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RADIAL VISCOSITY SOLUTIONS OF THE DIRICHLET PROBLEM FOR SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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1. Introduction

In this paper we are concerned with the Dirichlet problem (hereafter called (DP)) for the semilinear degenerate elliptic equation:

(1.1) $$\mathcal{F}[u](x) := -g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad B_R$$

(1.2) $$u = \beta \quad \text{on} \quad \partial B_R,$$

where $B_R = \{x \in \mathbf{R}^N; |x| < R\}$, $N \geq 2$, $g : [0, R] \to R^+ = [0, \infty)$ is a given continuous function, Δ is the Laplacian, and β is a given real number.

In 1981 symmetry properties of positive solutions of nonlinear elliptic equations were investigated by Gidas, Ni and Nirenberg [10, 11]. Since then there have been lots of works published in this direction (See, e.g. [1], [2], [9] and [14, 15]). There are also many works on the boundary value problem for elliptic equations whose coefficients are singular at the origin or on the boundary. See, e.g. Ebihara and Furusho [7] and Senba, Ebihara and Furusho [17], H. Egnell [8], Conti,Crotti and Pardo [3]. It is natural to ask whether solutions are radially symmetric for degenerate elliptic equations such as our (1.1). That is, we want to study the case when the degeneracy of equations arises in the interior of domains. In this case it is well known that in general we can not expect to have C^2-solutions.

In recent years, a considerable number of works have been done on the theory and applications of viscosity solutions. We refer the reader to the Monograph by Crandall, Ishii and Lions [4] for definitions, details and references. As regards earlier related works, in 1992, Crandall and Huan [5] studied the existence, uniqueness and non-uniqueness of viscosity solutions of the two-point boundary value problem for the linear ordinary differential equation:

$$\begin{cases} -a(x)u'' + c(x)u = f(x) & \text{in} \quad (-1, 1) \\ u(-1) = \gamma_-, \quad u(1) = \gamma_+ \end{cases}$$

under the assumption that $a, c, f \in C[-1, 1]$, $a \geq 0$ and $c \geq 0$. Soon after, Tomita

[19] generalized their work to the linear partial differential equation:

$$\begin{cases} -||x|-1|^{\lambda}\Delta u + c(x)u = f(x) & \text{in} \quad B_L = \{x \in \mathbf{R}^N; |x| < L\} \\ u(x) = \beta & \text{on} \quad |x| = L, \end{cases}$$

where $N \geq 2$, $L > 1$, $0 < \lambda < 2$ and β is a given number. It should be noted that Ishii and Ramaswamy [12] recently studied uniqueness and comparison results for a class of Hamilton-Jacobi equations with singular coefficients. See also Siconolfi [18].

The main purpose of this paper is to prove existence, uniqueness and nonuniqueness of viscosity solutions of (DP), and to study the symmetry property for viscosity solutions.

Our plan in this paper is as follows. In Section 2, we give a definition of standard viscosity solutions, and state our main results. Section 3 is devoted to the existence and uniqueness of radial standard viscosity solutions of (DP). In Section 4, we study the uniqueness and nonuniqueness of viscosity solutions for (DP), and prove that all viscosity solutions of (DP) must be standard, and hence radially symmetric under some assumption (cf. (H3) in Section 2). We also prove that if we do not assume (H3) then (DP) has infinitely many radial viscosity solutions.

2. Preliminaries and main results

Throughout this paper we make the following assumptions:
(H1) $f(t, y) \in C([0, R] \times \mathbf{R})$ is strictly increasing in y for each fixed $t \in [0, R]$.
(H2) There exists an implicit function $\varphi(t)$ of $f(t, y) = 0$.
It is clear that $\varphi(t)$ is continuous on $[0, R]$ by (H1) and (H2). To state a notion of weak solutions, we introduce the next notation:

$$\mathcal{Z}(g) = \{t \in (0, R]; g(t) = 0\}.$$

DEFINITION 2.1. $u = u(x) \in C(\overline{B_R})$ is called a standard viscosity solution of (DP) if (i) u satisfies (1.1) in B_R in the viscosity sense, (ii) $u(x) = \varphi(|x|)$ for all $|x| \in \mathcal{Z}(g)$, and (iii) $u(x) = \varphi(R)$ on ∂B_R.

REMARK 2.2. From the proofs given below it follows that if $g(R) > 0$ then the boundary condition (iii) may be replaced by

(iii)′ $$u(x) = \beta \quad \text{on} \quad \partial B_R,$$

where β is an arbitrary real number.

EXAMPLE 2.3. For g in the equation (1.1), we take the following examples into consideration.

1) $g(t) = \prod_{i=1}^{N_1} |t - a_i|^{\lambda_i}$, where $0 \le a_1 < a_2 < \cdots < a_{N_1} \le R$, $N_1 \le \infty$, and $\lambda_i > 0$, $i = 1, 2, \ldots, N_1$. In this case, $\mathcal{Z}(g) = \{a_1, a_2, \ldots, a_{N_1}\}$; in other words, (1.1) degenerates on spheres $|x| = a_i$, $i = 1, 2, \ldots, N_1$

2) $\mathcal{Z}(g) = \{a_1, a_2, \ldots, a_{N_1}\} \cup (\cup_{i=1}^{N_2} J_i)$, i.e., $g(t) = 0$ for $t = a_1, a_2, \ldots, N_1$ and $\forall t \in J_i = [p_i, q_i], i = 1, 2, \ldots, N_2$.

First we shall prove the existence of viscosity solutions of (DP). More precisely, we establish

Theorem 1. *Assume* (H1) *and* (H2). *Then there exists a unique standard viscosity solution* u^* *of* (DP). *Furthermore,* u^* *is radial.*

In order to state the uniqueness of viscosity solutions for (DP), we introduce some notation and an assumption on g. We denote the set of intervals $I = [c, d]$ such that $g(t) = 0$ for all $t \in I$ by $\mathcal{I}(g)$;

$$\mathcal{I}(g) = \{I \subset [0, R] \mid g(t) = 0 \quad \text{for} \quad \forall t \in I\}.$$

Of course, $\mathcal{I}(g)$ may be empty. For every $a \in \mathcal{Z}(g) \setminus \mathcal{I}(g)$ and $\delta > 0$, we put

$$I_\delta^+(a) := (a, a + \delta) \cap \mathcal{Z}(g)^c \quad \text{and} \quad I_\delta^-(a) := (a - \delta, a) \cap \mathcal{Z}(g)^c.$$

(H3) For every $a \in \mathcal{Z}(g) \setminus \mathcal{I}(g)$, we have either

$$\lim_{\delta \downarrow 0} \int_{I_\delta^-(a)} g(t)^{-1} dt = +\infty \quad \text{or} \quad \lim_{\delta \downarrow 0} \int_{I_\delta^+(a)} g(t)^{-1} dt = +\infty.$$

We shall next prove the uniqueness of viscosity solutions for (DP):

Theorem 2. *Suppose that* (H1)-(H3) *hold. Then there exists a unique viscosity solution* u *of* (DP). *Moreover,* u *is radial and standard.*

Finally we mention that uniqueness of viscosity solutions of (DP) does not hold without the assumption (H3). To see this, for simplicity, we assume
(H4) $g(a) = 0$, $0 < a < R$, $g(t) > 0$ for $\forall t \in [0, R] \setminus \{a\}$, and

$$\int^{a-0} g(s)^{-1} ds < \infty \quad \text{and} \quad \int_{a+0} g(s)^{-1} ds < \infty.$$

Example: $g(t) = |t - a|^\lambda$ with $0 < a < R$ and $0 < \lambda < 1$.

We shall at first prove the existence of a radial viscosity solution $\overline{u}(x) = \overline{y}(|x|) \in C^1(\overline{B_R}) \cap C^2((B_a \setminus \{x = 0\}) \cup A(a, R))$ of (DP). Denote the radial and standard viscosity solution of (DP) by $u^*(x)$. Then we have

Theorem 3. *Suppose* (H1), (H2) *and* (H4). *Let* u^* *be the standard viscosity solution and* $\overline{u}$ *be the* C^1*-viscosity solution of* (DP). *Then we have*

(i) *If* $\overline{u}(x) = u^*(x) = \varphi(a)$ *on* $|x| = a$ *then every viscosity solution is standard, hence viscosity solutions are unique.*

(ii) *If* $\overline{u}(x) > u^*(x) = \varphi(a)$ *on* $|x| = a$, *then* $\overline{u}$ *is the maximum viscosity solution of* (DP), *and* u^* *is the minimum viscosity solution of* (DP). *Furthermore, the problem* (DP) *has infinitely many radial viscosity solutions between* u^* *and* $\overline{u}$.

(iii) *If* $\overline{u}(x) < u^*(x) = \varphi(a)$ *on* $|x| = a$, *then* $\overline{u}$ (*resp.* u^*) *is the minimum* (*resp. maximum*) *viscosity solution of* (DP). *Furthermore, the problem* (DP) *has infinitely many radial viscosity solutions between* $\overline{u}$ *and* u^*.

3. The proof of Theorem 1

In this section we shall prove Theorem 1. In order to prove the existence of a radial and standard viscosity solution of (DP), we introduce two Dirichlet problems. Supposing $g(|x|) > 0$ in $A(a_i, b_i) = \{x \in \mathbf{R}^N; a_i < |x| < b_i\}$ and $g(|x|) = 0$ on $\partial A(a_i, b_i)$ with $0 < a_i < b_i < R$, we consider the Dirichlet problem:

$$(\mathrm{P}_i) \qquad \begin{cases} \mathcal{F}[u](x) = 0 & \text{in} \quad A(a_i, b_i) \\ u(x) = \varphi(a_i) \quad \text{on} \quad |x| = a_i & \text{and} \quad u(x) = \varphi(b_i) \quad \text{on} \quad |x| = b_i, \end{cases}$$

and under the conditions $g(|x|) > 0$ in $B_{b_0} \setminus \{0\}$ and $g(|x|) = 0$ on ∂B_{b_0} we consider the Dirichlet problem:

$$(\mathrm{P}_0) \qquad \begin{cases} \mathcal{F}[u](x) = 0 & \text{in} \quad B_{b_0} \\ u(x) = \varphi(b_0) & \text{on} \quad \partial B_{b_0}. \end{cases}$$

In what follows we consider (P_0) and (P_i) in the case $N = 2$, since we can treat them in case $N \geq 3$ by the same arguments. Clearly, $u(x) = y(|x|)$ is a classical radial solution of (P_i) if and only if $y \in C[a_i, b_i] \cap C^2(a_i, b_i)$ is a solution of the two-point boundary value problem:

$$(\mathrm{BVP}_i) \qquad \begin{cases} \mathcal{L}[y](t) := -\left(\ddot{y}(t) + \dfrac{1}{t}\dot{y}(t)\right) + g(t)^{-1} f(t, y(t)) = 0 \quad \text{in} \quad (a_i, b_i) \\ y(a_i) = \varphi(a_i) \quad \text{and} \quad y(b_i) = \varphi(b_i). \end{cases}$$

Similarly, in the case where we consider a radial $C^2(B_{b_0} \setminus \{0\})$-solution $u(x) = y(|x|)$ of (P_0), we are reduced to the boundary value problem:

$$(\mathrm{BVP}_0) \qquad \mathcal{L}[y](t) = 0 \quad \text{in} \quad (0, b_0)\ ; \ \ y(b_0) = \varphi(b_0).$$

3.1. To establish existence results for (BVP_i) and (BVP_0), we shall apply the following proposition.

Proposition 3.1. *Let* (H1) *and* (H2) *hold. Suppose* $g(t) > 0$ *in* (a_i, b_i) *and* $g(a_i) = g(b_i) = 0$. *Then there exists a unique solution* $y \in C[a_i, b_i] \cap C^2(a_i, b_i)$ *of* (BVP$_i$).

Proof. To prove this, we need the next lemma

Lemma 3.2. *Let* (H1) *and* (H2) *hold. Let* τ_1, τ_2 *be any numbers satisfying* $a_i < \tau_1 < \tau_2 < b_i$, *and let* α_1, α_2 *be any real numbers. Then there exists a unique classical solution* $y \in C[\tau_1, \tau_2] \cap C^2(\tau_1, \tau_2)$ *of* $\mathcal{L}[y] = 0$ *in* (τ_1, τ_2) *satisfying* $y(\tau_1) = \alpha_1$ *and* $y(\tau_2) = \alpha_2$.

Postponing the proof of Lemma 3.2 to Appendix, we proceed with the arguments to prove Proposition 3.1.

STEP 1. We first prove that for each fixed $\tau_0 \in (a_i, b_i)$ and any $\alpha \in \mathbf{R}$, the boundary value problem:

$$\text{(BVP)}^+ \qquad \mathcal{L}[y](t) = 0 \quad \text{in} \quad (\tau_0, b_i)\ ;\ y(\tau_0) = \alpha \quad \text{and} \quad y(b_i) = \varphi(b_i)$$

has a unique solution. To this end, let $\{\tau_n\} \uparrow b_i$. By Lemma 3.2, we get a solution y_n of $\mathcal{L}[y] = 0$ in (τ_0, τ_n) satisfying $y(\tau_0) = \alpha$ and $y(\tau_n) = \varphi(b_i)$. From the proof of Lemma 3.2 it follows that for each fixed n there exist constants M and K_n such that $|y_m(t)| \le M$, $|\dot{y}_m(t)| \le K_n$ for all $t \in [\tau_0, \tau_n]$, $m \ge n$ (cf. (A.5) in Appendix). Thus, applying the Ascoli-Arzera theorem, we find a subsequence $\{y_n(t)\}$ and a continuous function $y_\alpha^+(t)$ such that $y_n(t)$ converges to $y_\alpha^+(t)$ locally uniformly in $[\tau_0, b_i)$. Remarking that $y_{\alpha\beta}(t)$ is a C^2-solution of $\mathcal{L}[y] = 0$ satisfying $y(\tau_0) = \alpha$ and $\dot{y}(\tau_0) = \beta$ if and only if $y_{\alpha\beta}$ satisfies

$$\text{(3.1)} \quad y_{\alpha\beta}(t) = \alpha + \tau_0\beta(\log t - \log\tau_0) + \int_{\tau_0}^t (\log t - \log s) s g(s)^{-1} f(s, y_{\alpha\beta}(s)) ds$$

$$\text{(3.2)} \quad \dot{y}_{\alpha\beta}(t) = \frac{1}{t}\left(\tau_0\beta + \int_{\tau_0}^t s g(s)^{-1} f(s, y_{\alpha\beta}(s)) ds\right),$$

we see $y_\alpha^+(t)$ satisfies $\mathcal{L}[y_\alpha^+] = 0$ in $[\tau_0, b_i)$. By (H1) and the maximum principle, we see easily $y_\alpha^+(t)$ converges as $t \uparrow b_i$ provided we allow the limit to be $\pm\infty$. We now suppose $\lim_{t\to b_i} y_\alpha^+(t) = +\infty$. Then there is a $t^*, \tau_0 < t^* < b_i$, such that $y_\alpha^+(t^*) > M$. Thus, for sufficiently large m, $y_m(t^*) > M$ and $t^* < \tau_m < b_i$. Since $y_m(\tau_m) = \varphi(b_i)$, y_m has a local maximum $y_m(\hat{t}_m)$ with $\hat{t}_m < \tau_m$ satisfying $y_m(\hat{t}_m) > M$. This is a contradiction by the maximum principle. Similarly, the case $\lim_{t\to b_i} y_\alpha^+(t) = -\infty$ cannot occur. Thus $y_\alpha^+(t)$ converges to $y_\alpha^+(b_i)$ as $t \to b_i$.

We next show $y_\alpha^+(b_i) = \varphi(b_i)$. In case: $\lim_{t\to b_i} \int_{\tau_0}^t (\log t - \log s) s g(s)^{-1} ds < +\infty$, we easily see $y_\alpha^+(b_i) = \varphi(b_i)$ by (3.1) and (3.2). We consider the case:

$\lim_{t\to b_i}\int_{\tau_0}^t(\log t - \log s)sg(s)^{-1}ds = +\infty$. Suppose $y_\alpha^+(b_i) > \varphi(b_i)$. Then, by (3.1), $\lim_{t\to b_i} y_\alpha^+(t) = \infty$; this is a contradiction. Similarly, $y_\alpha^+(b_i) < \varphi(b_i)$ cannot occur.

Therefore, we conclude that $y_\alpha^+ \in C[\tau_0, b_i] \cap C^2(\tau_0, b_i)$ is a solution of (BVP)$^+$. We obtain the uniqueness of solutions of (BVP)$^+$ by (H1) and the maximum principle.

STEP 2. For each fixed $\tau_0 \in (a_i, b_i)$ and every $\alpha \in \mathbf{R}$, we denote the solution of (BVP)$^+$ obtained in Step 1 by $y_\alpha^+(t)$ in $[\tau_0, b_i]$. Put $p(\alpha) := \dot{y}_\alpha^+(\tau_0)$. In this step, we prove that

(i) If $\alpha_1 < \alpha_2$ then $y_{\alpha_1}^+(t) \le y_{\alpha_2}^+(t)$ for all $t \in [\tau_0, b_i]$ and $p(\alpha_1) > p(\alpha_2)$.

(ii) $p(\alpha)$ is continuous.

(iii) $\lim_{\alpha\to\infty} p(\alpha) = -\infty$ and $\lim_{\alpha\to-\infty} p(\alpha) = +\infty$.

Proof of (i). It is easy to see, by (H1) and the maximum principle, that $y_{\alpha_1}^+(t) \le y_{\alpha_2}^+(t)$ for all $t \in [\tau_0, b_i]$. To prove $p(\alpha_1) > p(\alpha_2)$ by contradiction, we suppose $p(\alpha_1) \le p(\alpha_2)$. Then, by (3.2), $\dot{y}_{\alpha_2}(t) - \dot{y}_{\alpha_1}(t) > 0$ for all $t \in [\tau_0, b_0]$. Hence $y_{\alpha_2}(b_0) - y_{\alpha_1}(b_0) > 0$; this contradicts $y_{\alpha_2}(b_0) - y_{\alpha_1}(b_0) = \varphi(b_0) - \varphi(b_0) = 0$. $\square$

Proof of (ii). Let $\{\alpha_n\} \downarrow \alpha$ as $n \to \infty$. By (i) above, $p(\alpha_n) \uparrow p^*(\le p(\alpha))$ and $y_{\alpha_n}^+(t) \to y^+(t)$ for all $t \in [\tau_0, b_i]$ as $n \to \infty$. Letting $n \to \infty$ in (3.1) replaced y by $y_{\alpha_n}^+$, we see that y^+ also satisfies (3.1) with $y = y^+, \alpha = \alpha$, and $\beta = p^*$. From $y_{\alpha_n}^+(b_i) = \varphi(b_i)(n = 1, 2, \ldots)$ it follows $y_{\alpha_n}^+(b_i) \to y^+(b_i) = \varphi(b_i)$ as $n \to \infty$. Thus y^+ is a C^2-solution of (BVP)$^+$, hence, by the uniqueness, $y_\alpha^+(t) = y^+(t)$ and $p^* = p(\alpha)$. This implies $p(\alpha_n) \uparrow p(\alpha)$. Similarly, if $\{\alpha_n\} \uparrow \alpha$ as $n \to \infty$ then $p(\alpha_n) \downarrow p(\alpha)$. Consequently $p(\alpha)$ is continuous. $\square$

Proof of (iii). Let $\{\alpha_n\} \uparrow +\infty$ as $n \to \infty$. To prove $p(\alpha_n) \downarrow -\infty$ by contradiction, we suppose $\{p(\alpha_n)\}$ is bounded from below, i.e., $p(\alpha_n) \ge -C_0, n = 1, 2, \ldots$. Put $\bar{M} = \max_{a_i\le t\le b_i}|\varphi(t)|$. Since $y_{\alpha_n}^+(\tau_0) = \alpha_n \to \infty$ as $n \to \infty$ and $y_{\alpha_n}^+(b_i) = \varphi(b_i)$, there exists a $\{\tilde{t}_n\} \subseteq (\tau_0, b_i)$ such that $y_{\alpha_n}^+(\tilde{t}_n) = \bar{M}$ and $y_{\alpha_n}^+(t) > \bar{M}$ for all $t \in [\tau_0, \tilde{t}_n)$. By (3.2), for $\tau_0 \le t < \tilde{t}_n, \dot{y}_{\alpha_n}^+(t) > (\tau_0/t)p(\alpha_n) \ge -C_0$. On the other hand, by the mean value theorem, there is a $\xi_n \in (\tau_0, \tilde{t}_n)$ such that $y_{\alpha_n}^+(\tilde{t}_n) - y_{\alpha_n}^+(\tau_0) = \dot{y}_{\alpha_n}^+(\xi_n)(\tilde{t}_n - \tau_0)$. Hence we have

$$-C_0 \le \dot{y}_{\alpha_n}^+(\xi_n) \le \frac{\bar{M} - \alpha_n}{b_i - \tau_0} \longrightarrow -\infty \quad (n \to \infty).$$

This is a contradiction; (iii) is proved. $\square$

STEP 3. We are going to complete the proof of Proposition 3.1. By similar arguments in Steps 1 and 2, there is a unique solution $y_\alpha^- \in C[a_i, \tau_0] \cap C^2(a_i, \tau_0)$ of

$$\text{(BVP)}^- \qquad \mathcal{L}[y](t) = 0 \quad \text{in} \quad (a_i, \tau_0)\ ;\ y(a_i) = \varphi(a_i) \quad \text{and} \quad y(\tau_0) = \alpha,$$

where τ_0 is the same number as in Step 2. Denote $\dot{y}_\alpha^-(\tau_0) = q(\alpha)$. By the same way as in Step 2, we see that

(i)′ If $\alpha_1 < \alpha_2$ then $y_{\alpha_1}^-(t) \le y_{\alpha_2}^-(t)$ for all $t \in [a_i, \tau_0]$ and $q(\alpha_1) < q(\alpha_2)$.

(ii)′ $q(\alpha)$ is continuous.

(iii)′ $\lim_{\alpha\to\infty} q(\alpha) = +\infty$ and $\lim_{\alpha\to-\infty} q(\alpha) = -\infty$.

By (i)-(iii) in Step 2 and (i)′-(iii)′ above, there is a unique α^* such that $p(\alpha^*) = q(\alpha^*)$. Putting

$$y_i^*(t) = \begin{cases} y_{\alpha^*}^-(t) & a_i \le t \le \tau_0 \\ y_{\alpha^*}^+(t) & \tau_0 \le t \le b_i, \end{cases}$$

we see that y_i^* is a unique C^2-solution of (BVP$_i$), because the uniqueness follows from (H1) and the maximum principle. The proof of Proposition 3.1 is complete. □

REMARK 3.3. For our later argument, it is important to note the following: For every $\tau \in (a_i, b_i)$ and every $\alpha \in \mathbf{R}$, let y_α be the unique solution of $\mathcal{L}[y](t) = 0$ in (τ, b_i) satisfying $y_\alpha(\tau) = \alpha$ and $y_\alpha(b_i) = \varphi(b_i)$. Then

1) In the case $\lim_{t\uparrow b_i} \int_\tau^t (\log t - \log s) s g(s)^{-1} ds = +\infty$. For every $\tau_1 \in (\tau, b_i)$ and $\gamma_1 > y_\alpha(\tau_1)$ (resp. $\gamma_1 < y_\alpha(\tau_1)$) there is a blowup (resp. blowdown) solution $y_1 \in C^2(\tau_0, T_{y_1})$ of $\mathcal{L}[y](t) = 0$ in $(\tau, T_{y_1}), \tau_1 < T_{y_1} \le b_i$, such that $y_1(\tau) = \alpha, y_1(\tau_1) = \gamma_1$, and $\lim_{t\uparrow T_{y_1}} y_1(t) = +\infty$ (resp. $\lim_{t\uparrow T_{y_1}} y_1(t) = -\infty$).

2) In the case $\lim_{t\uparrow b_i} \int_\tau^t (\log t - \log s) s g(s)^{-1} ds < \infty$ and $\int_\tau^{b_i} g(s)^{-1} ds = +\infty$. For every $\gamma_1 > \varphi(b_i)$ (resp. $\gamma_1 < \varphi(b_i)$) there is a solution $y_1 \in C[\tau, b_i] \cap C^2(\tau, b_i)$ of $\mathcal{L}[y] = 0$ in (τ, b_i) such that $y_1(\tau) = \alpha, y_1(b_i) = \gamma_1$, and $\lim_{t\uparrow b_i} \dot{y}_1(t) = +\infty$ (resp. $\lim_{t\uparrow b_i} \dot{y}_1(t) = -\infty$).

Of course, we can show an analogous remark in the direction of the left.

3.2. We next consider the boundary value problem (BVP$_0$) associated to (P$_0$).

Proposition 3.4. *Let* (H1) *and* (H2) *hold. Suppose* $g(t) > 0$ *in* $(0, b_0)$ *and* $g(b_0) = 0$. *Then there exists a unique solution* $y \in C[0, b_0] \cap C^2(0, b_0)$ *of* (BVP$_0$).

Proof. Let $\{\tau_n\} \downarrow 0$ as $n \to \infty$. Proceeding as in Step 1 in the proof of Proposition 3.1, we obtain solutions $y_n \in C[\tau_n, b_0] \cap C^2(\tau_n, b_0)$ of

$$\mathcal{L}[y](t) = 0 \quad \text{in} \quad (\tau_n, b_0)\ ; \ y(\tau_n) = \alpha_0 \quad \text{and} \quad y(b_0) = \varphi(b_0),$$

where $\alpha_0 > 0$ is a large constant so that $\alpha_0 > \max_{0\le t\le b_0} |\varphi(t)| + 1$. Since

$$\min_{0\le t\le b_0} \varphi(t) \le y_{n+1}(t) \le y_n(t) \quad \text{for} \quad \forall t \in [\tau_n, b_0],$$

$\{y_n(t)\}$ converges to a continuous function y_0^* locally uniformly in $(0, b_0)$. By (3.2),

$\{\dot{y}_n(t)\}$ also converges to $\dot{y}_0^*(t)$ locally uniformly in $(0, b_0)$. Now fix $0 < \tau_0 < b_0$. Letting $n \to \infty$ in (3.1) with $y = y_n$, $\alpha = y_n(\tau_0)$ and $\beta = \dot{y}_n(\tau_0)$, we obtain

$$\begin{aligned} y_0^*(t) = y_0^*(\tau_0) + \tau_0 \dot{y}_0^*(\tau_0) \log\left(\frac{t}{\tau_0}\right) \\ + \int_{\tau_0}^t (\log t - \log s) s g(s)^{-1} f(s, y_0^*(s)) ds \end{aligned} \tag{3.3}$$

for all $t \in (0, b_0)$, hence y_0^* satisfies $\mathcal{L}[y_0^*](t) = 0$ in $(0, b_0)$. By the same reason as before, $y_0^*(t)$ converges as $t \downarrow 0$. Denoting the limit by $y_0^*(0)$, we see $y_0^* \in C[0, b_0] \cap C^2(0, b_0)$ is a solution of (BVP$_0$).

It remains to prove the uniqueness for (BVP$_0$). To this end we divide our considerations into three cases.

CASE 1. $\int_{+0} \log s \cdot s g(s)^{-1} ds > -\infty$. Clearly, in this case, $\int_{+0} s g(s)^{-1} ds < +\infty$. Writing (3.3) in the form

$$\begin{aligned} y_0^*(t) = y_0^*(\tau_0) - \tau_0 \dot{y}_0^*(\tau_0) \log \tau_0 + \int_0^{\tau_0} \log s \cdot s g(s)^{-1} f(s, y_0^*(s)) ds \\ + C^* \log t + \int_0^t (\log t - \log s) s g(s)^{-1} f(s, y_0^*(s)) ds, \end{aligned}$$

where $C^* = \tau_0 \dot{y}_0^*(\tau_0) - \int_0^{\tau_0} s g(s)^{-1} f(s, y_0^*(s)) ds$, and noting $y_0^*(t)$ converges to $y_0^*(0)$ as $t \downarrow 0$, we get $C^* = 0$. Thus

$$\dot{y}_0^*(t) = \frac{1}{t} \int_0^t s g(s)^{-1} f(s, y_0^*(s)) ds. \tag{3.4}$$

We now suppose that (BVP$_0$) has two solutions $y_1(t), y_2(t)$, and $y_1(0) > y_2(0)$. Then by (H1) and (3.4) for sufficiently small $t > 0$, we have $\dot{y}_1(t) - \dot{y}_2(t) > 0$, and hence $z(t) := (y_1 - y_2)(t)$ is increasing at $t = 0$. Since $z(b_0) = \varphi(b_0) - \varphi(b_0) = 0$, z takes its positive maximum over $[0, b_0]$ at $\hat{t} \in (0, b_0)$; this contradicts the maximum principle.

CASE 2. $\int_{+0} \log s \cdot s g(s)^{-1} ds = -\infty$ and $\int_{+0} s g(s)^{-1} ds < \infty$. In this case we will prove $\lim_{t \downarrow 0} y_0^*(t) = y_0^*(0) = \varphi(0)$ by contradiction. First we suppose $y_0^*(0) > \varphi(0)$. Without loss of generality we may take $\tau_0 > 0$ as small as we like. Then we may assume $f(s, y_0^*(s)) \geq \delta_0 > 0$ on $[0, \tau_0]$ with some $\delta_0 > 0$. Let us write (3.3) in the form: for all $0 < t < \tau_0$

$$\begin{aligned} y_0^*(t) = y_0^*(\tau_0) - \tau_0 \dot{y}_0^*(\tau_0) \log \tau_0 + \int_t^{\tau_0} \log s \cdot s g(s)^{-1} f(s, y(s)) ds \\ + C^* \log t + \log t \int_0^t s g(s)^{-1} f(s, y_0^*(s)) ds, \end{aligned} \tag{3.5}$$

where C^* is the same constant as that in Case 1. We wish to show $C^* = 0$. Suppose $C^* > 0$. Then, for all $0 < t < \tau_0$,

$$\dot{y}_0^*(t) = \frac{1}{t}\left(C^* + \int_0^t sg(s)^{-1}f(s, y_0^*(s))ds\right) > \frac{C^*}{t},$$

because $f(s, y_0^*(s)) > 0$ for all $0 < s < \tau_0$. From this it follows $y_0^*(t) \le y_0^*(\tau_0) + C^* \log(t/\tau_0)$. Then $\lim_{t\downarrow 0} y_0^*(t) = -\infty$; this is a contradiction. Similarly, $C^* < 0$ cannot occur, whence $C^* = 0$. Thus, remarking the right-hand side of (3.5) $\longrightarrow -\infty$ as $t \downarrow 0$, we get a contradiction. Therefore, $y_0^*(0) \le \varphi(0)$. Similarly, we see $y_0^*(0) \ge \varphi(0)$, hence we conclude that $y_0^*(0) = \varphi(0)$. It is now easy to see, by (H1) and the maximum principle, that the uniqueness for (BVP_0) holds.

CASE 3. $\int_{+0} sg(s)^{-1}ds = +\infty$. In this case, we also prove $y_0^*(0) = \varphi(0)$. Suppose $y_0^*(0) > \varphi(0)$. As in Case 2, we may assume $f(s, y_0^*(s)) \ge \delta_0 > 0$ on $[0, \tau_0]$. Since

$$\dot{y}_0^*(t) \le \frac{1}{t}\left(\tau_0 \dot{y}_0^*(\tau_0) - \delta_0 \int_t^{\tau_0} sg(s)^{-1}ds\right) < -\frac{1}{t}$$

for all $0 < t < t^*$ with sufficiently small t^*, $y_0^*(t) > -\log t + \log t^* + y_0^*(t^*)$. This implies $\lim_{t\downarrow 0} y_0^*(t) = +\infty$; this is a contradiction. Thus $y_0^*(0) \le \varphi(0)$. Similarly, we see $y_0^*(0) \ge \varphi(0)$, hence $y_0^*(0) = \varphi(0)$. As usual, the uniqueness for (BVP_0) is now clear. The proof of Proposition 3.4 is complete. $\square$

REMARK 3.5. By the proof of the existence and uniqueness of solutions for (BVP_0), we can show the following which will be used in our later argument: Let y_0^* be the unique solution of (BVP_0). Then, for every $\tau_1 \in (0, b_0)$ and $\gamma_1 > y_0^*(\tau_1)$ (resp. $\gamma_1 < y_0^*(\tau_1)$), there is a blowup (resp. blowdown) C^2-solution y_1 of $\mathcal{L}[y] = 0$ in (T_{y_1}, b_0), $0 \le T_{y_1} < b_0$, such that $y_1(\tau_1) = \gamma_1$, $y_1(b_0) = \varphi(b_0)$, and $\lim_{t\downarrow T_{y_1}} y_1(t) = +\infty$ (resp. $\lim_{t\downarrow T_{y_1}} y_1(t) = -\infty$).

3.3. In this subsection we shall complete the proof of Theorem 1. We put

$$u^*(x) := \begin{cases} y_0^*(|x|) & (x \in B_{b_0}) \\ y_i^*(|x|) & (x \in \overline{A(a_i, b_i)}) \\ \varphi(|x|) & (|x| \in \mathcal{Z}(g)), \end{cases}$$

where y_i^* (resp. y_0^*) is the solution of (BVP_i) (resp. (BVP_0)) obtained in Proposition 3.1 (resp. Proposition 3.4). Then, it is clear that u^* is radial and standard, and satisfies $\mathcal{F}[u^*](x) = 0$ for all $x \in B_R \setminus \{0\}$ in the viscosity sense. Therefore, it suffices to verify that u^* satisfies $\mathcal{F}[u^*](0) = 0$ in the viscosity sense. It is obvious that if $0 \in \mathcal{Z}(g)$ is a cluster point of $\mathcal{Z}(g)$ then $u^*(0) = \varphi(0)$; hence $\mathcal{F}[u^*](0) = 0$ in the viscosity sense. Thus, it suffices to check that $u_0^*(x) = y_0^*(|x|)$ is a viscosity solution of (P_0).

Since, in the Cases 2 and 3 in the proof of Proposition 3.4, $g(0) = 0$ and $u_0^*(0) = \varphi(0)$, it is clear that u_0^* satisfies $\mathcal{F}[u_0^*](0) = 0$ in the viscosity sense. It remains to consider the Case 1 in the proof of Proposition 3.4.

CASE 1-1. $g(0) > 0$. In this case, by (3.4), we have $\lim_{t\downarrow 0} \dot{y}_0^*(t) = 0$. Moreover, noting $\lim_{t\downarrow 0} \dot{y}_0^*(t)/t = f(0, y_0^*(0))/(2g(0))$, we find

$$\lim_{t\downarrow 0} \ddot{y}_0^*(t) = \frac{1}{2g(0)} f(0, y_0^*(0)) := \ddot{y}_0^*(0).$$

For every $(p, X) \in J^{2,+}u_0^*(0)$, we have $p = Du_0^*(0) = 0$ and $u_0^*(x) \le u_0^*(0) + (1/2)\langle Xx, x\rangle + o(|x|^2)$. On the other hand, $u_0^*(x) = u_0^*(0) + (1/2)\ddot{y}_0^*(0)|x|^2 + o(|x|^2)$. Hence $X - \ddot{y}_0^*(0)E \ge O$. Thus $\mathrm{Tr}(X) \ge 2\ddot{y}_0^*(0)$ (Note $N = 2$). Since $-g(0)\,\mathrm{Tr}(X) + f(0, u_0^*(0)) \le -2g(0)\ddot{y}_0^*(0) + f(0, y_0^*(0)) = 0$, u_0^* satisfies $\mathcal{F}[u_0^*](0) \le 0$ in the viscosity sense. Similarly, we see that $-g(0)\,\mathrm{Tr}(X) + f(0, u_0^*(0)) \ge 0$ for all $(p, X) \in J^{2,-}u_0^*(0)$; this means u_0^* satisfies $\mathcal{F}[u_0^*](0) \ge 0$ in the viscosity sense. Consequently, u_0^* is a viscosity solution of (P_0).

CASE 1-2. $g(0) = 0$ and $\int_{+0} \log s \cdot sg(s)^{-1} ds > -\infty$. Without loss of generality, we may assume $\lim_{t\downarrow 0} y_0^*(t) = y_0^*(0) > \varphi(0)$. Taking $t_0 > 0$ sufficiently small, we have $f(s, y_0^*(s)) \ge \delta_0 > 0$ for all $t \in [0, t_0]$ with some $\delta_0 > 0$. In this case, we first note that u_0^* satisfies $\mathcal{F}[u_0^*](0) \ge 0$ in the viscosity sense, and $\underline{\lim}_{t\downarrow 0} \dot{y}_0^*(t) = \underline{\lim}_{t\downarrow 0} tg(t)^{-1} f(t, y_0^*(t)) \ge 0$ by our assumption: $y_0^*(0) > \varphi(0)$. It is easy to see that if $\underline{\lim}_{t\downarrow 0} \dot{y}_0^*(t) > 0$ then $J^{2,+}u_0^*(0) = \phi$, hence u_0^* satisfies $\mathcal{F}[u_0^*](0) \le 0$ in the viscosity sense. Therefore, u_0^* satisfies $\mathcal{F}[u_0^*](0) = 0$ in the viscosity sense. We next consider the case when $\underline{\lim}_{t\downarrow 0} \dot{y}_0^*(t) = 0$. In this case, from

$$y_0^*(t) \ge y_0^*(0) + \int_0^t (\log t - \log s) sg(s)^{-1} f(s, y_0^*(s)) ds,$$

it follows immediately

$$\begin{aligned}\lim_{t\downarrow 0} \frac{y_0^*(t) - y_0^*(0)}{t^2} &\ge \lim_{t\downarrow 0} \frac{\delta_0}{t^2} \int_0^t (\log t - \log s) sg(s)^{-1} ds \\ &= \frac{\delta_0}{4} \lim_{t\downarrow 0} g(t)^{-1} = +\infty.\end{aligned}$$

This implies $J^{2,+}u_0^*(0) = \phi$, hence u_0^* satisfies $\mathcal{F}[u_0^*](0) \le 0$ in the viscosity sense. Consequently, u_0^* is a viscosity solution of (P_0).

It remains to prove the uniqueness of standard viscosity solutions of (DP). To prove this, we use the following assertion which will be applied frequently in our later discussions.

Lemma 3.6. *Assume that Ω is a domain, $g \in C(\Omega)$ is nonnegative, and that $f(x,r) \in C(\Omega \times \mathbf{R})$ is increasing in r for each fixed $x \in \Omega$. Suppose that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a classical supersolution (resp. subsolution), and that $v \in C(\overline{\Omega})$ is a viscosity subsolution (resp. supersolution) of*

$$\begin{cases} -g(x)\Delta u + f(x,u) = 0 & in \quad \Omega \\ u(x) \geq v(x) \ (resp.\ u(x) \leq v(x)) & on \quad \partial\Omega. \end{cases}$$

Then $u(x) \geq v(x)$ (resp. $u(x) \leq v(x)$) on $\overline{\Omega}$.

The proof of this lemma is standard and easy, and so we omit giving it here.

Suppose that v is an arbitrary standard viscosity solution. It is easy to see by Lemma 3.6 that if $g(t) > 0$ in (a_i, b_i) and $g(a_i) = g(b_i) = 0$, then $v(x) = u_i^*(x)$ for all $x \in \overline{A(a_i, b_i)}$. In the case where $|x_0|$ is a cluster point of $\mathcal{Z}(g)$, noting that the solution $u_i^* \in C(\overline{A(a_i, b_i)}) \cap C^2(A(a_i, b_i))$ of (P_i) satisfies $|u_i^*(x)| \leq \max_{a_i \leq t \leq b_i} |\varphi(t)|$ for $\forall x \in \overline{A(a_i, b_i)}$, we have $u_i^*(x_0) = \lim_{t \to |x_0|} \varphi(t) = \varphi(|x_0|) = v(x_0)$.

Finally we claim $u_0^*(x) = v(x)$ for all $x \in B_{b_0}$ under the condition $g(t) > 0$ in $(0, b_0)$ and $g(b_0) = 0$. By virtue of Lemma 3.6 it suffices to verify $u_0^*(0) = v(0)$. Suppose $u_0^*(0) < v(0)$. Then, putting $\overline{V}(t) = \max_{|x|=t} v(x)$, we have $u_0^*(t_0) < \overline{V}(t_0)$ for $t_0 > 0$ small. By Remark 3.5, there is a C^2-solution $\hat{y}$ of $\mathcal{L}[\hat{y}] = 0$ in $(T_{\hat{y}}, b_0)$ such that $\hat{y}(b_0) = \varphi(b_0)$, $\hat{y}(t_0) = \gamma_0$ with $u_0^*(t_0) < \gamma_0 < \overline{V}(t_0)$, and $\lim_{t \downarrow T_{\hat{y}}} \hat{y}(t) = +\infty$ where $T_{\hat{y}}$ is the blowup time of $\hat{y}$. Putting $\hat{u}(x) = \hat{y}(|x|)$, we have $\hat{y}(\hat{t}) = \overline{V}(\hat{t})$ for some $T_{\hat{y}} < \hat{t} < t_0$. Hence $\hat{u}(x) \geq v(x)$ on $\partial A(\hat{t}, b_0)$. By Lemma 3.6, $\hat{u}(x) \geq v(x)$ for all $x \in \overline{A(\hat{t}, b_0)}$ which is impossible because $\hat{u}(x) = \hat{y}(t_0) = \gamma_0 < \overline{V}(t_0)$ on $|x| = t_0$. Thus $u_0^*(0) \geq v(0)$. Likewise, $u_0^*(0) \leq v(0)$. Hence, applying Lemma 3.6 again, we conclude $v = u_0^*$. The proof of Theorem 1 is complete.

4. The proofs of Theorems 2 and 3

In this section we are concerned with the uniqueness and nonuniqueness of viscosity solutions for (DP).

4.1. In this subsection we prove Theorem 2, that is, the uniqueness of viscosity solutions of (DP) under the assumption (H3). Let $u(x) \in C(\overline{B_R})$ be an arbitrary viscosity solution of (DP). Define for $x \in B_R$

$$\overline{U}(x) = \sup\{u(Qx); Q \in O(N)\} \quad \text{and} \quad \underline{U}(x) = \inf\{u(Qx); Q \in O(N)\},$$

where $O(N)$ denotes the set of orthogonal $N \times N$ matrices. Since $O(N)$ is compact and closed (in the matrix norm), we see $\overline{U}(x) = \max\{u(Qx); Q \in O(N)\}$ and $\underline{U}(x) = \min\{u(Qx); Q \in O(N)\}$. Denote

$$Q^+(x) = \{Q \in O(N); \overline{U}(x) = u(Qx)\}$$

$$Q^-(x) = \{Q \in O(N); \underline{U}(x) = u(Qx)\}.$$

In general, $Q^+(x)$ and $Q^-(x)$ may be multi-valued for $x \in B_R$.

Lemma 4.1. *$\overline{U}(x)$ and $\underline{U}(x)$ are continuous on $\overline{B_R}$.*

Proof. Let us prove that if $x_n \to x$ then $\overline{U}(x_n) \to \overline{U}(x)$. For simplicity, we denote $Q^+(x_n) = Q_n$. By the compactness, there is a subsequence (denote this by $\{Q_n\}$ again) such that $Q_n \to Q^* \in O(N)$ as $n \to \infty$. From

$$\begin{aligned} |u(Q_n x_n) - u(Q^* x)| &\leq |u(Q_n x_n) - u(Q^* x_n)| + |u(Q^* x_n) - u(Q^* x)|, \\ |Q_n x_n - Q^* x_n| &\leq \|Q_n - Q^*\| \cdot |x_n| \to 0 \quad (n \to \infty), \\ |Q^* x_n - Q^* x| &\leq \|Q^*\| \cdot |x_n - x| \to 0 \quad (n \to \infty), \end{aligned}$$

we see

$$\lim_{n\to\infty} u(Q_n x_n) = u(Q^* x). \tag{4.1}$$

On the other hand, it is clear, by the definition of $Q^+(x)$, that

$$u(Q^* x) \leq u(Q^+(x)x) \tag{4.2}$$

$$u(Q^+(x)x_n) \leq u(Q_n x_n). \tag{4.3}$$

Letting $n \to \infty$ in (4.3), we get $u(Q^+(x)x) \leq u(Q^* x)$. Thus, by (4.2), $u(Q^* x) = u(Q^+(x)x)$, that is, $Q^* \in Q^+(x)$. Therefore, by (4.1)

$$|\overline{U}(x_n) - \overline{U}(x)| = |u(Q_n x_n) - u(Q^* x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, $\underline{U}(x)$ is also continuous on $\overline{B_R}$. The proof of Lemma 4.1 is complete. □

Lemma 4.2. (i) *$\overline{U}$ is a radial viscosity subsolution of* (DP).
(ii) *$\underline{U}$ is a radial viscosity supersolution of* (DP).

Proof. We first show that $\overline{U}$ and $\underline{U}$ are radial. Let $x_1, x_2 \in \overline{B_R}$ be such that $|x_1| = |x_2|$. Then there exists $Q_1 \in O(N)$ such that $x_1 = Q_1 x_2$. From

$$\begin{aligned} \overline{U}(x_1) &= \max_{Q\in O(N)} u(Qx_1) = \max_{Q\in O(N)} u(QQ_1 x_2) \\ &= \max_{P\in O(N)} u(Px_2) = \overline{U}(x_2), \end{aligned}$$

here we used the fact $O(N)Q_1 = O(N)$, it follows that $\overline{U}$ is radial. Likewise, $\underline{U}$ is also radial.

We next prove that $\overline{U}$ is a viscosity subsolution. Let $z \in \mathbf{R}^N$ and $(p, X) \in J^{2,+}\overline{U}(z)$. Then

$$\begin{aligned}\overline{U}(y) &\le \overline{U}(z) + \langle p, y - z\rangle + \frac{1}{2}\langle X(y - z), y - z\rangle + o(|y - z|^2)\\ &= u(Q^+(z)z) + \langle p, y - z\rangle + \frac{1}{2}\langle X(y - z), y - z\rangle + o(|y - z|^2)\end{aligned}$$

near z. Noting $\overline{U}(y) \ge u(Q^+(z)y)$ and ${}^tQ^+(z)Q^+(z) = I$, we have

$$\begin{aligned}u(Q^+(z)y) \le\ & u(Q^+(z)z) + \langle Q^+(z)p, Q^+(z)(y - z)\rangle\\ &+ \frac{1}{2}\langle Q^+(z)X^tQ^+(z)(Q^+(z)(y - z)), Q^+(z)(y - z)\rangle\\ &+ o(|Q^+(z)y - Q^+(z)z|^2),\end{aligned}$$

hence $(Q^+(z)p, Q^+(z)X^tQ^+(z)) \in J^{2,+}u(Q^+(z)z)$. Since u is a viscosity solution,

$$-g(|Q^+(z)z|)\,\mathrm{Tr}(Q^+(z)X^tQ^+(z)) + f(|Q^+(z)z|, u(Q^+(z)z)) \le 0,$$

whence

$$-g(|z|)\,\mathrm{Tr}(X) + f(|z|, \overline{U}(z)) \le 0.$$

Therefore, $\overline{U}$ is a viscosity subsolution. Similarly, $\underline{U}$ is a viscosity supersolution. The proof of Lemma 4.2 is complete.

Now we are going to prove the following key lemmas.

Lemma 4.3. *Let* (H1), (H2) *and* (H3) *hold. Let* $x_0 \in B_R$ *be such that* $|x_0| \in \mathcal{Z}(g)$ *and* $g(t) > 0$ *for all* $t \in (|x_0| - \delta, |x_0| + \delta) \setminus \{|x_0|\}$ *with some* $\delta > 0$. *Suppose that either*

$$\int_{|x_0|-\delta}^{|x_0|} g(t)^{-1}dt = +\infty \quad \text{or} \quad \int_{|x_0|}^{|x_0|+\delta} g(t)^{-1}dt = +\infty.$$

holds. Then $\overline{U}(x_0) = \underline{U}(x_0) = \varphi(|x_0|)$.

Proof. For simplicity, we put $a = |x_0|$. We show $\overline{U}(a) \le \varphi(a) \le \underline{U}(a)$ from which it follows immediately $\overline{U}(a) = \underline{U}(a) = \varphi(a)$, under the assumption

$$\int_{a-\delta}^{a} g(t)^{-1}dt = +\infty.$$

We first prove $\overline{U}(a) \le \varphi(a) \le \underline{U}(a)$ in the case $\lim_{t\uparrow a}\int^t(\log t - \log s)sg(s)^{-1}ds = \infty$. Suppose $\overline{U}(a) > \varphi(a)$. By Proposition 3.1, we find C^2-solution $\overline{y}(t)$ of $\mathcal{L}[\overline{y}] = 0$

satisfying $\overline{y}(\tau_0) = \overline{U}(\tau_0)$ and $\overline{y}(a) = \varphi(a)$, where $a - \delta < \tau_0 < a$. Choosing $\tau_1(< a)$ sufficiently close to a so that $\overline{U}(\tau_1) > \overline{y}(\tau_1)$, we find a γ_1 such that $\overline{y}(\tau_1) < \gamma_1 < \overline{U}(\tau_1)$. Then, by Remark 3.3, there is a C^2 blowup solution $z(t)$ of $\mathcal{L}[z] = 0$ in (τ_0, T_z) satisfying $z(\tau_0) = \overline{U}(\tau_0)$ and $z(\tau_1) = \gamma_1$ and $z(t) \nearrow +\infty$ as $t \to T_z$. Thus z intersects $\overline{U}$ at some $\tau_2(< a)$. Remarking that $\overline{U}$ is a viscosity subsolution by Lemma 4.2, we have, by Lemma 3.6, $\overline{U}(t) \leq z(t)$ for all $t \in [\tau_0, \tau_2]$, hence $\overline{U}(\tau_1) \leq z(\tau_1)$; this is a contradiction. Thus $\overline{U}(a) \leq \varphi(a)$. Similarly, in this case, we have $\underline{U}(a) \geq \varphi(a)$.

We next prove $\overline{U}(a) \leq \varphi(a) \leq \underline{U}(a)$ in the case when $\lim_{t\uparrow a} \int^t (\log t - \log s) s g(s)^{-1} ds < \infty$. We again suppose $\overline{U}(a) > \varphi(a)$. In this case, by Remark 3.3, there is a unique solution z of $\mathcal{L}[z] = 0$ in (τ_0, a) satisfying $z(\tau_0) = \overline{U}(\tau_0)$ and $z(a) = \overline{U}(a)$. By Lemma 3.6, we have $\overline{U}(t) \leq z(t)$ for all $t \in [\tau_0, a]$. Since $\dot{z}(t) \to +\infty$ as $t \uparrow a$ by Remark 3.3,

$$\underline{\lim}_{t\uparrow a} \frac{\overline{U}(t) - \overline{U}(a)}{t - a} \geq \lim_{t\uparrow a} \frac{z(t) - z(a)}{t - a} = +\infty. \tag{4.4}$$

We now claim $J^{2,+}\overline{U}(a) \neq \phi$. Without loss of generality, we may assume that $\overline{U}$ is nondecreasing in $(a, a + \delta')$ with $\delta' > 0$ small, and that $\overline{U}(t_1) > \overline{U}(a)$ for some $t_1 \in (a, a + \delta')$. Define

$$w(t) := -M(t - t_1)^2 + \overline{U}(t_1) \quad \text{for} \quad \forall t \in [a, t_1],$$

where $M = (\overline{U}(t_1) - \overline{U}(a))/(t_1 - a)^2$. It is easy to see that $\mathcal{L}[w](t) \geq 0$ for all $t \in (a, t_1]$ if δ' is small so that $0 < \delta' < a$. Hence, by Lemma 3.6, $\overline{U}(t) \leq w(t)$ for all $t \in [a, t_1]$. Combining this with (4.4), we have $J^{2,+}\overline{U}(a) \neq \phi$. Recalling that $\overline{U}$ is a viscosity subsolution, we find that, for every $(p, X) \in J^{2,+}\overline{U}(a)$,

$$0 \geq -g(a)\,\mathrm{Tr}(X) + f(a, \overline{U}(a)) = f(a, \overline{U}(a)),$$

hence $\overline{U}(a) \leq \varphi(a)$; this is a contradiction. Similarly, we have $\underline{U}(a) \geq \varphi(a)$.

In a similar way we can see that $\overline{U}(|x_0|) \leq \varphi(|x_0|) \leq \underline{U}(|x_0|)$ holds under the assumption $\int_{|x_0|}^{|x_0|+\delta} g(t)^{-1} dt = +\infty$. The proof of Lemma 4.3 is complete. □

Lemma 4.4. *Let $x_0 \in B_R$ be such that $x_0 \in \overline{A(a', b')}$, where $g(t) \equiv 0$ in $I' = [a', b']$. Then $\overline{U}(x_0) = \underline{U}(x_0) = \varphi(|x_0|)$.*

Proof. To prove this, it suffices to note that $\mathcal{J}_+ := \{x \in A(a', b') | J^{2,+}\overline{U}(x) \neq \phi\}$ is dense in $A(a', b')$, (cf. [6]). Indeed, there exists a sequence $\{x_n\} \subset \mathcal{J}_+$ such that x_n converges to x_0 as $n \to \infty$. By definition of viscosity subsolutions, we have $f(x_n, \overline{U}(x_n)) \leq 0$, hence $\overline{U}(x_n) \leq \varphi(x_n)$. By the continuity of $\overline{U}$ and φ, we obtain $\overline{U}(x_0) \leq \varphi(|x_0|)$. Likewise, $\underline{U}(x_0) \geq \varphi(|x_0|)$. The proof is complete. □

Lemma 4.5. *Let* (H1), (H2) *and* (H3) *hold. Suppose* $a \in \mathcal{Z}(g)$ *is a cluster point of* $\mathcal{Z}(g)$. *Then*

$$\overline{U}(a) \leq \varphi(a) \leq \underline{U}(a).$$

Proof. In view of (H3) and Lemmas 4.3-4.4, we may assume that there exists a sequence $\{I_n = (a_n, b_n)\}$ of intervals such that

$$a_n < b_n < a_{n+1} \quad \text{and} \quad a_n \nearrow a \quad \text{as} \quad n \to \infty, \tag{4.5}$$

$$g(a_n) = g(b_n) = 0 \quad \text{and} \quad g(t) > 0 \quad \text{in} \quad I_n, n = 1, 2, \ldots. \tag{4.6}$$

$$\int_{a_n}^{b_n} g(t)^{-1} ds < \infty, n = 1, 2, \ldots, \text{and} \quad \sum_{n=1}^{\infty} \int_{a_n}^{b_n} g(t)^{-1} ds = +\infty. \tag{4.7}$$

We now suppose $\overline{U}(a) > \varphi(a)$. Without loss of generality, we may assume

$$f(t, \overline{U}(t)) \geq \delta_0 > 0 \quad \text{for} \quad a_1 \leq t \leq a. \tag{4.8}$$

By Remark 3.3, there exists a unique C^2-solution y_n of $\mathcal{L}[y_n](t) = 0$ in I_n satisfying $y_n(a_n) = \overline{U}(a_n)$ and $y_n(b_n) = \overline{U}(b_n)$. By Lemma 3.6, $\overline{U}(t) \leq y_n(t)$ on $\overline{I}_n$, hence

$$\overline{\lim}_{t\downarrow a_n} \frac{\overline{U}(t) - \overline{U}(a_n)}{t - a_n} \leq \dot{y}_n(a_n) \quad \text{and} \quad \underline{\lim}_{t\uparrow b_n} \frac{\overline{U}(t) - \overline{U}(b_n)}{t - b_n} \geq \dot{y}_n(b_n). \tag{4.9}$$

We may assume

$$\underline{\lim}_{t\downarrow a_1} \frac{\overline{U}(t) - \overline{U}(a_1)}{t - a_1} > -\infty \tag{4.10}$$

and for $t_n = a_n, b_n$

$$\underline{\lim}_{t\uparrow t_n} \frac{\overline{U}(t) - \overline{U}(t_n)}{t - t_n} \leq \overline{\lim}_{t\downarrow t_n} \frac{\overline{U}(t) - \overline{U}(t_n)}{t - t_n}, \quad n = 1, 2, \ldots. \tag{4.11}$$

In fact, if $\liminf_{t\downarrow a_1}(\overline{U}(t) - \overline{U}(a_1))/(t - a_1) = -\infty$ then $J^{2,+}\overline{U}(a_1) \neq \phi$. By definition of viscosity subsolution, $f(a_1, \overline{U}(a_1)) \leq 0$ which contradicts (4.8). If (4.11) does not hold for some n_0, then $J^{2,+}\overline{U}(t_{n_0}) \neq \phi$, hence $f(t_{n_0}, \overline{U}(t_{n_0})) \leq 0$ which contradicts (4.8).

Put

$$\gamma_n := \overline{\lim}_{t\downarrow b_n} \frac{\overline{U}(t) - \overline{U}(b_n)}{t - b_n}, \quad n = 1, 2, \ldots.$$

Clearly, by (3.2), (4.9) and (4.11) with $t_n = b_n$,

$$b_n \gamma_n \geq b_n \dot{y}_n(b_n) \geq a_n \dot{y}_n(a_n) + \int_{a_n}^{b_n} s g(s)^{-1} f(s, y_n(s)) ds. \tag{4.12}$$

For every $\delta_n > 0$, we define

$$w_n(t) := b_n(\gamma_n - \delta_n)\log\left(\frac{t}{b_n}\right) + \overline{U}(b_n) \quad \text{for} \quad b_n \le t \le a_{n+1}.$$

Since $\ddot{w}_n(t) + (1/t)\dot{w}_n(t) = 0$ for all $t \in [b_n, a_{n+1}]$ and $a_{n+1} - b_n$ may be assumed to be sufficiently small, we see w_n is a C^2 supersolution of $\mathcal{L}[y] = 0$ in (b_n, a_{n+1}). Noting that there is a t_n^* such that $0 < t_n^* - b_n \ll 1$ and $\overline{U}(t_n^*) > w_n(t_n^*)$, and that $\overline{U}(t) - w_n(t)$ is nondecreasing in $[t_n^*, a_{n+1}]$, we have

$$\begin{aligned} \underline{\lim}_{t\uparrow a_{n+1}} \frac{\overline{U}(t) - \overline{U}(a_{n+1})}{t - a_{n+1}} &\ge \lim_{t\uparrow a_{n+1}} \frac{w_n(t) - w_n(a_{n+1})}{t - a_{n+1}} \\ &= \dot{w}_n(a_{n+1}) = \frac{b_n(\gamma_n - \delta_n)}{a_{n+1}}. \end{aligned} \tag{4.13}$$

Using (4.9), (4.11) with $t_{n+1} = a_{n+1}$, and (4.13), we have

$$\frac{b_n(\gamma_n - \delta_n)}{a_{n+1}} \le \overline{\lim}_{t\downarrow a_{n+1}} \frac{\overline{U}(t) - \overline{U}(a_{n+1})}{t - a_{n+1}} \le \dot{y}_{n+1}(a_{n+1}). \tag{4.14}$$

Therefore, by (3.2), (4.12), and (4.14),

$$\begin{aligned} \dot{y}_{n+1}(b_{n+1}) &\ge \frac{1}{b_{n+1}}\left(b_n(\gamma_n - \delta_n) + \int_{a_{n+1}}^{b_{n+1}} sg(s)^{-1} f(s, y_{n+1}(s))ds\right) \\ &\ge \frac{1}{b_{n+1}}\left(a_n\dot{y}_n(a_n) + \int_{a_n}^{b_n} sg(s)^{-1} f(s, y_n(s))ds\right. \\ &\qquad\qquad \left. + \int_{a_{n+1}}^{b_{n+1}} sg(s)^{-1} f(s, y_{n+1}(s))ds - b_n\delta_n\right) \\ &\ge \\ &\vdots \\ &\ge (1/b_{n+1})\left(a_1\dot{y}_1(a_1) + \sum_{i=1}^{n+1}\int_{a_i}^{b_i} sg(s)^{-1} f(s, y_i(s))ds - \sum_{i=1}^{n} b_i\delta_i\right). \end{aligned}$$

Taking $\delta_n = 1/2^n$, we see, by (4.7) and (4.10), that $\dot{y}_{n+1}(b_{n+1}) \longrightarrow +\infty$ as $n \to \infty$. Thus, for arbitrary large $K > 0$, we find $\dot{y}_n(b_n) > K$ for large n. Noting that $\dot{y}_n(b_n) \le \gamma_n$, and that $\overline{U}(t) - w_n(t)$ is increasing in $(t_n^*, a), b_n < t_n^* < a$, we have

$$\underline{\lim}_{t\uparrow a} \frac{\overline{U}(t) - \overline{U}(a)}{t - a} \ge \lim_{t\uparrow a} \frac{w_n(t) - w_n(a)}{t - a} = \frac{b_n(\gamma_n - \delta)}{a} > \frac{K - \delta}{2},$$

which shows, from the arbitrariness of K, $\underline{\lim}_{t\uparrow a}(\overline{U}(t) - \overline{U}(a))/(t - a) = +\infty$. Hence, by usual arguments, $J^{2,+}\overline{U}(a) \ne \phi$, which implies $f(a, \overline{U}(a)) \le 0$; this is a contradiction.

In a similar way, $\underline{U}(a) \geq \varphi(a)$ is proved. The proof of Lemma 4.5 is complete. $\square$

It is now easy to complete the proof of Theorem 2. Indeed, Lemmas 4.3–4.5 imply that if $|x_0| \in \mathcal{Z}(g)$ then $u(x_0) = \varphi(|x_0|)$. By Lemmas 4.1, 4.2 and 3.6, we obtain $\overline{U}(x) \leq u^*(x) \leq \underline{U}(x)$ in $\overline{A(a_i, b_i)}$, hence $u(x) = u^*(x)$ on $\overline{A(a_i, b_i)}$. On the other hand, by recalling the proof of Theorem 1, we have $u(0) = u^*(0)$. Therefore, $u(x) = u^*(x)$ on $\overline{B_{b_0}}$. Consequently, we see that $u = u^*$ on $\overline{B_R}$. The proof of Theorem 2 is complete. $\square$

4.2. In this subsection we prove Theorem 3, that is, the nonuniqueness of viscosity solutions for (DP) under the assumption (H4). As before, for each given $\alpha \in \mathbf{R}$, let us consider two boundary value problems:

$$\text{(BVP)}^- \qquad \mathcal{L}[y](t) = 0 \quad \text{in} \quad (0, a) \; ; \; y(a) = \alpha$$

and

$$\text{(BVP)}^+ \qquad \mathcal{L}[y](t) = 0 \quad \text{in} \quad (a, R) \; ; \; y(a) = \alpha \quad \text{and} \quad y(R) = \beta.$$

By Proposition 3.1, we find a solution $y_\alpha^-(t)$ and $y_\alpha^+(t)$ of (BVP)^- and (BVP)^+, respectively. We put

$$p(\alpha) := \dot{y}_\alpha^-(a) \quad \text{and} \quad q(\alpha) := \dot{y}_\alpha^+(a).$$

As in the proof of Proposition 3.1, we verify that the following three assertions:
(i) If $\alpha_1 < \alpha_2$ then $y_{\alpha_1}^-(t) \leq y_{\alpha_2}^-(t)$ for $\forall t \in [0, a]$ (resp. $y_{\alpha_1}^+(t) \leq y_{\alpha_2}^+(t)$ for $\forall t \in [a, R]$) and $p(\alpha_1) < p(\alpha_2)$ (resp. $q(\alpha_1) > q(\alpha_2)$),
(ii) $p(\alpha)$ and $q(\alpha)$ are continuous.
(iii) $\lim_{\alpha\to-\infty} p(\alpha) < 0 < \lim_{\alpha\to\infty} p(\alpha)$, $\lim_{\alpha\to-\infty} q(\alpha) = +\infty$, and $\lim_{\alpha\to\infty} q(\alpha) = -\infty$.
Hence there is a unique $\alpha_0 \in \mathbf{R}$ such that $p(\alpha_0) = q(\alpha_0)$. Define

$$u_\alpha(x) = \begin{cases} y_\alpha^-(|x|) & (x \in \overline{B_a}) \\ y_\alpha^+(|x|) & (x \in \overline{A(a, R)}). \end{cases}$$

We now prove the assertions (i)–(iii) in Theorem 3. The assertion (i) is evident. To prove (ii), suppose $u_0(x) := u_{\alpha_0}(x) > u^*(x) = \varphi(a)$ on $|x| = a$, where u^* is the standard viscosity solution of (DP). Then from $\alpha_0 > \varphi(a)$ it follows that $p(\alpha) = \dot{y}_\alpha^-(a) < \dot{y}_\alpha^+(a) = q(\alpha)$ for $\varphi(a) \leq \forall\alpha < \alpha_0$. Thus $J^{2,+}u_\alpha(x) = \phi$ on $|x| = a$, hence u_α satisfies $\mathcal{F}[u_\alpha](x) \leq 0$ on $|x| = a$ in the viscosity sense. On the other hand, it is also clear that for $\varphi(a) < \alpha < \alpha_0, u_\alpha$ satisfies $\mathcal{F}[u_\alpha](x) \geq 0$ on $|x| = a$ in the viscosity sense. Therefore, $u_\alpha, \varphi(a) < \alpha < \alpha_0$, is a viscosity solution of (DP). By making use of the stability theorem [16], u_{α_0} is a viscosity solution of (DP). To prove that u_{α_0} (resp. u^*) is the maximum (resp. minimum) viscosity solution, we first note that if $\alpha > \alpha_0 > \varphi(a)$ then $\dot{y}_\alpha^-(a) > \dot{y}_\alpha^+(a)$, hence $J^{2,+}u_\alpha(x) \neq \phi$ for $|x| = a$ (cf. [19]), which implies that u_α could not be a viscosity subsolution of (DP). Remarking that if $\alpha < \varphi(a) < \alpha_0$ then $\dot{y}_\alpha^-(a) < \dot{y}_\alpha^+(a)$, we have $J^{2,-}u_\alpha(x) \neq \phi$ for $|x| = a$, hence u_α could not be a viscosity supersolution of (DP). Now, let v be an arbitrary viscosity solution of (DP). Define

$$\overline{V}(x) = \max_{|y|=|x|} v(y) \quad \text{and} \quad \underline{V}(x) = \min_{|y|=|x|} v(y).$$

By Lemma 4.2, $\overline{V}$ (resp. $\underline{V}$) is a viscosity subsolution (resp. supersolution). Therefore, as shown above,

$$\varphi(a) = y^*(a) = u^*(x) \leq \underline{V}(x) \leq \overline{V}(x) \leq u_{\alpha_0}(x) = y_{\alpha_0}(a) \quad \text{on} \quad |x| = a.$$

Therefore, we conclude that u^* (resp. u_{α_0}) is the minimum (resp. maximum) viscosity solution. Thus the proof of (ii) is complete. Similarly, we can prove the assertion (iii). The proof of Theorem 3 is complete.

Appendix. The proof of Lemma 3.2

We do not know that Lemma 3.2 has been proved elsewhere. For completeness, we give a proof by applying the known result:

Lemma A.1. *Suppose that $h \in C([\tau_1, \tau_2] \times \mathbf{R} \times \mathbf{R})$ is bounded. Then there exists a solution of the two-point boundary value problem*:

$$\ddot{y}(t) = h(t, y(t), \dot{y}(t)) \quad \textit{in} \quad (\tau_1, \tau_2)\ ;\ y(\tau_1) = \alpha_1 \quad \textit{and} \quad y(\tau_2) = \alpha_2.$$

Proof. See [13] for a proof. □

Proof of Lemma 3.2. We wish to prove the existence and uniqueness of C^2-solutions of

$$\text{(BVP)} \qquad \begin{cases} \ddot{y}(t) = -\dfrac{1}{t}\dot{y}(t) + g(t)^{-1} f(t, y(t)) & \text{in} \quad (\tau_1, \tau_2) \\ y(\tau_1) = \alpha_1 \quad \text{and} \quad y(\tau_2) = \alpha_2. \end{cases}$$

First let us introduce some constants:

$$M := \max\{|\alpha_1|, |\alpha_2|, \max_{\tau_1 \leq t \leq \tau_2} |\varphi(t)|\} \tag{A.1}$$

$$L := \max\left\{g(t)^{-1}|f(t, y)|; \tau_1 \leq t \leq \tau_2, |y| \leq M\right\} \tag{A.2}$$

$$\kappa := \max\left\{\frac{\tau_2}{\tau_1}, \frac{8M}{L\tau_1(\tau_2 - \tau_1)}\right\} \ (> 1) \tag{A.3}$$

$$K := \max\left\{L\kappa, \frac{|\alpha_1 - \alpha_2|}{\tau_2(\tau_2 - \tau_1)}\right\}. \tag{A.4}$$

Using these constants, we define functions $f_M(t, y)$ and $f_{MK}(t, y, p)$ as follows:

$$f_M(t, y) = \begin{cases} g(t)^{-1} f(t, -M) & (y \le -M) \\ g(t)^{-1} f(t, y) & (|y| \le M) \\ g(t)^{-1} f(t, M) & (y \ge M) \end{cases}$$

and

$$f_{MK}(t, y, p) = \begin{cases} \dfrac{\tau_2 K}{t} + f_M(t, y) & (p \le -\tau_2 K) \\ -\dfrac{p}{t} + f_M(t, y) & (|p| \le \tau_2 K) \\ -\dfrac{\tau_2 K}{t} + f_M(t, y) & (p \ge \tau_2 K). \end{cases}$$

Since $f_{MK}(t, y, p)$ is continuous and bounded, we see, by Lemma A.1 and (H1), that there exists a unique solution $y_{MK} \in C[\tau_1, \tau_2] \cap C^2(\tau_1, \tau_2)$ of

$$\ddot{y}(t) = f_{MK}(t, y(t), \dot{y}(t)) \quad \text{in} \quad (\tau_1, \tau_2) \ ; \ y(\tau_1) = \alpha_1 \quad \text{and} \quad y(\tau_2) = \alpha_2.$$

To see y_{MK} is a solution of (BVP), we show

$$|y_{MK}(t)| \le M, \ |\dot{y}_{MK}(t)| \le \tau_2 K \quad \text{for} \quad \forall t \in [\tau_1, \tau_2]. \tag{A.5}$$

It is easy to see $|y_{MK}(t)| \le M$ for all $t \in [\tau_1, \tau_2]$. Let us verify $|\dot{y}_{MK}(t)| \le \tau_2 K$. For the sake of simplicity, we denote y_{MK} by y. We first show that if there is a $t_1 \in [\tau_1, \tau_2]$ such that $\dot{y}(t_1) = \tau_2 K$ then $\dot{y}(t) \le \tau_2 K$ for all $t \in [t_1, \tau_2]$. Suppose there is a $t_2 \in (t_1, \tau_2]$ such that $\dot{y}(t_2) > \tau_2 K$. Then we find $\tilde{t} \in (t_1, t_2]$ such that $\dot{y}(\tilde{t}) > \tau_2 K$ and $\ddot{y}(\tilde{t}) > 0$. Thus, by (A.2) and (A.3), $\ddot{y}(\tilde{t}) \le -K + L < -K + L\kappa \le 0$; this is a contradiction.

We next claim $\dot{y}_{MK}(\tau_1) \le \tau_2 K$ by contradiction. Suppose $\dot{y}_{MK}(\tau_1) > \tau_2 K$. Then there is a $\overline{t} \in (\tau_1, \tau_2)$ such that $\dot{y}(\overline{t}) = \tau_2 K$. Indeed, if $\dot{y}(t) > \tau_2 K$ for all $t \in [\tau_1, \tau_2]$ then, by (A.4), $|\alpha_2 - \alpha_1| = |y(\tau_2) - y(\tau_1)| > |\alpha_2 - \alpha_1|$; this is a contradiction. Noting that $\dot{y}(t) \ge \tau_2 K \ge \tau_2 L\kappa$ for all $t \in [\tau_1, \overline{t}]$, we have by (A.3)

$$2M \ge |y(\overline{t}) - y(\tau_1)| \ge \tau_2 L\kappa(\overline{t} - \tau_1) > \frac{4M}{\tau_2 - \tau_1}(\overline{t} - \tau_1),$$

hence $\overline{t} - \tau_1 < (\tau_2 - \tau_1)/2$. Thus $\tau_2 - \overline{t} > (\tau_2 - \tau_1)/2$. Therefore, by (A.1)–(A.3),

$$2M \geq |y(\tau_2) - y(\bar{t})| \geq \bar{t}\dot{y}(\bar{t}) \log\left(\frac{\tau_2}{\bar{t}}\right) - L\int_{\bar{t}}^{\tau_2} (\log \tau_2 - \log s) s ds$$
$$> \bar{t}K(\tau_2 - \bar{t}) - \frac{L}{4}({\tau_2}^2 - \bar{t}^2) \geq \frac{L\tau_1\kappa}{4}(\tau_2 - \tau_1) \geq 2M;$$

this is a contradiction.

Similarly, $\dot{y}(t) \geq -\tau_2 K$ for all $t \in [\tau_1, \tau_2]$. We conclude, by (A.5), that y_{MK} is a solution of (BVP). The uniqueness is obvious. The proof of Lemma 3.2 is complete. □

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