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STRONGLY p-SUBHARMONIC FUNCTIONS AND VOLUME GROWTH PROPERTY OF COMPLETE RIEMANNIAN MANIFOLDS

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1. Introduction

Throughout this article we always denote by (M, g) a non-compact complete (connected) Riemannian manifold of dimension m. For a positive number $p > 1$, a smooth function u on M is said to be *strongly p-subharmonic* (resp. *p-subharmonic*) if u satisfies the following differential inequality on M:

$$\Delta_p u := \mathrm{div}_g(|\nabla u|^{p-2}\nabla u) \geq c > 0 \quad (\text{resp. } \Delta_p u \geq 0)$$

We note that Δ_2 is the ordinary Laplacian Δ defined by $\Delta := \mathrm{Trace}_g \nabla\nabla$. A few relations lying between the existence of non-constant bounded p-subharmonic functions on complete Riemannian manifolds and their volume growth property are known, and have been applied to show several Liouville type theorems for those functions (cf. [3], [7], [9], [10], [12], [13] etc.). For instance we can show the following volume growth estimate (see [13]), which is related to the p-parabolicity of (M, g) (cf. [6], [14]).

Theorem. *Suppose (M, g) admits a non-constant smooth p-subharmonic function bounded from above with $p > 1$. Then the following holds*:

$$\int_1^{+\infty} \left(\frac{r}{V_x(r)}\right)^{1/(p-1)} dr < +\infty \quad \textit{for any point } x \in M,$$

where $V_x(r)$ is the volume of geodesic ball $B_x(r)$ centered at x of radius $r > 0$. In particular if there exist a point $x_* \in M$ and a positive number $q > 1$ such that

$$\int_1^{+\infty} \left(\frac{r}{V_{x_*}(r)}\right)^{1/(q-1)} dr = +\infty,$$

then (M, g) admits no non-constant smooth p-subharmonic functions bounded from above with $p \geq q$.

In this article we continue to study such a kind of relations lying between the existence of a certain *strongly* p-subharmonic function and the volume growth property of (M, g) for the case $p \geq 2$. In the previous paper [11], we have studied the case $p = 2$ and observed that the relation is deeply related to a generalized maximum principle for the usual Laplacian. In this article it is verified that our argument used in [11] can be also developed to the case $p \geq 2$. However the case $1 < p < 2$ still remains. Furthermore we give a characterization of generalized maximum principle for the p-Laplacian Δ_p and a sufficient condition in terms of volume growth condition depending on p for the principle to hold. This yields a generalization of our previous result for the usual Laplacian (cf. [11]).

To formulate our result, for a smooth function u on M and given constants $\alpha > 0$, $\beta > 0$ and $\sigma \geq 0$, we set

$$\Omega_p(u, \alpha, \beta, \sigma) := \{\ x \in M\ ;\ u(x) \geq 0 \text{ and } \Delta_p u(x) \geq \beta K_\sigma(x) u(x)^{p+\alpha-1}\ \},$$

where K_σ is a positive continuous function on M satisfying the following condition for a fixed point $x_* \in M$:

$$K_\sigma(x) \geq \frac{C}{1 + d_M(x_*, x)^\sigma} \quad \text{for any point } x \in M \text{ and } C > 0,$$

and for a given constant $\gamma > 0$, we set

$$M(u, \gamma) := \{\ x \in M\ ;\ u(x) > \gamma\ \}.$$

For any $q \geq 0$, $x \in M$ and $r > 0$ we define the function $h_{q,x}(r)$ by

$$h_{0,x}(r) := \frac{\log V_x(r)}{\log r} \quad \text{and} \quad h_{q,x}(r) := \frac{\log V_x(r)}{r^q} \quad \text{if } q > 0.$$

First we state the following theorem which is a generalization of [11], Theorem 1.1.

Theorem 1. *Suppose* $M(u, \gamma)$ *is a non-empty subset of* $\Omega_p(u, \alpha, \beta, \sigma)$ *with* $\alpha \geq 1$, $p \geq 2$ *and* $p \geq \sigma \geq 0$. *Then the following assertions hold*:
(i) *If* $p > \sigma = 0$, *then for any point* $x \in M$ *there exist positive constants* $r_1 = r_1(\alpha, \beta, \gamma, p, x)$ *and* $C_1 = C_1(\beta, p)$ *such that*

$$\frac{\log \mathrm{Vol}(B_x(r) \cap M(u, \gamma))}{r^p} \geq C_1\ \gamma^{p\alpha/2}$$

for any $r \geq r_1$. *In particular, the following holds*:

$$\liminf_{r \to +\infty} h_{p,x}(r) = +\infty \quad \textit{for any } x \in M.$$

(ii) *If* $p > \sigma > 0$, *then there exist positive constants* $r_2 = r_2(\alpha, \beta, \gamma, \sigma, p, x_*)$ *and*

$C_2 = C_2(\beta, \sigma, p)$ *such that*

$$\frac{\log \mathrm{Vol}(B_{x_*}(r) \cap M(u, \gamma))}{r^{p-\sigma}} \geq C_2 \ \gamma^{p\alpha/2}$$

for any $r \geq r_2$. *In particular, the following holds*:

$$\lim_{r \to +\infty} \inf \ h_{p-\sigma, x_*}(r) = +\infty.$$

(iii) *If* $p = \sigma$, *then there exist positive constants* $r_3 = r_3(\alpha, \beta, \gamma, p, x_*)$, $C_3 = C_3(\beta, p)$ *and* $\gamma_* = \gamma_*(\alpha, \beta, p)$ *such that*

$$\frac{\log \mathrm{Vol}(B_{x_*}(r) \cap M(u, \gamma))}{\log r} \geq C_3 \ \gamma^{p\alpha/2}$$

for any $r \geq r_3$ *and* $\gamma \geq \gamma_*$. *In particular, the following holds*:

$$\lim_{r \to +\infty} \inf \ h_{0, x_*}(r) = +\infty.$$

From Theorem 1 we can induce the following non-existence result for non-negative smooth solutions satisfying a certain differetial inequality for the p-Laplacian (cf. [1], [2], [7], [8], [11]).

Corollary 2. *Let* (M, g) *be as above and let* $\alpha \geq 1$ *respectively.*
(i) *Suppose there exists a positive number* q *such that*

$$\lim_{r \to +\infty} \inf \ h_{q, x_*}(r) < +\infty.$$

Then any smooth solution $u \geq 0$ *satisfying the inequality* $\Delta_p u \geq \beta K_\sigma u^{p+\alpha-1}$ *outside a compact subset* T *of* M *satisfies* $u(x) \leq u_T^* := \sup_{y \in T} u(y)$ *for any* $x \in M$ *if* $p \geq \sigma + q$ *with* $p \geq 2$ *and* $\sigma \geq 0$, *where* $u_T^* := 0$ *if* $T = \phi$. *In particular there exists no non-zero smooth bounded solution* $u \geq 0$ *satisfying the inequality* $\Delta_p u \geq \beta K_\sigma u^\rho$ *on* M *if* $p \geq \sigma + q$ *with* $p \geq 2$, $\sigma \geq 0$ *and* $\rho \geq 0$.
(ii) *Suppose*

$$\lim_{r \to +\infty} \inf \ h_{0, x_*}(r) \ < \ +\infty.$$

Then any smooth solution $u \geq 0$ *satisfying the inequality* $\Delta_p u \geq \beta K_p u^{p+\alpha-1}$ *outside* T *satisfies* $u(x) \leq u_T^*$ *for any* $x \in M$ *if* $p \geq 2$. *In particular there exists no non-zero smooth bounded solution* $u \geq 0$ *satisfying the inequality* $\Delta_p u \geq \beta K_p u^\rho$ *on* M *if* $p \geq 2$ *and* $\rho \geq 0$.

REMARK 1. The range of α is not optimal in general and can be expected to be $\alpha > 0$. On the other hand, if the Ricci curvature of (M, g) satisfies $\mathrm{Ricci}_g(x) \geq$

$-C(1+r(x))^{2\nu}$ for $x \in M$, $C > 0$, $r(x) := d_M(x_*, x)$ and $\nu \leq -1$ (resp. $\nu > -1$), then we can verify that $V_{x_*}(r) \leq C_{\nu,1} r^{m+\delta(\nu)}$ with $C_{\nu,1} > 0$ and $0 \leq \delta(\nu) < +\infty$ (resp. $h_{\nu+1,x_*}(r) \leq C_{\nu,2} < +\infty$) for any $r \gg 0$ (cf. [4]).

As a corollary of the proof of Theorem 1 we get the following (see the proof of Theorem 2.1), which is a counterpart of Theorem.

Corollary 3. *Let (M, g) be as above and let $p \geq 2$ respectively. Suppose (M, g) admits a smooth strongly p-subharmonic function u, i.e., $\Delta_p u \geq c > 0$, bounded from above. Then the following holds*:

$$\liminf_{r \to +\infty} h_{p,x}(r) = +\infty \quad \textit{for any} \quad x \in M.$$

REMARK 2. For a given smooth monotone increasing function $h(r) > 0$ on a real line $\mathbf{R}$ such that $c_h := \int_1^{+\infty} dr/rh(r)^{1/(p-1)} < +\infty$ with $p \geq 2$, there exists a two dimensional complete Riemannian manifold (M, g_h) which admits a smooth bounded function $u \geq 0$ satisfying $\Delta_p u \equiv 1$ and a point $x \in M$ with $h_{p,x}(r) \sim h(r)$ for any $r \gg 0$. In fact let (M, g_h) be a two dimensional model M provided with a pole $x = 0$ and the metric $g_h = dr^2 + f(r)^2 d\theta^2$ on $M \setminus \{0\} \cong (0, +\infty) \times S^1$ such that (1) $f(0) = 0$, $f'(0) = 1$, $f(r) > 0$, $f'(r) > 0$, $f''(r) \geq 0$ if $r > 0$, and (2) $f(r) = c(\exp(r^p h(r)))'$ with $c > 0$ and $r \gg 0$. Setting $u(r) := \int_0^r \{(\int_0^t f(s) ds)^{1/(p-1)} / f(t)^{1/(p-1)}\} dt$, by a direct calculation it can be easily verified that $0 \leq \sup u \leq c_h < +\infty$, $\Delta_p u \equiv 1$ and $h(r)/2 \leq h_{p,0}(r) \leq h(r)$ for any $r \gg 0$ (cf. [11], Remark 2.4 and [4]).

The above example indicates us the following (cf. Remark 3 below): *if (M, g) admits a strongly p-subharmonic function bounded from above with $p \geq 2$, then*

$$\int_1^{+\infty} \frac{dr}{r h_{p,x}(r)^{1/(p-1)}} < +\infty \quad \textit{for any} \quad x \in M.$$

At least this is true in the case $p = 2$ because (M, g) is not stochastically complete (cf. [3]) if it admits a strongly subharmonic function bounded from above. The author thanks to Prof. A. Atsuji who pointed out the result to him.

The following is a generalization of [11], Theorem 2.3.

Theorem 4. *Let (M, g) be as above. Suppose there exists a point $x \in M$ and a positive number $q \geq 2$ such that*

$$\liminf_{r \to +\infty} h_{q,x}(r) < +\infty.$$

If $p \geq q$, then the following generalized maximum principle for the operator Δ_p holds: *for any smooth function f bounded from above, $\varepsilon > 0$ and $x \in M$, there exists a point*

$x_\varepsilon \in M$ depending on x such that

$$(1)\quad f(x) \le f(x_\varepsilon), \quad (2)\quad |\nabla f|(x_\varepsilon) < \varepsilon \quad \text{and} \quad (3)\quad \Delta_p f(x_\varepsilon) < \varepsilon.$$

REMARK 3. As a counterpart of Remark 2, it can be expected to hold the following stronger assertion, i.e., *the generalized maximum principle for Δ_p holds if $p \ge q > 1$ and*

$$\int_1^{+\infty} \frac{dr}{r h_{q,x}(r)^{1/(q-1)}} = +\infty \quad \text{for a certain point } x \in M.$$

2. A volume estimate for strong p-subharmonicity on complete Riemannan manifolds

This section is devoted to show Theorem 1. Using the same notations introduced in the section we restate Theorem 1 as follows.

Theorem 2.1. *Suppose $M(u,\gamma)$ is a non-empty subset of $\Omega_p(u,\alpha,\beta,\sigma)$ with $\alpha \ge 1$, $p \ge 2$ and $p \ge \sigma \ge 0$. Then the following assertions hold*:
(i) *If $p > \sigma = 0$, then for any point $x \in M$ there exists $r_1 = r_1(\alpha,\beta,\gamma,p,x) \gg 0$ such that*

$$\frac{\log \mathrm{Vol}(B_x(r) \cap M(u,\gamma))}{r^p} \ge \frac{\log 2}{2^{2p+1}} \left(\frac{\beta C_{**}}{2^4 C_* p^2}\right)^{p/2} \gamma^{p\alpha/2}$$

for any $r \ge r_1$, where C_* and C_{**} are positive constants not depending on $(\alpha,\beta,\gamma,p,x_*)$. In particular, the following holds:

$$\liminf_{r \to +\infty} h_{p,x}(r) = +\infty \quad \text{for any } x \in M.$$

(ii) *If $p > \sigma > 0$, then there exists $r_2 = r_2(\alpha,\beta,\gamma,\sigma,p,x_*) \gg 0$ such that*

$$\frac{\log \mathrm{Vol}(B_{x_*}(r) \cap M(u,\gamma))}{r^{p-\sigma}} \ge \frac{\log 2}{2^{2(p-\sigma)+1}} \left(\frac{\beta C_{**}}{2^{\sigma+4} C_* p^2}\right)^{p/2} \gamma^{p\alpha/2}$$

for any $r \ge r_2$, where C_* (resp. C_{**}) is a positive constant not depending on $(\alpha,\beta,\gamma,\sigma,p,x_*)$ (resp. depending only on σ). In particular, the following holds:

$$\liminf_{r \to +\infty} h_{p-\sigma,x_*}(r) = +\infty.$$

(iii) *If $p = \sigma$, then there exists $r_3 = r_3(\alpha,\beta,\gamma,p,x_*) \gg 0$ and $\gamma_* = \gamma_*(\alpha,\beta,p) \gg 0$ such that*

$$\frac{\log \mathrm{Vol}(B_{x_*}(r) \cap M(u,\gamma))}{\log r} \ge \frac{\log 2}{2^3} \left(\frac{\beta C_{**}}{2^{p+4} C_* p^2}\right)^{p/2} \gamma^{p\alpha/2}$$

for any $r \geq r_3$ *and* $\gamma \geq \gamma_*$, *where* C_* (*resp.* C_{**}) *is a positive constant not depending on* $(\alpha, \beta, \gamma, p, x_*)$ (*resp. depending only on* p). *In particular, the following holds*:

$$\lim_{r\to+\infty} \inf h_{0,x_*}(r) = +\infty.$$

Proof of Theorem 2.1. First we note

$$M(u,\gamma) = M\left(\frac{u}{\gamma}, 1\right) \quad \text{and} \quad \Omega_p(u,\alpha,\beta,\sigma) = \Omega_p\left(\frac{u}{\gamma}, \alpha, \beta\gamma^{\alpha}, \sigma\right).$$

From now on we replace u by u/γ and set $\delta := \beta\gamma^{\alpha}$. Hence we can see $M(u,1)(\neq \phi) \subset \Omega_p(u,\alpha,\delta,\sigma)$. For a fixed positive number $\rho > 1$ with $M(u,\rho) \neq \phi$, let λ be a smooth function defined on real line such that $\lambda(t) \equiv 0$ if $t \leq 1$, $\lambda(t) > 0, \lambda'(t) > 0, \lambda''(t) \geq 0$ if $t > 1$ and $\lambda(t) \equiv t$ if $t \geq \rho > 1$. Since the metric g is complete, for any fixed point $x \in M$ and $r > 0$ there exists a Lipschitz continuous function ω_r with $0 \leq \omega_r \leq 1$ on M such that $\omega_r \equiv 1$ on $B_x(r)$, supp $\omega_r \subset \overline{B_x(2r)}$ and $|\nabla\omega_r|^2 \leq C_*/r^2$, where $C_* > 0$ does not depend on x and r. For positive numbers k and q with $q > 1$, denoting $\omega = \omega_r$ a direct calculation shows

$$\begin{aligned}
&\operatorname{div}(\omega^{2k}|\nabla u|^{p-2}\nabla\lambda(u^q))\\
&\quad = q \operatorname{div}(\omega^{2k}\lambda'(u^q)u^{q-1}|\nabla u|^{p-2}\nabla u)\\
&\quad = q\Big\{q\lambda''(u^q)u^{2q-2}\omega^{2k}|\nabla u|^p + (q-1)\omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\\
&\qquad + \omega^{2k}\lambda'(u^q)u^{q-1}\Delta_p u + 2k\omega^{2k-1}\lambda'(u^q)u^{p-1}|\nabla u|^{p-2}\langle\nabla\omega,\nabla u\rangle\Big\}\\
&\quad \geq q\Big\{(q-1)\omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p + \delta\omega^{2k}\lambda'(u^q)K_\sigma u^{p+q+\alpha-2}\\
&\qquad - 2k\omega^{2k-1}\lambda'(u^q)u^{q-1}|\nabla u|^{p-1}|\nabla\omega|\Big\}.
\end{aligned}$$

By integrating the left hand side with respect to the measure dv_g induced by g and the hypothesis, for any $\varepsilon > 0$ and $B_x(2r,r) := B_x(2r)\setminus B_x(r)$ we obtain

$$\begin{aligned}
&(q-1)\int \omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\, dv_g + \delta\int K_\sigma\omega^{2k}\lambda'(u^q)u^{p+q+\alpha-2}\, dv_g\\
&\leq 2k\int \omega^{2k-1}\lambda'(u^q)u^{q-1}|\nabla u|^{p-1}|\nabla\omega|\, dv_g\\
&\leq \varepsilon\int \omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\, dv_g + \frac{k^2}{\varepsilon}\int_{B_x(2r,r)} \omega^{2(k-1)}\lambda'(u^q)u^q|\nabla u|^{p-2}|\nabla\omega|^2\, dv_g.
\end{aligned}$$

Taking $\varepsilon = (q-1)/2 > 0$ in the above inequality the following holds for any $p \geq 2$:

$$\int \omega^{2k}\lambda'(u^q)K_\sigma u^{p+q+\alpha-2}\ dv_g + \frac{q-1}{2}\int \omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\ dv_g \le \frac{2C_* k^2}{\delta(q-1)r^2}\int_{B_x(2r,r)} \omega^{2(k-1)}\lambda'(u^q)u^q|\nabla u|^{p-2}\ dv_g.$$

Especially if $2k > p > 2$, then

$$\int_{B_x(2r,r)} \omega^{2(k-1)}\lambda'(u^q)u^q|\nabla u|^{p-2}\ dv_g \le \left(\int_{B_x(2r,r)} \omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\ dv_g\right)^{(p-2)/p}\left(\int_{B_x(2r,r)} \omega^{2k-p}\lambda'(u^q)u^{p+q-2}\ dv_g\right)^{2/p},$$

which implies the following for $p > 2$

$$\int \omega^{2k}\lambda'(u^q)u^{q-2}|\nabla u|^p\ dv_g \le \left(\frac{4C_* k^2}{\delta(q-1)^2 r^2}\right)^{p/2}\int_{B_x(2r,r)} \omega^{2k-p}\lambda'(u^q)u^{p+q-2}\ dv_g.$$

Hence for any k and p with $2k > p \ge 2$, we can induce the following estimate from the above estimates:

$$\int \omega^{2k}\lambda'(u^q)K_\sigma u^{p+q+\alpha-2}\ dv_g \le \frac{2C_* k^2}{\delta(q-1)r^2}\left(\frac{4C_* k^2}{\delta(q-1)^2 r^2}\right)^{(p-2)/2}\int_{B_x(2r,r)} \omega^{2k-p}\lambda'(u^q)u^{p+q-2}\ dv_g.$$

Since $\alpha \ge 1$ and $\lambda'(u^q) > 0$ if and only if $u > 1$, setting $k = p(p+q+\alpha-2)/2 > 0$, we get

$$\begin{aligned}\int_{B_x(2r,r)} &\omega^{2k-p}\lambda'(u^q)u^{p+q-2}\ dv_g \\ &\le \left(\int \omega^{2k}\lambda'(u^q)K_\sigma u^{2k(p+q-2)/(2k-p)}dv_g\right)^{(2k-p)/2k}\left(\int_{B_x(2r,r)} K_\sigma^{-(2k-p)/p}\lambda'(u^q)dv_g\right)^{p/2k} \\ &\le \left(\int \omega^{2k}\lambda'(u^q)K_\sigma u^{2k/p}\ dv_g\right)^{(2k-p)/2k}\left(\int_{B_x(2r,r)} K_\sigma^{-(2k-p)/p}\lambda'(u^q)\ dv_g\right)^{p/2k}.\end{aligned}$$

Therefore the following holds:

$$\int \omega^{2k}\lambda'(u^q)K_\sigma u^{2k/p}\ dv_g \le \left(\frac{2C_* k^2}{\delta(q-1)r^2}\right)^{2k/p}\left(\frac{4C_* k^2}{\delta(q-1)^2 r^2}\right)^{k(p-2)/p}\int_{B_x(2r,r)} K_\sigma^{-(2k-p)/p}\lambda'(u^q)\ dv_g$$

for any $q > 1$, $p \ge 2$ and $r \ge r_0 = r_0(x,\gamma)$ with $B_x(r_0) \cap M(u,1) \neq \phi$. If $\inf_{y\in B_{x_*}(2r,r)} K_\sigma(y) \ge C_\sigma/(2r)^\sigma$ for any $r > r_0$ and $C_\sigma > 0$, then the above estimate implies

$$\int \omega^{2k}\lambda'(u^q)u^{2k/p}\ dv_g \leq \left(\frac{2^{1+\sigma}C_*k^2}{\delta C_\sigma(q-1)r^{2-\sigma}}\right)^{2k/p}\left(\frac{4C_*k^2}{\delta(q-1)^2r^2}\right)^{k(p-2)/p}\int_{B_x(2r,r)}\lambda'(u^q)\ dv_g$$

for any $x \in M$. Taking $q > 0$ so that $q \geq \max\{p+\alpha-2, p\} \geq 2$, we get the following:

$$\frac{2C_*k^2}{\delta(q-1)} \leq \frac{4C_*p^2q}{\delta} \quad \text{and} \quad \frac{4C_*k^2}{\delta(q-1)^2} \leq \frac{16C_*p^2}{\delta}.$$

Hence setting $F(q,r) := \int_{B_x(r)}\lambda'(u^q)\ dv_g \geq 0$ and $C_{**} := \min\{C_\sigma, 1\}$, we can see

$$F(q,r) \leq \left(\frac{2^{\sigma+4}C_*p^2}{\delta C_{**}}\right)^{p(p+q+\alpha-2)/2}\left(qr^{\sigma-p}\right)^{p+q+\alpha-2}F(q,2r).$$

For $p \geq \sigma$ we put

$$q = q(r) := \frac{1}{2}\left(\frac{\delta C_{**}}{2^{\sigma+4}C_*p^2}\right)^{p/2} r^{p-\sigma} \quad \left(\geq \max\{p+\alpha-2, p\}\right) \text{ and } F(r) := F(q(r), r).$$

Finally there exists $r_0 := r(\alpha, \beta, \gamma, \sigma, p, x) \gg 1$ such that $F(r)$ satisfies the following:

$$F(r) \leq \left(\frac{1}{2}\right)^{q(r)+\alpha} F(2r)$$

for any $r \geq r_0$. Suppose $p > \sigma$ and take any r with $r \geq 2r_0$. Since there exists $k \geq 1$ such that $2^{-(k+1)} < r_0/r \leq 2^{-k}$, by putting $r_i = 2^i r_0$, we obtain for any $r \geq r_1$

$$F(r_0) \leq \left(\frac{1}{2}\right)^{\sum_{i=0}^{k-1}q(r_i)+k\alpha} F(r_k) \leq \left(\frac{1}{2}\right)^{\lambda r^{p-\sigma}}\left(\frac{r_1}{r}\right)^{\alpha} F(r),$$

where

$$\lambda := \frac{1}{2^{2(p-\sigma)+1}}\left(\frac{\beta C_{**}}{2^{\sigma+4}C_*p^2}\right)^{p/2} > 0.$$

Therefore there exists $r(\alpha, \beta, \gamma, \sigma, p, x) > 0$ such that

$$\frac{\log\ F(r)}{r^{p-\sigma}} \geq \frac{\log 2}{2^{2(p-\sigma+1)}}\left(\frac{\beta C_{**}}{2^{\sigma+4}C_*p^2}\right)^{p/2}\gamma^{p\alpha/2}$$

for any $r \geq r(\alpha, \beta, \gamma, p, x)$. Since we have replaced u by u/γ in the beginning and may assume $\sup_{\mathbf{R}}\lambda'(t) = 1$, $F(r) \leq \mathrm{Vol}(B_x(r) \cap M(u,\gamma))$ for any $r \gg 0$. Therefore we can obtain the desired estimate. The case $p = \sigma$ can be shown similarly.

With respect to the divergence of volume if the function u is unbounded, then the assertion is trivial respectively. If $u^* := \sup_M u < +\infty$ and satisfies $\Delta_p u \geq \beta K_\sigma u^t$ with $t \geq 0$, then we may assume that $u^* > 1$ and u does not attain u^* on M by the hypothesis $p \geq 2$. Hence $v := 1/(u^* - u)$ is unbounded on M and satisfies $\Delta_p v \geq \beta\gamma^t K_\sigma v^p$ on $M(v, 1/(u^* - \gamma))(= M(u, \gamma))$ with $\gamma \geq u^* - 1$. Therefore we can attain the conclusion similarly. This completes the proof of Theorem 2.1. □

3. A characterization of generalized maximum principle for the operator Δ_p on complete Riemannian manifolds

Let (M, g) be a non-compact complete (connected) Riemannian manifold of dimension m. Generalized maximum principle for the operator Δ_p is formulated and characterized as follows.

Theorem 3.1. *For a fixed positive number $p \geq 2$ the following two statements are equivalent*:
(i) *For any smooth function u on M with $\{x \in M;\ u(x) > 0\} \neq \phi$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, $M(u, \gamma)(\neq \phi)$ can not be contained in $\Omega_p(u, \alpha, \beta) := \{x \in M;\ u(x) > 0$ and $\Delta_p u(x) \geq \beta u(x)^{p+\alpha-1}\}$.*
(ii) *For any smooth function f bounded from above, $\varepsilon > 0$ and $x \in M$, there exists a point $x_\varepsilon \in M$ such that*

$$(1)\ f(x) \leq f(x_\varepsilon), \quad (2)\ |\nabla f|(x_\varepsilon) < \varepsilon \quad \textit{and} \quad (3)\ \Delta_p f(x_\varepsilon) < \varepsilon.$$

REMARK. To show the indication (i) $\Longrightarrow$ (ii) it is sufficient to assume $\alpha \geq 1$.

Proof of (i) $\Longrightarrow$ (ii). We need two lemmas to show our claim. First the following lemma follows from the hypothesis (i) immediately.

Lemma 3.2. *Let u be a smooth function on M such that $0 < u^* := \sup_M u \leq +\infty$ and u does not attain u^* on M. Suppose the assertion* (i) *in* Theorem 3.1 *holds. Then for constants $\alpha \geq 1, \beta > 0, \sigma > 0$ with $p \geq \sigma$, the following holds*:
(1) *$\Gamma_p(u, \alpha, \beta) := \{x \in M\ ;\ \Delta_p u(x) < \beta u(x)^{p+\alpha-1}\}$ is a non-empty unbounded open subset of M.*
(2) *$u(x) \leq u^*(\alpha, \beta) := \sup_{y \in \Gamma_p(u,\alpha,\beta)} u(y)$ for any $x \in M$. Especially if $u^*(\alpha, \beta)$ is finite for a certain pair (α, β), then $u^*(\alpha, \beta)$ is independent of α and β, and hence $u^* = u^*(\alpha, \beta) < +\infty$.*

The following lemma has been proved in a special case in [5]. Since the proof for general case is essentially the same, we state it without proof here.

Lemma 3.3. *Let (X, h) be a complete Riemannian manifold and let f be a smooth function bounded from above on X. For any $\varepsilon > 0$ take a point $y_\varepsilon \in X$ with* $\sup_X f - \varepsilon^2 < f(y_\varepsilon)$. *Then there exists a point $x_\varepsilon \in X$ such that* (i) $f(y_\varepsilon) \leq f(x_\varepsilon)$, (ii) $d_X(x_\varepsilon, y_\varepsilon) \leq \varepsilon$ *and* (iii) $|\nabla f|(x_\varepsilon) \leq \varepsilon$, *where d_X is the distance function relative to h.*

We are now in a position to begin the proof. Since $p \geq 2$, the assertion is trivial if f attains $f^* := \sup_M f$. We suppose that f does not attain f^* on M. For any given point $x \in M$, we put $\varepsilon_* := \min\{\varepsilon, f^* - f(x)\}/(1+\min\{\varepsilon, f^* - f(x)\}) > 0$. Set $w := 1/(1 + f^* - f) > 0$ and $M_q := M(w^q, 1 - \varepsilon_*^2)$ for any positive integer q. Then clearly $M_{q_1} \subset M_{q_2}$ and $\partial M_{q_1} \cap \partial M_{q_2} = \phi$ if $q_1 > q_2 \geq 1$. On the other hand, setting $\alpha := p-1 \geq 1$, $\Gamma_q := \Gamma_p(w^q, \alpha, \varepsilon_*)$ is non-empty by Lemma 3.2, (i). By using the fact $\Delta_p w^q \geq w^{p-1}(q/q-1)^{p-1}\Delta_p w^{q-1}$ for any $q \geq 2$ repeatedly and $0 < w < 1$ on M, we obtain $\Gamma_{q_1} \subset \Gamma_{q_2}$ and $\partial\Gamma_{q_1} \cap \partial\Gamma_{q_2} = \phi$ if $q_1 > q_2 \geq 1$. Setting $\Sigma_q := \Gamma_q \cap M_q$, Σ_q is also non-empty and $\sup_{\Sigma_q} w^q = 1$ by Lemma 3.2, (ii). In particular Σ_q is unbounded for any $q \geq 1$ because f does not attain f^*, and it can be verified that $\Sigma_{q_1} \subset \Sigma_{q_2}$ and $\partial\Sigma_{q_1} \cap \partial\Sigma_{q_2} = \phi$ if $q_1 > q_2 \geq 1$. Suppose Σ_q converges to a non-empty subset $\Sigma_\infty \subset M$ containing a point x_∞ as q tends to infinity. Then w should attain 1 at x_∞. This is a contradiction. Hence $M \setminus \Sigma_q$ converges to the whole space M as q tends to infinity. This implies that $d_M(x_*, \Sigma_q)$ is unbounded for a fixed point $x_* \in M$. Setting $\lambda_q := \sup_{y \in \partial\Sigma_q} d_M(y, \partial\Sigma_1) \in (0, +\infty]$ for any $q > 1$, $\lim_{q\to+\infty} \lambda_q = +\infty$ by the above observation. Since λ_q is non-decreasing in q, there exists a large positive integer q_* such that $\varepsilon_* < \lambda_q \leq +\infty$ for any integer q with $q \geq q_*$. For a fixed $q \geq q_*$, there exists a point $y_* \in \partial\Sigma_q$ with $d_M(y_*, \partial\Sigma_1) > \varepsilon_*$. Clearly such a point admits a small positive constant δ_* such that $\overline{B_z(\varepsilon_*)} \subset \Sigma_1$ if $z \in B_{y_*}(\delta_*) \cap \Sigma_q$. Now we take a point $z_\varepsilon \in B_{y_*}(\delta_*) \cap \Sigma_q$. By Lemma 3.3, there exists a point $x_\varepsilon \in \overline{B_{z_\varepsilon}(\varepsilon_*)} \cap M_q \subset \Sigma_1$ such that $|\nabla w^q|(x_\varepsilon) \leq \varepsilon_*$. If q is large enough, then x_ε is the desired point.

Proof of (ii) $\Longrightarrow$ (i). Suppose $M(u, \gamma) \subset \Omega_p(u, \alpha, \beta)$ with $\alpha > 0$. Let λ be a smooth function defined on real line such that $\lambda(t) = 0$ for $t < \gamma$, $\lambda'(t) > 0$, $\lambda''(t) \geq 0$ for $t \geq \gamma$ and $\lambda'(t) = 1$ for $t \geq \gamma + \delta$ with $\delta > 0$. Taking δ arbitrarily we may assume that $v := \lambda(u)$ satisfies $\Delta_p v \geq \beta v^{p+\alpha-1}$ on $\{v > \gamma^*\} \neq \phi$ with $\gamma^* := \lambda(\gamma + \delta) > 0$. Set $w := -1/(1 + v)^q$ with $q := \alpha/p > 0$ and $\varepsilon_* := \min\{\sup_M w - 1/(1 + \gamma^*)^q, 1\} > 0$. By the hypothesis for any $\varepsilon > 0$ with $0 < \varepsilon < \varepsilon_*$, there exists a point $x_\varepsilon \in M$ such that (1) $\sup_M w - \varepsilon < w(x_\varepsilon)$, (2) $|\nabla w|(x_\varepsilon) < \varepsilon$, (3) $\Delta_p w(x_\varepsilon) < \varepsilon$. Since $\Delta_p v(x_\varepsilon) \geq \beta v^{p+\alpha-1}(x_\varepsilon)$, by a direct calculation there exists a constant $C(\alpha, \beta, p) > 0$ not depending on $\varepsilon > 0$ such that

$$\left(\frac{v(x_\varepsilon)}{1 + v(x_\varepsilon)}\right)^{p+\alpha-1} \leq C(\alpha, \beta, p)\varepsilon$$

This implies $v^* := \sup_M v < +\infty$ and so there exists $C > 0$ not depending on ε such that $v(x_\varepsilon)^{p+\alpha-1} \leq C\varepsilon$. Letting $\varepsilon \to 0$ we obtain $v^* = 0$, which implies $u \leq \gamma$ on $M(u,\gamma) = \{u > \gamma\} \neq \phi$. This is a contradiction. This completes the proof of Theorem 3.1. □

As a byproduct of Theorem 3.1, we get the following non-existence theorem for non-negative solution satisfying a certain differential inequality (cf. [8]).

Corollary 3.4. *For a positive number $p \geq 2$ suppose the generalized maximum principle for Δ_p holds on (M, g). Then any smooth solution $u \geq 0$ satisfying the inequality $\Delta_p u \geq \beta u^q$ outside a compact subset T of M satisfies $u(x) \leq u_T^* := \sup_{y\in T} u(y)$ for any $x \in M$ if $q > p-1$, where $u_T^* := 0$ if $T = \phi$. In particular there exists no non-zero smooth bounded solution $u \geq 0$ satisfying the inequality $\Delta_p u \geq \beta u^\rho$ on M if $\rho \geq 0$.*

Now it is clear that Theorems 2, 3 and 4 follow from Theorems 2.1 and 3.1 immediately.

References

[1] K-S. Cheng and J-T. Lin: *On the elliptic equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$*, Trans. Amer. Math.Soc. **304** (1987), 639–668.

[2] S.Y. Cheng and S.T. Yau: *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[3] A.A. Grigor'yan: *On stochastically complete manifolds*, Soviet Math. Dokl. **34** (1987), 310–313.

[4] R.E. Greene and H. Wu: Function theory on manifolds which possess a pole, Lecture Notes in Math. **699**, Springer-Verlag.

[5] J.B. Hiriart-Urruty: *A short proof of the variational principle for approximate solutions of a minimaization problem*, Amer. Math. Monthly, **90** (1983), 206–207.

[6] I. Holopainen: *Volume growth, Green's functions, and parabolicity of ends*, Duke Math. J. **97** (1999), 319–346.

[7] L. Karp: *Subharmonic functions on real and complex manifolds*, Math. Z. **179** (1982), 535–554.

[8] Y. Naito and H. Usami: *Entire solutions of the inequality* $\operatorname{div}(A(|Du|)Du) \geq f(u)$, Math. Z. **225** (1997), 167–175.

[9] M. Rigoli, M. Salvatori and M. Vignati: *A note on p-subharmonic functions on complete manifolds*, Manuscripta Math. **92** (1997), 339–359.

[10] M. Rigoli, M. Salvatori and M. Vignati: *Volume growth and p-subharmonic functions on complete manifolds*, Math. Z. **227** (1998), 367–375.

[11] K. Takegoshi: *A volume estimate for strong subharmonicity and maximum principle on complete Riemannian manifolds*, Nagoya Math. J. **151** (1998), 25–36.

[12] K. Takegoshi: *A maximum principle for p-harmonic maps with L^q finite energy*, Proc. Amer. Math. Soc. **126** (1998), 3749–3753.

[13] K. Takegoshi: *A divergence property of L^q-integral of P-subharmonic functions on complete Riemannian manifolds*, preprint.

[14] K. Takegoshi: *A note on divergence of L^p-integral of subharmonic functions and its applications*, preprint.

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