\nu_1,\ldots,\nu_k$ be distinct natural numbers satisfying $B(p_{\nu_j},r_1)\cap B(p,r_1)\neq\phi$. Then we have $\cup_{\nu=1}^{k} B(p_\nu, r_3) \subset B(p, r_0)$. Hence

$$C_2 \geq \mathrm{vol}(B(p, r_0)) \geq \sum_{j=1}^{k} \mathrm{vol}(B(p_{\nu_j}, r_3)) \geq \frac{k}{C_2}$$

holds. Hence we may take a natural number N with $N > C_2^2$. $\square$

Proof of (3-c). Suppose that $Cl(\widetilde{D}) \setminus (\cup_{\nu\in\mathbf{N}} B(p_\nu, r_2)) \neq \phi$. Then there exists a constant $r > 0$ satisfying $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \setminus (\cup_{\nu\in\mathbf{N}} B(p_\nu, r_2)) \neq \phi$ from (3-b). The point $o = p_1$ is contained by $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \cap B(o, r_2)$. On the other hand, there exists a natural number $\mu \in \mathbf{N}$ satisfying $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \cap B(p_\nu, r_2) = \phi$ for $\nu \geq \mu$. Then we have $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \cap (\cup_{k=1}^{\nu-1} B(p_{\nu_k}, r_2)) \neq \phi$, and $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \setminus (\cup_{k=1}^{\nu-1} B(p_{\nu_k}, r_2)) \neq \phi$. Hence $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \cap \partial(\cup_{k=1}^{\nu-1} B(p_{\nu_k}, r_2)) \neq \phi$ holds. Therefore we have $d(o, p_\nu) < r$ since p_ν is contained by $\partial(\cup_{k=1}^{\nu-1} B(p_{\nu_k}, r_2))$. On the other hand, we have $d(o, p_\nu) \geq r + r_2$ since $\big(Cl(\widetilde{D}) \cap B(o, r)\big) \cap B(p_{\nu_k}, r_2)) = \phi$ holds. It leads to contradiction.

Let $\lambda : E(0, R_1) \longrightarrow [0, 1]$ be a C^∞-function with $\mathrm{supp}(\lambda) \subset E(0, R_1)$ and $\lambda \equiv 1$ on $E(0, R_2)$. we put $\Psi_\nu := \Psi_{p_\nu}$ and

$$\lambda_\nu := \begin{cases} \lambda(\Psi_\nu) & \text{on } U(p_\nu, R_1), \\ 0 & \text{on } \widetilde{Y} \setminus U(p_\nu, R_1). \end{cases}$$

Then the function λ_ν is a C^∞-function on $\widetilde{Y}$ with $\mathrm{supp}(\lambda_\nu) \subset U(p_\nu, R_1)$ and $\lambda_\nu \equiv 1$ on $U(p_\nu, R_2)$. We put $\tau' := \sum \exp(r(p_\nu)) \cdot \lambda_\nu(p)$. Then the following holds.

There exists a positive constant C_3 satisfying

(4-a) $(\exp r)/C_3 \leq \tau' \leq C_3 \cdot \exp r$,

(4-b) $\|d\tau'\|_{\widetilde{g}} \leq C_3 \cdot \exp r$,

(4-c) $-C_3\widetilde{g} \cdot \exp r \leq i\partial\overline{\partial}\tau' \leq C_3\widetilde{g} \cdot \exp r$. $\square$

Proof of (4-a). Let $p \in Cl(\widetilde{D})$ with $p \in B(p_\nu, r_1)$ for $\nu \in \mathbf{N}$. Then we have $r(p) - r_1 \leq r(p_\nu) \leq r(p) + r_1$. Hence

$$\exp(-r_1 + r(p)) \leq \exp r(p_\nu) \leq \exp(r_1 + r(p))$$

holds. On the other hand, we have $\lambda_\nu(p) = 0$ if $p \notin B(p_\nu, r_1)$. Hence

$$\exp(-r_1 + r(p)) \left(\sum_{\nu=1}^{\infty} \lambda_\nu(p) \right) \leq \tau' \leq \exp(r_1 + r(p)) \left(\sum_{\nu=1}^{\infty} \lambda_\nu(p) \right)$$

holds. From (3-b) and (3-c), we have $1 \leq \tau' \leq N$. Hence $\exp(-r_1) \cdot \exp r \leq \tau' \leq N \exp(r_1) \cdot \exp r$ holds. $\square$

Proof of (4-b). For $p \in U(p_\nu, R_1) \cap Cl(\widetilde{D})$ and $v \in T_pY$,

$$|(\partial\lambda_\nu)(p)(v)| = |(\partial\lambda_\nu)(\Psi_\nu(p))((\Psi_\nu)_*(v))| \leq C_4\|v\|_g$$

holds for a positive constant C_4. Hence we have $\|(d\lambda_\nu)(p)\|_g \leq C_4$. Therefore we obtain

$$\|d\tau'(p)\|_g \leq \sum \exp(r(p_\nu)) \cdot |(d\lambda_\nu)(p)|_g \leq NC_4 \exp r_1 \cdot \exp r(p)$$

since $r(p_\nu) \leq r_1 + r(p)$ holds. □

Proof of (4-c). For $p \in U(p_\nu, R_0) \cap Cl(\widetilde{D})$ and $v \in T_pY$,

$$\begin{aligned}\left|i\left(\partial\overline{\partial}\lambda_\nu\right)(p)(v,v)\right| &= \left|i\left(\partial\overline{\partial}\lambda_\nu\right)(\Psi_\nu(p))((\Psi_\nu)_*(v),(\Psi_\nu)_*(v))\right| \\ &\leq C_5\|(\Psi_\nu)_*(v)\|_{g_e} \leq C_5\|v\|_{\Psi_\nu^* g_e} \leq C_5'\|v\|_g\end{aligned}$$

holds for positive constants C_5 and C_5'. Hence we have

$$\left|i\left(\partial\overline{\partial}\tau'\right)(p)(v,v)\right| = \left|i\partial\overline{\partial}\left(\sum_{\nu=1}^{\infty}\exp(r(p_\nu)\cdot\lambda_\nu)(p)(v,v)\right)\right|$$

$$\leq \exp r_1 \cdot \exp r(p)\sum_{\nu=1}^{\infty}\left|i\left(\partial\overline{\partial}\lambda_\nu\right)(p)(v,v)\right| \leq C_5'N\exp r_1\cdot\exp r(x)\cdot\|v\|_g.$$

Therefore (4-c) holds. □

We put

$$\tau := \log\tau' + C,$$

where C is a positive constant. Then the function τ satisfies properties of Lemma 4.4′ for a sufficiently large C from (4-a) and (4-b) and (4-c). □

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