Alekseevsky, D.V. and Marchiafava, S.
Osaka J. Math.
38 (2001), 869–904

HERMITIAN AND KÄHLER SUBMANIFOLDS OF A QUATERNIONIC KÄHLER MANIFOLD

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(Received January 19, 2000)

Introduction

Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a quaternionic Kähler manifold where $\widetilde{g}$ is the metric and $Q \subset \mathrm{End}T\widetilde{M}$ is the quaternionic structure. We will always assume that $\widetilde{M}^{4n}$ *has non zero scalar curvature*. A submanifold $M^{2m} \subset \widetilde{M}^{4n}$ with induced metric g is called an *almost Hermitian submanifold* if it is given a g-orthogonal almost complex structure J on M which is induced from a section J_1 of the bundle $Q_{|M} \to M$. This means that $J_1T_xM = T_xM \quad \forall x \in M$ and $J = J_{1|TM}$.

An almost Hermitian submanifold (M^{2m}, J, g) is called *Hermitian* if the almost complex structure J is integrable, *almost Kähler* if the Kähler form $F = g \circ J$ is closed and *Kähler* if F is parallel. In the first section we study an almost Hermitian submanifold (M^{2m}, J, g) of a quaternionic Kähler manifold $\widetilde{M}^{4n}$. We give different conditions for an almost Hermitian submanifold to be Hermitian. For example, we prove that any analytic complete almost Hermitian submanifold M of the quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ with positive scalar curvature is Hermitian if $\dim_{\mathbb{R}} M = 4k$ (Theorem 1.4). We prove that any almost Kähler submanifold $M^{2m}, m \neq 3$, of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ is Kähler and, hence, a minimal submanifold and give some local characterizations of such submanifold. In particular, by completing a known result of K. Tsukada, we prove that an almost Hermitian submanifold M is Kähler if and only if it is *totally complex*, i.e. it satisfies the condition

$$J_2T_xM \perp T_xM \quad \forall x \in M$$

where J_2 is a section of $Q_{|M} \to M$ which anticommutes with J_1. In Section 2 we study Kähler submanifolds M^{2n} in a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$. Using the isomorphism $J_2 : TM \to T^{\perp}M$ between the tangent and the normal bundle, we identify the second fundamental form h of M with a tensor $C = J_2 \circ h \in TM \otimes S^2T^*M$. This tensor at any point $x \in M$ belongs to the first prolongation of the space $S_J \subset \mathrm{End}\, T_xM$ of symmetric endomorphisms anticommuting with J and the associated covariant tensor $g \circ C$ has the form $gC = q + \overline{q}$ where $q \in S^3T_x^{*1,0}M$ is a holomorphic

Work done under the program of G.N.S.A.G.A. of C.N.R.; partially supported by M.U.R.S.T (Italy) and E.S.I. (Vienna).

cubic form.

The Gauss-Codazzi equations written in terms of the tensor C take a simple form. We show that the second Gauss-Codazzi equation is equivalent to the first. In Subsection 2.3 we study the case when M^{2n} is a Kähler submanifold of a (locally) symmetric quaternionic Kähler space $\widetilde{M}^{4n}$. We get the necessary and sufficient conditions for M^{2n} to be a locally symmetric manifold in terms of the tensor C. In particular, if M is curvature invariant, i.e. if the curvature $\widetilde{R}$ of $\widetilde{M}$ at a point $x \in M$ satisfies the condition

$$\widetilde{R}(T, T)T \subset T, \qquad T = T_x M$$

then M is a (locally Hermitian) symmetric manifold if and only if the $\mathfrak{u}(n)$-valued 2-form

$$[C, C] : X \wedge Y \mapsto [C_X, C_Y], \quad X, Y \in TM$$

(which satisfies the first and the second Bianchi identity) is parallel.

If $\widetilde{M}^{4n}$ is a quaternionic space form, then any Kähler submanifold M is curvature invariant. Hence, it is symmetric if and only if the tensor $[C, C]$ is parallel.

The Section 3 is devoted to a classification of Kähler submanifolds M^{2n} of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ with parallel non zero second fundamental form h, or shortly, *parallel Kähler submanifolds*. In terms of the tensor C, this means that

$$\nabla_X C = \omega(X) J \circ C, \qquad X \in TM$$

where ω is the 1-form defined by (2.2) and ∇ is the Levi-Civita connection of M. We prove that any parallel submanifold (M, J, g) which is not totally geodesic admits a parallel holomorphic line subbundle L of the bundle $S^3 T^{*1,0} M$ such that the connection induced on L has the curvature $R^L = i\nu g \circ J$, where ν is the reduced scalar curvature of $\widetilde{M}^{4n}$. We give the classification of all such Kähler manifolds M^{2n} with parallel holomorphic line bundle of cubic form. All of them are Hermitian symmetric spaces. Moreover, the remarkable Tsukada results [20] show that all these manifolds M^{2n} admit (an explicitly described) realization as non totally geodesic parallel Kähler submanifolds of the quaternionic projective space $\mathbb{H}P^n$. The similar problem of realization of M^{2n} as parallel Kähler submanifolds of other Wolf spaces (i.e. symmetric quaternionic Kähler spaces) remains open.

1. Almost Hermitian submanifolds of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$

1.1. The structure equations of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ Let $(\tilde{M}^{4n}, Q, \tilde{g})$ be a *quaternionic Kähler* manifold that is a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ of dimension $4n$ with parallel quaternionic structure Q, i.e. a rank-3 subbundle of the bundle of endomorphisms locally spanned by a triple of locally defined anticommuting

g-orthogonal almost complex structures $H = (J_1, J_2, J_3 = J_1 J_2)$. H is called a *local basis* of Q. Since Q is parallel, one can write

$$\widetilde{\nabla} J_\alpha = \omega_\gamma \otimes J_\beta - \omega_\beta \otimes J_\gamma \tag{1.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection, the ω_α, $\alpha = 1, 2, 3$, are locally defined 1-forms and (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. Moreover, if $n > 1$ then the metric $\widetilde{g}$ is Einstein [5] and the 1-forms ω_α satisfy the following structure equations (see [1, Th. 5.7]):

$$d\omega_\alpha + \omega_\beta \wedge \omega_\gamma = -\nu F_\alpha, \tag{1.2}$$

where ν is the reduced scalar curvature related to the scalar curvature K by $\nu = K/4n(n+2)$, $F_\alpha = \widetilde{g} \circ J_\alpha$, $\alpha = 1, 2, 3$, are Kähler forms and the exterior differential of a 1-form ω is given by $d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X, Y])$, $X, Y \in T\widetilde{M}$. By taking the exterior derivative of (1.2) we get

$$\nu(dF_\alpha - F_\beta \wedge \omega_\gamma + \omega_\beta \wedge F_\gamma) = 0. \tag{1.3}$$

We recall also that the following identities for the curvature tensor $\widetilde{R}$ hold:

$$[\widetilde{R}(X, Y), J_\alpha] = -\nu\Big(F_\gamma(X, Y)J_\beta - F_\beta(X, Y)J_\gamma\Big). \tag{1.4}$$

For $n = 1$ the formula (1.2) and all the following results remain true if we assume that *the metric* $\widetilde{g}$ *is Einstein and anti-self-dual* (i.e. the self-dual part W_+ of the Weyl tensor vanishes). This will be assumed in the following.

1.2. Almost Hermitian submanifolds of $\widetilde{M}^{4n}$ Let (M^{2m}, g) be a submanifold of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ with induced metric $g = \widetilde{g}_{|M}$ and J is a g-orthogonal almost complex structure on M^{2m}. The manifold (M^{2m}, J, g) is called an *almost Hermitian submanifold* of $\widetilde{M}$ if there is a section $J_1 : M \to Q_{|M}$ such that

$$J_1 T_x M = T_x M \qquad \forall\, x \in M$$

and $J = J_1|TM$.

If the complex structure J is integrable, then (M, J, g) is called an *Hermitian submanifold*.

REMARK. Note that the section J_1 of $Q_{|M}$ is uniquely defined by J.

For any point $x \in M^{2m}$ we can always choose a local basis $H = (J_1, J_2, J_3 = J_1 J_2)$ of Q defined in a neighborhood $\widetilde{U}$ of x in $\widetilde{M}^{4n}$ such that $J_{1|T(M\cap\tilde{U})} = J$. We will call it an *adapted basis* for (M^{2m}, J, g). Since our considerations are local, we will assume for simplicity that $\widetilde{U} \supset M^{2m}$ and we put

$$F = F_{1|M}, \quad \omega = \omega_{1|M} .$$

For any $x \in M$ we denote $\overline{T}_x M$ the maximal quaternionic (i.e. Q-invariant) subspace of the tangent space $T_x M$ and write

$$T_x M = \overline{T}_x M + \mathcal{D}_x$$

where $\mathcal{D}_x$ is the orthogonal complement. Note that if (J_1, J_2, J_3) is an adapted basis in a point $x \in M$ then $\overline{T}_x M = T_x M \cap J_2 T_x M$.

Recall that if M is a submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $T_x \widetilde{M} = T_x M + T_x^{\perp} M$ is the orthogonal decomposition of the tangent space $T_x \widetilde{M}$ at point $x \in M$ then the covariant derivative $\widetilde{\nabla}_X$ in the direction of a vector $X \in T_x M$ can be written as:

$$\widetilde{\nabla}_X \equiv \begin{pmatrix} \nabla_X & -A_X \\ A_X^t & \nabla_X^{\perp} \end{pmatrix}, \tag{1.5}$$

that is,

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \widetilde{\nabla}_X \xi = -A^{\xi} X + \nabla_X^{\perp} \xi$$

for any tangent vector field Y and any normal vector field ξ on M. Here ∇_X is the covariant derivative of the induced metric g on M, $\nabla_X^{\perp}$ is the normal covariant derivative in the normal bundle $T^{\perp} M$ which preserves the *normal metric* $g^{\perp} = \tilde{g}|T^{\perp} M$, $A_X^t Y = h(X, Y) \in T^{\perp} M$ is the *second fundamental form* and $A_X \xi = A^{\xi} X$, where $A^{\xi} \in \operatorname{End} TM$ is the *shape operator* associated with a normal vector ξ.

We will use this notation in the sequel.

Theorem 1.1 ([2]). *Let (M^{2m}, J, g), $m > 1$, be an almost Hermitian submanifold of the quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$. Then*
(1) *the almost complex structure J is integrable if and only if the local 1-form $\psi = \omega_3 \circ J - \omega_2$ on M^{2m} associated with an adapted basis $H = (J_\alpha)$ vanishes.*
(2) *J is integrable if one of the following condition holds*:
a) $\dim(\mathcal{D}_x) > 2$ *on an open dense set $U \subset M$,*
b) *(M, J) is analytic and* $\dim(\mathcal{D}_x) > 2$ *at some point $x \in M$.*

Proof. (1) Remark that if M is an almost complex submanifold of an almost complex manifold (N, J_1) then the restriction of the Nijenhuis tensor N_{J_1} to the submanifold M coincides with the Nijenhuis tensor N_J of the almost complex structure $J = J_{1|TM}$.

Using this remark, we can write

$$4N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$$
$$= 4N_{J_1}(X,Y) = (\widetilde{\nabla}_{JX}J_1)Y - (\widetilde{\nabla}_{JY}J_1)X + J(\widetilde{\nabla}_Y J_1)X - J(\widetilde{\nabla}_X J_1)Y$$

and hence

$$\begin{aligned} 4N_J(X,Y) \ = \ & [\omega_3(JX) - \omega_2(X)]J_2Y - [\omega_2(JX) + \omega_3(X)]J_3Y \\ & -[\omega_3(JY) - \omega_2(Y)]J_2X + [\omega_2(JY) + \omega_3(Y)]J_3X \end{aligned}$$

for any $X, Y \in TM$, where (J_1, J_2, J_3) is a local adapted basis. This implies (1).

(2) We assume that J is not integrable. Then the 1-form

$$\psi = (\omega_3 \circ J - \omega_2)|_{TM}$$

is not identically zero, by (1). Denote by $a = g^{-1}\psi$ the local vector field on M associated with the 1-form ψ and by $\overline{a}, a'$ the projections of a onto $\overline{T}M$ and $\mathcal{D}$ respectively. Now we need the following Lemma.

Lemma 1.2. *Let (M^{2m}, J, g), $m > 1$, be an almost Hermitian submanifold of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$. Then in any point $x \in M^{2m}$ where the Nijenhuis tensor $N(J)_x \neq 0$, or equivalently the vector $a_x \neq 0$, the subspace $\mathcal{D}_x$ is spanned by a'_x and Ja'_x:*

$$\mathcal{D}_x = \mathrm{span}\{a'_x, Ja'_x\}.$$

In particular $\mathcal{D}_x = 0$ if $\dim M$ is divisible by 4.

Proof. Remark that

$$4N_J(X,Y) = J_2\left[\psi(X)Y - \psi(JX)JY - \psi(Y)X + \psi(JY)JX\right] \in J_2TM \cap TM = \overline{T}M$$

for any $X, Y \in TM$. This shows that for any $X, Y \in TM$ the vector

$$\left[\psi(X)Y - \psi(JX)JY - \psi(Y)X + \psi(JY)JX\right] \in \overline{T}M\,.$$

For $X = a = g^{-1}\psi$, the last condition says that

$$b_Y := |a|^2 Y - \psi(Y)a + \psi(JY)Ja \in \overline{T}M \qquad \forall \ Y \in TM.$$

By projecting the vector b_Y to $\mathcal{D}$ for $Y = \overline{Y} \in \overline{T}M$ and $Y = Y' \in \mathcal{D}$ respectively we get the equations:

$$-\overline{\psi}(\overline{Y})a' + \overline{\psi}(J\overline{Y})Ja' = 0\,, \qquad \forall \ \overline{Y} \in \overline{T}M \tag{1.6}$$

and

$$|a|^2 Y' - \psi'(Y')a' + \psi'(JY')Ja' = 0 \qquad \forall \ Y' \in \mathcal{D}. \tag{1.7}$$

where $\psi' = g \circ a'$, $\overline{\psi} = g \circ \overline{a}$. The last equation shows that $\mathcal{D}_x = \{a', Ja'\}$ when $a \neq 0$. □

The Lemma implies statements (2)a) and (2)b) since in the analytic case the set U of points where $a \neq 0$ is open and dense and $\dim \mathcal{D}_x \leq 2$ on U. □

Corollary 1.3 ([2]). *Let (M^{4k}, J, g) be an almost Hermitian submanifold of dimension $4k$ of a quaternionic Kähler manifold $\widetilde{M}^{4n}$. Assume that the set U of points $x \in M$ where the Nijenhuis tensor of J is not zero is open and dense. Then M is a totally geodesic quaternionic Kähler submanifold.*

Proof. By Lemma 1.2, in a point $x \in U$ one has $\dim \mathcal{D}_x = 0$ or 2. The second case is excluded by dimensional reason. Then U is a quaternionic, hence totally geodesic, submanifold of $\widetilde{M}$. This implies that M is quaternionic. □

As another corollary we get the following theorem.

Theorem 1.4 ([2]). *Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a complete quaternionic Kähler manifold with positive scalar curvature. Then any analytic almost Hermitian submanifold (M, J, g) of dimension $4k$ with complete induced metric is a Hermitian submanifold.*

Proof. Assume that the almost complex structure J is not integrable. Then by Theorem 1.1 and Corollary 1.3, M is a totally geodesic quaternionic Kähler submanifold. It is known that it has the same (positive) reduced scalar curvature as $\widetilde{M}$. Hence it is a compact quaternionic Kähler manifold. By [4, Theorem 3.8] such manifold has no almost complex structure. Contradiction. □

1.3. Almost Kähler, Kähler and totally complex submanifolds

DEFINITION 1.5. An almost Hermitian submanifold (M^{2m}, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is called *almost Kähler* (resp., *Kähler*) if the Kähler form $F = g \circ J$ is closed (resp., parallel).

Theorem 1.6. *Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a quaternionic Kähler manifold with non vanishing reduced scalar curvature ν. Then any almost Kähler submanifold (M^{2m}, J, g), $m \neq 3$, of $\widetilde{M}$ is Kähler.*

If $m = 3$, then $TM = \overline{T}M + \mathcal{D}$ where $\mathcal{D}$ is two-dimensional distribution and $\omega_2 \circ J_2|_{TM} = -\omega_3 \circ J_3|_{TM}$.

Proof. We will show that $N_J = 0$ on M. Assume that there exists a point $x \in M$ where $N_{J|_x} \neq 0$. We will prove that this leads to a contradiction. By identity (1.3), the condition that the Kähler form $F = F_{1|_M}$ is closed can be written as

$$F_2^T \wedge \omega_3^T = F_3^T \wedge \omega_2^T\,, \tag{1.8}$$

where $F_\alpha^T, \omega_\alpha^T$ are the restriction of the forms F_α, ω_α to M.

CLAIM 1. If $\dim \overline{T}_x M > 4$ then $\omega_2^T(\overline{T}_x M) = \omega_3^T(\overline{T}_x M) = 0$; if $\dim \overline{T}_x M = 4$ then $(\omega_3^T \circ J + \omega_2^T)_x(\overline{T}_x M) = 0$.

It follows from the lemma below.

Lemma 1.7. *Let (V, g) be an Euclidean vector space with a (constant) quaternionic structure $Q = \mathrm{span}(J_1, J_2, J_3)$ and $F_\alpha = g \circ J_\alpha$ Kähler forms. Then the equation*

$$F_2 \wedge \xi = F_3 \wedge \eta$$

for 1-forms ξ, η has a non trivial solution if and only if $\dim V = 4$ and all solutions are given by

$$(\xi = \eta \circ J_1, \eta) \qquad \forall\, \eta \in V^*.$$

Proof. Assume that $\dim V = 4k > 4$ and (ξ, η) is a non trivial solution. Then, for any $Y \in V$ there exists a unit vector X such that $Y \perp QX$. Then

$$\begin{aligned}(F_2 \wedge \xi)(X, J_2X, Y) &= \|X\|^2 \xi(Y) = \xi(Y)\\ &= (F_3 \wedge \eta)(X, J_2X, Y) = F_3(X, J_2X)\eta(Y) = 0\end{aligned}$$

and

$$\begin{aligned}(F_2 \wedge \xi)(X, J_3X, Y) &= 0\\ &= (F_3 \wedge \eta)(X, J_3X, Y) = \|X\|^2 \eta(Y) = \eta(Y).\end{aligned}$$

Hence $\xi = \eta = 0$ and we get a contradiction. If $\dim V = 4$ then any solution (ξ, η) satisfies the identity

$$\begin{aligned}(F_2 \wedge \xi)(X, J_2X, J_1X) &= \|X\|^2 \xi(J_1X)\\ = (F_3 \wedge \eta)(X, J_2X, J_1X) &= F_3(J_2X, J_1X)\eta(X) = -\|X\|^2 \eta(X).\end{aligned}$$

Hence $\xi = \eta \circ J$. It is also easy to check that $(\xi = \eta \circ J, \eta)\,, \eta \in V^*$, is always a solution.

CLAIM 2. If $\dim \overline{T}_x M \geq 4$ then $\omega_2^T(\mathcal{D}_x) = \omega_3^T(\mathcal{D}_x) = 0$.

To prove it we calculate both sides of equation (1.8) on vectors X, J_2X, Y, where X is a unit vector from $\overline{T}_x M$ and $Y \in \mathcal{D}_x$. We get

$$(F_2^T \wedge \omega_3^T)(X, J_2X, Y) = \omega_3^T(Y) = F_3(X, J_2X)\omega_2^T(Y) = 0.$$

Hence, $\omega_3^T(\mathcal{D}_x) = 0$ and similarly $\omega_2^T(\mathcal{D}_x) = 0$.

CLAIM 3. If $a_x = g^{-1}(\omega_3 \circ J - \omega_2)_x \neq 0$ then $\dim(\mathcal{D}_x) = 2$.

It follows from Lemma 1.2.

Now we can finish the proof of Theorem 1.6. Denote by U the open submanifold of $\widetilde{M}$ where the Nijenhuis tensor of J is not zero. For $x \in U$ $\dim \mathcal{D}_x = 2$ by Claim 3 and $\dim \overline{T}_x M = 0\,,4$ by Claims 1 and 2. Hence if $U \neq \emptyset$ then $\dim M = 2$ or 6. But it is well known that any 2-dimensional almost Kähler submanifold is Kähler. This shows that if the almost complex structure J on M is not integrable, then $\dim M = 6$ with the stated properties. $\square$

Theorem 1.8. *Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a quaternionic Kähler manifold with $\nu \neq 0$. An almost Hermitian submanifold (M^{2m}, J, g), $m > 1$, of $\widetilde{M}$ is Kähler if and only if one of the following equivalent conditions holds*:

$$k_1) \qquad \omega_{2|T_xM} = \omega_{3|T_xM} = 0 \qquad \forall\, x \in M,$$
$$k_2) \qquad J_2 T_x M \perp T_x M \qquad \forall\, x \in M$$

where ω_α are 1-forms associated to an adapted basis (J_α) by (1.1).

Proof. Let (M^{2m}, J, g) be an almost Hermitian submanifold of $\widetilde{M}$. Using (1.1) we get

$$(1.9) \qquad \begin{aligned}(\widetilde{\nabla}_X J_1)Y &= (\nabla_X J)Y + h(X, JY) - J_1 h(X, Y)\\ &= \omega_3(X)J_2Y - \omega_2(X)J_3Y, \qquad X, Y \in TM.\end{aligned}$$

Taking the orthogonal projection on T_xM we conclude that

$$(1.10) \qquad (\nabla_X J)Y = 0 \Longleftrightarrow \big[\omega_3(X)J_2Y - \omega_2(X)J_3Y\big]^T = 0$$

where $[\quad]^T$ means the tangent part. It is clear that any one of the conditions k_1) or k_2) implies $\nabla J_{|_x} = 0\, \forall\, x \in M$, that is (M, J, g) is Kähler. To prove that the conditions k_1), k_2) are also necessary for (M, J) to be Kähler, we first show that at a point $x \in M$ where $\nabla J_{|_x} = 0$ at least one of them must hold: in fact, from the identities

$(\nabla_X J)Y = \left[\omega_3(X)J_2Y - \omega_2(X)J_3Y\right]^T = 0$, $(\nabla_X J)(JY) = -\left[\omega_3(X)J_3Y + \omega_2(X)J_2Y\right]^T = 0 \quad \forall\, X, Y \in T_xM$, one gets

$$\left[\omega_2^2(X) + \omega_3^2(X)\right]\left[J_2Y\right]^T = 0, \qquad \forall\, X, Y \in T_xM$$

and the claim follows immediately. Now we assume that (M, J, g) is Kähler and prove that both k_1) and k_2) must hold on M.

1) Suppose that k_1) does not hold at $x \in M$. Then k_2) holds on an open neighbourhood U_x of x in M and the structure equation (1.3) for $\alpha = 2, 3$ gives $(\omega_3 \wedge F_1)_{T_xM} = (\omega_2 \wedge F_1)_{T_xM} = 0$ which imply (since $\dim T_xM > 2$) $\omega_{3|T_xM} = \omega_{2|T_xM} = 0$, by contradicting the assumption.

2) On the other hand, assume that k_2) does not hold at $x \in M$. Hence k_1) holds on an open neighbourhood V_x of x and the structure equation (1.2) for $\alpha = 2, 3$ gives $\nu F_{2|_{T_xM}} = \nu F_{3|_{T_xM}} = 0$. Since $\nu \neq 0$ these give a contradiction. □

Theorem 1.9. *Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a quaternionic Kähler manifold with non vanishing reduced scalar curvature ν and (M^{2m}, J, g), $m > 1$, an almost Hermitian submanifold of $\widetilde{M}^{4n}$.*
(1) *If (M^{2m}, J, g) is Kähler then the second fundamental form h of M satisfies the identity*

$$h(X, JY) = h(JX, Y) = J_1h(X, Y) \qquad \forall\, X, Y \in TM. \tag{1.11}$$

In particular, M is minimal (see [7]).
(2) *Conversely, if the identity (1.11) holds on an almost Hermitian submanifold M^{2m} of $\widetilde{M}^{4n}$ then it is either a Kähler submanifold or a quaternionic submanifold and these cases cannot happen simultaneously.*

Proof. (1) From identity (1.9) and Theorem 1.8 it is clear that if (M, J) is Kähler then (1.11) holds. It implies that $h(JX, JY) = -h(X, Y)$. This shows that the mean curvature vector of M

$$\mathbf{n} = \mathrm{tr}(h) = \sum_{i=1}^{m} h(E_i, E_i) + \sum_{i=1}^{m} h(JE_i, JE_i) = 0$$

where $E_1, \ldots, E_m, JE_1, \ldots, JE_m$ is an orthonormal basis of T_xM. Hence, M is minimal.

(2) Conversely, let assume that (1.11) holds on the almost Hermitian submanifold (M, J, g). Then for any $X, Y \in T_xM$ we have

$$\begin{aligned}(\nabla_X J)Y &= \nabla_X(JY) - J\nabla_X Y \\ &= \widetilde{\nabla}_X(JY) - h(X, JY) - J_1\left[\widetilde{\nabla}_X Y - h(X, Y)\right]\end{aligned}$$

$$= \widetilde{\nabla}_X(J_1 Y) - J_1(\widetilde{\nabla}_X Y) = (\widetilde{\nabla}_X J_1)Y.$$

Hence

$$(\nabla_X J)Y = \omega_3(X)J_2 Y - \omega_2(X)J_3 Y \in T_x M \qquad \forall\, X, Y \in T_x M$$

and also

$$(\nabla_X J)(JY) = -\omega_3(X)J_3 Y - \omega_2(X)J_2 Y \in T_x M \qquad \forall\, X, Y \in T_x M.$$

These imply that

$$\left[\omega_2^2(X) + \omega_3^2(X)\right]\left[J_2 Y\right]^N = 0\,, \qquad \forall\, X, Y \in TM\,, \tag{1.12}$$

where $[\quad]^N$ means the normal part. We set

$$M_1 = \{x \in M\,,\ J_2 T_x M = T_x M\}, \quad M_2 = \{x \in M - M_1\,,\ \omega_{2|T_x M} = \omega_{3|T_x M} = 0\}.$$

Then by (1.12) $M = M_1 \cup M_2$, $M_1 \cap M_2 = \emptyset$ and M_1 is a closed subset and M_2 is an open subset of M. We prove that M_2 is also closed. The structure equations (1.2), $\alpha = 2, 3$ show that $F_2|M_2 = F_3|M_2 = 0$. Hence $M_2 \subset M_3$ where

$$M_3 = \{x \in M\,,\ J_2 T_x M \perp T_x M\}$$

is a closed subset of M with $M_1 \cap M_3 = \emptyset$. This shows that $M_2 = M_3$ is a closed subset of M. Then, either $M_1 = \emptyset$, $M_2 = M$ is a Kähler submanifold or $M_2 = \emptyset$ and $M = M_1$ is a quaternionic Kähler (totally geodesic) submanifold. Since the set of points x of M where $J_2 T_x M \neq T_x M$ is open the conclusion follows. $\square$

Corollary 1.10. *A totally geodesic almost Hermitian submanifold (M, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ with $\nu \neq 0$ is either a Kähler submanifold or a quaternionic submanifold and these conditions cannot happen simultaneously.*

Proof. The first statement follows directly from Theorem 1.9) since (1.11) certainly holds for a totally geodesic submanifold ($h = 0$). To prove the last statement we remark that a quaternionic submanifold $(M^{4k}, g = \widetilde{g}_{|M})$ of $\widetilde{M}^{4n}$ is a quaternionic Kähler manifold with the same reduced scalar curvature ν. If M is also Kähler, $\nabla J = 0$, then it must be $\nu = 0$ (see [4]). $\square$

DEFINITION 1.11. An almost Hermitian submanifold (M, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is called *totally complex* if

$$J_2 T_x M \perp T_x M \qquad \forall\, x \in M$$

where (J_1, J_2, J_3) is an adapted basis.

Theorem 1.12 (see also [20]). (1) *A totally complex submanifold* (M^{2m}, J, g) *of a quaternionic Kähler manifold* $(\widetilde{M}, Q, \widetilde{g})$ *is Kähler.*
(2) *Conversely, if* $\widetilde{M}^{4n}$ *has* $\nu \neq 0$ *then any Kähler submanifold* (M^{2m}, J, g) *of* $\widetilde{M}^{4n}$ *is totally complex.*

Proof. (1) Let (M^{2m}, J, g) be a totally complex submanifold of $\widetilde{M}^{4n}$. Then using (1.1) for $\alpha = 1$ we get

$$(\nabla_X J)Y = \left[\omega_3(X)J_2Y - \omega_2(X)J_3Y\right]^T = 0 \qquad \forall\, X, Y \in TM\,.$$

(2) is a corollary of Theorem 1.8 (see condition k_2). □

REMARK. The local 1-form $\omega = \omega_1|_M$ on the Kähler submanifold (M^{2n}, J, g) of $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is the connection form of the circle bundle $\{\sin t J_2 + \cos t J_3\}$ orthogonal to $\widetilde{J}$ in Q and the global 2-form $-d\omega = \nu F$ is the curvature. In particular, the Chern form $c_1 = \nu/(2\pi)F$ is an integer.

2. Maximal Kähler submanifolds of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$

2.1. The shape tensor C_X Let (M^{2n}, J, g) be a Kähler submanifold of maximal possible dimension $2n$ of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$. We fix an adapted basis (J_1, J_2, J_3) of Q. For simplicity we will assume that it is defined on a neighbourhood of M^{2n} in $\widetilde{M}^{4n}$. We have the orthogonal decomposition

$$T_x\widetilde{M} = T_xM \oplus J_2T_xM \qquad \forall\, x \in M. \tag{2.1}$$

Then the following equations hold:

$$\widetilde{\nabla}_X J_1 = 0\,, \quad \widetilde{\nabla}_X J_2 = \omega(X)J_3\,, \quad \widetilde{\nabla}_X J_3 = -\omega(X)J_2 \qquad \forall\, X \in TM \tag{2.2}$$

where ω is a 1-form.

We identify the normal bundle $T^\perp M$ with the tangent bundle TM using J_2:

$$\begin{aligned} \varphi = J_{2|T^\perp M} : T_x^\perp M &\to T_xM \\ \xi &\mapsto J_2\xi\,. \end{aligned}$$

Then the second fundamental form h of M is identified with the tensor field

$$C = J_2 \circ h \in TM \otimes S^2T^*M$$

and the normal connection $\nabla^\perp$ on $T^\perp M$ is identified with a linear connection $\nabla^N = J_2 \circ \nabla^\perp \circ J_2^{-1}$ on TM.

We will call C the *shape tensor* of the Kähler submanifold M.

Note that C depends on the adapted basis (J_α) and it is defined only locally. If (J'_α) is another adapted basis and $J'_2 = \cos\theta J_2 + \sin\theta J_3$ then the shape tensor transforms as

$$C \mapsto C' = \cos\theta C + \sin\theta J \circ C\,.$$

Lemma 2.1. *One has*:
(1) *For any $X \in TM$ the endomorphism C_X of TM is symmetric and $C_X = A^{\varphi^{-1}X} = -A^{J_2 X}$ where A^ξ is the shape operator, defined in* (1.5).
(*Note that $C_{J_2\xi} = A^\xi,\ \forall \xi \in T^\perp M$.*)
(2) $\nabla^N_X = \nabla_X - \omega(X)J,\ \ X \in TM$.
(3) *The curvature of the connection ∇^N is given by*

$$R^N_{XY} = R_{XY} - d\omega(X,Y)J.$$

(4) $\{C_X, J\} = C_X \circ J + J \circ C_X = 0$ *and, hence,* $\operatorname{tr} C_. = \sum_{2n} C_{E_i} E_i = 0$, *where* (E_i) *is an orthonormal basis of $T_x M$.*
(5) *The tensors $gC, gC \circ J$ defined by*

$$gC(X,Y,Z) = g(C_X Y, Z)\,, \qquad (gC \circ J)(X,Y,Z) = gC(JX,Y,Z)$$

are symmetric, i.e. $gC, gC \circ J \in S^3 T^* M$.

Proof. (1) Using (2.1) and (2.2), for any $X, Y, Z \in TM$ one has

$$\begin{aligned}\langle C_X Z, Y\rangle &= -\langle h(X,Z), J_2 Y\rangle = \langle \widetilde{\nabla}_X (J_2 Y), Z\rangle \\ &= \langle J_2 \widetilde{\nabla}_X Y, Z\rangle = -\langle \widetilde{\nabla}_X Y, J_2 Z\rangle = -\langle h(X,Y), J_2 Z\rangle \\ &= \langle C_X Y, Z\rangle\,.\end{aligned}$$

Moreover

$$\begin{aligned}\langle -A^{J_2 X} Y, Z\rangle &= -\langle h(Y,Z), J_2 X\rangle = \langle J_2 h(Y,Z), X\rangle = \langle C_Y Z, X\rangle \\ &= \langle Z, C_X Y\rangle.\end{aligned}$$

This proves (1).

(2) We have

$$\begin{aligned}\langle \nabla^N_X Y, Z\rangle &= \langle J_2 \circ \nabla^\perp_X \circ (J_2^{-1} Y), Z\rangle = \langle \nabla^\perp_X (J_2 Y), J_2 Z\rangle = \langle \widetilde{\nabla}_X (J_2 Y), J_2 Z\rangle \\ &= \langle \omega(X) J_3 Y, J_2 Z\rangle + \langle J_2(\nabla_X Y + h(X,Y)), J_2 Z\rangle \\ &= \langle -\omega(X) J_1 Y, Z\rangle + \langle \nabla_X Y, Z\rangle.\end{aligned}$$

(3) $$R^N_{XY} = [\nabla_X - \omega(X)J, \nabla_Y - \omega(Y)J] - \nabla_{[X,Y]} + \omega([X,Y])J.$$
$$= R_{XY} - \{X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y])\}J = R_{XY} - d\omega(X,Y)J.$$

(4) By using (1.10) we have

$$C_X JY = J_2 h(X, JY) = J_2 J_1 h(X,Y) = -J_1 J_2 h(X,Y) = -JC_X Y\,.$$

(5) The first statement follows from (1). Using (4) we prove the second one:

$$\begin{aligned} gC \circ J(X,Y,Z) &= gC(JX,Y,Z) = \langle C_{JX}Y, Z\rangle = \langle C_Y(JX), Z\rangle \\ &= -\langle JC_Y X, Z\rangle = \langle C_Y X, JZ\rangle = \langle C_Y JZ, X\rangle = \langle C_{JZ}Y, X\rangle \\ &= gC \circ J(Z,Y,X). \end{aligned}$$ □

We denote by ∇' the linear connection in a tensor bundle which is a tensor product of a tangent tensor bundle of M and a normal tensor bundle defined by ∇ and $\nabla^\perp$. For example, if k is a section of the bundle $T^\perp M \otimes S^2T^*M$ then

$$(\nabla'_X k)(Y,Z) = \nabla^\perp_X(k(Y,Z)) - k(\nabla_X Y, Z) - k(Y, \nabla_X Z).$$

Then using (2) of Lemma 2.1, we get the following expression for the covariant derivative of the second fundamental form.

$$\begin{aligned} J_2(\nabla'_X h)(Y,Z) &= (\nabla_X C)_Y Z - \omega(X)J \circ C_Y Z \\ &= (\nabla^N_X C)(Y,Z) + 2\omega(X)J \circ C_Y Z. \end{aligned} \tag{2.3}$$

Denote by

$$S_J = \{A \in \operatorname{End} TM, \{A, J\} = 0, g(AX,Y) = g(X,AY)\}$$

the bundle of symmetric endomorphisms of TM, which anticommute with J and by

$$S_J^{(1)} = \{A \in \operatorname{Hom}(TM, S_J) = T^*M \otimes S_J, A_X Y = A_Y X\}$$

its first prolongation. Then conditions (4), (5) can be reformulated as follows.

Corollary 2.2. *The tensor $C = J_2 h$ belongs to the space $S_J^{(1)}$ and its covariant derivative is given by*

$$\nabla_X C = J_2 \nabla'_X h + \omega(X)J \circ C\,.$$

2.2. Gauss-Codazzi equations Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ and

$$\widetilde{R}_{XY} = R^{TT}_{XY} + R^{\perp T}_{XY} + R^{T\perp}_{XY} + R^{\perp\perp}_{XY}$$

the decomposition of the curvature operator $\widetilde{R}_{XY}$, $X, Y \in T_xM$ of the manifold $\widetilde{M}$ according to the decomposition

$$\mathrm{End}(T_x\widetilde{M}) = \mathrm{End}(T_xM) + \mathrm{Hom}(T_xM, T_x^{\perp}M) + \mathrm{Hom}(T_x^{\perp}M, T_xM) + \mathrm{End}(T_x^{\perp}M).$$

Using (1.5) and calculating the curvature operator $\widetilde{R}_{XY} = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$ of the connection $\widetilde{\nabla}$, we get the following *Gauss-Codazzi equations*:

$$\begin{array}{rcll} R^{\top\top}_{XY} &=& R_{XY} - A_X A^t_Y + A_Y A^t_X & (\top\top) \\ &=& R_{XY} - \sum_i A^{\xi_i} X \wedge A^{\xi_i} Y & \\ R^{\perp\perp}_{XY}\xi &=& R^{\perp}_{XY}\xi - \sum_i \langle X, [A^{\xi_i}, A^{\xi}]Y\rangle \xi_i & (\perp\perp) \\ R^{\top\perp}_{XY}\xi &=& -(\nabla_X A^{\xi} - A^{\nabla^{\perp}_X \xi})Y + (\nabla_Y A^{\xi} - A^{\nabla^{\perp}_Y \xi})X & (\top\perp) \\ R^{\perp\top}_{XY} Z &=& (\nabla'_X h)(Y, Z) - (\nabla'_Y h)(X, Z) & (\perp\top) \end{array}$$

where ξ_i is an orthonormal basis of $T^{\perp}M$, $X, Y \in TM$, $\xi \in T^{\perp}M$, R, $R^{\perp}$ are the curvature tensors of the connections ∇, $\nabla^{\perp}$. (We identify a bivector $X \wedge Y$ with the skew-symmetric operator $Z \mapsto \langle Y, Z\rangle X - \langle X, Z\rangle Y$.) Recall that ∇' is the connection in $T^{\perp}M \otimes S^2TM$ induced by $\nabla^{\perp}$ and ∇.

DEFINITION 2.3. Let M be a submanifold M of a Riemannian manifold $\widetilde{M}$. Then
(1) M is called *curvature invariant* if

$$\widetilde{R}_{XY} Z \in TM, \quad \forall X, Y, Z \in TM,$$

or equivalently,

$$R^{\top\perp} = R^{\perp\top} = 0.$$

(2) ([15]) M is called *strongly curvature invariant* if it is curvature invariant and moreover

$$\widetilde{R}_{\xi\eta}\zeta \in T^{\perp}M, \quad \forall \xi, \eta, \zeta \in T^{\perp}M.$$

(3) M is called *parallel* if the second fundamental form is parallel: $\nabla' h = 0$.

Let us recall the following known result.

Proposition 2.4. *A parallel submanifold M of a locally symmetric manifold $\widetilde{M}$ is curvature invariant and locally symmetric.*

Proof. First statement follows from $(\perp\top)$. The second statement follows by covariant derivation of $(\top\top)$ and remark that $\nabla R^{TT} = 0$ since $\widetilde{\nabla}\widetilde{R} = R^{\perp\top} = 0$, see also the formula (2.5.1) below. $\square$

2.2.1. Gauss-Codazzi equations for a Kähler submanifold By specifying the previous formulas to the totally complex submanifold $M^{2n} \subset \widetilde{M}^{4n}$ of a quaternionic Kähler manifold and using Lemma 2.1 and (2.3) we get the following Proposition.

Proposition 2.5. *The Gauss-Codazzi equations for a maximal totally complex submanifold (M^{2n}, J) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ can be written as*

(1) $R^{TT}_{XY} = R_{XY} - [C_X, C_Y]$

(2) $J_2 R^{\perp\perp}_{XY} J_2^{-1} = R^N_{XY} - [C_X, C_Y] = R_{XY} - [C_X, C_Y] - d\omega(X, Y)J$

(3) $J_2 R^{\perp T}_{XY} = P_{XY} - P_{YX}$

where (J_α) is an adapted basis of (M^{2n}, J), $C = J_2 h$ is the shape operator and $P_{XY} := (\nabla_X C)_Y - \omega(X)J \circ C_Y$.

Note that $R^{T\perp}_{XY} J_2$ is the adjoint of $J_2 R^{\perp T}_{XY}$.

Corollary 2.6. *The Ricci tensor Ric_M of the Kähler submanifold $M^{2n} \subset \widetilde{M}^{4n}$ is given by*

$$\mathrm{Ric}_M = \mathrm{Ric}(R^{TT}) - \mathrm{tr}_g \langle C_\cdot, C_\cdot \rangle = \mathrm{Ric}(R^{TT}) - \left\langle \sum_i C^2_{E_i} \cdot, \cdot \right\rangle$$

or, more precisely,

$$\mathrm{Ric}_M(X, Y) = \mathrm{Ric}(R^{TT})(X, Y) - \sum_{i=1}^{2n} \langle C_{E_i} X, C_{E_i} Y \rangle \qquad X, Y \in TM$$

where $\mathrm{Ric}(R^{TT})$ is the Ricci tensor of the tangential part R^{TT} of $\widetilde{R}$, that is, $\mathrm{Ric}(R^{TT})(X, Y) = \mathrm{tr}(Z \mapsto R^{TT}_{ZX} Y)$ and $(E_i), i = 1, \dots, 2n$, is an orthonormal basis of TM. In particular, the Ricci curvature $\mathrm{Ric}_M(X, X)$ of M in the direction of a unit vector $X \in TM$ is not bigger then $\mathrm{Ric}(R^{TT})(X, X)$. If $\mathrm{Ric}(R^{TT}) \leq 0$ then $Ric_M \leq 0$.

Proof. It is a straightforward consequence of (1) above and Lemma 2.1(4). □

Corollary 2.7. *Let $\widetilde{M}^{4n}$ be a quaternionic Kähler manifold of non positive sectional curvature. Then any totally complex submanifold M^{2n} of $\widetilde{M}^{4n}$ has non positive Ricci curvature. Moreover the second fundamental form of M^{2n} vanishes at any point x where the Ricci curvature of M^{2n} vanishes.*

Proposition 2.8. *Let M^{2n} be a Kähler submanifold of a quaternionic Kähler manifold $\widetilde{M}^{4n}$. Then*

(1) *M^{2n} is parallel if and only if $P_{XY} := (\nabla_X C)_Y - \omega(X)J \circ C_Y = 0$;*

(2) *M^{2n} is curvature invariant if and only if the tensor P_{XY} belongs to the second prolongation $S_J^{(2)}$ of the space S_J, where*

$$S_J^{(2)} = \{A \in \mathrm{Hom}(TM, S_J^{(1)}),\ A_{XY} = A_{YX}\}.$$

Then M^{2n} is strongly curvature invariant.

Proof. 1) follows from (2.3). First statement of 2) follows from (3) of Proposition 2.5. The last statement follows from the general identity for the curvature tensor $\widetilde{R}$ of $\widetilde{M}^{4n}$:

$$\langle \widetilde{R}(J_2X, J_2Y), J_2Z, J_2W\rangle = \langle \widetilde{R}(X,Y), Z, W\rangle \qquad \forall\, X, Y, Z, W \in T\widetilde{M}$$

and remark that $J_2T_xM = T_x^{\perp}M$, $\forall\, x \in M$. □

Proposition 2.9. *For a Kähler submanifold M^{2n} of a quaternionic Kähler manifold $\widetilde{M}^{4n}$, the second Gauss-Codazzi equation follows from the first.*

Proof. For any $X, Y, U, V \in TM$, by using (1.4), one has:

$$\begin{aligned}\langle J_2R^{\perp\perp}_{XY}J_2U, V\rangle &= \langle J_2\widetilde{R}_{XY}J_2U, V\rangle\\ &= -\langle \widetilde{R}_{XY}U, V\rangle - \nu\langle F(X,Y)J_1U + F_3(X,Y)J_3U, V\rangle\\ &= -\langle R^{TT}_{XY}U, V\rangle - \nu\langle F(X,Y)J_1U, V\rangle,\end{aligned}$$

that is,

$$R^{\perp\perp}_{XY} = J_2R^{TT}_{XY}J_2^{-1} - \nu F(X,Y)J_1. \tag{2.4}$$

Moreover $d\omega(X,Y) = -\nu F(X,Y)$. □

2.3. Maximal Kähler submanifolds of a quaternionic symmetric space Now we assume that the quaternionic Kähler manifold $\widetilde{M}^{4n}$ is a (locally) symmetric manifold, i.e. $\widetilde{\nabla}\widetilde{R} = 0$.

Proposition 2.10. *Let M^{2n} be a Kähler submanifold of a quaternionic locally symmetric space $\widetilde{M}^{4n}$. Then the covariant derivatives of the tangential part R^{TT}, the normal part $R^{\perp\perp}$ and mixed part $R^{\perp T}$ of the curvature tensor $\widetilde{R}_{|M}$ can be expressed in terms of these tensors and the shape operator $C = J_2 \circ h$ as follows*:

$$\begin{aligned}\langle(\nabla_XR^{TT})(Y,Z)U, V\rangle \;=\; &-\langle R^{\perp T}(Y,Z)U, J_2C_XV\rangle + \langle R^{\perp T}(Y,Z)V, J_2C_XU\rangle\\ &-[\langle J_2R^{\perp T}(U,V)C_XY, Z\rangle + \langle R^{\perp T}(U,V)Y, J_2C_XZ\rangle],\end{aligned} \tag{2.5.1}$$

$$
\begin{aligned}
(\nabla'_X R^{\perp T})(Y,Z)U &= J_2 C_X R^{TT}(Y,Z)U - R^{\perp\perp}(Y,Z)J_2 C_X U \\
&\quad - [\widetilde{R}(J_2 C_X Y, Z)U + \widetilde{R}(Y, J_2 C_X Z)U]^{\perp} \\
&= J_2 C_X R^{TT}(Y,Z)U - J_2 R^{TT}(Y,Z)C_X U + \nu F(Y,Z)J_3 C_X U \\
&\quad - [\widetilde{R}(J_2 C_X Y, Z)U + \widetilde{R}(Y, J_2 C_X Z)U]^{\perp},
\end{aligned}
\tag{2.5.2}
$$

$$
\begin{aligned}
(\nabla'_X R^{T\perp})(Y,Z)\xi &= -\left[\widetilde{R}(J_2 C_X Y, Z)\xi + \widetilde{R}(Y, J_2 C_X Z)\xi\right]^T \\
&\quad - R^{TT}(Y,Z)C_{J_2\xi}X + C_X J_2 R^{\perp\perp}(Y,Z)\xi,
\end{aligned}
\tag{2.5.3}
$$

(2.5.4)

$$
\begin{aligned}
\langle(\nabla'_X R^{\perp\perp})(Y,Z)J_2U, J_2V\rangle &= \langle R^{\perp T}(U,V)Z, J_2 C_X Y\rangle - \langle R^{\perp T}(U,V)Y, J_2 C_X Z\rangle \\
&\quad + \langle R^{\perp T}(Y,Z)C_U X, J_2 V\rangle + \langle C_X R^{T\perp}(Y,Z)J_2 U, V\rangle
\end{aligned}
$$

for any $X, Y, Z, U, V \in TM$, $\xi \in T^{\perp}M$.

Proof. For $Y, Z, U \in TM$ we have the decomposition $\widetilde{R}(Y,Z)U = R^{TT}(Y,Z)U + R^{\perp T}(Y,Z)U$. Then it follows

$$
\begin{aligned}
0 = (\widetilde{\nabla}_X \widetilde{R})(Y,Z)U &= (\nabla_X R^{TT})(Y,Z)U + (\nabla'_X R^{\perp T})(Y,Z)U \\
&\quad + h(X, R^{TT}(Y,Z)U) - A_X R^{\perp T}(Y,Z)U \\
&\quad - [\widetilde{R}(h(X,Y),Z)U + \widetilde{R}(Y, h(X,Z))U + \widetilde{R}(Y,Z)h(X,U)].
\end{aligned}
$$

By taking the tangential and the normal part of the equation, we get

$$
\begin{aligned}
\text{(a)}\quad (\nabla_X R^{TT})(Y,Z)U &= A_X R^{\perp T}(Y,Z)U \\
&\quad - [\widetilde{R}(J_2 C_X Y, Z)U + \widetilde{R}(Y, J_2 C_X Z)U + \widetilde{R}(Y,Z)J_2 C_X U]^T, \\
\text{(b)}\quad (\nabla'_X R^{\perp T})(Y,Z)U &= J_2 C_X R^{TT}(Y,Z)U \\
&\quad - [\widetilde{R}(J_2 C_X Y, Z)U + \widetilde{R}(Y, J_2 C_X Z)U + \widetilde{R}(Y,Z)J_2 C_X U]^{\perp}.
\end{aligned}
$$

The scalar product of (a) with V gives

$$
\begin{aligned}
\langle(\nabla_X R^{TT})(Y,Z)U, V\rangle &= -\langle R^{\perp T}(Y,Z)U, J_2 C_X V\rangle \\
&\quad - [\langle\widetilde{R}(J_2 C_X Y, Z)U, V\rangle + \langle\widetilde{R}(Y, J_2 C_X Z)U, V\rangle + \langle\widetilde{R}(Y,Z)J_2 C_X U, V\rangle].
\end{aligned}
$$

Now we take into account that for any tangent vectors $X, Y, U, V \in TM$ one has $\langle J_2 R^{\perp T}(U,V)X, Y\rangle = \langle\widetilde{R}(J_2X, Y)U, V\rangle$. In fact, let us recall the identity

$$
[\widetilde{R}(U,V), J_2] = \nu(\langle U, J_1 V\rangle J_3 - \langle U, J_3 V\rangle J_1).
$$

Then one has, for example,

$$\begin{aligned}\langle \widetilde{R}(J_2C_XY,Z)U,V\rangle &= \langle \widetilde{R}(U,V)J_2C_XY,Z\rangle\\ &= \langle J_2\widetilde{R}(U,V)C_XY,Z\rangle + \nu\langle U,J_1V\rangle\langle J_3C_XY,Z\rangle\\ &= \langle J_2\widetilde{R}(U,V)C_XY,Z\rangle = \langle J_2R^{\perp T}(U,V)C_XY,Z\rangle.\end{aligned}$$

Hence (2.5.1) follows.

The first equality in (2.5.2) coincides with (b). To get the second equality it is sufficient to use (1.4).

The other two identities are proved analogously, as follows. We have

$$\begin{aligned}0 = (\widetilde{\nabla}_X\widetilde{R})(Y,Z)\xi &= (\nabla' R)^{T\perp}(Y,Z)\xi + (\nabla' R^{\perp\perp})(Y,Z)\xi\\ &+ \widetilde{R}(J_2C_XY,Z)\xi + \widetilde{R}(Y,J_2C_XZ)\xi + \widetilde{R}(Y,Z)C_{J_2\xi}X\\ &+ h(X,R^{T\perp}(Y,Z)\xi) - A_XR^{\perp\perp}(Y,Z)\xi.\end{aligned}$$

Hence, by passing to the tangential and normal part, we get

$$\text{(c)}\qquad \begin{aligned}(\nabla'_XR^{T\perp})(Y,Z)\xi = &- \left[\widetilde{R}(J_2C_XY,Z)\xi + \widetilde{R}(Y,J_2C_XZ)\xi\right]^T\\ &- R^{TT}(Y,Z)C_{J_2\xi}X + C_XJ_2R^{\perp\perp}(Y,Z)\xi,\end{aligned}$$

$$\text{(d)}\qquad \begin{aligned}(\nabla'_XR^{\perp\perp})(Y,Z)\xi = &- \left[\widetilde{R}(J_2C_XY,Z)\xi + \widetilde{R}(Y,J_2C_XZ)\xi\right]^\perp\\ &- R^{\perp T}(Y,Z)C_{J_2\xi}X + J_2C_XR^{T\perp}(Y,Z)\xi.\end{aligned}$$

(c) is (2.5.3). If we take $\xi = J_2U$, $\eta = J_2V$ then (d) is equivalent to the identity

$$\begin{aligned}\langle(\nabla'_XR^{\perp\perp})(Y,Z)J_2U,J_2V\rangle = &- \langle R(U,V)J_2C_XY,Z\rangle - \langle R(U,V)Y,J_2C_XZ\rangle\\ &- \langle R^{\perp T}(Y,Z)C_{J_2\xi}X,J_2V\rangle + \langle C_XR^{T\perp}(Y,Z)\xi,V\rangle\end{aligned}$$

that is (2.5.4). □

By (2.5.1) we get immediately the following result.

Proposition 2.11. *If the Kähler submanifold $M^{2n}\subset\widetilde{M}^{4n}$ is curvature invariant, i.e. $R^{\perp T}=0$, then the tensor field R^{TT} is parallel,*

$$\nabla R^{TT} = 0\,,$$

and satisfies the identity

$$\text{(2.6)}\qquad \begin{aligned}-C_XR^{TT}(Y,Z) + R^{TT}(Y,Z)C_X + \nu F(Y,Z)J_1C_X\\ = \left[J_2(\widetilde{R}(J_2C_XY,Z) + \widetilde{R}(Y,J_2C_XZ))\right]^{TT}\end{aligned}$$

where $(A)^{TT}$ denotes the $\mathrm{End}(T_xM)$ *component of an endomorphism A of $T_x\widetilde{M}$.*

Denote by $[C, C]$ the $\mathrm{End}(T_x M)$-valued two-form, given by

$$[C, C](X, Y) = [C_X, C_Y] \qquad \forall\, X, Y \in TM.$$

(One can easily check that it is globally defined on M.)

For a subspace $\mathcal{G} \subset \mathrm{End}(T_x M)$ we define the space $\mathcal{R}(\mathcal{G})$ of the *curvature tensors of type* $\mathcal{G}$ by

$$\mathcal{R}(\mathcal{G}) = \{R \in \mathcal{G} \otimes \Lambda^2 T_x^* M \quad | \quad \mathrm{cycl}\, R(X, Y)Z = 0,\ \forall\, X, Y, Z \in T_x M\}$$

where cycl is the sum of cyclic permutations of X, Y, Z.

As another corollary of Proposition 2.10 and Proposition 2.5 1) we have the following result.

Proposition 2.12. *Under the assumptions of* Proposition 2.11 *the tensor field* $[C, C] = R - R^{TT}$ *belongs to the space* $\mathcal{R}(\mathfrak{u}_n)$ *and satisfies the second Bianchi identities*:

$$\mathrm{cycl}\, \nabla_Z [C_X, C_Y] = 0.$$

Proof. The tensor $[C, C]$ satisfies the first Bianchi identity since R and R^{TT} do it. Since $[C_X, C_Y]$ commutes with J and it is skew-symmetric with respect to the metric g, the tensor $[C, C]$ belongs to the space $\mathcal{R}(\mathfrak{u}_n)$ of the $\mathfrak{u}_n-$curvature tensors. The last statement follows from remark that $\nabla R^{TT} = 0$ and that R satisfies the second Bianchi identity. □

As another corollary of Proposition 2.10 we get the following result.

Proposition 2.13. *A maximal Kähler submanifold* M^{2n} *of a locally symmetric quaternionic Kähler manifold* $\widetilde{M}^{4n}$ *is locally symmetric* (*that is* $\nabla R = 0$) *if and only if the following identity holds*:

$$\begin{aligned}\langle \nabla_X\ [C, C]_{Y,Z} U, V\rangle &= \langle R^{\perp T}(Y, Z)U, J_2 C_X V\rangle - \langle R^{\perp T}(Y, Z)V, J_2 C_X U\rangle \\ &\quad + [\langle J_2 R^{\perp T}(U, V) C_X Y, Z\rangle + \langle R^{\perp T}(U, V)Y, J_2 C_X Z\rangle]\,. \end{aligned} \tag{2.7}$$

If, moreover, M *is curvature invariant then* (2.7) *reduces to the condition that the tensor field* $[C, C]$ *is parallel* ($\nabla[C, C] = 0$).

Proof. The proof follows directly from the Gauss-Codazzi equations, see Proposition 2.5(1), and (2.5.1). □

2.4. Maximal totally complex submanifolds of quaternionic space forms

Now *we assume that* $(\widetilde{M}^{4n}, Q, \widetilde{g})$ *is a non flat quaternionic space form,* i.e. a quaternionic Kähler manifold which is locally isometric to the quaternionic projective space

$\mathbb{H}P^n$ or the dual quaternionic hyperbolic space $\mathbb{H}H^n$ with standard metric of reduced scalar curvature ν. Recall that the curvature tensor of $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is given by $\widetilde{R} = \nu R_{\mathbb{H}P^n}$ where

$$R_{\mathbb{H}P^n}(X,Y) = \frac{1}{4}\Big(X \wedge Y + \sum_\alpha J_\alpha X \wedge J_\alpha Y - \sum_\alpha 2\langle J_\alpha X, Y\rangle J_\alpha \Big) .$$

We denote by $R_{\mathbb{C}P^n}$ the curvature tensor of the complex projective space $\mathbb{C}P^n$ (normalized such that the holomorphic curvature is equal to 1):

$$R_{\mathbb{C}P^n}(X,Y) = \frac{1}{4}\Big(X \wedge Y + JX \wedge JY - 2\langle JX, Y\rangle J \Big) .$$

Proposition 2.14. *Let* (M^{2n}, J, g) *be a totally complex submanifold of the quaternionic space form* $\widetilde{M}^{4n}$*. Then*:

(1) $R^{TT}_{XY} = \nu(R_{\mathbb{C}P^n})_{XY} = (\nu/4)(X \wedge Y + J_1 X \wedge J_1 Y - 2\langle J_1 X, Y\rangle J_1)$.

(2) $\mathrm{Ric}(R^{TT}) = (\nu/2)(n+1)g \,,\ g = \widetilde{g}_{|M}$.

(3) $R^{\perp T} = R^{T\perp} = 0$.

(4) $R^{\perp\perp}_{XY} = (\nu/4)(J_2 X \wedge J_2 Y + J_3 X \wedge J_3 Y - 2\langle J_1 X, Y\rangle J_1)$.

Proof. It is a straightforward verification. □

As a consequence of Corollary 2.6 and Proposition 2.14 we get

Proposition 2.15. *Let* M^{2n} *be a Kähler submanifold of a quaternionic space form* $\widetilde{M}^{4n}$ *with reduced scalar curvature* ν*. Then*

$$\mathrm{Ric}_M(X,X) = \frac{\nu}{2}(n+1)g(X,X) - \operatorname{tr} C_X^2 \leq \frac{\nu}{2}(n+1)g(X,X) \,, \qquad X \in T_x M.$$

Moreover the second fundamental form h_x *of* M *at point* $x \in M$ *vanishes if and only if* $(\mathrm{Ric}_M)_x = (\nu/2)(n+1)g$*. In particular* M *is a totally complex totally geodesic submanifold if and only if*

$$\mathrm{Ric}_M = \frac{\nu}{2}(n+1)g \,.$$

From Proposition 2.13 we get

Proposition 2.16. *A maximal Kähler submanifold* (M^{2m}, J, g) *of a non flat quaternionic space form is locally symmetric if and only if the tensor field* $[C, C]$ *is parallel. In particular, any maximal Kähler submanifold with parallel second fundamental form is* (*locally*) *symmetric.*

Proof. It is sufficient to prove only the last statement. Assume that $\nabla' h = 0$. Then $\nabla_X C = \omega(X) JC$ and

$$\begin{aligned}\nabla_X[C, C] &= [\nabla_X C, C] + [C, \nabla_X C] \\ &= \omega(X)\Big([JC, C] + [C, JC]\Big) = 0\end{aligned}$$

since C_Y anticommute with J. $\square$

Conjecture. let (M^{2n}, J, g) be a Kähler manifold. Any tensor field $C \in S_J^{(1)}$ which satisfies conditions

$$1) \qquad \nu(R_{\mathbb{C}P^n})_{XY} = R_{XY} - [C_X, C_Y]$$

and

$$2) \qquad (\nabla_X C)_Y - \omega(X) J \circ C_Y \in S_J^{(2)},$$

where ω is a 1-form such that $d\omega = -\nu F$, defines a totally complex embedding in $\mathbb{H}P^n$.

3. Classification of parallel Kähler submanifolds M^{2n} of $\widetilde{M}^{4n}$

3.1. The parallel cubic line bundle We will assume that $\widetilde{M}^{4n}$ is a quaternionic Kähler manifold with the reduced scalar curvature $\nu \neq 0$ and (M^{2n}, J) is a *parallel totally complex submanifold* of $\widetilde{M}$, that is $\nabla' h = 0$ or, equivalently,

$$P_{XY} := (\nabla_X C)_Y - \omega(X) J \circ C_Y = 0 \qquad X, Y \in TM. \tag{3.1}$$

We will assume that M is not a totally geodesic submanifold, i.e. $h \neq 0$.

By Proposition 2.8 M is a curvature invariant submanifold ($R^{\perp T} = 0$). We denote by $T^{\mathbb{C}}M = T^{1,0}M + T^{0,1}M$ the decomposition of the complexified tangent bundle into holomorphic and antiholomorphic parts and by $T^{*\mathbb{C}}M = T^{*1,0}M + T^{*0,1}M$ the dual decomposition of the cotangent bundle.

Denote by $S_J^{(1)\mathbb{C}}$ the complexification of the bundle $S_J^{(1)}$ (see Corollary 2.2) and by $g \circ S^{(1)\mathbb{C}}$ the associated subbundle of the bundle $S^3(T^*M)^{\mathbb{C}}$. We will call $S^3(T^*M)^{\mathbb{C}}$ the *bundle of cubic forms*.

Proposition 3.1. *Let (M^{2n}, J) be a parallel Kähler submanifold of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ with $\nu \neq 0$. If it is not totally geodesic then on M there is a canonically defined parallel complex line subbundle L of the bundle $S^3(T^{*1,0}M)$ of holomorphic cubic forms such that the curvature of the connection ∇^L induced by the Levi-Civita connection ∇ has the curvature form*

$$R^L = i\nu F, \tag{3.2}$$

where $F = g \circ J$ is the Kähler form of M.

Proof. We first prove the following lemma.

Lemma 3.2. $g \circ S_J^{(1)\mathbb{C}} = S^3 T^{*1,0} M + S^3 T^{*0,1} M$.

Proof. Since $J|T^{1,0}M = i$ and $J|T^{0,1}M = -i$, the space of complex endomorphisms of $T_x^{\mathbb{C}} M$ which anticommute with J is

$$\mathrm{Hom}(T^{1,0}M, T^{0,1}M) + \mathrm{Hom}(T^{0,1}M, T^{1,0}M)\,.$$

Hence the space $g \circ S_J^{\mathbb{C}}$ of symmetric bilinear forms, associated with $S_J^{\mathbb{C}}$ is

$$g \circ S_J^{\mathbb{C}} = S^2(T^{*1,0}M) + S^2(T^{*0,1}M)\,,$$

which proves the lemma. □

Using Lemma 3.2 we can decompose the cubic form $gC \in g \circ S_J^{(1)}$ associated with the shape operator $C = J_2 h$ into holomorphic and antiholomorphic parts:

$$gC = q + \overline{q} \in S^3 T^{*1,0} M + S^3 T^{*0,1} M\,.$$

Since, by assumption,

$$\nabla_X C = \omega(X) J \circ C$$

we have

$$g \nabla_X C = \nabla_X g C = \nabla_X q + \nabla_X \overline{q} = \omega(X) g(J \circ C).$$

For $Y, Z \in T^{1,0}M$, we get

$$\nabla_X (gC)(Y, Z) = \omega(X) g(JC(Y, Z)) = -i\omega(X) gC(Y, Z)$$

since $C(Y, Z) \in T^{0,1}M$ and $JC(Y, Z) = -iC(Y, Z)$. This shows that

$$\nabla_X q = -i\omega(X) q. \tag{3.3}$$

Using (2.3), one check that under the changing of adapted basis $(J_\alpha) \to (J'_\alpha)$ with

$$J'_2 = \cos\theta J_2 - \sin\theta J_3$$

the cubic form q changes by

$$q \to q' = (\cos\theta - \sin\theta i) q\,.$$

Note also that the cubic form $q \neq 0$ at any point, since by assumption the second fundamental form h is parallel and not zero. These show that the complex line bundle $L = \mathrm{span}_{\mathbb{C}}(q) \subset S^3T^{*1,0}$ is globally defined and parallel, i.e. the Levi-Civita connection ∇ preserves L and defines a connection ∇^L in L. Using (3.3), we calculate the curvature of ∇^L as follows:

$$\begin{aligned}
R^L(X,Y)q &= \Big([\nabla^L_X, \nabla^L_Y] - \nabla^L_{[X,Y]}\Big)q \\
&= \Big([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\Big)q \\
&= -\nabla_X\Big(\omega(Y)iq\Big) + \nabla_Y\Big(\omega(X)iq\Big) + \omega([X,Y])iq \\
&= -d\omega(X,Y)iq = \nu F(X,Y)iq\,.
\end{aligned}$$

This proves Proposition 3.1. □

DEFINITION 3.3. A parallel subbundle $L \subset S^3(T^{*1,0}M)$ with the curvature form (3.2) on a Kähler manifold M is called a *parallel cubic line bundle* of type ν.

Corollary 3.4. *A parallel maximal Kähler not totally geodesic submanifold M of a quaternionic Kähler manifold $\widetilde{M}$ with $\nu \neq 0$ has a parallel cubic line bundle L of type ν.*

3.2. de Rham decomposition of Kähler manifolds with parallel cubic line bundle Let M be a complete simply connected Kähler manifold with the de Rham decomposition

$$M = M_0 \times M_1 \times \cdots \times M_p \tag{3.4}$$

into product of the flat Kähler manifold M_0 and of the irreducible Kähler manifolds M_i, $i = 1, \dots, p$.

We will assume that M admits a parallel cubic line bundle L of type $\nu \neq 0$.

Lemma 3.5. *Assume that M admits a parallel cubic line bundle of type $\nu \neq 0$. Then there is no flat factor M_0 in* (3.4).

Proof. If M_0 exists, for a non zero $X \in TM_0$, using (3.2), we get $0 = R_{XJX} = R^L_{XJX} = \nu F(X, JX) \neq 0$. Contradiction. □

Denote by $\mathfrak{h} = \mathfrak{h}_1 + \cdots + \mathfrak{h}_p$ the direct sum decomposition of the holonomy Lie algebra $\mathfrak{h}$ at a point $x \in M$ associated to the de Rham decomposition. Then the commutator $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ is a semisimple Lie algebra with the direct sum decomposition

$$\mathfrak{h}' = \mathfrak{h}'_1 + \cdots + \mathfrak{h}'_p\,, \qquad \mathfrak{h}'_i = [\mathfrak{h}_i, \mathfrak{h}_i]\,.$$

REMARK 3.6. The subalgebra $\mathfrak{h}_i' = 0$ if $\dim_{\mathbb{R}} M_i = 2$ and $\mathfrak{h}_i'$ acts irreducibly on the holomorphic tangent space $T_x^{1,0}M_i$ if $\dim_{\mathbb{R}} M_i > 2$. (See the Table 1 below.)

Let $V = T_x^{1,0}M = V_1 + \cdots + V_p$ be the decomposition of the holomorphic tangent space associated to the de Rham decomposition. Then the space S^3V^* of cubic forms on V has $\mathfrak{h}$-invariant decomposition

$$S^3V^* = \sum_{i,j,k=1}^{p} V_i^* V_j^* V_k^* \tag{3.5}$$

where for simplicity $V_i^* V_j^* V_k^*$ denotes the symmetric tensor product and the subalgebra $\mathfrak{h}_i$ acts non trivially only on V_i^*.

Remark that if L is a parallel line bundle the commutator $\mathfrak{h}'$ acts trivially on its fiber $L_x \subset S^3V^*$, that is,

$$L_x \subset (S_3V^*)^{\mathfrak{h}'} \tag{3.6}$$

where we always denote by $V^{\mathfrak{h}}$ the subspace of $\mathfrak{h}$-invariant vectors of an $\mathfrak{h}$-module V. Denote by $q = \sum q_{ijk}$, $q_{ijk} \in V_i^* V_j^* V_k^*$ the decomposition of a non zero element $q \in L_x$.

Lemma 3.7. *If $q_{ijk} \neq 0$, then the set $\{i, j, k\}$ contains all indexes $\{1, \ldots, p\}$.*

Proof. Assume that $l \notin \{i, j, k\}$ and take vectors $X, Y \in T_xM_l$ such that $F(X, Y) = 1$. Since the curvature operator R_{XY} acts trivially on V_i, V_j, V_k and preserves the decomposition (3.4) this contradicts to the identity (3.2). □

Corollary 3.8. 1) $p \leq 3$.
2) *If $p = 3$, then $L_x \subset V_1^* V_2^* V_3^*$ and* $\dim_{\mathbb{C}} V_i = 1$, $i = 1, 2, 3$.
3) *If $p = 2$, then $L_x \subset V_1^{*2} V_2^* + V_1^* V_2^{*2}$ and one of the spaces V_1, V_2 has dimension* 1.

Proof. It is sufficient to prove statement about dimension. Let $p = 3$, then

$$L_x \subset (V_1^* V_2^* V_3^*)^{\mathfrak{h}'} = (V_1^*)^{\mathfrak{h}_1'}(V_2^*)^{\mathfrak{h}_2'}(V_3^*)^{\mathfrak{h}_3'} \neq 0\,.$$

By Remark 3.6 $(V_i^*)^{\mathfrak{h}_i'} \neq 0$ if and only if $\dim_{\mathbb{C}} V_i = 1$ and hence $\mathfrak{h}_i' = 0$. The proof in case $p = 2$ is similar. □

This implies the following proposition.

Proposition 3.9. *Let M be a simply connected complete Kähler manifold with parallel cubic line bundle L of type $\nu \neq 0$. If M is reducible, then either*

$$M = M_1 \times M_2 \times M_3, \qquad \dim_{\mathbb{R}} M_i = 2 \qquad (i = 1, 2, 3) \tag{3.7.1}$$

or

$$M = M_1 \times M_2 \tag{3.7.2}$$

where M_1 is an irreducible Kähler manifold and $\dim_{\mathbb{R}} M_2 = 2$.

The following Proposition specifies the structure of such a reducible manifold M.

Proposition 3.10. *Let M be a simply connected, complete, reducible Kähler manifold with parallel cubic line bundle L of type $\nu \neq 0$. Then either*

$$M = M_\nu^2 \times M_\nu^2 \times M_\nu^2 \tag{3.8.1}$$

where $M_\nu^2 (= \mathbb{C}P^1$ or $\mathbb{C}H^1)$ is the 2-dimensional manifold of constant curvature ν, or

$$M = M_1 \times M_\nu^2 \tag{3.8.2}$$

where M_1 is a complete simply connected irreducible Kähler-Einstein manifold with $\mathrm{Ric}_{M_1} = \nu(m/2)g_{(1)}$, *where* $\dim M_1 = 2m$, *such that*

$$(S^2V^*)^{\mathfrak{h}'_1} \neq 0$$

where $\mathfrak{h}'_1$ is the commutator of the holonomy Lie algebra $\mathfrak{h}_1$ of M_1 at a point $x \in M_1$ and $V = T_x^{1,0}M_1$.

Conversely, any manifold M given by (3.8.1) *or* (3.8.2) *has a parallel cubic line bundle of type ν.*

Proof. First we consider the case (3.8.1). Denote by $g_{(i)}, J_{(i)}, F_{(i)} = g_{(i)} \circ J_{(i)}$ respectively the metric, the complex structure and the Kähler form of the de Rham factor M_i of M, $i = 1, 2, 3$. Then $J = \sum J_{(i)}$ and $F = \sum_i F_{(i)}$. Denote by z_i a basis of the holomorphic cotangent space $T_{x_i}^{*1,0} M_i \cong \mathbb{C}$ at $x_i \in M_i$. Then at the point $x = (x_1, x_2, x_3)$ the fiber of the line bundle L is given by $L_x = \mathbb{C}q$, $q = z_1 z_2 z_3$. The curvature operator $R_{XY}^{(i)}$, $X, Y \in T_{x_i}M_i$ of M_i is given by

$$R_{XY}^{(i)} = -k_{(i)} F_{(i)}(X, Y) J_{(i)} = -k_{(i)} F(X, Y) J_{(i)} \qquad i = 1, 2, 3$$

where $k_{(i)}$ is the curvature of M_i. Using this, we obtain from (3.2)

$$R_{XY}^L q = R_{XY}^{(i)}(z_1 z_2 z_3) = -k_{(i)} F(X, Y) J_{(i)}(z_1 z_2 z_3)$$

$$= -k_{(i)}F(X,Y)(J_{(i)}z_i)(z_jz_k) = k_{(i)}F(X,Y)iq$$
$$= \nu F(X,Y)iq$$

where $i = 1,2,3$ and (i,j,k) is a cyclic permutation of $(1,2,3)$. Hence, $k_{(i)} = \nu = const$ and $M_i = M^2_\nu$. Conversely it is clear that (3.8.1) has parallel cubic line bundle of type ν generated by $q = z_1z_2z_3$.

Now we consider the case (3.8.2). We will assume that $\dim_{\mathbb{R}} M_1 > 2$. (The case $\dim_{\mathbb{R}} M_1 = \dim_{\mathbb{R}} M_2 = 2$ can be treated similarly.) Then Corollary 3.8 3) shows that at any point $x \in M$ the fiber $L_x = \mathbb{C}q = \mathbb{C}(bz_2)$ where z_2 is a basis of the holomorphic cotangent space $T^{*1,0}_x M_2 \cong \mathbb{C}$ and $b \in (S^2V^*)^{\mathfrak{h}'}$ is an $\mathfrak{h}'$-invariant symmetric bilinear form on $V_1 = T^{1,0}_x M_1$. The same calculation as before shows that (3.2) is equivalent to the following conditions

(1) the curvature $k_{(2)}$ of M_2 is equal to ν, $k_{(2)} = \nu$

(2) $R^{(1)}_{XY} = -(\nu/2)F(X,Y)J_{(1)} + R'_{(1)XY}$

where $R^{(1)}_{XY}$ is the curvature operator of M_1 and $R'_{(1)XY}$ is its projection on $\mathfrak{h}'_1 = [\mathfrak{h}_1,\mathfrak{h}_1]$. Now, since $2\,\mathrm{Ric}_{M_1}(X, J_{(1)}Y) = -\operatorname{tr} J_{(1)}R^{(1)}_{XY} = -(\nu/2)F(X,Y)(2m)$ where $2m = \dim_{\mathbb{R}} M_1$, the condition (2) is equivalent to the condition

(2') $\mathrm{Ric}_{M_1} = \nu(m/2)g_{(1)}$.

The converse statement is clear now. □

REMARK 3.11. The above proof still works if we drop the assumption of completeness and leads to the conclusion that M will be locally isomorphic to (3.8.1) or (3.8.2).

The next proposition gives necessary and sufficient conditions for an irreducible Kähler manifold to admit a parallel cubic line bundle of type $\nu \neq 0$.

Proposition 3.12. *A complete simply connected irreducible Kähler manifold M^{2n} with holonomy Lie algebra $\mathfrak{h}$ at a point x admits a parallel cubic line bundle of type ν if and only if it is Kähler-Einstein with*

$$\mathrm{Ric}_M = \frac{\nu}{3}ng \tag{3.9}$$

and

$$(S^3V^*)^{\mathfrak{h}'} \neq 0$$

where $V = T^{1,0}_x M$ is the holomorphic tangent space with the natural action of the Lie algebra $\mathfrak{h}' = [\mathfrak{h},\mathfrak{h}]$.

Proof. The proof is similar to the reducible case. The condition $(S^3V^*)^{\mathfrak{h}'} \neq 0$ is equivalent to the existence of a parallel cubic line bundle $L \subset S^3T^{*1,0}M$ (which

Table 1.
List of holonomy of irreducible Kähler manifolds M^{2n}

n^0.	H	$\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$	$T^{1,0}$	$\frac{G}{H}$	$n = \dim_{\mathbb{C}} M$
1	$S(U_p \times U_q)$	$\mathfrak{su}_\mathbf{p} + \mathfrak{su}_\mathbf{q}$	$V(\pi_1) \otimes V(\pi_1)$	$\frac{SU_{p+q}}{S(U_p \times U_q)} = \mathrm{Gr}_p(\mathbb{C}^{p+q})$	pq
2	$SO_2 \cdot SO_m$	$\mathfrak{so}_\mathbf{m}$	$V(\pi_1) = \mathbb{C}^m$	$\frac{SO_{2+m}}{SO_2 \cdot SO_m} = \mathrm{Gr}_2(\mathbb{R}^{2+m})$	m
3	U_{l+1}	$\mathfrak{su}_{l+1}$	$V(2\pi_1) = S^2\mathbb{C}^{l+1}$	$\frac{Sp_{l+1}}{U_{l+1}}$	$\frac{(l+1)(l+2)}{2}$
4	U_{l+1}	$\mathfrak{su}_{l+1}$	$V(\pi_2) = \Lambda^2\mathbb{C}^{l+1}$	$\frac{SO_{2(l+1)}}{U_{l+1}}$	$\frac{l(l+1)}{2}$
5	$SO_2 \cdot SO_{10}$	$\mathfrak{so}_{10}$	$V(\pi_5) = \mathbb{C}^{16}$	$\frac{E_6}{\mathrm{Spin}_{10} \cdot SO_2}$	16
6	$T^1 \cdot E_6$	$\mathfrak{e}_6$	$V(\pi_1) = \mathbb{C}^{27}$	$\frac{E_7}{T^1 \cdot E_6}$	27
7*	U_{l+1}	$\mathfrak{su}_{l+1}$	$V(\pi_1) = \mathbb{C}^{l+1}$	$\frac{SU_{l+2}}{U_{l+1}} = \mathbb{C}P^{l+1}$	$l+1$
8*	SU_{l+1}	$\mathfrak{su}_{l+1}$	$V(\pi_1) = \mathbb{C}^{l+1}$	$---$	$l+1$
9*	Sp_l	$\mathfrak{sp}_l$	$V(\pi_1) = \mathbb{C}^{2l}$	$---$	$2l$

We indicate by * the groups H which are holonomy groups of non-symmetric irreducible Kähler manifolds.

is obtained by parallel translation of the line $L_x = \mathbb{C}q$, $0 \neq q \in (S^3V^*)^{\mathfrak{h}'})$ and the Kähler-Einstein condition means that the curvature R^L of the induced connection ∇^L satisfies (3.2). $\square$

Propositions 3.10 and 3.12 reduce the classification of Kähler manifolds with parallel cubic line bundle to the determination of the irreducible holonomy Lie algebras $\mathfrak{h}$ of Kähler manifolds such that the representation of $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ in the holomorphic tangent space $V = T_x^{1,0}M$ has non trivial invariant quadratic or cubic form, i.e. such that

$$S^2(V^*)^{\mathfrak{h}'} \neq 0 \qquad \text{or} \qquad S^3(V^*)^{\mathfrak{h}'} \neq 0\,.$$

We give such description in the next subsection.

3.3. Quadratic and cubic invariants of the holonomy representation $\mathfrak{h}'$ on $V = T_x^{1,0}M$ In the previous Table 1 we give the list of all irreducible holonomy groups H of simply connected Kähler manifolds M. We indicate also the semisimple part $\mathfrak{h}'$ of the holonomy Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ and its representation in the holomorphic tangent space $V = T_x^{1,0}$ and the compact Hermitian symmetric space G/H with holonomy group H if it exists.

For $\mathfrak{h}'$-modules we use notations according to [17] and denote by $\pi_1, \ldots, \pi_l$ the fundamental weights associated with simple roots $\alpha_1, \ldots, \alpha_l$ of the Lie algebra $\mathfrak{h}'$. We denote by $V(\lambda)$ the irreducible $\mathfrak{h}'$-module with highest weight λ. In particular $V(\pi_1)$ is the simplest representation of a simple Lie algebra $\mathfrak{h}'$.

The following proposition describes all irreducible Kähler manifolds which admit a parallel quadratic or cubic line bundle.

Proposition 3.13. *Let M be a simply connected compact irreducible Kähler manifold with holonomy Lie algebra $\mathfrak{h}$ such that the semisimple part $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ has a non trivial quadratic or cubic invariant, i.e. $S^2(V^*)^{\mathfrak{h}'} \neq 0$ or $S^3(V^*)^{\mathfrak{h}'} \neq 0$, where $V = T_x^{1,0}M$. Then M is one of the following Hermitian symmetric spaces from* Table 1.

I. $\underline{S^2(V^*)^{\mathfrak{h}'} \neq 0}$

$\boxed{n^0 1 \qquad (p = q = 2) \qquad M^8 = \mathrm{Gr}_2(\mathbb{C}^4) = \dfrac{SU_4}{S(U_2 \times U_2)}}$

$V = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathrm{Mat}_2(\mathbb{C})$ *is the* $\mathfrak{h}' = (\mathfrak{su}_2 + \mathfrak{su}_2)$*-module with the action* $\mathfrak{h}' \ni (A, B) : X \to AX + XB^t$;

$$S^2(V^*)^{\mathfrak{h}'} = \mathbb{C}d \ , \qquad d(X) = \det X \, .$$

$\boxed{n^0 2 \qquad M^{2m} = \mathrm{Gr}_2(R^{m+2}) = \dfrac{SO_{2+m}}{SO_2 \times SO_m}}$

$V = \mathbb{C}^m$ *is the standard* $\mathfrak{h}' = \mathfrak{so}_m$*-module*;

$$S^2(V^*)^{\mathfrak{h}'} = \mathbb{C}g \quad \text{where } g \text{ is the complex Euclidean metric.}$$

REMARK. It is known that $\mathrm{Gr}_2(\mathbb{C}^4) \cong \mathrm{Gr}_2(\mathbb{R}^6)$.

$\boxed{n^0 3 \qquad (l = 1) \qquad M^6 = \dfrac{Sp_2}{U_2}}$

$V = S^2\mathbb{C}^2 = \mathrm{Mat}_2^{\mathrm{Sym}}(\mathbb{C})$ *is the* $\mathfrak{h}' = \mathfrak{su}_2$*-module with the action* $\mathfrak{h}' \ni A : X \mapsto AX + XA^t$;

$$S^2(V^*)^{\mathfrak{h}'} = \mathbb{C}d \, , \qquad d(X) = \det X \, .$$

II. $\underline{S^3(V^*)^{\mathfrak{h}'} \neq 0}$

$\boxed{n^0 1 \qquad (p = q = 3) \qquad M^{18} = \mathrm{Gr}_3(\mathbb{C}^6) = \dfrac{SU_6}{S(U_3 \times U_3)}}$

$V = \mathbb{C}^3 \otimes \mathbb{C}^3 = \mathrm{Mat}_3(\mathbb{C})$ *is the* $\mathfrak{h}' = \mathfrak{su}_3 + \mathfrak{su}_3$*-module with the action* $\mathfrak{h}' \ni (A, B) : X \mapsto AX + XB^t$;

$$S^3(V^*)^{\mathfrak{h}'} = \mathbb{C}d \, , \quad d(X) = \det X \, .$$

$$\boxed{n^0 3 \quad (l=2) \qquad M^{12} = \frac{Sp_3}{U_3}}$$

$V = S^2\mathbb{C}^3 = \mathrm{Mat}_3^{\mathrm{Sym}}(\mathbb{C})$ *is* $\mathfrak{h}' = \mathfrak{su}_3$*-module with the action* $\mathfrak{h}' \ni A : X \mapsto AX + XA^t$;

$$S^3(V^*)^{\mathfrak{h}'} = \mathbb{C}d\,, \qquad d(X) = \det X\,.$$

$$\boxed{n^0 4 \quad (l=5) \qquad M^{30} = \frac{SO_{12}}{U_6}}$$

$V = \Lambda^2\mathbb{C}^6 = \mathrm{Mat}_6^{\mathrm{skew}}(\mathbb{C})$ *is* $\mathfrak{h}' = \mathfrak{su}_6$*-module with the action* $\mathfrak{h}' \ni A : X \mapsto AX + XA^t$;

$$S^3(V^*)^{\mathfrak{h}'} = \mathbb{C}Pf\,, \quad Pf(X) = \text{pfaffian of } X\,.$$

$$\boxed{n^0 6 \qquad M^{54} = \frac{E_7}{T^1 \cdot E_6}}$$

$V = \mathbb{C}^{27} = \mathrm{Herm}_3(\mathbb{O})$ (*Hermitian matrices of order* 3 *over the octonians*) *is a* $\mathfrak{h}' = \mathfrak{e}_6$*-module* (*see* [17]);

$$S^3(V^*)^{\mathfrak{h}'} = \mathbb{C}d\,, \quad d(X) = \det(X).$$

Proof. Using criterion for existence of a symmetric bilinear invariant for an irreducible representation $V(\lambda)$ of a semisimple Lie algebra $\mathfrak{h}'$ (see [17], pp. 195–196) we get that only the following Lie algebras $\mathfrak{h}'$ from Table 1 have such invariant:

$$n^o\,1) \quad \text{for} \quad p=q=2, \qquad n^o\,2), \qquad n^o\,3) \quad \text{for} \quad l=1.$$

Now the proof is straightforward. To prove the second statement, we remark that for any two irreducible $\mathfrak{h}'$-modules U, V we have $(U \otimes V)^{\mathfrak{h}'} = \mathrm{Hom}(U^*, V)^{\mathfrak{h}'}$. In particular, if $S^3(V^*)^{\mathfrak{h}'} \neq 0$ then

$$0 \neq (V^* \otimes S^2V^*)^{\mathfrak{h}'} = \mathrm{Hom}(V, S^2V^*)^{\mathfrak{h}'}.$$

This implies that the $\mathfrak{h}'$-module S^2V^* has an irreducible submodule isomorphic to V. The decomposition of S^2V^* into irreducible submodules for all $\mathfrak{h}'$-modules V from Table 1 is described in [17]. They are the following:

$n^o\,1)$ $V = V(\pi_1) \otimes V(\pi_1)$, $V^* = V(\pi_{p-1}) \otimes V(\pi_{q-1})$, $S^2(V^*) = V(2\pi_{p-1}) \otimes V(2\pi_{q-1}) + V(\pi_{p-2}) \otimes V(\pi_{q-2})$;

$V \subset S^2V^* \Leftrightarrow p = q = 3$

$n^o\,2)$ $V = V(\pi_1) = V^*$, $S^2(V^*) = V(2\pi_1) + \mathbb{C}g$ (g is the Euclidean metric)

$V \not\subseteq S^2V^*$

n^o 3) $V = V(2\pi_1)$, $V^* = V(2\pi_l)$, $S^2(V^*) = V(4\pi_l) + V(2\pi_{l-1})$
$V \subset S^2V^* \Leftrightarrow l = 2$
n^o 4) $V = V(\pi_2)$, $V^* = V(\pi_{l-1})$, $S^2(V^*) = V(2\pi_{l-1}) + V(\pi_{l-3})$
$V \subset S^2(V^*) \Leftrightarrow l = 5$
n^o 5) $V = V(\pi_5)$, $V^* = V(\pi_4)$; $S^2(V^*) = V(2\pi_4) + V(\pi_1)$
$V \not\subseteq S^2(V^*)$
n^o 6) $V = V(\pi_1)$, $V^* = V(\pi_5)$; $S^2(V^*) = V(2\pi_5) + V(\pi_1)$
$V \subset S^2(V^*)$
n^o 7 – 8) $V = V(\pi_1)$, $V^* = V(\pi_l)$, $S^2(V^*) = V(2\pi_l)$
$V \not\subseteq S^2(V^*)$
n^o 9) $V = V(\pi_1) = V^*$, $S^2(V^*) = V(2\pi_1)$
$V \not\subseteq S^2V^*$.
It follows that $V \subset S^2(V^*)$ only in the cases:

$$n^o\ 1)\quad p = q = 3,\quad n^o\ 3)\quad l = 2,\quad n^o\ 4)\quad l = 5,\quad n^o 6).$$

We can easily describe the cubic invariant in all these cases as it was stated in the proposition. □

Propositions 3.10, 3.12, 3.13 imply the following theorem.

Theorem 3.14. *Let M^{2n} be a simply connected complete Kähler manifold which admits a parallel cubic line bundle of type $\nu \neq 0$.*

If $\nu > 0$ then M^{2n} is one of the Hermitian symmetric spaces described in Table 2, *where also the representation of the semisimple part $\mathfrak{h}'$ of the holonomy Lie algebra $\mathfrak{h}$ of M on the holomorphic tangent space $V = T_x^{1,0}M$ is given as well as the description of the fiber of line bundle L. The metric of irreducible M^{2n} is normalized such that the scalar curvature $K = 2/3\nu n^2$. The metric of $P := \mathbb{C}P^1 = SU_2/T^1$ has constant curvature ν. If $M = M_1^{2(n-1)} \times \mathbb{C}P^1$ then the metric of M_1 is normalized such that the scalar curvature $K_1 = \nu(n-1)^2$.*

If $\nu < 0$ then M^{2n} is the (non compact) dual space of one of the symmetric spaces of Table 2.

3.4. Classification of parallel totally complex not totally geodesic submanifolds M^{2n} of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ Let M^{2n} be a parallel totally complex not totally geodesic submanifold of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ with the reduced scalar curvature $\nu \neq 0$. Then by Theorem 1.8 M is Kähler and by Proposition 3.1 and Corollary 3.4 it has a canonically defined parallel cubic line bundle of type ν. By applying Theorem 3.14 we get the following theorem.

Theorem 3.15. *Let M^{2n} be a simply connected complete parallel totally complex not totally geodesic submanifold of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ with reduced*

Table 2.
List of simply connected Kähler manifolds M^{2n}
with parallel cubic line bundle L of type $\nu > 0$

Case of reducible M^{2n}

n^0	n	M^{2n}	$\mathfrak{h}'$	$T^{1,0}$	L
I	n	$\frac{SO_{n+1}}{SO_2 \cdot SO_{n-1}} \times P$	$\mathfrak{so}_{\mathbf{n-1}} + \mathbb{R}$	$V(\pi_1) \oplus \mathbb{C}$	$g_{SO_{n-1}} \cdot T^{1,0}(P^1)$
1	2	$P \times P'$	0	$\mathbb{C} \oplus \mathbb{C}$	$S^2 T^{1,0}(P) \cdot T^{1,0}(P')$
2	3	$P \times P' \times P''$	0	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$T^{1,0}(P) \cdot T^{1,0}(P') \cdot T^{1,0}(P'')$
3	4	$\frac{Sp_2}{U_2} \times P$	$\mathfrak{su}_{\mathbf{2}}$	$S^2\mathbb{C}^2 \otimes \mathbb{C}$	$(\mathbb{C}\det) \cdot \mathbb{C}$

Case of irreducible M^{2n}

n^0	n	M^{2n}	$\mathfrak{h}'$	$T^{1,0}$	L
4	1	P	0	$\mathbb{C}$	$S^3\mathbb{C}$
5	6	$\frac{Sp_3}{U_3}$	$\mathfrak{su}_{\mathbf{3}}$	$V(2\pi_1) = S^2 V(\pi_1)$	$\mathbb{C}\det$
6	9	$\frac{SU_6}{S(U_3 \times U_3)}$	$\mathfrak{su}_{\mathbf{3}} + \mathfrak{su}_{\mathbf{3}}$	$V(\pi_1) \otimes V(\pi_1)$	$\mathbb{C}\det$
7	15	$\frac{SO_{12}}{U_6}$	$\mathfrak{su}_{\mathbf{6}}$	$V(\pi_1) = \Lambda^2 V(\pi_1)$	$\mathbb{C}\mathrm{pf}$
8	27	$\frac{E_7}{T^1 \cdot E_6}$	$\mathfrak{e}_{\mathbf{6}}$	$V(\pi_1)=\mathbb{C}^{27}$ $=\mathrm{Herm}_3(\mathbb{O})$	$\mathbb{C}\det$

where P, P', P'' denote copies of $\mathbb{C}P^1$ and dot means the symmetric product.

REMARK. *$n^0 3$ of reducible case is isomorphic to $n^0 I$ for $n = 4$.*

scalar curvature ν. Assume that $\nu > 0$. Then M^{2n} is one of the compact Hermitian symmetric spaces described in Table 2 *and the scalar curvature of each factor is described as in previous theorem. For $\nu < 0$ the submanifold M^{2n} is one of the dual symmetric spaces.*

Corollary 3.16. *Under the hypothesis of the theorem assume that the complex dimension n of the parallel totally complex submanifold $M^{2n} \subset \widetilde{M}^{4n}$ is different from* 1, 2, 3, 4, 6, 9, 15, 27*. Then M is isometric to the compact symmetric space $SO_{n+1}/(SO_2 \cdot SO_{n-1}) \times \mathbb{C}P^1$ or its non compact dual.*

Remark that Tsukada constructed the explicit realization of all these manifolds as parallel totally complex submanifolds in the quaternionic projective space $\mathbb{H}P^n$ ([20]). On the other hand he proved that in dual hyperbolic quaternionic space $\mathbb{H}H^n$, $n \geq 2$, any parallel totally complex submanifold M^{2n} is totally geodesic (see [20] Th. 7.2). The problem of realization of these manifolds as parallel totally complex submanifolds in other Wolf spaces is still open. For this purpose let us consider the following results.

4. Curvature invariant Kähler submanifolds in a quaternionic Kähler symmetric space

Let $\widetilde{M}^{4n}$ be a locally symmetric quaternionic Kähler manifold with non zero scalar curvature, $\nu \neq 0$.

We will prove some result on non existence of non totally geodesic curvature invariant Kähler submanifold M^{2n} in the manifold $\widetilde{M}^{4n}$ which is not a quaternionic space form.

We need the following lemma.

Lemma 4.1. *On a curvature invariant Kähler submanifold M^{2n} of a locally symmetric quaternionic Kähler manifold $\widetilde{M}^{4n}$ the following identity holds*:

$$
\begin{aligned}
&2\langle \widetilde{R}(J_2C_XY, JY)U, J_2V\rangle = \\
&\Big\langle \Big[C_X R^{TT}(Y, JY) - R^{TT}(Y, JY)C_X - \nu\|Y\|^2 JC_X\Big]U, V\Big\rangle \\
&-\nu\Big(\langle JU, V\rangle\langle C_XY, Y\rangle + \langle U, V\rangle\langle JC_XY, Y\rangle\Big) \qquad \forall X, Y, U, V \in TM.
\end{aligned} \tag{4.1}
$$

Proof. By using the usual properties of the curvature tensor of a Riemannian manifold, the anticommutation property of J with any C_X and J_2, and finally (1.4) we have the following identities:

$$
\begin{aligned}
&\langle \widetilde{R}(Y, J_2C_XJY)U, J_2V\rangle = \langle \widetilde{R}(U, J_2V)Y, J_2C_XJY\rangle = \\
&\langle \widetilde{R}(U, J_2V)Y, JJ_2C_XY\rangle = -\langle \widetilde{R}(U, J_2V)JJ_2C_XY, Y\rangle = \\
&-\langle J\widetilde{R}(U, J_2V)J_2C_XY, Y\rangle + \nu\Big(\langle F_3(U, J_2V)J_2^2C_XY - F_2(U, J_2V)J_3J_2C_XY, Y\rangle\Big) = \\
&-\langle J\widetilde{R}(U, J_2V)J_2C_XY, Y\rangle + \nu\Big(\langle JU, V\rangle\langle C_XY, Y\rangle + \langle U, V\rangle\langle JC_XY, Y\rangle\Big) = \\
&\langle \widetilde{R}(U, J_2V)J_2C_XY, JY\rangle + \nu\Big(\langle JU, V\rangle\langle C_XY, Y\rangle + \langle U, V\rangle\langle JC_XY, Y\rangle\Big).
\end{aligned}
$$

That is:

$$
\begin{aligned}
&\langle \widetilde{R}(Y, J_2C_XJY)U, J_2V\rangle = \\
&\langle \widetilde{R}(J_2C_XY, JY)U, J_2V\rangle + \nu\Big(\langle JU, V\rangle\langle C_XY, Y\rangle + \langle U, V\rangle\langle JC_XY, Y\rangle\Big).
\end{aligned}
$$

On the other hand (2.6) for $Z = JY$ is equivalent to the identity

$$
\begin{aligned}
&\langle \widetilde{R}(J_2C_XY, JY)U, J_2V\rangle + \langle \widetilde{R}(Y, J_2C_XJY)U, J_2V\rangle = \\
&\Big\langle \Big[C_X R^{TT}(Y, JY) - R^{TT}(Y, JY)C_X - \nu\|Y\|^2 JC_X\Big]U, J_2V\Big\rangle.
\end{aligned}
$$

Then by substituting the previous identity in this last one (4.1) follows immediately. □

Now we prove the following result.

Theorem 4.2. *Let M^{2n}, $n > 1$, be a curvature invariant, Kähler submanifold of the locally symmetric quaternionic Kähler manifold $\widetilde{M}^{4n}$. Assume that M^{2n} has the holonomy group SU_n or U_n. Then*

$$R^{TT} = \alpha R_{\mathbb{C}P^n}$$

for some real number α and one of the following possibilities holds.
(1) $\alpha \neq \nu$: *then M^{2n} is totally geodesic ($C \equiv 0$),*
(2) $\alpha = \nu$: *then for any $x \in M^{2n}$ one has the identity*

$$\langle \widetilde{R}(J_2C_XY, JY)U, \xi\rangle = \nu\langle R_{\mathbb{H}P^n}(J_2C_XY, JY)U, \xi\rangle \quad \forall X, Y, U \in T^xM, \xi \in T_xM^{\perp}.$$

Moreover, if for some $x \in M^{2n}$ there exists a vector $X \in T_xM$ such that C_X is non degenerate, one has $\widetilde{R}_x = (\nu R_{\mathbb{H}P^n})_x$ and $\widetilde{M}^{4n}$ is a quaternionic space form.

Proof. STEP 1. By proposition 2.11 the $\mathfrak{u}_n$-curvature tensor field $R^{TT} = R - [C, C] \in \mathcal{R}(\mathfrak{u}_n)$ is parallel, hence invariant under the holonomy group SU_n or U_n. Since the SU_n-invariants in $\mathcal{R}(\mathfrak{u}_n)$ are spanned by $R_{\mathbb{C}P}$, we get

$$R^{TT} = \alpha R_{\mathbb{C}P^n}$$

for some constant α.

STEP 2. The (4.1) becomes

$$\begin{aligned}2\langle \widetilde{R}(J_2C_XY, JY)U, J_2V\rangle = {} & (\alpha - \nu)\|Y\|^2\langle JC_XU, V\rangle \\ & +\frac{\alpha}{2}\Big[\langle JY, U\rangle\langle C_XY, V\rangle + \langle Y, U\rangle\langle JC_XY, V\rangle \\ & + \langle JC_XY, U\rangle\langle Y, V\rangle + \langle C_XY, U\rangle\langle JY, V\rangle\Big] \\ & -\nu\Big[\langle C_XY, Y\rangle\langle JU, V\rangle + \langle JC_XY, Y\rangle\langle U, V\rangle\Big].\end{aligned} \tag{4.2}$$

Let $\nu \neq \alpha$ and assume that $C \neq 0$ at a point $x \in M^{2n}$. If there exist non zero vectors $X, Y \in T_xM$ such that $C_XY = 0$ and hence

$$(\alpha - \nu)\|Y\|^2\langle JC_XU, V\rangle = 0 \qquad \forall U, V \in T_xM,$$

it implies $C_XU = 0$, $\forall U \in T_xM$. Since by exchanging X with U one gets $C_XU = 0$, $\forall X, U \in T_xM$, that is $C = 0$, there is a contradiction. On the other hand, let C_X be non singular for any non zero vector $X \in T_xM$. Then for any fixed non zero vector $Y \in T_xM$ and arbitrary $\xi \in T_xM^{\perp}$ one can compute

$$2\langle \widetilde{R}(\xi, JY)JY, \xi\rangle = \left(\frac{3}{2}\alpha - \nu\right)\|Y\|^2\|\xi\|^2 + \left(\nu + \frac{\alpha}{2}\right)\left[\langle J_2\xi, Y\rangle^2 + \langle J_2\xi, JY\rangle^2\right] \tag{4.3}$$

since ξ can be always written as $\xi = J_2C_XY$ for some $X \in T_xM$. Let us also take into account that for any $W \in T_xM$ one has

$$\mathrm{Ric}(\widetilde{R})(W, W) = \mathrm{Ric}(R^{TT})(W, W) + \sum_i^{2n}\langle \widetilde{R}(J_2E_i, W)W, J_2E_i\rangle$$

where (E_i) is an orthonormal basis of T_xM. That is

$$\mathrm{Ric}(\widetilde{R})(W, W) = \alpha\,\mathrm{Ric}(R_{\mathbb{C}P^n})(W, W) + \sum_i^{2n}\langle \widetilde{R}(J_2E_i, W)W, J_2E_i\rangle.$$

Moreover one has

$$\begin{aligned}\mathrm{Ric}(R_{\mathbb{C}P^n})(W, W) &= \frac{n+1}{2}\|W\|^2,\\ \mathrm{Ric}(\widetilde{R})(W, W) &= \mathrm{Ric}(R_{\mathbb{H}P^n})(W, W) = (n+2)\|W\|^2\end{aligned}$$

and, by (4.3),

$$2\sum_i\langle \widetilde{R}(J_2E_i, W)W, J_2E_i\rangle = \Big[(3n+1)\alpha - 2(n-1)\nu\Big]\|W\|^2.$$

Hence the previous identity reduces to

$$(n+2)\nu\|W\|^2 = \left[\alpha\left(\frac{n+1}{2}\right) + \left(\frac{3n+1}{2}\right)\alpha - \frac{2(n-1)}{2}\nu\right]\|W\|^2$$

that is $\alpha = \nu$, which gives again a contradiction.

Hence, $\nu \neq \alpha$ implies that M^{2n} is totally geodesic. It remains to study the case $\nu = \alpha$.

STEP 3. In case $\nu = \alpha$ one has the identity

$$\langle \widetilde{R}(J_2C_XY, JY)U, J_2V\rangle = \nu\langle R_{\mathbb{H}P^n}(J_2C_XY, JY)U, J_2V\rangle.$$

It remains to show that in fact if there exist a point $x \in M^{2n}$ and a vector $X \in T_xM$ such that C_X is non degenerate then the identity obtained by putting W instead of C_XY,

$$\langle \widetilde{R}(J_2W, JY)U, J_2V\rangle = \nu\langle R_{\mathbb{H}P^n}(J_2W, JY)U, J_2V\rangle$$

holds. This is easily proved by using the fact that if C_X is non degenerate for some X then this is true for vectors on an open neighbourhood of X. Then it follows that

$$\widetilde{R}_x = \nu(R_{\mathbb{H}P^n})_x .$$

In this case $\widetilde{M}^{4n}$, which is assumed to be symmetric, is locally isometric to a quaternion space form of reduced scalar curvature ν. □

ACKNOWLEDGEMENT. The authors would like to thank heartily the referee for very careful reading of the manuscript and for pointing out misprints and some mistakes.

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