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CONSTRUCTION OF THE EVOLUTION OPERATOR OF PARABOLIC TYPE

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1. Introduction and Main Theorem

In this note we construct the evolution operator of parabolic type, or the fundamental solution of the linear ordinary differential equation

$$\frac{du(t)}{dt} + A(t)u(t) = f(t), \quad a < t < b, \tag{1.1}$$

of parabolic type in a Banach space X. The equation (1.1) is said to be "of parabolic type" if it satisfies the condition:
(A1) $-A(t)$ *is a linear operator with dense domain, and there exist constants* $\kappa > \pi/2$ *and* C_0 *such that the resolvent set of* $-A(t)$ *contains the sector* $\Sigma_\kappa := \{\lambda \in \mathbb{C}; |\arg\lambda| \le \kappa\}$ *for any* $t \in I := [a, b]$ *and* $\|\lambda(\lambda + A(t))^{-1}\|_{X\to X} \le C_0$ *holds for any* $\lambda \in \Sigma_\kappa$ *and any* $t \in I$.

$-A(t)$ generates an analytic semi-group $\{e^{-\tau A(t)}; \tau \ge 0\}$ on X.

Our result is stated as follows:

Main Theorem. *Assume* (A1), *and the following hypotheses* (A2), (A3):
(A2) *The domain* $\mathcal{D}(A(t)) = Y$ *for any* $t \in I$ *and* $A(\cdot) \in \mathcal{C}(I; \mathcal{L}(Y, X))$, *where* Y *is a Banach space continuously imbedded in* X.
(A3) *Defining*

$$\omega(h) := \sup\{\|A(t + h) - A(t)\|_{Y\to X}; a \le t \le b - h\}, \tag{1.2}$$

$\omega(h)/h$ *is integrable on* $(0, \delta)$ *for some positive* δ. *Then, there exists the evolution operator to the equation* (1.1), *i.e., there exists a strongly continuous* $\mathcal{L}(X)$*-valued function* $U(t, s)$, $a \le s \le t \le b$, *having the following properties*:
(a) $U(t, r)U(r, s) = U(t, s)$ *for* $a \le s \le r \le t \le b$,
(b) $U(t, t) = I$ *for* $a \le t \le b$,
(c) $(\partial/\partial t)U(t, s)x = -A(t)U(t, s)x$ *for any* $x \in X$ *and* $a \le s < t < b$,
(d) $(\partial/\partial s)U(t, s)x = U(t, s)A(s)x$ *for any* $x \in Y$ *and* $a < s < t \le b$.

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Moreover, the evolution operator $U(t,s)$ *is uniquely determined by* $\{A(t)\}_{a\le t\le b}$, *and satisfies the estimates*

$$\|A(t)U(t,s)\|_{X\to X}\le\frac{M}{t-s},\quad \|U(t,s)A(s)\|_{Y\to Y}\le\frac{M}{t-s} \tag{1.3}$$

for any $a\le s<t\le b$, *where* M *is a constant.*

It is well known that any strong solution $u(t)$ to (1.1) with the initial date $u(a)=u_0$ must be of the form $u(t)=U(t,a)u_0+F(t)$, where

$$F(t):=\int_a^t U(t,s)f(s)ds. \tag{1.4}$$

It is also known that the condition $f\in\mathcal{C}(I;X)$ does not guarantee differentiability of $F(t)$. Regard to this we have

Theorem 1.1. *Assume* (A1), (A2), (A3), $f\in L^1(I;X)\cap B^0_{\infty,1}((a,b);X)_{\text{loc}}$, *and define* F *by* (1.4). *Then* $F\in\mathcal{C}(I;X)\cap\mathcal{C}^1((a,b);X)$, $F(t)\in\mathcal{D}(A(t))$ *for any* $t\in(a,b)$, *and* $u(t)=U(t,s)u_0+F(t)$ *is the unique strong solution to* (1.1) *with the initial condition* $u(a)=u_0$.

Study of the evolution operator of parabolic type has a rather long history, but we recall here only a few articles related to our result. Tanabe [7] constructed the evolution operator under the hypotheses (A1) (A2) and

$$\omega(h)\le Ch^{\theta},\quad 0<\theta\le 1. \tag{A3$'$}$$

(i.e., $A(t)$ is a Hölder continuous $\mathcal{L}(Y,X)$-valued function.) It is easy to see that (A3) is a true improvement of (A3′). Kawatsu [2] gave also an improvement of (A3′), i.e., under the assumption that "$\omega(h)|\log h|/h$ is integrable on $(0,\delta)$" he proved the existence of the evolution operator. Our assumption is better than that of Kawatsu, and we hope that our theorem will be useful in studying non-linear problems.

Our result was announced in [3]. The proof given by one of the authors eleven years ago is based on the approximation theory of integral equations with operator-valued unknown function and it is rather long. In this note we will give a simple and straightforword proof which contains some new methods to investigate abstract differential equations in a Banach space.

The result corresponding to Theorem 1.1 for the case where $A(t)$ is independent of t has been given in [4].

NOTATION. $\|x\|_X$ denotes the norm of x in a space X.
$\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y, whose norm is denoted by $\|U\|_{X\to Y}$, $\mathcal{L}(X) := \mathcal{L}(X, X)$.
$\mathcal{C}(\Omega; X)$ denotes the space of X-valued continuous functions on a domain Ω.
$L^p(\Omega; X)$ denotes the space of X-valued strongly measurable functions $f(t)$ with $\|f(t)\|_X \in L^p(\Omega)$.

2. Preliminary observation

We first observe that Main Theorem follows from the following fact:
For some small positive number δ there exists a strongly continuous $\mathcal{L}(X)$-valued function $U(t, s)$ on the area $T_\delta := \{(t, s); a \le s \le t \le b, t-s \le \delta\}$ satisfying the conditions (b), (c), (d) *in Main Theorem and the inequality*

$$\|U(t, s)\|_{X\to Y} \le \frac{M_1}{t-s} \quad \text{for} \quad a \le s < t \le b \quad \text{with} \quad t - s \le \delta. \tag{2.1}$$

In fact, when $(t, s) \in T_\delta$, the derivative of $U(t, r)U(r, s)$ with respect r vanishes in the interval (s, t). Therefore, $U(t, r)U(r, s)$ is independent of $r \in (s, t)$. Together with the strong continuity of $U(t, r)U(r, s)$, this implies that $U(t, r)U(r, s) = U(t, s)$ holds when $(t, s) \in T_\delta$. (1.3) follows dirctly from (2.1), since $\sup_{t\in I} \|A(t)\|_{X\to Y} < \infty$.

When $\delta \le t - s < 2\delta$, we define $U(t, s) := U(t, r)U(r, s)$, where r is a point with $\max\{s, t - \delta\} < r < \min\{s + \delta, t\}$. $U(t, s)$ is independent of the choice of r, since for any $\max\{s, t - \delta\} < r < r_1 < \min\{s + \delta, t\}$ we have $U(t, r)U(r, s) = U(t, r_1)U(r_1, r)U(r, s) = U(t, r_1)U(r_1, s)$. Thus, the evolution operator $U(t, s)$ can be defined when $t - s < 2\delta$. The fact that $U(t, s)$ has the properties (a), (b), (c) and (d) in Main Theorem is a simple consequence of the definition.

Repeating this argument, we can finally construct the evolution operator for any point (t, s) with $a \le s \le t \le b$, and we see easily that (1.3) holds for any $a \le s < t \le b$.

Finally, if $\tilde{U}(t, s)$ is another $\mathcal{L}(X)$-valued strongly continuous function satisfying (b) and (c) in Main Theorem, the derivative of $U(t, r)\tilde{U}(r, s)$ with respect r vanishes in the interval (s, t). So, $U(t, r)\tilde{U}(r, s)$ is independent of r, which implies that $U(t, s) = U(t, r)\tilde{U}(r, s) = \tilde{U}(t, s)$. This gives the uniqueness of the evolution operator, which completes the proof of Main Theorem.

3. Lemmas

Lemma 3.1. *If $f \in L^1([\alpha, \beta]; Z)$, then $\int_\alpha^\beta f(s)ds \in Z$. Here Z is a Banach space.*

Proof. See Yosida [10] p. 133. □

Lemma 3.2. *If $F(\lambda)$ is holomolphic and satisfies $\|F(\lambda)\|_X \le C|\lambda|^\alpha$ in $\Sigma_\kappa \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge 1\}$, $\|\int_\Gamma e^{t\lambda}F(\lambda)d\lambda\|_X \le C(\alpha, c_0)Ct^{-\alpha-1}$ holds for any $0 < t \le c_0 < \infty$, where $C(\alpha, c_0)$ is a constant depend only on α, c_0 and κ. Here, Γ denotes a path $\lambda = \lambda(\sigma)$ ($\sigma \in \mathbb{R}$) contained in $\Sigma_\kappa \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge 1\}$ such that $|\lambda(\sigma)| \to \infty$, $0 < \varepsilon \le \pm \arg \lambda(\sigma) - \pi/2$ as $\sigma \to \pm\infty$.*

Lemma 3.3. *From* (A1) *and* (A2) *it follows that*

$$\|(\lambda + A(t))^{-1}\|_{X\to Y} \le C_1(1 + |\lambda|^{-1}), \tag{3.1}$$

$$\|\lambda(\lambda + A(t))^{-1}\|_{Y\to Y} \le C_2 \tag{3.2}$$

hold for any $\lambda \in \Sigma_\kappa$ and any $t \in I := [a, b]$. Here C_1 and C_2 are constants.

Proof. Assume (A1) and (A2). Since the identity

$$(1 + A(t))^{-1} = (1 + A(t_0))^{-1}\sum_{n=0}^{\infty}\{(A(t_0) - A(t))(1 + A(t_0))^{-1}\}^n$$

holds if $\|A(t) - A(t_0)\|_{Y\to X} < \|(1 + A(t_0))^{-1}\|_{X\to Y}^{-1}$, we see that $(1 + A(t))^{-1} \in \mathcal{C}(I; \mathcal{L}(X, Y))$, which implies that $C' := \sup_{a\le t\le b}\|(1 + A(t))^{-1}\|_{X\to Y}$ is finite. Hence, by the identity $(\lambda + A)^{-1} = \{1 + (1-\lambda)(\lambda + A)^{-1}\}(1 + A)^{-1}$ we have (3.1). Also, by the identity $(\lambda + A)^{-1} = (1 + A)^{-1}(\lambda + A)^{-1}(1 + A)$ we have (3.2). □

Lemma 3.2, (3.1), (3.2) and the identities $e^{-\tau A(t)} = (1/(2\pi i))\int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1}d\lambda$,

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}\{A(s) - A(t)\}(\lambda + A(s))^{-1}$$

give the following

Lemma 3.4. *Assume* (A1) *and* (A2). *Then,*

$$\|e^{-\tau A(t)}\|_{X\to X} \le M_0, \tag{3.3}$$

$$\|e^{-\tau A(t)}\|_{Y\to Y} \le M_1, \tag{3.4}$$

$$\|e^{-\tau A(t)}\|_{X\to Y} \le M\tau^{-1}, \tag{3.5}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{X\to X} \le P_0\|A(t) - A(s)\|_{Y\to X}, \tag{3.6}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{Y\to Y} \le P_1\|A(t) - A(s)\|_{Y\to X}, \tag{3.7}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{X\to Y} \le P\tau^{-1}\|A(t) - A(s)\|_{Y\to X}, \tag{3.8}$$

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{Y\to X} \le P'\tau\|A(t) - A(s)\|_{Y\to X}, \tag{3.9}$$

hold for any $a \le s \le t \le b$ and $0 < \tau \le c_0$. Here M_0, M_1, M, P_0, P_1, P and P' are constants independent of t, s and τ.

By (3.6), (3.7), (3.8) and the strong continuity of semi-group $e^{-\tau A(s)}$ we see that $e^{-\tau A(t)} - e^{-\sigma A(s)} = \{e^{-\tau A(t)} - e^{-\tau A(s)}\} + \{e^{-\tau A(s)} - e^{-\sigma A(t)}\} \to 0$ as $(\tau, t) \to (\sigma, s)$. Hence, we have

Lemma 3.5. *Let* $0 < c < \infty$, *and assume* (A1) *and* (A2). *Then,* $e^{-\tau A(t)}$ *is an* $\mathcal{L}(X)$*-valued* (*and* $\mathcal{L}(Y)$*-valued*) *strongly continuous function of* $(\tau, t) \in [0, c] \times [a, b]$. $e^{-\tau A(t)}$ *is also an* $\mathcal{L}(X, Y)$*-valued strongly continuous function of* $(\tau, t) \in (0, c] \times [a, b]$.

4. The series giving the evolution operator

According to Tanabe [7], to construct the evolution operator $U(t, s)$ we make use of the series

$$U(t, s) = \sum_{n=0}^{\infty} W_n(t, s) := W_0(t, s) + \sum_{n=1}^{\infty} \int_s^t W_0(t, r) R_n(r, s) dr \tag{4.1}$$

where $W_0(t, s) := e^{-(t-s)A(s)}$, $R_1(t, s) := -\{A(t) - A(s)\}e^{-(t-s)A(s)}$ and

$$R_{n+1}(t, s) = \int_s^t R_1(t, r) R_n(r, s) dr \quad \text{for } n = 1, 2, \cdots . \tag{4.2}$$

To prove the convergence of the series (4.1) we start with

Lemma 4.1. *Let* $\omega(t)$ *be a non-negative bounded measurable function of* $t \in (0, \delta_0)$ *such that*

$$\gamma(t) := \int_0^t \omega(s) \frac{ds}{s} < \infty \tag{4.3}$$

for $0 < t \le \delta_0$. *Then, putting* $\omega_1 = \omega$,

$$\omega_{n+1}(t) := t \int_0^t \frac{\omega(t-s)}{t-s} \frac{\omega_n(s)}{s} ds \quad \textit{for } n = 1, 2, \cdots , \tag{4.4}$$

can be defined inductively, and

$$\int_0^t \frac{\omega_n(s)}{s} ds \le \gamma(t)^n, \tag{4.5}$$

$$\omega_n(t) \le n^2 M' \gamma(t)^{n-1}, \tag{4.6}$$

hold for $n = 1, 2, \cdots$ *and* $0 < t \le \delta_0$, *where* $M' := \sup_{0<t\le\delta_0} \omega(t)$.

Proof. Clearly (4.5) and (4.6) hold for $n = 1$. Assume that (4.5) and (4.6) hold for n. Then, noting that γ is a increasing function, by Fubini's theorem we have

$$\int_0^t \frac{\omega_{n+1}(s)}{s} ds = \int_0^t \left\{ \int_r^t \frac{\omega(s-r)}{s-r} ds \right\} \frac{\omega_n(r)}{r} dr \leq \gamma(t)^{n+1}.$$

Also, taking $r = nt/(n+1)$, we have

$$\omega_{n+1}(t) \leq t \int_0^r \frac{M'}{t-r} \frac{\omega_n(s)}{s} ds + t \int_r^t \frac{\omega(t-s)}{t-s} \frac{M' n^2 \gamma(t)^{n-1}}{r} ds \leq M' \gamma(t)^n (n+1)^2.$$

This gives (4.6) for $n+1$. □

In the following of this note we always assume that (A1), (A2) and (A3) hold, and by ω we denote the function defined by (1.2).

Lemma 4.2. *Let $a \leq s < t \leq b$. Then,*

$$\|R_n(t,s)\|_{X \to X} \leq M^n \frac{\omega_n(t-s)}{t-s}, \quad n = 1, 2, \cdots, \tag{4.7}$$

$$\|W_n(t,s)\|_{X \to X} \leq M_0 (M\gamma(t-s))^n, \quad n = 0, 1, \cdots. \tag{4.8}$$

Proof. As $\|A(t) - A(s)\|_{Y \to X} \leq \omega_1(t-s)$, (3.5) implies (4.7) for $n = 1$. Assume that (4.7) holds for n. Then, by (4.2) we have

$$\|R_{n+1}(t,s)\|_{X \to X} \leq M^{n+1} \int_s^t \frac{\omega(t-r)}{t-r} \frac{\omega_n(r-s)}{r-s} dr = M^{n+1} \frac{\omega_{n+1}(t-s)}{t-s}.$$

Clearly (4.8) holds for $n = 0$. (4.1), (4.5) and (4.7) imply (4.8) for $n \geq 1$. □

5. Norm of $W_n(t,s)$

We make use of the symbols: $Z_0(t,s) := e^{-(t-s)A(t)}$, $Q_1(t,s) := Z_0(t,s)\{A(t) - A(s)\}$, $H(t,\sigma,s) := \{e^{-(t-\sigma)A(t)} - e^{-(t-\sigma)A(s)}\} e^{-(\sigma-s)A(s)}$, $G(t,s) := H(t,s,s)$.

Lemma 5.1. *Let $a \leq s \leq \sigma \leq t \leq b$. Then*

$$\begin{aligned} W_1(t,s) &= \int_\sigma^t \{Q_1(t,r)W_0(r,s) - G(t,r)R_1(r,s)\} dr \\ &\quad + H(t,\sigma,s) + \int_s^\sigma W_0(t,r)R_1(r,s) dr, \end{aligned} \tag{5.1}$$

$$\begin{aligned} W_{n+1}(t,s) &= \int_\sigma^t [Q_1(t,r)W_n(r,s) + (G(t,r)\{R_n(r,s) - R_{n+1}(r,s)\}] dr \\ &\quad + \int_s^\sigma [W_0(t,r)R_{n+1}(r,s) + H(t,\sigma,r)R_n(r,s)] dr \quad (n \geq 1). \end{aligned} \tag{5.2}$$

Proof. By the formula

$$Z_0(t,r)R_1(r,s) = Q_1(t,r)W_0(r,s) - \frac{\partial}{\partial r}\left\{e^{-(t-r)A(t)}e^{-(r-s)A(s)}\right\}$$

we have

$$\int_\sigma^t Z_0(t,r)R_1(r,s)dr = \int_\sigma^t Q_1(t,r)W_0(r,s)dr + H(t,\sigma,s), \tag{5.3}$$

which implies (5.1), for $W_0(t,s) = Z_0(t,s) - G(t,s)$. By (5.3) we have

$$\begin{aligned}
&\int_\sigma^t Z_0(t,r)R_{n+1}(r,s)dr \\
&= \int_\sigma^t \Big[\int_\tau^t Z_0(t,r)R_1(r,\tau)dr\Big] R_n(\tau,s)d\tau \\
&\quad + \int_s^\sigma \Big[\int_\sigma^t Z_0(t,r)R_1(r,\tau)dr\Big] R_n(\tau,s)d\tau \\
&= \int_\sigma^t \Big[\int_\tau^t Q_1(t,r)W_0(r,\tau)dr + G(t,\tau)\Big] R_n(\tau,s)d\tau \\
&\quad + \int_s^\sigma \Big[\int_\sigma^t Q_1(t,r)W_0(r,\tau)dr + H(t,\sigma,\tau)\Big] R_n(\tau,s)d\tau \\
&= \int_\sigma^t \{Q_1(t,r)W_n(r,s) + G(t,r)R_n(r,s)\}dr + \int_s^\sigma H(t,\sigma,\tau)R_n(\tau,s)d\tau,
\end{aligned} \tag{5.4}$$

which gives (5.2). □

The estimate $\|W_0(t,s)\|_{X\to Y} \le M/(t-s)$ follows from (3.5). For the case $n \ge 1$ we have

Lemma 5.2. *$W_n(t,s) \in \mathcal{L}(X,Y)$ when $a \le s < t \le b$, $A(t)W_n(t,s)$ is continuous with respect to $(t,s) \in \{(t,s); a \le s < t \le b\}$, and the inequality*

$$\|W_n(t,s)\|_{X\to Y} \le \frac{Kn^3(M\gamma(t-s))^{n-1}}{t-s} \tag{5.5}$$

holds for $n = 1, 2, \cdots$. Here $\gamma(t)$ is the function given by (4.3).

Proof. Case where $n = 1$. Since it follows from (3.8) that

$$\|G(t,s)\|_{X\to Y} \le \frac{P\omega(t-s)}{t-s} \le \frac{PM'}{t-s}, \tag{5.6}$$

and $\|H(t,\sigma,s)\|_{X\to Y} \le PM'M_0(t-\sigma)^{-1}$, by (5.1) with $\sigma = (t+s)/2$, with the aid of (3.3), (3.5), and (4.7), we obtain

$$\begin{aligned}\|W_1(t,s)\|_{X\to Y} &\le \int_\sigma^t \|Q_1(t,r)W_0(r,s)-G(t,r)R_1(r,s)\|_{X\to Y}dr\\ &\quad+\|H(t,\sigma,s)\|_{X\to Y}+\int_s^\sigma \|W_0(t,r)R_1(r,s)\|_{X\to Y}dr\\ &\le M\int_\sigma^t \left\{ \frac{M\omega(t-r)}{(t-r)(r-s)}+\frac{P\omega(t-r)\omega(r-s)}{(t-r)(r-s)} \right\} dr\\ &\quad+\frac{2M'PM_0}{t-s}+\int_s^\sigma \frac{M^2\omega(r-s)}{(t-r)(r-s)}dr\\ &\le \frac{2M+2PM'}{t-s}M\gamma(b-a)+\frac{2M'PM_0}{t-s}+\frac{2M^2\gamma(b-a)}{t-s}\le \frac{K}{t-s}.\end{aligned}$$

Here we take K so that $K \ge 4M^2\gamma(b-a)+2PM'(M\gamma(b-a)+M_0)$.

Since $A(t)$ is closed, we also see that

$$\begin{aligned}A(t)W_1(t,s) &= \int_\sigma^t A(t)\{Q_1(t,r)W_0(r,s)-G(t,r)R_1(r,s)\}dr\\ &\quad+A(t)H(t,\sigma,s)+\int_s^\sigma A(t)W_0(t,r)R_1(r,s)dr.\end{aligned}$$

Hence $A(t)W_1(t,s)$ is continuous. In view of Lemma 3.1, we see that the conclusion of the lemma holds for $n=1$.

Assume that (5.5) holds for n. Hence, taking $\sigma=(nt+s)/(n+1)$, by (5.2), (5.6), (4.7) and (4.6) we have

$$\begin{aligned}&\|W_{n+1}(t,s)\|_{X\to Y}\\ &\le \int_\sigma^t \frac{\omega(t-r)}{t-r}\left[\frac{KM^n\gamma^{n-1}n^3}{r-s}+P\left\{ \frac{M^n\omega_n(r-s)}{r-s}+\frac{M^{n+1}\omega_{n+1}(r-s)}{r-s} \right\}\right] dr\\ &\quad+\int_s^\sigma \left[\frac{M}{t-r}\frac{M^{n+1}\omega_{n+1}(r-s)}{r-s}+\frac{P\omega(t-r)}{t-\sigma}M_0\frac{M^n\omega_n(r-s)}{r-s}\right]dr\\ &\le \frac{KM^n\gamma^n n^3+PM'M^n\gamma^n\{n^2+(n+1)^2M\gamma\}}{\sigma-s}+\frac{M(M\gamma)^{n+1}+PM_0M'M^n\gamma^n}{t-\sigma}\\ &\le \frac{(n+1)M^n\gamma^n}{t-s}\{Kn^2+PM'(2M\gamma+1)n+PM'(3M\gamma+M_0)+M^2\gamma\}\\ &\le \frac{(n+1)^3KM^n\gamma^n}{t-s}. \quad (\text{Here } \gamma=\gamma(t-s).)\end{aligned}$$

Here, we take $K := 4M^2\gamma(b-a)+PM'(3M\gamma(b-a)+2M_0+1)$. This estimate gives that $W_{n+1}(t,s)\in\mathcal{L}(X,Y)$. The same argument as for W_1 gives that $A(t)W_n(t,s)$ is continuous in (t,s) when $a\le s<t\le b$. □

Construction of $U(t,s)$ when $t-s$ is small. Take δ small enough so that $\gamma(\delta)<1/M$, where $\gamma(\delta)$ is given by (4.3). Then, with help of the estimate (4.8) and (4.5), we can define $U(t,s)$ by (4.1) when $t-s \leq \delta$. Since $W_n(t,s)$, $n = 0, 1, \cdots$ are strongly continuous function and the series (4.1) converges uniformly, we see that $U(t,s)$ is strongly continuous.

By (5.5) we see that the series (4.1) converges in $\mathcal{L}(X,Y)$ when $0 < t-s \leq \delta$, since $\sum_{n=1}^{\infty}(M\gamma(\delta))^{n-1}n^3 < \infty$. Hence, $U(t,s)$ is a strongly continuous $\mathcal{L}(X,Y)$-valued function of $(t,s) \in \{(t,s); a \leq s < t \leq b, t-s \leq \delta\}$, and satisfies (2.1).

6. Proof of differentiability with respect to *t*

Lemma 6.1. *Let* $g \in \mathcal{C}([c,b];X)$, $a \leq c < b$, *define* $G(t) := \int_c^t W_0(t,r)g(r)dr$, *and assume that* $G \in \mathcal{C}((c,b);Y)$. *Then,* $G \in \mathcal{C}^1((c,b);X)$, *and*

$$\frac{dG}{dt}(t) = g(t) - \int_c^t R_1(t,r)g(r)dr - A(t)G(t). \tag{6.1}$$

Proof. Let $c < t < b$, $0 < h < b-t$. Then, we have

$$\begin{aligned}\frac{1}{h}\{G(t+h)-G(t)\} &= \frac{e^{-hA(t)}-1}{h}G(t) + \int_0^1 e^{-(h-h\sigma)A(t+h\sigma)}g(t+h\sigma)d\sigma \\ &\quad + \int_s^t \frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r)dr.\end{aligned}$$

Because of the fact that $G(t) \in \mathcal{D}(A(t))$, the fisrt term in the right-hand side converges to $-A(t)G(t)$ as $h \to +0$. Since $e^{-\tau A(r)}$ is a strongly continuous function of $(\tau, r) \in [0,\tau_0] \times [a,b]$ (see Lemma 3.5), it follows that $e^{-\tau A(r)}g(t)$ is a uniformly continuous function of $(\tau, r, t) \in [0,\tau_0] \times [a,b] \times [c.b]$. Hence, $e^{-(h-h\sigma)A(t+h\sigma)}g(t+h\sigma) \to g(t)$ as $h \to +0$ uniformly with respect to $\sigma \in [0,1]$, which implies that the second term in the right-hand side converges to $g(t)$. Lebesgue's dominated convergence theorem implies that the third term in the right-hand side converges to the second term of the formula (14) as $h \to +0$, since by the estimate (3.9) we have

$$\left\|\frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r)\right\|_X \leq C\frac{\omega(t-r)}{t-r}\|g(r)\|_X \in L^1,$$

and since

$$\frac{1}{h}\{e^{-hA(r)}-e^{-hA(t)}\}e^{-(t-r)A(r)}g(r) \to -R_1(t,r)g(r) \text{ as } h \to +0$$

for any $r \in [c,t]$. Thus we can conclude that $G(t)$ is right-differentiable, and its right-derivative is strongly contimuous. From a well-known lemma (see Yosida [10], p. 239) it follows that $G(t)$ is differentiable and (14) holds, which completes the proof of Lemma 6.1. □

Lemma 6.2. *Let* $a \leq s < b$. *Then,* $U(t,s)x \in \mathcal{D}(A(t))$ *is strongly differentiable in* $t \in (s, \min\{s+\delta, b\})$ *and its derivative is* $-A(t)U(t,s)x$ *for any* $x \in X$.

Proof. Assume that $a \leq s < t \leq \min\{s+\delta, b\}$. Then, by (5.5) we see that $\sum_{n=0}^{m} W_n(t,s)$ converges to $U(t,s)$, and $\sum_{n=0}^{m} A(t)W_n(t,s)$ converges as $m \to \infty$. As $A(t)$ is closed, it follows that $U(t,s)x \in \mathcal{D}(A(t))$ and $A(t)U(t,s)x = \sum_{n=0}^{\infty} A(t)W_n(t,s)x$ for any $x \in X$. Moreover, the estimate (5.5) implies that the above series converges uniformly with respect to $t \in [s+\varepsilon, \min\{s+\delta, b\}]$. Hence $A(t)U(t,s)x$ is continuous. On the other hand it follows from the above lemma that $W_n(t,s)x$ is differentiable in $t \in (s,b)$ and its derivative with respect t is equal to $R_n(t,s)x - R_{n+1}(t,s)x - A(t)W_n(t,s)x$ for any $x \in X$. Thus, we have

$$\frac{\partial}{\partial t}\sum_{n=0}^{m} W_n(t,s)x = -R_{n+1}(s,t)x - \sum_{n=0}^{m} A(t)W_n(t,s)x \to -A(t)U(t,s)x$$

uniformly with respect to $t \in [s+\varepsilon, \min\{s+\delta, b\})$ as $m \to \infty$, which completes the proof of the lemma. □

7. Proof of differentiability with respect to *s*

To prove differentiability of $U(t,s)$ in s we make use of another series which expresses $U(t,s)$. (See Tanabe [8].)

$$V(t,s) = \sum_{n=0}^{\infty} Z_n(t,s) := e^{-(t-s)A(t)} + \sum_{n=1}^{\infty}\int_s^t Q_n(t,r)W_0(r,s)dr, \tag{7.1}$$

where $Q_1(t,s) := e^{-(t-s)A(t)}\{A(t) - A(s)\}$ and

$$Q_{n+1}(t,s) := \int_s^t Q_n(t,r)Q_1(r,s)dr, \quad \text{for } n = 1, 2, \cdots. \tag{7.2}$$

Lemma 7.1. *If $a \leq s < t \leq b$,*

$$\|Q_n(t,s)\|_{Y\to Y} \leq \frac{M^n\omega_n(t-s)}{t-s} \quad \textit{for} \quad n = 1, 2, \cdots, \tag{7.3}$$

$$\|Q_n(t,s)\|_{Y\to X} \leq M_0M'(M\gamma(t-s))^{n-1} \quad \textit{for} \quad n = 1, 2, \cdots, \tag{7.4}$$

$$\|Z_n(t,s)\|_{Y\to Y} \leq M_1M^n\gamma(t-s)^n, \quad \textit{for} \quad n = 0, 1, \cdots, \tag{7.5}$$

$$\|Z_n(t,s)\|_{X\to X} \leq Kn(M\gamma(t-s))^{n-1} \quad \textit{for} \quad n = 1, 2, \cdots. \tag{7.6}$$

Proof. By (3.5) and $\|A(t)-A(s)\|_{Y\to X} \leq \omega(t-s)$ we have (7.3) for $n = 1$. The inequality (7.3) can be proved in the same way as (4.7). It is clear that $\|Q_1(t,s)\|_{Y\to X} \leq M_0\omega(t-s) \leq M_0M'$. Assume that (7.4) holds for n. Then, by (7.2) and (7.3) we have

$$\|Q_{n+1}(t,s)\|_{Y\to X} \leq \int_s^t M_0M'M^n\gamma(t-s)^{n-1}\frac{\omega(r-s)}{r-s}dr \leq M_0M'(M\gamma(t-s))^n.$$

Also, (7.3) and $\|Z_0(t,s)\|_{Y\to Y} \leq M_1$ implies (7.5).

Next, it follows that $\|Z_0(t,s)\|_{X\to X} \le M_0$, and it follows from the identity

$$Z_1(t,s) = \int_s^t \{Q_1(t,r)G(r,s) + Z_0(t,r)R_1(r,s)\}dr - G(t,s)$$

and the estimate

$$\|Q_1(t,r)G(r,s) + Z_0(t,r)R_1(r,s)\|_{X\to X} \le M_0\{M'P + M\}\frac{\omega(r-s)}{r-s}$$

that $Z_1(t,s) \in \mathcal{L}(X)$ and (7.6) for $n = 1$ holds. Assuming that $Z_n(t,s) \in \mathcal{L}(X)$ and (7.6) holds for n, by the identity

$$Z_{n+1}(t,s) = \int_s^t [\{Q_{n+1}(t,r) - Q_n(t,r)\}G(r,s) + Z_n(t,r)R_1(r,s)]dr$$

we see that $\|Z_{n+1}(t,s)\|_{X\to X}$ is estimated by

$$\begin{aligned}
&\int_s^t \{M_0M'(M\gamma(t-s))^{n-1}(M\gamma(t-s)+1)P + KnM^n\gamma(t-s)^{n-1}\}\frac{\omega(r-s)}{r-s}dr \\
&\le M_0M'PM^{n-1}\gamma(t-s)^n(M\gamma(t-s)+1) + KnM^n\gamma(t-s)^n \\
&\le K(n+1)(M\gamma(t-s))^n.
\end{aligned}$$

Hence $Z_{n+1}(t,s) \in \mathcal{L}(X)$ and (7.5) holds for $n+1$. Thus the lemma has been completely proved. □

Lemma 7.2. *$Z_n(t,s)y \in C^1((a,t);X)$ for any $a < t \le b$ and any $y \in Y$, and its derivative with respect to s is equal to $-Q_n(t,s)y + Q_{n+1}y + Z_n(t,s)A(s)y$.*

Proof. This follows from the identity

$$\begin{aligned}
\frac{Q_n(t,s-h)y - Q_n(t,s)y}{-h} &= -\int_0^1 Q_n(t,s-h\sigma)e^{-h(1-\sigma)A(s-h\sigma)}yd\sigma \\
&\quad -\int_s^t Q_n(t,r)e^{-(r-s)A(r)}\frac{e^{-hA(r)} - e^{-hA(s)}}{h}ydr \\
&\quad -Z_n(t,s)\frac{e^{-hA(s)}-1}{h}y
\end{aligned}$$

and the argument which led to Lemma 6.1. □

In similar way as Lemma 6.2, from Lemma 7.1 and Lemma 7.2 we obtain

Lemma 7.3. *Take δ so that $M\gamma(\delta) < 1$ holds. Then, the series (7.1) converges when $a \le s \le t \le b$, $t - s \le \delta$, $V(t,s)y$ is differentiable with respect to s in the interval $(\max\{t-\delta, a\}, t)$ and its derivative is $V(t,s)A(s)y$ if $a < t \le b$ and if $y \in Y$.*

Now, the fact that the derivative of $V(t,r)U(r,s)$ with respect to r vanishes implies that $V(t,r)U(r,s)$ is independent of $r \in (s,t)$. Since $U(t,s)$ and $V(t,s)$ are strongly continuous, this gives that $V(t,s) = V(t,r)U(r,s) = U(t,s)$. Thus, by Lemma 7.3 we know that $U(t,s)y$ is differentiable with respect to s in the interval $(\max\{t-\delta, a\}, t)$ for any $t \in (a,b]$ and for any $y \in Y$.

Thus, the facts stated at the beginning of §2 have been completely proved.

8. Proof of Theorem 1.1

Lemma 8.1. *Let $f \in B^{0}_{\infty,1}((a,b);X)$, and define $F_0(t) := \int_a^t W_0(t,s)f(s)ds$. Then $F_0 \in \mathcal{C}(I;Y) \cap \mathcal{C}^1((a,b);X)$, $A(\cdot)F_0(\cdot) \in \mathcal{C}(I;X)$, and the inequalities*

$$\|A(t)F_0(t)\|_X \le C\|f\|_{B^{0}_{\infty,1}((a,b);X))}, \quad \|F_0(t)\|_Y \le \tilde{C}\|f\|_{B^{0}_{\infty,1}((a,b);X)} \tag{8.1}$$

hold for any $t \in I$, where C and $\tilde{C}$ are constants independent of f.

Proof. We first prove that $E(t) := \int_a^t Z_0(t,s)f(s)ds \in \mathcal{D}(A(t))$ for any $t \in I$, $\|A(t)E(t)\|_X \le C'\|f\|_{B^{0}_{\infty,1}((a,b);X))}$ for any $t \in I$, and $A(\cdot)E(\cdot) \in \mathcal{C}(I;X)$.

If $f \in \mathcal{C}^1(I;X)$, we have that

$$A(t)\int_a^t Z_0(t,s)f(s)ds = f(t) - Z_0(t,a)f(a) - \int_a^t Z_0(t,s)f'(s)ds \tag{8.2}$$

holds for any $t \in I$, where $f'(s) = df(s)/ds$ (see Proof of Lemma 5 in [4]). Hence, according to the theory of Besov spaces (see [4] §3), it suffices to consider the case where

$$f(t) = \int_0^c \frac{d\tau}{\tau}\int \frac{1}{\tau}\varphi\left(t, \frac{t-s}{\tau}\right)u(\tau,s)ds, \quad u \in L^1([0,c];L^\infty(I;X)).$$

Here, $\varphi(t,s) = (\partial\psi/\partial s)(t,s)$, $\psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $\psi(t,s) = 0$ if $s-(2t-a-b)/(b-a) \ge 1$. Let η be a C^∞-function such that

$$\eta(t) = 0 \text{ when } t \le 1,\ \eta(t) = 1 \text{ when } t \ge 2 \text{ and } 0 \le \eta(t) \le 1.$$

Then, by Fubini's theorem we have

$$\begin{aligned} E(t) \;&=\; \int_0^c \frac{d\tau}{\tau}\int \{\Phi_1(\tau,t,r) + \Phi_2(\tau,t,r)\}u(\tau,r)dr, \\ \Phi_1(\tau,t,r) \;&:=\; \int_a^t \left\{1-\eta\left(\frac{t-s}{\tau}\right)\right\}Z_0(t,s)\frac{1}{\tau}\varphi\left(s, \frac{s-r}{\tau}\right)ds, \\ \Phi_2(\tau,t,r) \;&:=\; \int_a^t \eta\left(\frac{t-s}{\tau}\right)Z_0(t,s)\frac{1}{\tau}\varphi\left(s, \frac{s-r}{\tau}\right)ds. \end{aligned} \tag{8.3}$$

As $A(t)Z_0(t,s)=(\partial/\partial s)Z_0(t,s)$, an integration by parts shows that

$$\begin{aligned} A(t)\Phi_1(\tau,t,r) = {} & \frac{1}{\tau}\varphi\left(t,\frac{t-r}{\tau}\right) - \left\{1-\eta\left(\frac{t-a}{\tau}\right)\right\} Z_0(t,a)\frac{1}{\tau}\varphi\left(a,\frac{a-r}{\tau}\right) \\ & - \sum_{j=1,2}\int_a^t \left\{1-\eta\left(\frac{t-s}{\tau}\right)\right\} Z_0(t,s)\frac{1}{\tau^j}\varphi_j\left(s,\frac{s-r}{\tau}\right)ds \\ & - \int_a^t \eta'\left(\frac{t-s}{\tau}\right) Z_0(t,s)\frac{1}{\tau^2}\varphi\left(s,\frac{s-r}{\tau}\right)ds, \end{aligned}$$

where $\varphi_1(t,s)=(\partial/\partial t)\varphi(t,s)$, $\varphi_2(t,s)=(\partial/\partial s)\varphi(t,s)$. Hence we have

$$\int \|A(t)\Phi_1(\tau,t,r)\|_{X\to X}dr \le C_0 + \sum_{j=1,2} C_j \int_{t-2\tau}^t ds\tau^{1-j} + C_3\frac{1}{\tau}\int_{t-2\tau}^{t-\tau} ds \le C_4.$$

Since

$$\varphi\left(s,\frac{s-r}{\tau}\right) = \tau\frac{\partial}{\partial s}\left\{\psi\left(s,\frac{s-r}{\tau}\right)\right\} - \tau\psi_1\left(s,\frac{s-r}{\tau}\right),$$

where $\psi_1(t,s) := (\partial\psi/\partial t)(t,s)$, we also have

$$\begin{aligned} A(t)\Phi_2(\tau,t,r) = {} & -\eta\left(\frac{t-a}{\tau}\right)A(t)Z_0(t,a)\psi\left(a,\frac{a-r}{\tau}\right) \\ & - \int_a^t \eta\left(\frac{t-s}{\tau}\right)A(t)^2 Z_0(t,s)\psi\left(s,\frac{s-r}{\tau}\right)ds \\ & + \int_a^t \frac{1}{\tau}\eta'\left(\frac{t-s}{\tau}\right)A(t)Z_0(t,s)\psi\left(s,\frac{s-r}{\tau}\right)ds \\ & - \int_a^t \eta\left(\frac{t-s}{\tau}\right)A(t)Z_0(t,s)\psi_1\left(s,\frac{s-r}{\tau}\right)ds, \end{aligned}$$

which implies, together with the fact that $\eta(t)/t \le 1$, that

$$\begin{aligned} \int \|A(t)\Phi_2(\tau,t,r)\|_{X\to X}dr \le {} & C_5\eta\left(\frac{t-a}{\tau}\right)\frac{\tau}{t-a} + C_6\tau\int_a^{t-\tau}(t-s)^{-2}ds \\ & + C_7\int_{t-2\tau}^{t-\tau}\frac{ds}{t-s} + C_8\int_a^t \eta\left(\frac{t-s}{\tau}\right)\frac{\tau}{t-s}ds \\ \le {} & C_9. \end{aligned}$$

As $\|u(\tau,\cdot)\|_{L^\infty(I;X)} \in L^1((0,c))$ and $A(t)$ is closed, these results and (8.3) give that

$$E(t)\in\mathcal{D}(A(t)),\ A(t)E(t) = \int_0^c \frac{d\tau}{\tau}\int A(t)\{\Phi_1(\tau,t,r)+\Phi_2(\tau,t,r)\}u(\tau,r)dr$$

and $\|A(t)E(t)\|_X \le C'\|f\|_{B^0_{\infty,1}((a,b);X)}$. Since this integral converges uniformly with respect to $t\in I$, we also see that $A(t)E(t)$ is continuous.

The results proved above imply, with the aid of the following lemma and the identity $F_0(t) = \int_a^t Z_0(t,s)f(s)ds - \int_a^t G(t,s)f(s)ds$, that $F_0(t) \in \mathcal{D}(A(t))$ for any $t \in I$, the first inequality in (8.1) and $A(\cdot)F_0(\cdot) \in \mathcal{C}(I;X)$. The second inequality in (8.1) and $F_0 \in \mathcal{C}(I;Y)$ follow from these facts together with the identity $F_0(t) = (I + A(t))^{-1}(I + A(t))F_0(t)$.

Finally, these facts and Lemma 6.1 imply that $F_0 \in \mathcal{C}^1((a,b);X)$. □

Lemma 8.2. *Let $f \in \mathcal{C}(I;X)$ and put $G(t) := \int_a^t G(t,s)f(s)ds$. Then $G \in \mathcal{C}(I;Y)$ and $\|G(t)\|_Y \le P\gamma(t-a)\|f\|_{L^\infty((a,t);X)}$.*

Proof. This follows from the inequality (5.6). □

Now, let us proceed *to prove* Theorem 1.1.

STEP 1. Consider the case where $f \in B^0_{\infty,1}((a,b);X)$ and $M\gamma(b-a) < 1$, where γ is the function defined by (4.3). The estimate (4.8) implies that the series $\sum_{n=0}^\infty W_n(t,s)f(s)$ converges to $U(t,s)f(s)$ in X uniformly in $(t,s) \in T := \{(t,s); a \le s \le t \le b\}$. Hence we have

$$F(t) = \int_a^t U(t,s)f(s)ds = \sum_{n=0}^\infty \int_a^t W_n(t,s)f(s)ds = \sum_{n=0}^\infty F_n(t). \tag{8.4}$$

Using Fubini's theorem, by (4.1) we have

$$F_n(t) := \int_a^t W_n(t,s)f(s)ds = \int_a^t W_0(t,s)H_n(s)ds \tag{8.5}$$

for $n = 0, 1, \cdots$, where $H_0(t) := f(t)$ and

$$H_n(t) := \int_a^t R_n(t,s)f(s)ds \quad \text{for } n = 1, 2, \cdots . \tag{8.6}$$

By Lemma 8.1 we have $F_0 \in \mathcal{C}(I;Y)$ and $\|F_0(t)\|_Y \le \tilde{C}\|f\|_{B^0_{\infty,1}(I;X)}$. Assume that

$$F_n \in \mathcal{C}(I;Y) \text{ and } \|F_n(t)\|_Y \le K(n+1)M^n\gamma(t-a)^n\|f\|_{B^0_{\infty,1}(I;X)}. \tag{8.7}$$

Here K is a constant which will be chosen later on. The identity

$$F_{n+1}(t) = \int_a^t [Q_1(t,s)F_n(s) + G(t,s)\{H_n(s) - H_{n+1}(s)\}]ds \tag{8.8}$$

for $n = 0, 1, \cdots$, which is a consequence of (5.1) and (5.2) with $\sigma = s$, together with (7.3), Lemma 8.2 and the inequality

$$\|H_n(t)\|_X \le M^n\gamma(t-a)^n\|f\|_{L^\infty(I;X)} \tag{8.9}$$

for $n = 0, 1, \cdots$, which follows from (4.7), implies that

$$\|F_{n+1}(t)\|_Y \le \{ K(n+1)(M\gamma(t-a))^{n+1} + 2PM^n\gamma(t-a)^{n+1} \} \|f\|_{B^0_{\infty,1}(I;X)}.$$

Taking $K = \max\{\tilde{C}, 2P/M\}$, this gives (8.7) for $n+1$. By (8.7) we see that $\sum_{n=0}^{\infty} F_n$ converges in $\mathcal{C}(I;Y)$, so that $F \in \mathcal{C}(I;Y)$.

Furthermore, by (8.5) and Lemma 6.1 we see that $F_n \in \mathcal{C}^1(I;X)$ and its derivative is $F_n'(t) = H_n(t) - H_{n+1}(t) - A(t)F_n(t)$. Since the right-hand side of the identity

$$\frac{d}{dt}\sum_{j=0}^{n} F_j(t) = f(t) - H_{n+1}(t) - A(t)\sum_{j=0}^{n} F_j(t)$$

converges to $f(t) - A(t)F(t)$ uniformly in t as $n \to \infty$, we can conclude that $F \in \mathcal{C}^1(I;X)$ and $F'(t) = f(t) - A(t)F(t)$.

STEP 2. Consider now the general case. Let $f \in L^1(I;X) \cap B^0_{\infty,1}((a,b);X)_{\mathrm{loc}}$, and let $a < t < b$. Take α and β so that $a < \alpha < t < \beta < b$ and $M\gamma(\beta - \alpha) < 1$, and put

$$F(t) = \int_a^{\alpha} U(t,s)f(s)ds + \int_{\alpha}^{t} U(t,s)f(s)ds = F_1(t) + F_2(t). \tag{8.10}$$

Since $f \in B^0_{\infty,1}((\alpha,\beta);X)$, by the results in Step 1 we see that $F_2(t) \in \mathcal{D}(A(t))$, $A(\cdot)F_2(\cdot)$ is continuous, F_2 is differentiable, and $F_2'(t) = f(t) - A(t)F_2(t)$. Since $U(t,\alpha)$ is differentiable and $\{\partial/\partial t\}U(t,\alpha) = -A(t)U(t,\alpha)$, it follows that $F_1(t) = U(t,\alpha)F(\alpha)$ is differentiable, $F_1(t) \in \mathcal{D}(A(t))$ and $A(t)F_1(t)$ is continuous. Thus, $F(t)$ is differentiable and $F'(t) = -A(t)F_1(t) + f(t) - A(t)F_2(t) = f(t) - A(t)F(t)$. This completes the proof of Theorem 1.1.

REMARK. The condition $f \in B^0_{\infty,1}((a,b);X)$ follows from $f \in \mathcal{C}(I;X)$ and

$$\rho(h;f) := \sup_{a \le s < s+h \le b} \|f(s+h) - f(s)\| \in L^1\Big((0,\delta), \frac{dh}{h}\Big) \tag{8.11}$$

for some δ. In fact, let $\varphi(t,s)$ be a C$^{\infty}$-function such that $\int \varphi(t,s)ds = 0$ and $\varphi(t,s) = 0$ when $|s - (2t-a-b)/(b-a)| \ge 1$. Then we have

$$\begin{aligned}
&\int_0^c \Big\| \int \frac{1}{\tau}\varphi\Big(t, \frac{t-s}{\tau}\Big) f(s)ds \Big\|_{L^\infty(I;X)} \frac{d\tau}{\tau} \\
&= \int_0^c \Big\| \int \frac{1}{\tau}\varphi\Big(t, \frac{h}{\tau}\Big)\{f(t-h) - f(t)\}dh \Big\|_{L^\infty(I;X)} \frac{d\tau}{\tau} \\
&\le C_0 \int_0^c \frac{d\tau}{\tau^2} \int_{|h| \le \ell\tau, b-a} \rho(|h|;f)dh \le 2C_0\ell \int_0^{b-a} \rho(h;f)\frac{dh}{h} < \infty.
\end{aligned}$$

Thus we have $f \in B^0_{\infty,1}((a,b);X)$ by Thoerem 1 in [4].

From this fact and Theorem 1.1 we see that $F(t)$ is differentiable if $f \in L^1((a,b);X)$ and the condition (8.11) is satisfied locally. This result has been directly proved by Tojima [9] (The case where $A(t)$ is independent of t has been discussed by Crandal-Pazy [1]).

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