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ESTIMATES OF HITTING PROBABILITIES FOR A 1-DIMENSIONAL REINFORCED RANDOM WALK

MASAKI TAKESHIMA

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1. Introduction

In this paper, we will discuss the recurrence of a matrix type reinforced random walk $\overrightarrow{X} = \{X_n\}_{n\geq 0}$, with initial weights $\{w(0, j)\}_{j\in\mathbf{Z}}$ and a reinforcing matrix $A = \{a(n, j)\}_{n\in\mathbf{N},\, j\in\mathbf{Z}}$ of non-negative numbers. The transition mechanism of this walk is defined through its weight process $\overrightarrow{W} = \{w(n, j)\}_{n\geq 0,\, j\in\mathbf{Z}}$ in the following manner.

$$(1.1)\qquad \begin{aligned} P[X_{n+1} = j+1 \mid X_n = j,\ \{w(n,i)\}_{i\in\mathbf{Z}}] &= \frac{w(n, j)}{w(n, j-1) + w(n, j)}, \\ P[X_{n+1} = j-1 \mid X_n = j,\ \{w(n,i)\}_{i\in\mathbf{Z}}] &= \frac{w(n, j-1)}{w(n, j-1) + w(n, j)}. \end{aligned}$$

The weight process $\overrightarrow{W}$ is a family of additive functional of $\overrightarrow{X}$, which are defined in terms of A.

$$(1.2)\qquad w(n, j) = w(0, j) + \sum_{l=1}^{\phi(n,j)} a(l, j),$$

where $\phi(n, j)$ is the total number that $\overrightarrow{X}$ crosses the edge $\{j,\ j+1\}$ up to time n;

$$(1.3)\qquad \phi(n, j) = \sum_{l=1}^{n} 1_{\{\{X_{l-1},\, X_l\}=\{j,\, j+1\}\}}$$

where 1_A is an indicator function of a set A. Throughout this paper we call the pair $[\overrightarrow{X}, \overrightarrow{W}]$ simply by a reinforced random walk. We shall abbreviate a reinforced random walk to RRW.

We call a path $\overrightarrow{X}$ recurrent if for every $j \in \mathbf{Z}$, X_n visits j infinitely often, and transient if for every $j \in \mathbf{Z}$, X_n visits j only finitely many times. If there exist $\alpha < \beta$ such that $\alpha \leq X_n \leq \beta$ for all n, then we say that the path $\overrightarrow{X}$ has finite range. We will introduce convergence tests of the a following function. Let $\Phi : (0, \infty)^{\mathbf{N}\cup\{0\}} \to (0, \infty]$ be given by

$$\Phi(\overrightarrow{\alpha}) = \sum_{k=0}^{\infty} \alpha(k)^{-1} \tag{1.4}$$

for every infinite dimensional positive vector $\overrightarrow{\alpha} = \{\alpha(k)\}_{k=0}^{\infty}$. This function plays an important role in this paper. We define the column vector $\overrightarrow{v}_j = \{v(m, j)\}_{m\geq 0}$ by

$$v(m, j) = w(0, j) + \sum_{l=1}^{m} a(l, j), \tag{1.5}$$

and initial weights vectors $\overrightarrow{w}_+ = \{w(0, j)\}_{j\geq 0}$, $\overrightarrow{w}_- = \{w(0, -j-1)\}_{j\geq 0}$, $\overrightarrow{w}_{2,+} = \{w(0, j)^2\}_{j\geq 0}$, $\overrightarrow{w}_{2,-} = \{w(0, -j-1)^2\}_{j\geq 0}$.

Our first result concerns the transience of a RRW.

Theorem 1.1. *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW with a reinforcing matrix* A. *Assume that* $\Phi(\overrightarrow{v}_j) = \infty$ *for all* $j \in \mathbf{Z}$, *where* $\Phi(\overrightarrow{v}_j)$ *is defined through* (1.4) *and* (1.5).

(1) *If* $\Phi(\overrightarrow{w}_+) < \infty$, *then*

$$P[\lim_{n\to\infty} X_n = \infty] > 0.$$

(2) *If* $\Phi(\overrightarrow{w}_-) < \infty$, *then*

$$P[\lim_{n\to\infty} X_n = -\infty] > 0.$$

On the other hand we have a 0-1 law by Sellke.

Theorem (Sellke's 0-1 Law). *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW with a reinforcing matrix* A. *If* $\Phi(\overrightarrow{v}_j) = \infty$ *for all* $j \in \mathbf{Z}$, *then*

$$P[\overrightarrow{X} \text{ is transient}] = 1 \quad \text{or} \quad P[\overrightarrow{X} \text{ is recurrent}] = 1.$$

The proof of Sellke's 0-1 law will be given in Section 5.

From Theorem 1.1 and Sellke's 0-1 law, we obtain the following corollary.

Corollary 1.2. *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW with a reinforcing matrix* A. *Assume that* $\Phi(\overrightarrow{v}_j) = \infty$ *for all* $j \in \mathbf{Z}$. *If* $\Phi(\overrightarrow{w}_+) < \infty$ *or* $\Phi(\overrightarrow{w}_-) < \infty$, *then*

$$P[\overrightarrow{X} \text{ is transient}] = 1.$$

In order to explain this, we will focus on a special class of RRW's. One is a RRW with reinforcement only when the walk moves further away from its starting point. This is expressed by the following condition in terms of A;

(1.6) $$a(2l, j) = 0 \qquad \text{for all } l \in \mathbf{N},\ j \in \mathbf{Z}.$$

We call this RRW by an 'out only' RRW and this matrix A by an 'out only' matrix. Naturally in a similar way we can define an 'in only' RRW and an 'in only' matrix. In this case we have

(1.7) $$a(2l-1, j) = 0 \qquad \text{for all } l \in \mathbf{N},\ j \in \mathbf{Z}.$$

Note that an 'out only' RRW has a tendency to come back, and an 'in only' RRW has a tendency to go out. This is nearly an intuition at the present stage, but it will become clear that these notions are essentially important. In fact we show below that an 'in only' RRW is more likely to be transient than an 'out only' RRW.

The next theorem tells us that if $\Phi(\overrightarrow{w}_+) = \Phi(\overrightarrow{w}_-) = \infty$, then we can find an reinforcing matrix A such that the corresponding RRW is either recurrent or finite range a. s.

Theorem 1.3. *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW. We assume that* $\Phi(\overrightarrow{w}_+) = \infty$, $\Phi(\overrightarrow{w}_-) = \infty$ *and that there exists a positive constant* $C_{1,j}$ *for every* $j \in \mathbf{Z}$ *such that* $\sum_{j=-\infty}^{\infty} C_{1,j} w(0, j)^{-1} < \infty$ *and* $a(2m, j) \le C_{1,j}$ *for all* $m \in \mathbf{N}$ *and* $j \in \mathbf{Z}$. *Then we have*

$$P[\overrightarrow{X} \text{ is recurrent}] + P[\overrightarrow{X} \text{ has finite range}] = 1.$$

REMARK. It is clear that an 'out only' RRW satisfies the above condition. Combining Theorem 1.1 and Theorem 1.3, we can obtain the following corollary.

Corollary 1.4. *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW. Then the following two conditions are equivalent.*
(1) $P[\overrightarrow{X}$ *is recurrent*$] + P[\overrightarrow{X}$ *has finite range*$] = 1$ *for every 'out only' matrix* A.
(2) $\Phi(\overrightarrow{w}_+) = \Phi(\overrightarrow{w}_-) = \infty$.

This contrasts with the following Davis' result.

Theorem ([2]). *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW. Then the following two conditions are equivalent.*
(1) $P[\overrightarrow{X}$ *is recurrent*$] + P[\overrightarrow{X}$ *has finite range*$] = 1$ *for every reinforcing matrix* A.
(2) $\Phi(\overrightarrow{w}_{2,+}) = \Phi(\overrightarrow{w}_{2,-}) = \infty$.

To be more precise, Davis proved the following two statements:
(i) If $\Phi(\overrightarrow{w}_{2,+}) = \Phi(\overrightarrow{w}_{2,-}) = \infty$, then whatever a reinforcing matrix A is, we have

$$P[\overrightarrow{X} \text{ is recurrent}] + P[\overrightarrow{X} \text{ has finite range}] = 1.$$

(ii) Let $\{w(0, j)\}_{j\in\mathbf{Z}}$ be such that $\Phi(\overrightarrow{w}_{2,+}) < \infty$ and $w(0, j) = 1$ for $j < 0$. Set

$$a(l, j) = \begin{cases} w(0, j) & j \geq 0,\ l : \text{even}, \\ 0 & \text{otherwise}. \end{cases} \tag{1.8}$$

Then the corresponding RRW $[\overrightarrow{X}, \overrightarrow{W}]$ satisfies that

$$P[\overrightarrow{X} \text{ is transient}] > 0.$$

In this case, it turns out that $\overrightarrow{X}$ is transient a. s.

Note that the matrix A in (1.8) defines an 'in only' RRW. If $\Phi(\overrightarrow{w}_{+}) < \infty$, then by Corollary 1.2, we know that $P[\overrightarrow{X}$ is transient $] = 1$. An interesting case is that $\Phi(\overrightarrow{w}_{+}) = \infty$ and $\Phi(\overrightarrow{w}_{2,+}) < \infty$. By the above construction we also have $\Phi(\overrightarrow{w}_{-}) = \infty$. Now we change the reinforcing matrix A in (1.8) to 'out only' type by a trivial manner.

$$\tilde{a}(l, j) = \begin{cases} w(0, j) & j \geq 0,\ l : \text{odd}, \\ 0 & \text{otherwise}. \end{cases} \tag{1.9}$$

Then our new RRW $[\tilde{X}, \tilde{W}]$ is 'out only' type, and by Corollary 1.4,

$$P[\tilde{X} \text{ is recurrent}] + P[\tilde{X} \text{ has finite range}] = 1.$$

Further, the definition of $\tilde{A}$ in (1.9) implies that $\Phi(\overrightarrow{v}_j) = \infty$ for all $j \in \mathbf{Z}$, and by Theorem 3.1 of [4] (originally Theorem 6 of [3]) we know that $\tilde{X}$ has a. s. infinite range. Thus $[\tilde{X}, \tilde{W}]$ is a. s. recurrent.

2. Hitting Probability

Let $\tau(l, j)$ be the l-th hitting time of $\overrightarrow{X}$ to j for all $l \in \mathbf{N}$, $j \in \mathbf{Z}$. Namely,

$$\begin{aligned} \tau(0, j) &= 0, \\ \tau(l, j) &= \inf\{n > \tau(l-1, j) \mid X_n = j\} \text{ for } l \in \mathbf{N}. \end{aligned}$$

If the above set is empty, then we put $\tau(l, j) = \infty$.

We can assume without loss of generality that $X_0 = 0$ and $X_1 = 1$. Because if $X_0 = j_0 \neq 0$, we consider a shifted RRW $Y_n = X_n - j_0$, whose weight process is given by $w'(n, j) = w(n, j + j_0)$ for all $n \geq 0$, $j \in \mathbf{Z}$. Obviously $\overrightarrow{Y} = \{Y_n\}_{n\geq 0}$ starts at 0, and recurrence of $\overrightarrow{Y}$ is equivalent to recurrence of $\overrightarrow{X}$. So we can assume that $X_0 = 0$. Then at the first jump X_1 visits either 1 or -1. Since the argument is symmetric we can assume that $X_1 = 1$, too.

Let $k > 0$ be fixed arbitrarily. We look for an estimate of the following probability

$$P[\tau(1, k) < \tau(1, 0) \mid (X_0, X_1) = (0, 1)].$$

Following Davis[1], we divide each path of $\overrightarrow{X}$ into excursions from k. Let $\overrightarrow{j} = (j_m)_{m\geq 0}$ be a possible path of $\overrightarrow{X}$. Since we assume that $X_0 = 0$ and $X_1 = 1$, we have $j_0 = 0$ and $j_1 = 1$. If $\tau(l,k) < \infty$ and $\tau(l+1,k) < \infty$ for some $l \in \mathbf{N}$, $J(l) = (j_{\tau(l,k)+1}, \ldots, j_{\tau(l+1,k)})$ forms an excursion of $\overrightarrow{j}$ starting from a nearest neighbor site of k and ending at k when this sequence first hits the site k. If $\tau(l,k) < \infty$ but $\tau(l+1,k) = \infty$ for some $l \in \mathbf{N}$, we put $J(l) = (j_{\tau(l,k)+1}, \ldots)$, this excursion has an infinite length and never returns to k. In this case, $\overrightarrow{j}$ is divided into $l-1$ excursions of a finite length and a excursion of an infinite length. $J(l)$ is called either an up excursion or a down excursion, according to whether $j_{\tau(l,k)+1} = k+1$ or $j_{\tau(l,k)+1} = k-1$. We put $|J(l)| = \tau(l+1,k) - \tau(l,k)$ i.e, $|J(l)|$ is the length of l-th excursion $J(l)$.

Let $S_k(l)$ denote the direction that the l-th excursion starts, i.e,

$$S_k(l) = u \text{ if the } l\text{-th excursion } J(l) \text{ is an up excursion,}$$
$$S_k(l) = d \text{ if the } l\text{-th excursion } J(l) \text{ is a down excursion.}$$

For each $l \geq 1$ the conditional distribution of $S_k(l+1)$ given the condition that $\#\{1 \leq m \leq l \mid S_k(m) = u\} = n$ is in the following form.

$$\begin{aligned} & P[S_k(l+1) = u \mid \#\{1 \leq m \leq l \mid S_k(m) = u\} = n] \\ = \ & \frac{v(2n,k)}{v(2l-2n+1,k-1) + v(2n,k)} \\ = \ & 1 - P[S_k(l+1) = d \mid \#\{1 \leq m \leq l \mid S_k(m) = u\} = n]. \end{aligned}$$

Let $\mathcal{U}(k)$ and $\mathcal{D}(k)$ be the set of all up and down excursions from k, respectively. Each element $J \in \mathcal{U}(k)$ with $|J| < \infty$ is a sequence $J = (j_1, \ldots, j_n)$, such that $j_1 = k+1$, $j_n = k$, $|j_m - j_{m-1}| = 1$ for $2 \leq m \leq n$, and $j_m \neq k$ unless $m = n$. If $|J| = \infty$ and $J \in \mathcal{U}(k)$, $J = (j_1, \ldots)$, such that $j_1 = k+1$, $|j_m - j_{m-1}| = 1$ for $m \geq 2$, and $j_m \neq k$ for all $m \in \mathbf{N}$. The same is true for $J \in \mathcal{D}(k)$ except that $j_1 = k-1$.

For a given path $\overrightarrow{j} = (j_m)_{m\geq 0}$, let $\overrightarrow{D_k}(l) \in \mathcal{D}(k)$ be the l-th down excursion of $\overrightarrow{j}$ form k, and $\overrightarrow{U_k}(l) \in \mathcal{U}(k)$ be the l-th up excursion of $\overrightarrow{j}$. If the walk starts at a point smaller than k, then we need some additional notation for the part of $\overrightarrow{j}$ before it hits k for the first time.

If $\tau(1,k) < \infty$, we write $\overrightarrow{B_k}$ for the part of $\overrightarrow{j}$ until it hits k, namely $\overrightarrow{B_k} = (j_1, j_2, \ldots, j_{\tau(1,k)})$. Also we write $\overrightarrow{D_k}(0)$ for the part $(j_{\tau(1,k-1)}, \ldots, j_{\tau(1,k)})$ of the path $\overrightarrow{j}$ after first hitting $k-1$ till it hits k for the first time. If $\tau(1,k) = \infty$, $\overrightarrow{B_k}$ is the total path $\overrightarrow{j}$.

It is not difficult to see that $\{(\overrightarrow{B_k}, \overrightarrow{D_k}), (\overrightarrow{S_k}), (\overrightarrow{U_k})\}$ are independent, where $\overrightarrow{D_k} = \{\overrightarrow{D_k}(l)\}_{l\in\mathbf{N}}$, $\overrightarrow{U_k} = \{\overrightarrow{U_k}(l)\}_{l\in\mathbf{N}}$ and $\overrightarrow{S_k} = \{S_k(l)\}_{l\in\mathbf{N}}$ (see [1], Lemma 4.1).

We are mainly interested in estimating the probability

$$P[\tau(1,k) < \tau(1,0) \mid (X_0, X_1) = (0,1)]$$

and its limit as $k \to \infty$. This probability can be expressed as

$$P[\overrightarrow{B_k} \not\ni 0 \mid (X_0, X_1) = (0, 1)]$$

where for a portion $J = (j_l, \dots, j_n)$ of a possible path $\overrightarrow{j} = (j_m)_{m \geq 0}$, $\{J \not\ni 0\}$ implies that $j_m \neq 0$ for $m = l, \dots, n$; the walk does not hit 0 during the time interval $\{l, \dots, n\}$.

Then this probability can be divided into a product of consecutive conditional probabilities:

$$\begin{aligned} &P[\overrightarrow{B_k} \not\ni 0 \mid (X_0, X_1) = (0, 1)] \\ &= P[\overrightarrow{B_2} \not\ni 0 \mid (X_0, X_1) = (0, 1)] \prod_{j=3}^{k} P[\overrightarrow{B_j} \not\ni 0 \mid \{\overrightarrow{B_{j-1}} \not\ni 0\} \cap \{(X_0, X_1) = (0, 1)\}] \\ &= \frac{v(0, 1)}{v(1, 0) + v(0, 1)} \prod_{j=3}^{k} P[\overrightarrow{D_j}(0) \not\ni 0 \mid \{\overrightarrow{B_{j-1}} \not\ni 0\} \cap \{(X_0, X_1) = (0, 1)\}], \end{aligned} \tag{2.1}$$

where the product $\prod_{n=a}^{b} x(n)$ is understood as 1 if $a > b$, for any sequence of real numbers $\{x(n)\}_{n \geq 0}$. In order to estimate the right hand side, we condition on a set with a more detailed information. Let

$$\Lambda_{k,p,l} = \left\{ \begin{matrix} \text{the walk has passed the edge } \{k-1, k\} \text{ just} \\ 2p + 1 \text{ times before the excursion } \overrightarrow{D_{k+1}}(l) \text{ begins} \end{matrix} \right\}.$$

Then dividing $\overrightarrow{D_{k+1}}(l)$ into down excursions from k, we have

$$\begin{aligned} &P\left[\overrightarrow{D_{k+1}}(l) \not\ni 0 \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{q=0}^{l-1} \{\overrightarrow{D_{k+1}}(q) \not\ni 0\} \cap \{(X_0, X_1) = (0, 1)\} \cap \Lambda_{k,p,l} \right] \\ &= \sum_{n \geq p} P\left[\bigcap_{m=p+1}^{n} \{\overrightarrow{D_k}(m) \not\ni 0\} \cap \bigcap_{r=p+1+l}^{n+l} \{S_k(r) = d\} \cap \{S_k(n + l + 1) = u\} \right. \\ &\qquad \left. \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{m=1}^{p} \{\overrightarrow{D_k}(m) \not\ni 0\} \cap \Lambda_{k,p,l} \cap \{(X_0, X_1) = (0, 1)\} \right], \end{aligned}$$

where the intersection $\bigcap_{j=a}^{b} A_j = \Omega$ if $a > b$ for any sequence of subsets of Ω. By the independence the right hand side of the above equality is equal to

$$(2.2)\quad \sum_{n=p}^{\infty} P\left[\{S_k(n+l+1)=u\} \cap \bigcap_{r=p+1+l}^{n+l} \{S_k(r)=d\} \;\middle|\; \Lambda_{k,p,l}\right]$$

$$\times \prod_{m=p+1}^{n} P\left[\overrightarrow{D_k}(m) \not\ni 0 \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{q=0}^{m-1}\{\overrightarrow{D_k}(q) \not\ni 0\} \cap \{(X_0, X_1)=(0,1)\}\right]$$

$$= \sum_{n=p}^{\infty} \left(\prod_{m=p+1}^{n} \frac{v(2m-1,k-1)}{v(2m-1,k-1)+v(2l,k)}\right) \frac{v(2l,k)}{v(2n+1,k-1)+v(2l,k)}$$

$$\times \prod_{m=p+1}^{n} P\left[\overrightarrow{D_k}(m) \not\ni 0 \;\middle|\; \{\overrightarrow{B_{k-1}} \not\ni 0\} \cap \bigcap_{q=0}^{m-1}\{\overrightarrow{D_k}(q) \not\ni 0\} \cap \{(X_0, X_1)=(0,1)\}\right].$$

This allows us to use induction in k to estimate our hitting probability.

3. Positivity of Escape Probability

In this section we prove Theorem 1.1. Since the arguments are similar, we only prove (1) of Theorem 1.1. To this end we will show that

$$(3.1)\qquad P[\overrightarrow{B_k} \not\ni 0 \,|\, (X_0, X_1)=(0,1)] \geq \prod_{j=1}^{k-1} \frac{v(0,j)}{v(0,j)+v(1,0)}.$$

If we have (3.1), then the condition that $\Psi(\overrightarrow{w}_+) = \sum_{j\geq 0} w(0,j)^{-1} < \infty$ implies that the right hand side of (3.1) converges to some positive constant as $k \to \infty$. This proves that

$$(3.2)\qquad P[\limsup_{n\to\infty} X_n = \infty,\ \tau(1,0)=\infty \,|\, (X_0, X_1)=(0,1)] > 0.$$

On the other hand by Sellke's Theorem, the condition that $\Phi(\overrightarrow{v}_j) = \sum_{n\geq 0} v(n,j)^{-1} = \infty$ assumes that with probability 1, every sample path of $\overrightarrow{X}$ is either transient or recurrent. So, (3.2) implies that

$$P[\lim_{n\to\infty} X_n = \infty \,|\, (X_0, X_1)=(0,1)] > 0.$$

To prove (3.1), we prepare an elementary lemma.

Lemma 3.1. *Let c_1, c_2 be positive constants, and let $\{b_m\}_{m\geq 0}$ be an increasing sequence of positive numbers and $\sum_{m=0}^{\infty} b_m^{-1} = \infty$. Then for all $p \geq 0$, we have*

$$\sum_{n=p}^{\infty} \frac{c_1}{b_{2n+1}+c_1} \prod_{m=p+1}^{n} \frac{b_{2m-1}}{b_{2m-1}+c_1} \cdot \frac{b_{2m}}{b_{2m}+c_2} \geq \frac{c_1}{c_1+c_2}. \tag{3.3}$$

Proof. Note that

$$\begin{aligned} &\sum_{n=p}^{\infty} \left(1 - \frac{b_{2n+1}}{b_{2n+1}+c_1} \cdot \frac{b_{2n+2}}{b_{2n+2}+c_2}\right) \prod_{m=p+1}^{n} \left(\frac{b_{2m-1}}{b_{2m-1}+c_1} \cdot \frac{b_{2m}}{b_{2m}+c_2}\right) \\ &= \lim_{N\to\infty} \left(1 - \prod_{m=p+1}^{N} \frac{b_{2m-1}}{b_{2m-1}+c_1} \cdot \frac{b_{2m}}{b_{2m}+c_2}\right), \end{aligned}$$

which exists and is equal to 1 if $\sum_{m=0}^{\infty} b_m^{-1} = \infty$. Thus we look at the infimum of

$$\begin{aligned} \Gamma_n &= \frac{c_1 \cdot (b_{2n+1}+c_1)^{-1}}{1 - b_{2n+1} \cdot (b_{2n+1}+c_1)^{-1} \cdot b_{2n+2} \cdot (b_{2n+2}+c_2)^{-1}} \\ &= \frac{c_2^{-1} + b_{2n+2}^{-1}}{b_{2n+1} \cdot b_{2n+2}^{-1} \cdot c_1^{-1} + c_2^{-1} + b_{2n+2}^{-1}} \end{aligned}$$

Since $\{b_m\}_{m\geq 0}$ is increasing, Γ_n is not smaller than

$$\frac{c_2^{-1}}{c_1^{-1}+c_2^{-1}} = \frac{c_1}{c_1+c_2}. \qquad \square$$

Now we go back to the equation (2.2) and estimate the right hand side from above.

Lemma 3.2. *Under the same condition as in* Theorem 1.1, and the condition that $X_0 = 0$ and $X_1 = 1$, we have for $k \geq 2, l \geq 0$,

$$\begin{aligned} &P\left[\overrightarrow{D_k}(l) \not\ni 0 \;\middle|\; \{\overrightarrow{B_{k-1}} \not\ni 0\} \cap \bigcap_{q=0}^{l-1} \{\overrightarrow{D_k}(q) \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\}\right] \\ &\geq \frac{v(2l, k-1)}{v(2l, k-1) + v(1,0)}. \end{aligned} \tag{3.4}$$

Before proving this lemma, we note that (3.4) actually implies (3.1). Because by (2.1) and (3.4), we have

$$\begin{aligned}
& P[\overrightarrow{B_k} \not\ni 0 \,|\, (X_0, X_1) = (0,1)] \\
= \; & P[\overrightarrow{B_2} \not\ni 0 \,|\, (X_0, X_1) = (0,1)] \\
& \times \prod_{j=3}^{k} P[\overrightarrow{D_j}(0) \not\ni 0 \,|\, \{\overrightarrow{B_{j-1}} \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\}] \\
\geq \; & \frac{v(0,1)}{v(0,1)+v(1,0)} \prod_{j=3}^{k} \frac{v(0,j-1)}{v(0,j-1)+v(1,0)} = \prod_{j=1}^{k-1} \frac{v(0,j)}{v(0,j)+v(1,0)}.
\end{aligned}$$

Here, we used

$$P[\overrightarrow{D_j}(0) \not\ni 0 \,|\, \{\overrightarrow{B_{j-1}} \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\}] \geq \frac{v(0,j-1)}{v(0,j-1)+v(1,0)}, \tag{3.5}$$

which is much simpler than (3.4). But we need to prove a more general inequality if we want to show (3.5) by induction.

Now first we look at the case that $k = 2$. In this case, the left hand side conditional probability in (3.4) is exactly equal to

$$\frac{v(2l,1)}{v(2l,1)+v(1,0)}.$$

Therefore (3.4) is true for $k = 2$.

Now assume the inequality for every $l \geq 0$ at k. Then by (2.2) we have for every $p \geq 0$,

$$\begin{aligned}
& P\left[\overrightarrow{D_{k+1}}(l) \not\ni 0 \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{q=0}^{l-1} \{\overrightarrow{D_{k+1}}(q) \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\} \cap \Lambda_{k,p,l}\right] \qquad (3.6) \\
& = \sum_{n=p}^{\infty} \left(\prod_{m=p+1}^{n} \frac{v(2m-1,k-1)}{v(2m-1,k-1)+v(2l,k)} \right) \frac{v(2l,k)}{v(2n+1,k-1)+v(2l,k)} \\
& \quad \times \prod_{m=p+1}^{n} P\left[\overrightarrow{D_k}(m) \not\ni 0 \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{q=1}^{m-1} \{\overrightarrow{D_k}(q) \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\}\right] \\
& \geq \sum_{n=p}^{\infty} \frac{v(2l,k)}{v(2n+1,k-1)+v(2l,k)} \prod_{m=p+1}^{n} \frac{v(2m-1,k-1)}{v(2m-1,k-1)+v(2l,k)} \\
& \quad \times \prod_{m=p+1}^{n} \frac{v(2m,k-1)}{v(2m,k-1)+v(1,0)}
\end{aligned}$$

by our induction hypothesis. Now we use Lemma 3.1 to obtain that the right hand side of (3.6) is not less than

$$\frac{v(2l,k)}{v(2l,k)+v(1,0)},$$

since $\Phi(\vec{v}_k) = \sum_{l=0}^{\infty} v(l,k)^{-1} = \infty$. This value is independent of p, and we completed the induction. □

4. No Probability of Transience

In this section we prove Theorem 1.3. We will show that if $\Phi(\vec{w}_+)=\Phi(\vec{w}_-)=\infty$, $\sum_{j=-\infty}^{\infty} C_{1,j} w(0,j)^{-1} < \infty$, $a(2m,j) \leq C_{1,j}$ for all $m \in \mathbf{N}$ and $j \in \mathbf{Z}$, then

$$P[\{\vec{X} \text{ visits 0 infinitely often}\} \cup \{\vec{X} \text{ has finite range}\}] = 1. \tag{4.1}$$

This implies that the probability that $\vec{X}$ is transient is equal to zero. The event

$$G = \{\vec{X} \text{ visits 0 infinitely often}\}$$

can be divided into three events:

$$\begin{aligned} G_1 &= \{\text{there exists a number } N \text{ such that } X_n \in \{0,1\} \text{ for all } n \geq N\}, \\ G_2 &= \{\text{there exists a number } N \text{ such that } X_n \in \{0,-1\} \text{ for all } n \geq N\} \end{aligned}$$

and

$$G_3 = \{\vec{X} \text{ is recurrent}\}.$$

To be more precise, the symmetric difference of the events G and $G_1 \cup G_2 \cup G_3$ has probability zero. (This fact is mentioned in the proof of Theorem 3.2 in [4].) Then (1) of Theorem 1.3 holds.

In order to prove (4.1), we will show that

$$P[\{\tau(1,0) < \infty\} \cup \{\vec{X} \text{ has finite range}\} \mid (X_0, X_1) = (0,1)] = 1 \tag{4.2}$$

if $\Phi(\vec{w}_+) = \Phi(\vec{w}_-) = \infty$. By symmetry of the argument this implies that

$$P[\{\tau(1,0) < \infty\} \cup \{\vec{X} \text{ has finite range}\} \mid (X_0, X_1) = (0,-1)] = 1 \tag{4.3}$$

under the same condition that if $\Phi(\vec{w}_+) = \Phi(\vec{w}_-) = \infty$. Then (4.2) and (4.3) shows that

$$P[\{\tau(1,0) < \infty\} \cup \{\vec{X} \text{ has finite range}\} \mid X_0 = 0] = 1. \tag{4.4}$$

Conditioned on the path $\{X_0, X_1, \ldots, X_{\tau(1,0)}\}$, the new RRW $\hat{X} = \{X_{\tau(1,0)+n}\}_{n\geq 0}$ is again a RRW starting at the origin 0, and with initial weights

$\hat{w} = \{w(\tau(1,0), j)\}_{j\in\mathbf{Z}}$ which satisfy the conditions that $\Phi(\hat{w}_+) = \Phi(\hat{w}_-) = \infty$. Repeating the above argument, we have

$$P\left[\begin{array}{l}\{\hat{X} \text{ returns to the origin eventually}\}\\ \cup\{\hat{X} \text{ has finite range}\}\end{array}\middle| X_0, X_1, \dots, X_{\tau(1,0)}\right] = 1. \tag{4.5}$$

Combining (4.4) with (4.5), we obtain that the event

$$\{\overrightarrow{X} \text{ returns at least twice to the orgin}\} \cup \{\overrightarrow{X} \text{ has finite range}\}$$

has probability one.

We can iterate this argument as many times as we want and for every $m \geq 1$, we have

$$P[\{\tau(m,0) < \infty\} \cup \{\overrightarrow{X} \text{ has finite range}\}] = 1. \tag{4.6}$$

Letting $m \to \infty$, we have (4.1). Thus, all that we have to prove is (4.2).

To show (4.2), we will prove that

$$P[\overrightarrow{B_k} \not\ni 0 \,|\, (X_0, X_1) = (0,1)] \leq \left(v(1,0)\Xi_k^+(w)\right)^{-1} \tag{4.7}$$

where

$$\Xi_k^+(w) = v(1,0)^{-1} + \sum_{j=1}^{k-1} v(0,j)^{-1} C_{3,j},$$

and where

$$C_{3,j} = \prod_{i=1}^{j-1} \left(1 + C_{1,i} v(0,i)^{-1}\right)^{-1}.$$

If (4.7) is true, then

$$\sum_{j\geq 0} v(0,j)^{-1} C_{3,j} \geq C_3 \sum_{j\geq 0} v(0,j)^{-1}$$

where

$$C_3 = \lim_{j\to\infty} C_{3,j} = \prod_{i=1}^{\infty} \left(1 + C_{1,i} v(0,i)^{-1}\right)^{-1}.$$

Since $\sum_{j\geq 1} C_{1,j} w(0,j)^{-1} < \infty$, $C_3 > 0$, and since $\Phi(\overrightarrow{w}_+) = \infty$ we have $\sum_{j\geq 0} v(0,j)^{-1} = \infty$. Therefore $\lim_{k\to\infty} \Xi_k^+ = \infty$. Letting $k \to \infty$ in (4.7), we obtain (4.2).

In order to prove (4.7), we prepare the following lemma, which in essence is similar to Lemma 3.1, but we get an opposite inequality.

Lemma 4.1. *Let c_1, c_2, c_3 be positive numbers, and $\{b_m\}_{m\geq 0}$ be an increasing sequence of positive numbers such that $b_{2m} - b_{2m-1} \leq c_3$ for all $m \geq 1$. Then for all $p \geq 0$,*

$$\sum_{n=p}^{\infty} \frac{c_1}{b_{2n+1}+c_1} \prod_{m=p+1}^{n} \frac{b_{2m-1}}{b_{2m-1}+c_1} \cdot \frac{b_{2m}}{b_{2m}+c_2} \leq \frac{b_0^{-1}+c_2^{-1}}{(1+c_3 b_0^{-1})^{-1} c_1^{-1} + b_0^{-1} + c_2^{-1}}. \tag{4.8}$$

Proof. We will have a similar argument as in the proof of Lemma 3.1. We first note that

$$\begin{aligned}&\sum_{n=p}^{\infty}\left(\left(1-\frac{b_{2n+1}}{b_{2n+1}+c_1}\cdot\frac{b_{2n+2}}{b_{2n+2}+c_2}\right)\prod_{m=p+1}^{n}\frac{b_{2m-1}}{b_{2m-1}+c_1}\cdot\frac{b_{2m}}{b_{2m}+c_2}\right)\\ &=\lim_{N\to\infty}\left(1-\prod_{m=p}^{N}\frac{b_{2m-1}}{b_{2m-1}+c_1}\cdot\frac{b_{2m}}{b_{2m}+c_2}\right)\leq 1.\end{aligned}$$

Therefore we only have to estimate

$$\frac{c_1\cdot(b_{2n+1}+c_1)^{-1}}{1-b_{2n+1}\cdot(b_{2n+1}+c_1)^{-1}\cdot b_{2n+2}\cdot(b_{2n+2}+c_2)^{-1}} = \frac{c_2^{-1}+b_{2n+2}^{-1}}{b_{2n+1}\cdot b_{2n+2}^{-1}\cdot c_1^{-1}+c_2^{-1}+b_{2n+2}^{-1}}.$$

Since $\{b_m\}_{m\geq 0}$ is increasing and positive, the last term is not larger than

$$\frac{c_2^{-1}+b_0^{-1}}{(1+c_3 b_0^{-1})^{-1} c_1^{-1} + b_0^{-1} + c_2^{-1}}. \qquad \square$$

Lemma 4.2. *Let $[\overrightarrow{X}, \overrightarrow{W}]$ be a RRW such that $X_0 = 0$ and $a(2n, j) \leq C_{1,j}$ for all $n \in \mathbf{N}$, $j \in \mathbf{Z}$. Then we have for all $k \geq 2, l \geq 0$,*

$$\begin{aligned}&P\left[\overrightarrow{D_k}(l)\not\ni 0 \;\middle|\; \overrightarrow{B_{k-1}}\not\ni 0,\ \bigcap_{q=0}^{l-1}\{\overrightarrow{D_k}(q)\not\ni 0\}\cap\{(X_0,X_1)=(0,1)\}\right]\\ &\leq \frac{v(2l,k-1)}{v(2l,k-1)+C_{3,k-1}\{\Xi_{k-1}^{+}(w)\}^{-1}}.\end{aligned} \tag{4.9}$$

Proof. We prove (4.9) by induction. As before it is straightforward to check (4.9) for every $l \geq 0$ when $k = 2$, since $\Xi_{k-1}^{+}(w) = \Xi_1^{+}(w) = v(1,0)^{-1}$ and $C_{3,k-1} = C_{3,1} = 1$. So we assume that (4.9) is true for k. Now from (2.2) and by the induction hypothesis, we have for every $p \geq 0$,

$$(4.10) P\left[\overrightarrow{D_{k+1}}(l) \not\ni 0 \;\middle|\; \{\overrightarrow{B_k} \not\ni 0\} \cap \bigcap_{q=0}^{l-1} \{\overrightarrow{D_{k+1}}(q) \not\ni 0\} \cap \{(X_0, X_1) = (0,1)\} \cap \Lambda_{k,p,l}\right]$$
$$\leq \sum_{n=p}^{\infty} \left(\prod_{m=p+1}^{n} \frac{v(2m-1,k-1)}{v(2m-1,k-1)+v(2l,k)} \right) \frac{v(2l,k)}{v(2n+1,k-1)+v(2l,k)}$$
$$\times \prod_{m=p+1}^{n} \frac{v(2m,k-1)}{v(2m,k-1)+C_{3,k-1}\{\Xi^+_{k-1}(w)\}^{-1}}.$$

We can use Lemma 4.1 to obtain that the right hand side of (4.10) is not larger than

$$\frac{C^{-1}_{3,k-1}\Xi^+_{k-1}(w)+v(0,k-1)^{-1}}{C^{-1}_{3,k-1}\Xi^+_{k-1}(w)+v(0,k-1)^{-1}+(1+C_{1,k-1}v(0,k-1)^{-1})^{-1}v(2l,k)^{-1}}$$
$$= \frac{\Xi^+_{k-1}(w)+C_{3,k-1}v(0,k-1)^{-1}}{\Xi^+_{k-1}(w)+C_{3,k-1}v(0,k-1)^{-1}+C_{3,k}v(2l,k)^{-1}}$$
$$= \frac{v(2l,k)}{v(2l,k)+C_{3,k}\{\Xi^+_k(w)\}^{-1}}.$$

We completed the induction.

By using Lemma 4.2, from (2.1) we obtain that

$$P[\overrightarrow{B_k} \not\ni 0 \,|\, (X_0,X_1)=(0,1)]$$
$$= \frac{v(0,1)}{v(1,0)+v(0,1)} \prod_{j=3}^{k} P[\overrightarrow{D_j}(0) \not\ni 0 \,|\, \{\overrightarrow{B_{j-1}} \not\ni 0\} \cap \{(X_0,X_1)=(0,1)\}]$$
$$\leq \frac{v(0,1)}{v(1,0)+v(0,1)} \prod_{j=3}^{k} \frac{\Xi^+_{j-1}(w)}{\Xi^+_{j-1}(w)+C_{3,j-1}v(0,j-1)^{-1}}$$
$$= \frac{v(1,0)^{-1}}{v(1,0)^{-1}+v(0,1)^{-1}} \prod_{j=3}^{k} \frac{\Xi^+_{j-1}(w)}{\Xi^+_j(w)} = v(1,0)^{-1}(\Xi^+_k(w))^{-1}.$$

Thus we have (4.7). □

5. Appendix: Sellke's 0-1 law

In this section, we show the following result.

Theorem 5.1 (Sellke). *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW with a reinforcing matrix* A.

If $\Phi(\overrightarrow{v}_j) = \infty$ *for all* $j \in \mathbf{Z}$, *then*

$$P[\overrightarrow{X} \text{ is transient}] = 1 \quad \text{or} \quad P[\overrightarrow{X} \text{ is recurrent}] = 1.$$

The proof of this theorem can be found in [3], but for the sake of completeness, we give a proof.

For simplicity, we assume that $P[X_0 = 0] = 1$. Before going into the proof of Theorem 5.1, we introduce a new notation. Let F and G be subsets of Ω. We use the notation $F \sqsubset G$ if $P[F \setminus G] = 0$.

First, we will prove the following lemma.

Lemma 5.2. *Let* $[\overrightarrow{X}, \overrightarrow{W}]$ *be a matrix type RRW with a reinforcing matrix* A. *If* $\Phi(\overrightarrow{v}_j) = \infty$ *for all* $j \in \mathbf{Z}$, *then*

$$P[\overrightarrow{X} \text{ is transient}] + P[\overrightarrow{X} \text{ is recurrent}] = 1. \tag{5.1}$$

Hence $P[\overrightarrow{X}$ *has finite range*$] = 0$.

Proof of Lemma 5.2. We may show that

$$P[\{\overrightarrow{X} \text{ is not transient}\} \cap \{\overrightarrow{X} \text{ is recurrent}\}^c] = 0. \tag{5.2}$$

Let B_j R_j, L_j, I_j be events defined by
$B_j = \{\overrightarrow{X}$ visits j infinitely often $\}$,
$R_j = \{$there exists $N_j \in \mathbf{N}$ such that if $X_n = j$, then $X_{n+1} = j+1$ for all $n \geq N_j$ $\}$,
$L_j = \{$there exists $N_j \in \mathbf{N}$ such that if $X_n = j$, then $X_{n+1} = j-1$ for all $n \geq N_j$ $\}$,
$I_j = \Omega \setminus \{R_j \cup L_j\}$.
In order to prove (5.1), we only have to show that

$$B_j \sqsubset \left\{ \bigcap_{i \in \mathbf{Z}} B_i \right\} \tag{5.3}$$

for all $j \in \mathbf{Z}$, under the condition that $\Phi(v_i) = \infty$ for all $i \in \mathbf{Z}$.

From the assumption that $\Phi(v_{j-1}) = \Phi(v_j) = \infty$ and Rubin's theorem (See Corollary 3.5 of [4]), it is easy to see that

$$B_j \sqsubset B_j \cap I_j \subset B_{j+1}$$

and

$$B_j \sqsubset B_j \cap I_j \subset B_{j-1}.$$

By induction and the assumption, we see that

$$B_j \sqsubset B_i$$

for all $i \in \mathbf{Z}$. □

To prove Theorem 5.1, we need some more notations.

Let $\{Y_j(m)\}_{j\in\mathbf{Z}, m\geq 0}$ be independent exponential random variables such that $E[Y_j(m)] = v(m, j)^{-1}$ for all $m \geq 0$.

Now, we construct a sequence $\{S_j(l)\}_{l\geq 1}$ using $Y_j^u = \{Y_j^u(m)\}_{m\geq 0} = \{Y_j(2m)\}_{m\geq 0}$ and $Y_j^d = \{Y_j^d(m)\}_{m\geq 0} = \{Y_{j-1}(2m+1)\}_{m\geq 0}$ for all $j > 0$.

We put

$$G_j^u = \left\{ \sum_{m=0}^{n} Y_j(2m) \right\}_{n\geq 0},$$

$$G_j^d = \left\{ \sum_{m=0}^{n} Y_{j-1}(2m+1) \right\}_{n\geq 0}$$

and

$$G_j = G_j^u \cup G_j^d.$$

We note that

$$P\left[\sum_{m=0}^{n} Y_j(2m) = \sum_{m=0}^{l} Y_{j-1}(2m+1) \right] = 0$$

for all $n \geq 0, l \geq 0$.

Let $g_j(n)$ be the n-th smallest number in G_j. Then for each $n \in \mathbf{N}$, we put

$$S_j(n) = \begin{cases} u & \text{if } g_j(n) \in G_j^u, \\ d & \text{if } g_j(n) \in G_j^d. \end{cases}$$

Using this construction, we can obtain for $n \geq 0$, $0 \leq m \leq n$,

$$P[S_j(n+1) = u \mid \Lambda_{m,n}] = \frac{v(2m, j)}{v(2n-2m+1, j-1) + v(2m, j)}$$

where $\Lambda_{m,n} = \{\omega \in \Omega \mid \#\{1 \leq l \leq n \mid S_j(l) = u\} = m\}$.

In a similar way, we can construct a sequence $\{S_j(l)\}_{l\geq 1}$ using $\{Y_j(l)\}_{l\geq 0, j\in\mathbf{Z}}$ for every $j \leq 0$. We put $Y_j^u = \{Y_j(2m)\}_{m\geq 0}$ and $Y_j^d = \{Y_{j-1}(2m)\}_{m\geq 0}$ if $j = 0$. We put $Y_j^u = \{Y_j(2m+1)\}_{m\geq 0}$ and $Y_j^d = \{Y_{j-1}(2m)\}_{m\geq 0}$ if $j < 0$. From construction of $\overrightarrow{S}_j = \{S_j(n)\}_{n\geq 1}$, we remark that $\{\overrightarrow{S}_j\}_{j\in\mathbf{N}}$ are independent.

Given $\{Y_j(l)\}_{l\geq 0, j\in\mathbf{Z}}$, we define a RRW $\overrightarrow{X}$ starting from 0 using only informations of $\{Y_j(l)\}_{l\geq 0, j\in\mathbf{Z}}$ in the following way.

1. For the case $X_0 = 0$, if $Y_0^u(0) < Y_0^d(0)$, then we put $X_1 = 1$, $\phi(1,0) = 1$ and $\phi(1, j) = 0$ for every $j \neq 0$, and if $Y_0^u(0) > Y_0^d(0)$, then we put $X_1 = -1$, $\phi(1,-1) = 1$ and $\phi(1, i) = 0$ for every $i \neq -1$.
2. Assume that we have constructed $[\overrightarrow{X}, \overrightarrow{W}]$ up to time n, and assume that $X_n = j$. Let $m(n, j) = [\phi(n, j)/2]$, where for a real number x, $[x]$ denotes the largest integer which does not exceed x. If $\sum_{l=0}^{m(n,j)} Y_j^u(l) < \sum_{m=0}^{m(n,j-1)} Y_j^d(l)$, then we put $X_{n+1} = j+1$, $\phi(n+1, j) = \phi(n, j)+1$ and $\phi(n+1, i) = \phi(n, i)$ for every $i \neq j$, and if $\sum_{l=0}^{m(n,j)} Y_j^u(l) > \sum_{m=0}^{m(n,j-1)} Y_j^d(l)$, then we put $X_{n+1} = j - 1$, $\phi(n + 1, j - 1) = \phi(n, j - 1) + 1$ and $\phi(n + 1, i) = \phi(n, i)$ for every $i \neq j - 1$.

Proof of Theorem 5.1. From (5.3), we will only show that $\{\overrightarrow{X}$ is transient$\}$ is a tail event under the condition (5.1). We fixed $i \in \mathbf{N}$ and put $F_i = [-i + 1, i]$. Set

$$\mathcal{F}^i = \sigma\{Y_j(m) | m \geq 0,\ |j| \geq i\}.$$

We assume that $\mathcal{F}^i$ contains all P-negligible sets.

From (5.1), we obtain

$$P[X_n \notin F_i \text{ for some } n \geq m \mid X_m = j] = 1$$

for all $m \geq 0$ and $j \in (-i + 1, i)$.

On the other hand, from [1] Lemma 4.2 we see that $\overrightarrow{U_i}(l) \in \mathcal{F}^i$ and $\overrightarrow{D_{-i+1}}(l) \in \mathcal{F}^i$ for every $l \in \mathbf{N}$. Thus we obtain

$$\begin{aligned} H_i &= \{X_m \in F_i \text{ only finitely often}\} \\ &= \{|\overrightarrow{U_{i+1}}(l)| = \infty, \text{ for some } l \in \mathbf{N}\} \cup \{|\overrightarrow{D_{-i}}(l)| = \infty, \text{ for some } l \in \mathbf{N}\} \in \mathcal{F}^i. \end{aligned}$$

But using (5.1) again, we have

$$H_i \sqsubset \{\overrightarrow{X} \text{ is transient}\}.$$

It is clear that

$$\{\overrightarrow{X} \text{ is transient}\} \subset H_i.$$

This means that

$$\{\overrightarrow{X} \text{ is transient}\} \in \mathcal{F}^i$$

for all $i \in \mathbf{N}$. Then we have

$$\{\overrightarrow{X} \text{ is transient}\} \in \bigcap_{i \in \mathbf{N}} \mathcal{F}^i. \qquad \square$$

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Department of Mathematics
Kobe University
Rokkoudai-chou 1-1, Nada-ku
Kobe-City 657-8501, Japan

Current address:
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama-chou 1-1
Toyonaka, Osaka 560-0043, Japan
e-mail: takesima@math.sci.osaka-u.ac.jp