

## A CHARACTERIZATION OF FOUR-GENUS OF KNOTS

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### Introduction

We shall work in piecewise linear category. All knots and links will be assumed to be oriented in a 3-sphere  $S^3$ .

The 4-genus  $g^*(K)$  of a knot  $K$  in  $S^3 = \partial B^4$  is the minimum genus of orientable surfaces in  $B^4$  bounded by  $K$  [1]. The *nonorientable 4-genus*  $\gamma^*(K)$  is the minimum first Betti number of nonorientable surfaces in  $B^4$  bounded by  $K$  [3]. For a slice knot, it is defined to be zero instead of one. The first author [4] defined the *4-dimensional clasp number*  $c^*(K)$  to be the minimum number of the double points of transversely immersed 2-disks in  $B^4$  bounded by  $K$ . He gave an inequality  $g^*(K) \leq c^*(K)$  [4, Lemma 9] and asked whether an equality  $g^*(K) = c^*(K)$  holds or not. For this question, H. Murakami and the second author [3] gave a negative answer, i.e., they proved that there is a knot  $K$  such that  $g^*(K) < c^*(K)$ . Thus  $c^*(K)$  is not enough to characterize  $g^*(K)$ . In this paper we give characterizations of 4-genus and nonorientable 4-genus by using certain 4-dimensional numerical invariants.

The local move as illustrated in Fig. 1(a) (resp. 1(b)) is called an *H-move* (resp. *H'-move*) for some positive integer  $n$ . Both an *H-move* and an *H'-move* realize a crossing change when  $n = 1$ . Thus these moves are certain kinds of unknotting operations of knots. Let  $L_n$  (resp.  $L'_n$ ) be a link as illustrated in Fig. 2(a) (resp. 2(b)). Then we easily see that an *H-move* (resp. *H'-move*) can be realized by a *fusion/fission* [2, p. 95] of  $L_n$  (resp.  $L'_n$ ); see Fig. 3. Therefore, for a knot  $K$  in  $\partial B^4$ , there is a *singular disk*  $D$  in  $B^4$  with  $\partial D = K$  that satisfies the following:

- (1)  $D$  is a locally flat except for points  $p_1, p_2, \dots, p_{m(D)}$  in the interior of  $D$ .
- (2) For each  $p_i$  ( $i = 1, 2, \dots, m(D)$ ) there is a small neighborhood  $N(p_i)$  of  $p_i$  in  $B^4$  such that  $(\partial N(p_i), \partial(N(p_i) \cap D))$  is a link  $L_{n_i}$  (resp.  $L'_{n_i}$ ) for some integer  $n_i$ .

We call these points  $p_1, p_2, \dots, p_{m(D)}$  *singularities of type H* (resp. *type H'*). Among these disks satisfying the above,  $c_H^*(K)$  (resp.  $c_{H'}^*(K)$ ) is the minimum number of  $m(D)$ . Note that  $c_H^*(K) \leq c^*(K)$  and  $c_{H'}^*(K) \leq c^*(K)$ .

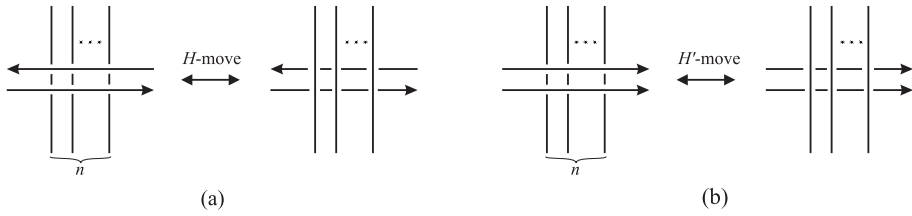


Fig. 1.

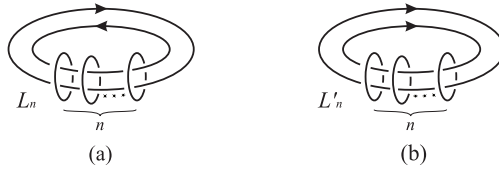


Fig. 2.

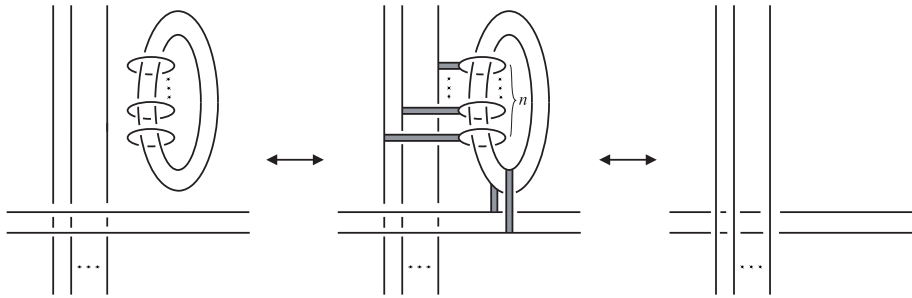


Fig. 3.

In this paper, we shall prove the following.

**Theorem 1.** For any knot  $K$ , the following equalities hold.

- (1)  $c_H^*(K) = g^*(K)$ .
- (2)  $c_{H'}^*(K) = \lceil (\gamma^*(K) + 1)/2 \rceil$ .

Here  $\lceil x \rceil$  is the maximum integer that is not greater than  $x$ .

Since the inequality  $\gamma^*(K) \leq 2g^*(K) + 1$  holds for any knot  $K$  [3, Proposition 2.2], by Theorem 1, we have the following corollary.

**Corollary 2.** For any knot  $K$ ,  $c_{H'}^*(K) \leq c_H^*(K) + 1$ .

REMARK. Let  $K_n$  be a  $(2, 2n + 1)$ -torus knot ( $n = 1, 2, \dots$ ). It is known that  $g^*(K_n) = n$ . On the other hand, we note that  $K_n$  bounds a Möbius band in a 4-ball and that  $K_n$  is not a slice knot. This implies  $\gamma^*(K_n) = 1$ . Therefore, by Theorem 1, we have  $c_{H'}^*(K_n) = 1$  and  $c_H^*(K_n) = n$ .

**Proof of Theorem 1**

In order to prove Theorem 1, we shall show the following lemma.

**Lemma 3.** *Let  $K$  (resp.  $-K'$ ) be a knot in  $S^3 \times \{0\}$  (resp.  $S^3 \times \{1\}$ ). Suppose that  $K$  and  $-K'$  cobound a twice punctured surface  $F$  in  $S^3 \times [0, 1]$  such that  $F$  has neither maximal points nor minimal points. Then the following hold.*

- (1) *If  $F$  is orientable and oriented so that  $\partial F = K \cup (-K')$ , then  $K$  is obtained from  $K'$  by  $g(F)$   $H$ -moves.*
- (2) *If  $F$  is nonorientable, then  $K$  is obtained from  $K'$  or  $-K'$  by  $[\beta_1(F)/2]$   $H'$ -moves.*

Here  $-K'$  denotes the knot  $K'$  with reversed orientation,  $g(F)$  the genus of  $F$  and  $\beta_1(F)$  the first Betti number of  $F$ .

Proof. Suppose that  $F$  is orientable. Then  $2g(F) = \beta_1(F) - 1$ . We regard each saddle point as a saddle band in the sense of [2, p. 107]. We can deform  $F$  so that all saddle bands lie in  $S^3 \times \{1/2\}$ ; see [2]. Note that  $F \cap (S^3 \times \{1/2\})$  is a 2-complex that consists of  $K$  and  $2g(F)$  bands  $b_1, b_2, \dots, b_{2g(F)}$ , and that  $K'$  is obtained from  $K$  by hyperbolic transformations [2, Definition 1.1] along the bands  $b_1, b_2, \dots, b_{2g(F)}$ . Moreover we may assume that  $F \cap (S^3 \times \{1/2\})$  is homeomorphic to a 2-complex as illustrated in Fig. 4(a). Hence  $K$ ,  $F \cap (S^3 \times \{1/2\})$  and  $K'$  can be given as shown in Fig. 5(a). Then we can deform  $F \cap (S^3 \times \{1/2\})$  into a 2-complex as illustrated in Fig. 6(a) by combining the three kinds of local moves; (1) changing a crossing of  $b_{2i}$  and  $b_j$ , (2) changing a crossing of  $b_{2i}$  and  $K$ , and (3) adding a  $\pm 1$ -full twist to  $b_{2i}$ , where  $i = 1, 2, \dots, g(F)$  and  $j = 1, 2, \dots, 2g(F)$ . We note that this deformation is realized by  $g(F)$  local moves as illustrated in Fig. 7. Since the result of hyperbolic transformations along the bands in Fig. 6(a) is  $K$ ,  $K$  is obtained from  $K'$  by  $g(F)$  local moves as illustrated in Fig. 8. It is not hard to see that the local move as in Fig. 8 is realized by a single  $H$ -move; see Fig. 9 for example. Thus  $K$  is obtained from  $K'$  by  $g(F)$   $H$ -moves.

Suppose  $F$  is nonorientable and that  $\beta_1(F) - 1$  is even. Set  $\beta_1(F) - 1 = 2\gamma$ . In the above arguments, by replacing  $g(F)$ ,  $K'$ , Fig. 4(a), 5(a), 6(a) and  $H$ -move with  $\gamma$ ,  $\pm K'$ , Fig. 4(b), 5(b), 6(b) and  $H'$ -move respectively, we have the required result.

In the case that  $\beta_1(F) - 1$  is odd, we have the conclusion by the following. By attaching a small half-twisted band to  $F \cap (S^3 \times \{1/2\})$ , we find a new surface  $F'$  in  $S^3 \times [0, 1]$  such that  $K$  and  $-K'$  cobound  $F'$ ,  $\beta_1(F') = \beta_1(F) + 1$  and  $F'$  has neither maximal points nor minimal points. □

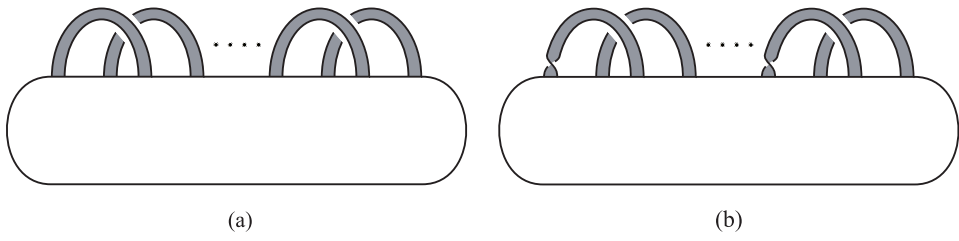


Fig. 4.

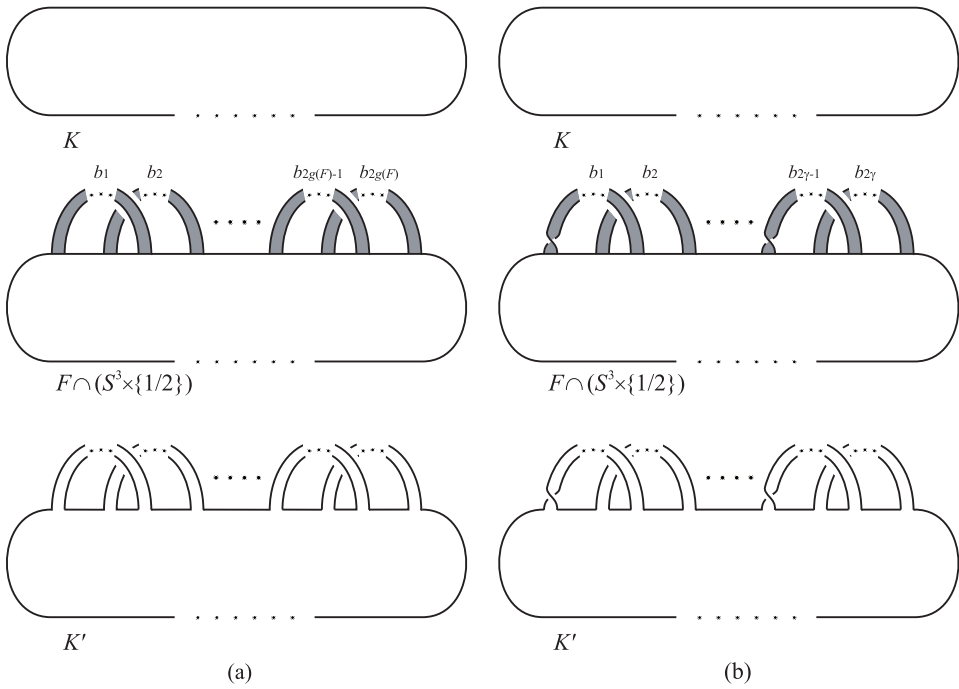


Fig. 5.

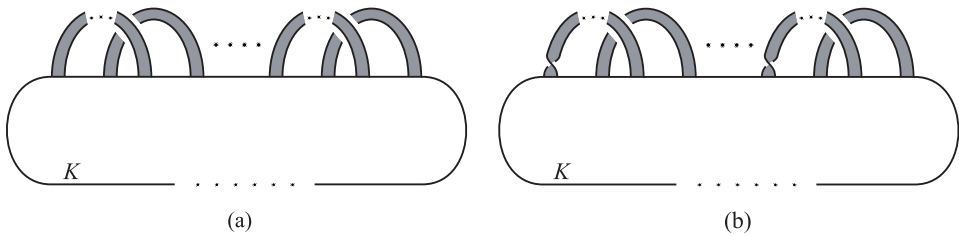


Fig. 6.

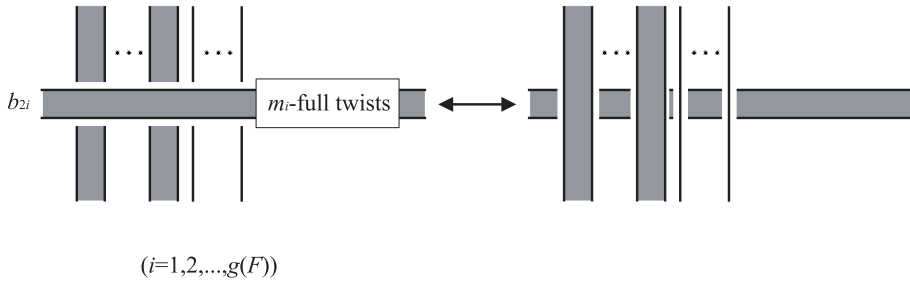


Fig. 7.

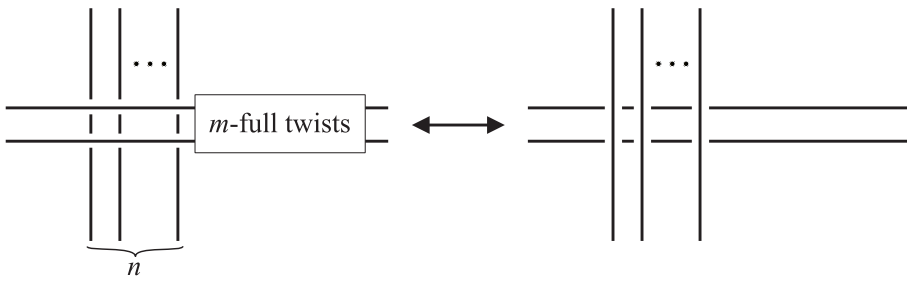


Fig. 8.

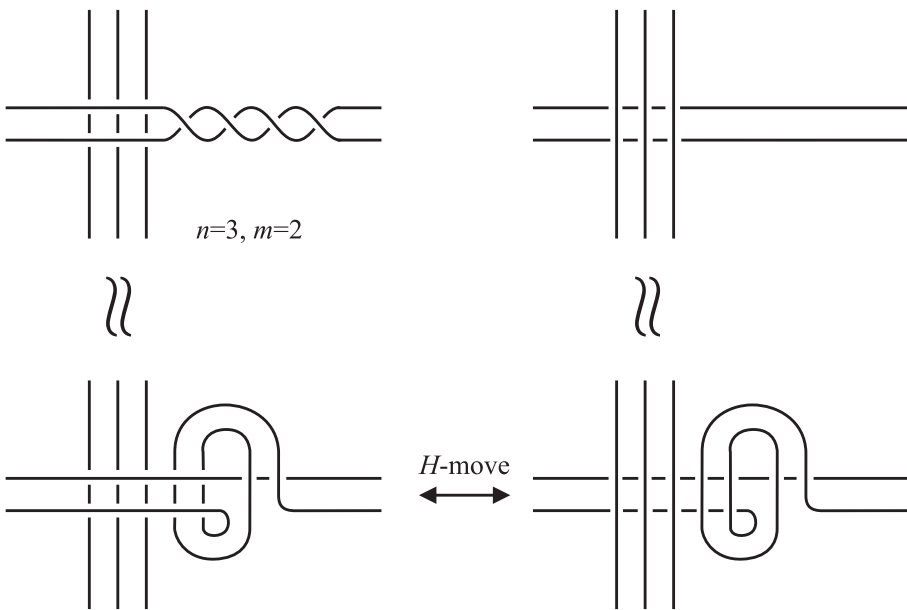


Fig. 9.

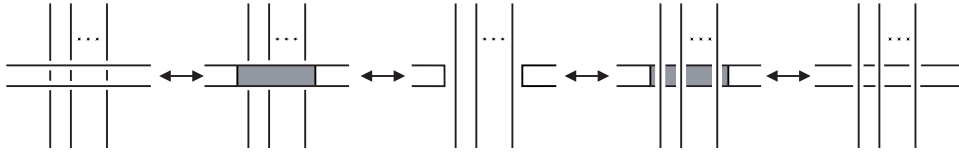


Fig. 10.

Proof of Theorem 1. An  $H$ -move ( $H'$ -move) is realized by twice hyperbolic transformations as illustrated in Fig. 10. Hence we have  $c_H^*(K) \geq g^*(K)$  and  $2c_{H'}^*(K) \geq \gamma^*(K)$ . Note that  $2c_{H'}^*(K) \geq \gamma^*(K)$  implies  $c_{H'}^*(K) \geq [(\gamma^*(K) + 1)/2]$ .

Suppose that a knot  $K$  in  $\partial B^4$  bounds a surface  $F$  in  $B^4$ . We assume that  $B^4 = (S^3 \times [0, \infty)) \cup \{1\text{pt}\}$ . We can deform  $F$  so that the following conditions are satisfied [2]:

- (1)  $F \cap (S^3 \times [0, 1])$  is an annulus that does not have maximal points.
- (2)  $F \cap (S^3 \times [1, 2])$  is a surface that has neither maximal points nor minimal points.
- (3)  $F \cap (S^3 \times [2, \infty))$  is a disk that does not have minimal points, i.e.,  $F \cap (S^3 \times \{2\})$  is a ribbon knot.

Set  $\partial(F \cap (S^3 \times [0, 1])) \setminus K = -K'$  and  $\partial(F \cap (S^3 \times [2, \infty))) = K''$ . If  $F$  is orientable (resp. nonorientable), then by Lemma 3, we have that the ribbon knot  $K''$  is obtained from  $K'$  by  $g(F)$   $H$ -moves (resp. from  $K'$  or  $-K'$  by  $[(\beta_1(F) + 1)/2]$   $H'$ -moves). This implies that  $K'$  and  $-K''$  (resp.  $\pm K''$ ) cobound a singular annulus in  $S^3 \times [1, 2]$  with  $g(F)$  singularities of type  $H$  (resp.  $[(\beta_1(F) + 1)/2]$  singularities of type  $H'$ ). Hence we have  $c_H^*(K) \leq g^*(K)$  and  $c_{H'}^*(K) \leq [(\gamma^*(K) + 1)/2]$ . This completes the proof.  $\square$

Since both  $H$ -move and  $H'$ -move are unknotting operations, we can define 4-dimensional unknotting numbers,  $u_H^*(K), u_{rH}^*(K), u_{H'}^*(K)$  and  $u_{rH'}^*(K)$ , of a knot  $K$  by the similar ways to those of  $u^*(K)$  and  $u_r^*(K)$  in [4]. Namely  $u_H^*(K)$  (resp.  $u_{rH}^*(K)$ ) is the minimum number of  $H$ -moves that is needed to transform  $K$  into a slice knot (resp. a ribbon knot), and  $u_{H'}^*(K)$  (resp.  $u_{rH'}^*(K)$ ) is the minimum number of  $H'$ -moves that is needed to transform  $K$  into a slice knot (resp. a ribbon knot). The *ribbon 4-genus*  $g_r^*(K)$  of a knot  $K$  in  $S^3 = \partial B^4$  is the minimum genus of orientable surfaces in  $B^4$  bounded by  $K$  that has no minimal points [4]. The *nonorientable ribbon 4-genus*  $\gamma_r^*(K)$  is the minimum first Betti number of nonorientable surfaces in  $B^4$  bounded by  $K$  that has no minimal points. For a ribbon knot, it is defined to be 0 instead of 1. We define  $c_{rH}^*$  (resp.  $c_{rH'}^*$ ) to be the minimum number of type  $H$  (resp. type  $H'$ ) singular points of singular disks in  $B^4$  bounded by  $K$  that has no minimal points and whose singularities are of type  $H$  (resp. type  $H'$ ). From the proof of Theorem 1, we have the following theorem.

**Theorem 4.** *For any knot  $K$ , the following equalities hold.*

- (1)  $c_{rH}^*(K) = g_r^*(K) = u_{rH}^*(K)$ .
- (2)  $c_{rH'}^*(K) = [(\gamma_r^*(K) + 1)/2] = u_{rH'}^*(K)$ .

Since the trivial knot in  $\partial B^4$  bounds a Möbius band in  $B^4$  without minimal points, we have  $\gamma_r^*(K) \leq 2g_r^*(K) + 1$  for any knot  $K$ . By Theorem 4, we have the following corollary.

**Corollary 5.** *For any knot  $K$ ,  $u_{rH'}^*(K) \leq u_{rH}^*(K) + 1$ .*

REMARK. Let  $K_n$  be a  $(2, 2n + 1)$ -torus knot ( $n = 1, 2, \dots$ ). Since  $g^*(K) \leq g_r^*(K) \leq g(K)$  [4, Lemma 2], we have  $g_r^*(K_n) = n$ , where  $g(K)$  is the genus of  $K$ . On the other hand, since  $K_n$  is not a ribbon knot and  $K_n$  bounds a Möbius band in a 4-ball that has no minimal points, we have  $\gamma_r^*(K_n) = 1$ . Therefore, by Theorem 4, we have  $c_{rH'}^*(K_n) = 1$  and  $c_{rH}^*(K_n) = n$ .

By the definitions of  $c_H^*(K)$ ,  $c_{H'}^*(K)$ ,  $u_H^*(K)$  and  $u_{H'}^*(K)$ , we have  $c_H^*(K) \leq u_H^*(K)$  and  $c_{H'}^*(K) \leq u_{H'}^*(K)$ .

**Conjecture.** *For any knot  $K$ ,  $c_H^*(K) = u_H^*(K)$  and  $c_{H'}^*(K) = u_{H'}^*(K)$ .*

REMARK. If  $g^*(K) = g_r^*(K)$ , then by Theorems 1 and 4,  $c_H^*(K) = g^*(K) = g_r^*(K) = u_{rH}^*(K) \geq u_H^*(K)$ . If  $\gamma^*(K) = \gamma_r^*(K)$ , then by Theorems 1 and 4,  $c_{H'}^*(K) = [(\gamma^*(K) + 1)/2] = [(\gamma_r^*(K) + 1)/2] = u_{rH'}^*(K) \geq u_{H'}^*(K)$ . Thus if  $g^*(K) = g_r^*(K)$  and  $\gamma^*(K) = \gamma_r^*(K)$  for any knot  $K$ , then the conjecture above is true.

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**References**

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