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LOGARITHMIC JET BUNDLES AND APPLICATIONS

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0. Introduction

Hyperbolic complex manifolds have been studied extensively during the last 30 years (see, for example, [10], [11]). However, it is still an important problem in hyperbolic geometry to understand the algebro-geometric and the differential-geometric meanings of hyperbolicity. The use of jet bundles has become a powerful tool to attack this problem. For example, Green and Griffiths ([5]) explained an approach to establish Bloch's Theorem on the algebraic degeneracy of holomorphic maps into abelian varieties by constructing negatively curved pseudometrics on jet bundles and by applying Ahlfors' Lemma. Siu and Yeung ([22]) succeeded in carrying out this approach by generalizing the notion of strictly negative curvature. Moreover, a Second Main Theorem for divisors in abelian varieties was claimed in ([22]) but properly proved only recently by Noguchi, Winkelmann and Yamanoi ([18]), who also generalized the claim to the case of semi-tori.

Demailly ([2]) presented a new construction of projective jets and pseudo-metrics on them which realizes directly the approach to Bloch's theorem given in [5]. These projective jets are closer to the geometry of holomorphic curves than the usual jets, since the action of the group of reparametrizations of germs of curves, which is geometrically redundant, is divided out. Using these pseudometrics on projective jets, Demailly and El Goul ([3], see also McQuillan ([13])) were able to show that a (very) generic surface X in $\mathbf{P}^3$ of degree $d \geq 21$ is Kobayashi hyperbolic. As a corollary one obtains that the complement of a (very) generic curve in $\mathbf{P}^2$ of degree $d \geq 21$ is hyperbolic and hyperbolically embedded, a result first proved by Siu and Yeung ([20]) for much higher degree, using jet bundles and value distribution theory. In both papers this quasi-projective case is treated by proving hyperbolicity of a branched cover over the compactification.

However, it is desirable to have also a direct approach to deal with quasiprojective varieties, since one can hope to get easier proofs and even better results[1]. So one should also consider the case of logarithmic jet bundles. Noguchi generalized the Green-Griffiths jet bundles to the logarithmic case in [16], as a first in this regard, and used it to prove the logarithmic version of Bloch's theorem with an approach that

[1]Actually, El Goul told the first named author that, using the results of this paper, he succeeded to drop the degree in [3] from 21 to 15.

started in [14] independently from and earlier than that of Green-Griffiths ([5]).

The main purpose of the present paper is to generalize Demailly's construction of projective jet bundles and strictly negatively curved pseudometrics on them to the logarithmic case. In Sections 1 to 3, we establish this logarithmic generalization of Demailly's construction explicitly via coordinates, just as Noguchi's generalization of the jets used by Green-Griffiths. These explicit coordinates should be very useful for further applications. We also have another, more intrinsic way to obtain the same generalization in [4], which is much shorter, but does not give coordinates right away. In Section 4 we prove the Ahlfors Lemma and the Big Picard Theorem for logarithmic projective jet bundles.

In Section 5, we use our method to give a metric proof of Lang's Conjecture for semi-abelian varieties and of a Big Picard analogue of it, the most relevant previous works for the latter are by Noguchi ([15]) and by Lu ([12]). The first result is due to Siu and Yeung ([21]) and Noguchi ([17]), who used value distribution theory while we use negatively curved jet metrics. However, a common ingredient, due to Siu and Yeung ([21]), is to construct a special jet differential (which naturally lives on the jet space constructed by Demailly) from theta functions on an abelian variety, the existence of which on a semi-abelian variety we cite from Noguchi ([17]). Hence, the main importance of this section is the method of proof. In fact, we have to overcome some small technical difficulties to make our method work in this case: For example, we have to introduce a d-operator for sections over logarithmic projective jet bundles, and we have to deal with the case of a divisor which can have worse singularities than normal crossing, and with the precise relations between two different logarithmic structures (the one coming from the boundary divisor of a semi-abelian variety, the other coming from its union with an arbitrary reduced algebraic divisor). In this way, Section 5 can also serve as a complement to Sections 1 to 4.

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1. Log-directed jet bundles

1.1. Logarithmic jet bundles In this subsection we recall some basic setup and results of Noguchi in [16]. For the proofs we refer to this article. Furthermore, in Sections 1 to 3 we denote open subsets of a manifold by O, in order to distinguish them from open neighborhoods of a given point, usually with fixed coordinates centered at this point, which we denote by U.

Let X be a complex manifold. Let $x \in X$. We consider germs $f : (\mathbf{C}, 0) \to (X, x)$ of holomorphic curves through x. Two such germs are considered to be equivalent if they have the same Taylor expansions of order k in some (and hence, any) local coordinate around x. Denote the equivalence class of f by $j_k(f)$. We define $J_kX_x = \{j_k(f) | f : (\mathbf{C}, 0) \to (X, x)\}$ and $J_kX = \cup_{x \in X} J_kX_x$. Let $\pi : J_kX \to X$ be the natural projection. Then J_kX carries the structure of a holomorphic fiber bundle over X. It is called the *k-jet bundle* over X. If no confusion arises, we will denote the sheaf of sections of J_kX also by J_kX. There exist, for $k \geq l$, canonical projection maps

$$\pi_{l,k} : J_kX \to J_lX \ ; \ \ j_k(f) \to j_l(f), \tag{1.1}$$

and J_1X is canonically isomorphic to the holomorphic tangent bundle TX over X. If $F : X \to Y$ is a holomorphic map to another complex manifold Y, then it induces a holomorphic map

$$F_k : J_kX \to J_kY \ ; \ \ j_k(f) \to j_k(F \circ f) \tag{1.2}$$

over F.

Let ΩX be the holomorphic cotangent bundle over X. Take a holomorphic section $\omega \in H^0(O, \Omega X)$ for some open subset $O \subset X$. For $j_k(f) \in J_kX|_O$ we put $f^*\omega = Z(t)dt$. Then the derivatives $(d^jZ/dt^j)(0)$, $0 \leq j \leq k-1$ are well defined, independently of the representative f for $j_k(f)$. Hence, we have a well defined mapping

$$\tilde{\omega} : J_kX|_O \to \mathbf{C}^k \ ; \ \ j_k(f) \to \left(\frac{d^jZ}{dt^j}(0)\right)_{0 \leq j \leq k-1} \tag{1.3}$$

which is holomorphic. If, moreover, $\omega^1, \ldots, \omega^n$ with $n = \dim X$ are holomorphic 1-forms on O such that $\omega^1 \wedge \cdots \wedge \omega^n$ does not vanish anywhere, then we have a biholomorphic map

$$(\tilde{\omega}^1, \ldots, \tilde{\omega}^n) \times \pi : J_kX|_O \to \left(\mathbf{C}^k\right)^n \times O \tag{1.4}$$

which we call the *trivialization* associated with $\omega^1, \ldots, \omega^n$. More generally, if ω is a section over O in the sheaf of meromorphic 1-forms, then the map $\tilde{\omega}$ defined as in equation (1.3) induces a meromorphic vector valued function

$$\tilde{\omega} : J_kX|_O \to \mathbf{C}^k. \tag{1.5}$$

Let $\overline{X}$ be a complex manifold with a normal crossing divisor D. This means that around any point x of $\overline{X}$, there exist local coordinates $z_1, \ldots, z_n$ centered at x such that D is defined by $z_1z_2 \ldots z_l = 0$ in some neighborhood of x and for some $l \leq n$. We note that l depends on x, which is implicitly assumed. The pair $(\overline{X}, D)$ will be called a *log-manifold*. Let $X = \overline{X} \setminus D$.

Following Iitaka ([7]), we define the logarithmic cotangent sheaf $\overline{\Omega}X = \Omega(\overline{X}, \log D)$ as the locally free subsheaf of the sheaf of meromorphic 1-forms on $\overline{X}$, whose restriction to X is ΩX (where we identify vector bundles and their sheaves of sections) and whose localization at any point $x \in D$ is given by

$$\overline{\Omega} X_x = \sum_{i=1}^{l} \mathcal{O}_{\overline{X},x} \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{\overline{X},x} dz_j, \tag{1.6}$$

where the local coordinates $z_1, \ldots, z_n$ around x is chosen as before. Its dual, the logarithmic tangent sheaf $\overline{T}X = T(\overline{X}, -\log D)$, is a locally free subsheaf of the holomorphic tangent bundle $T\overline{X}$ over $\overline{X}$. Its restriction to X is identical to TX, and its localization at any $x \in D$ is given by

$$\overline{T}X_x = \sum_{i=1}^{l} \mathcal{O}_{\overline{X},x} z_i \frac{\partial}{\partial z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{\overline{X},x} \frac{\partial}{\partial z_j}. \tag{1.7}$$

Given log-manifolds $(\overline{X'}, D')$ and $(\overline{X}, D)$, a holomorphic map $F : \overline{X'} \to \overline{X}$ such that $F^{-1}D \subset D'$ will be called a *log-morphism* from $(\overline{X'}, D')$ to $(\overline{X}, D)$. If no confusion arises, we will simply write $F : X' \to X$ for the log-morphism $F : (\overline{X'}, D') \to (\overline{X}, D)$. It induces (see [7]) vector bundle morphisms,

$$F^* : \overline{\Omega} X \to F^{-1}\overline{\Omega} X' \to \overline{\Omega} X' \text{ and } F_* : \overline{T}X' \to F^{-1}\overline{T}X \to \overline{T}X, \tag{1.8}$$

where we have again identified locally free sheaves and vector bundles.

Let $s \in H^0(O, J_k\overline{X})$ be a holomorphic section over an open subset $O \subset \overline{X}$. We say that s is a *logarithmic k-jet field* if the map $\tilde{\omega} \circ s|_{O'} : O' \to \mathbf{C}^k$ is holomorphic for all $\omega \in H^0(O', \overline{\Omega} X)$ for all open subsets O' of O and where the map $\tilde{\omega}$ is defined as in equation (1.5). The set of logarithmic k-jet fields over open subsets of $\overline{X}$ defines a subsheaf of the sheaf $J_k\overline{X}$, which we denote by $\overline{J}_k X$. By a) of the following proposition, $\overline{J}_k X$ is the sheaf of sections of a holomorphic fiber bundle over $\overline{X}$, which we denote again by $\overline{J}_k X$, and which we call the *logarithmic k-jet bundle* of $(\overline{X}, D)$.

Proposition 1.1 (see [16]). a) *$\overline{J}_k X$ is the sheaf of sections of a holomorphic fiber bundle over $\overline{X}$. (However, it is only a subsheaf and not a subbundle of $J_k\overline{X}$.)*
b) *We have a canonical identification of $(\overline{J}_k X)|_X$ with $J_k X$.*
c) *Let $O \subset \overline{X}$ be an open set and θ be any meromorphic function on O such that the support of its divisor (θ) is contained in D. Let $d^l \log\theta$ be the l-th component of the map $\widetilde{\Theta} : \overline{J}_k X|_O \to \mathbf{C}^k$, where $\Theta = d\log\theta$ (see* equation (1.3) *and* (1.5)). *Then the differentials $d^l \log\theta$, $l = 1, \ldots, k$, define holomorphic functions on $\overline{J}_k X|_O$. Moreover, outside the support of (θ), we have $(d^l \log\theta)(j_k(f)) = \{(d^l \log(\theta \circ f))/dt^l\}(0)$.*
d) *There exists, for $k \geq l$, a canonical projection map $\pi_{l,k} : \overline{J}_k X \to \overline{J}_l X$, which extends the map $(\pi_{l,k}|_{J_k X}) : J_k X \to J_l X$ (see* equation (1.1)), *and $\overline{J}_1 X$ is canonically isomorphic to $\overline{T}X$.*
e) *A log-morphism $F : X' \to X$ induces a canonical map $F_k : \overline{J}_k X' \to \overline{J}_k X$, which extends the map $F_k|_{J_k X} : J_k X' \to J_k X$ (see* equation (1.2)).

Finally, we want to express the local triviality of $\overline{J}_k X$ explicitly in terms of coordinates. Let $z_1, \ldots, z_n$ be coordinates in an open set $U \subset \overline{X}$ in which $D = \{z_1 z_2 \ldots z_l = 0\}$. Let $\omega^1 = dz_1/z_1, \ldots, \omega^l = dz_l/z_l,\ \omega^{l+1} = dz_{l+1}, \ldots, \omega^n = dz_n$. Then we have a biholomorphic map (see equations (1.4) and (1.5))

$$(1.9) \qquad (\tilde{\omega}^1, \ldots, \tilde{\omega}^n) \times \pi : \overline{J}_k X|_U \to \left(\mathbf{C}^k\right)^n \times U.$$

Let $s \in H^0(U, \overline{J}_k X)$ be given by $s(x) = (Z(x); x)$ in this trivialization with

$$Z = \left(Z^i_j\right)_{i=1,\ldots,n;\, j=1,\ldots,k} ,$$

where the $Z^i_j(x)$ are holomorphic functions on U and the indices j correspond to the orders of derivatives. Then the same s, considered as an element of $H^0(U, J_k\overline{X})$ and trivialized by $\omega^1 = dz_1, \ldots, \omega^n = dz_n$ (see equation (1.4)) is given by $s(x) = (\hat{Z}(x); x)$ with $\hat{Z} = (\hat{Z}^i_j)_{i=1,\ldots,n; j=1,\ldots,k}$, where

$$(1.10) \qquad \hat{Z}^i_j = \begin{cases} z_i \cdot \left(Z^i_j + g_j\left(Z^i_1, \ldots, Z^i_{j-1}\right)\right) & : \quad i \le l \\ Z^i_j & : i \ge l+1 \end{cases} .$$

Here, the g_j are polynomials in the variables $Z^i_1, \ldots, Z^i_{j-1}$ with constant coefficients and without constant terms (in particular $g_1 = 0$), which are obtained by expressing first the different components Z^i_j of $((dz_i/z_i)\tilde{\ }) \circ s(x)$ in terms of the components $\hat{Z}^i_j$ of $\widetilde{dz_i} \circ s(x)$ by using the chain rule, and then by expressing the Z^i_j in terms of the $\hat{Z}^i_j$ by inverting this system of polynomial equations. This clarifies equation (1.13) in [16], where, for $i \le l$, only the leading term $z_i Z^i_j$ is given. This also exhibits the sheaf inclusion $\overline{J}_k X|_U \subset J_k\overline{X}|_U$ explicitly in terms of coordinates. By abuse of notation, we also consider the Z^i_j's as the holomorphic functions defined on $\overline{J}_k X|_U$ given by equation (1.9), so that $Z^1_1, \ldots, Z^n_k; z_1, \ldots, z_n$ form a holomorphic coordinate system on $\overline{J}_k X|_U$.

We remark that a trivialization of $\overline{J}_k X|_U$ is also obtained if we replace the special ω's used in equation (1.9) by any $\omega^1, \ldots, \omega^n \in H^0(U, \overline{\Omega} X)$ with

$$(1.11) \qquad \omega^1 \wedge \cdots \wedge \omega^n = \frac{a(x)}{z_1 z_2 \ldots z_l} dz_1 \wedge \cdots \wedge dz_n,$$

where $a(x)$ is a nowhere vanishing holomorphic function on U.

1.2. Log-directed jet bundles We first follow Demailly ([2]). Let X be a complex manifold together with a holomorphic subbundle $V \subset TX$. The pair (X, V) is called a *directed manifold*. If (X, V) and (Y, W) are two such manifolds, then a holomorphic map $F : X \to Y$ which satisfies $F_*(V) \subset W$ is called a *directed morphism*.

Let (X, V) be a directed manifold. The subset $J_k V$ of $J_k X$ is defined to be the set of k-jets $j_k(f) \in J_k X$ for which there exists a representative $f : (\mathbf{C}, 0) \to (X, x)$

such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0. We will show in the next subsection that $J_k V$ is a fiber bundle over X, which we call the *directed k-jet bundle* $J_k V$ of (X, V). If $F : (X, V) \to (Y, W)$ is a directed morphism, then equation (1.2) induces a holomorphic map

$$F_k : J_k V \to J_k W \ ; \quad j_k(f) \to j_k(F \circ f) \tag{1.12}$$

over F, since the restriction of $F_k : J_k X \to J_k Y$ to $J_k V$ maps to $J_k W$ as $(F \circ f)'(t) = F_*(f'(t)) \in W_{F \circ f(t)}$ if $f'(t) \in V_{f(t)}$.

We now generalize Demailly's directed k-jet bundles to the logarithmic context. We define a *log-directed manifold* to be the triple $(\overline{X}, D, \overline{V})$, where $(\overline{X}, D)$ is a log-manifold together with a subbundle $\overline{V}$ of $\overline{T}X$. A *log-directed morphism* between log-directed manifolds $(\overline{X'}, D', \overline{V}')$ and $(\overline{X}, D, \overline{V})$ is a log-morphism $F : (\overline{X}', D') \to (\overline{X}, D)$ such that $F_*(\overline{V}') \subset \overline{V}$.

Let $(\overline{X}, D, \overline{V})$ be a log-directed manifold and set $V = \overline{V}|_X$. By Proposition 1.1 we can canonically identify $(\overline{J}_k X)|_X$ with $J_k X$. Hence, the directed k-jet bundle $J_k V$ of (X, V) can be considered as a subset of the logarithmic k-jet bundle $\overline{J}_k X$ over $\overline{X}$. We define the *log-directed k-jet bundle* $\overline{J}_k V$ of $(\overline{X}, D, \overline{V})$ to be the topological closure $\overline{J_k V} \subset \overline{J}_k X$ of $J_k V$ in $\overline{J}_k X$. If $F : (\overline{X}', D', \overline{V}') \to (\overline{X}, D, \overline{V})$ is a log-directed morphism, it induces a map

$$F_k : \overline{J}_k V' \to \overline{J}_k V \tag{1.13}$$

over F which is holomorphic. It is the restriction of the canonical map $F_k : \overline{J}_k X \to \overline{J}_k X'$ (see Proposition 1.1) to $\overline{J}_k V'$ and is also an extension of the map $F_k|_X : J_k V' \to J_k V$ (see equation (1.12)) to $\overline{J}_k V'$.

1.3. Structure of log-directed jet bundles In this subsection, we study the local structure of $\overline{J}_k V \subset \overline{J}_k X$ over $\overline{X}$. In particular, we show that $\overline{J}_k V \subset \overline{J}_k X$ is a submanifold of $\overline{J}_k X$ which itself is a locally trivial bundle. This justifies the name of log-directed k-jet bundle for $\overline{J}_k V$ introduced in the previous subsection.

First, we consider the directed manifold (X, V). For any point $x_0 \in X$, there is a coordinate system $(z_1, \ldots, z_n)$, centered at x_0, on a neighborhood U of x_0 such that the fibers V_x for $x \in U$ can be defined by linear equations

$$V_x = \left\{ \xi = \sum_{1 \le i \le n} \xi_i \frac{\partial}{\partial z_i} \;\middle|\; \xi_i = \sum_{1 \le m \le r} a_{im}(x) \xi_m \text{ for } i = r+1, \ldots, n \right\} . \tag{1.14}$$

We fix x_0, U and these coordinates from now on. If we trivialize $TX = J_1 X$ by $\omega^1 = dz_1, \ldots, \omega^n = dz_n$ over U as in equation (1.4), we obtain

$$V_x = \left\{ \left(Z_1^1, \ldots, Z_1^n\right) \;\middle|\; Z_1^i = \sum_{1 \le m \le r} a_{im}(x) Z_1^m \text{ for } i = r+1, \ldots, n \right\} . \tag{1.15}$$

If we trivialize $J_k X$ by the same forms, we obtain more generally:

Proposition 1.2. a) *Let P_h^i be the polynomials in the variables Z_j^i with coefficients depending holomorphically on x obtained by formally differentiating the equations*

$$f_i'(t) = \sum_{1 \leq m \leq r} a_{im}(f(t)) f_m'(t)$$

$h-1$ times with respect to t, using the fact that $Z_j^i(j_k(f)) = f_i^{(j)}(t)$ in our trivialization. Then we have

$$\begin{aligned}(J_k V)_x = \{(Z_j^i)_{i=1,\dots,n;j=1,\dots,k} \mid\ & Z_h^i = P_h^i(x, Z_1^1, \dots, Z_1^n, \dots, Z_{h-1}^1, \dots \\ & \dots, Z_{h-1}^n, Z_h^1, \dots, Z_h^r) \text{ for } h = 1, \dots, k, \quad i = r+1, \dots, n\}.\end{aligned} \tag{1.16}$$

b) *$J_k V \subset J_k X$ is a submanifold, and the canonical projection*

$$K : U \to \mathbf{C}^r \ ; \ \ (z_1, \dots, z_n) \to (z_1, \dots, z_r)$$

induces a bundle isomorphism

$$K_k : J_k V|_U \to K^{-1}(J_k \mathbf{C}^r).$$

Proof for a). Let $j_k(f) \in J_k V$. By definition, there exists a representative f such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of $0 \in \mathbf{C}$, namely

$$f_i'(t) = \sum_{1 \leq m \leq r} a_{im}(f(t)) f_m'(t) \, .$$

Now it follows from the chain rule that $j_k(f)$ satisfies equations of the form $Z_h^i = P_h^i$, $h = 1, \dots, k$, $i = r+1, \dots, n$, given in equation (1.16).

Conversely let $Z_j^i \in \mathbf{C}$, $i = 1, \dots, n$, $j = 1, \dots, k$ be given satisfying the equations $Z_h^i = P_h^i$, $h = 1, \dots, k$, $i = r+1, \dots, n$ of equation (1.16). For $x \in X$ fixed, define, for $i = 1, \dots, r$, holomorphic functions

$$f_i : \mathbf{C} \to \mathbf{C}; \ t \to z_i(x) + \sum_{\nu=1}^{k} \frac{Z_\nu^i}{\nu!} t^\nu .$$

Now we integrate the system of differential equations

$$f_i'(t) = \sum_{1 \leq m \leq r} a_{im}(f_1(t), \dots, f_n(t)) f_m'(t) \ \ i = r+1, \dots, n \, ,$$

to obtain a germ $f : (\mathbf{C}, 0) \to (X, x)$ with $z_i(f) = f_i$, $i = 1, \ldots, n$. We see by construction (as $f'(t) \in V_{f(t)}$) that

$$(\tilde{\omega}^1, \ldots, \tilde{\omega}^n)(j_k(f)) = (Z^i_j)_{i=1,\ldots,n; j=1,\ldots,k} .$$

Proof for b). If one replaces successively in the P^i_h all the Z^i_j with $i \geq r+1$ and $j \leq h-1$ by their expressions in terms of the Z^i_j with $i \leq r$ via equation (1.16), we get from a) that $J_k V|_U$ is the graph of these new functions P^i_h, $i = r+1, \ldots, n$; $h = 1, \ldots, k$ in the variables $z_1, \ldots, z_n$ and Z^i_j, $i = 1, \ldots, r$; $h = 1, \ldots, k$. These are in turn coordinates for $K^{-1}(J_k \mathbf{C}^r)$. □

Let now $(\overline{X}, D, \overline{V})$ be a log-directed manifold. Let $x_0 \in \overline{X}$ and let $z_1, \ldots, z_n$ be a coordinate system centered at x_0 on a neighborhood U of x_0 where D is defined by $z_1 z_2 \ldots z_l = 0$ for some $l \leq n$. If we trivialize $\overline{T}X = \overline{J}_1 X$ over U by $\omega^1 = dz_1/z_1, \ldots, \omega^l = dz_l/z_l$, $\omega^{l+1} = dz_{l+1}, \ldots, \omega^n = dz_n$ as in equations (1.9) and (1.10), we obtain

$$V_x = \left\{ (Z^1_1, \ldots, Z^n_1) \mid Z^i_1 = \sum_{m \in A} a_{im}(x) Z^m_1 \text{ for } i \in B \right\} \tag{1.17}$$

for all $x \in U$, where, after permuting $z_1, \ldots, z_l$ respectively $z_{l+1}, \ldots, z_n$, we have $A = \{1, \ldots, a, l+1, \ldots, l+b\}$ and $B = \{1, \ldots, n\} \setminus A$ with $a + b = r = \operatorname{rank} V$. We fix this setup for the rest of this section.

First, we generalize the projection K to log-directed manifolds:

Proposition 1.3. *With $E = \{z_1 \ldots z_a = 0\}$, the log-directed projection*

$$K : (\overline{X}, D, \overline{V})|_U \to (\mathbf{C}^r, E, \overline{T}\mathbf{C}^r)\ ; \quad (z_1, \ldots, z_n) \to (z_1, \ldots z_a, z_{l+1}, \ldots, z_{l+b})$$

has bijective differential map $(K_*)_x$ for all $x \in U$.

Proof. We trivialize $\overline{T}\mathbf{C}^r$ by the forms $\omega^1 = dz_1/z_1, \ldots, \omega^a = dz_a/z_a$, $\omega^{l+1} = dz_{l+1}, \ldots, \omega^{l+b} = dz_{l+b}$. We claim that K_* is given by the projection map

$$(K_*)_x : (\overline{T}X)_x \to (\overline{T}\mathbf{C}^r)_x;\ (Z^1_1, \ldots, Z^n_1) \to (Z^1_1, \ldots, Z^a_1, Z^l_1, \ldots, Z^{l+b}_1) \tag{1.18}$$

in these coordinates. In fact, by analytic continuation it suffices to prove equation (1.18) for $x \in X = \overline{X} \setminus D$. Let $(Z^1_1, \ldots, Z^n_1) \in (\overline{T}X)_x = (TX)_x$ be a vector in the logarithmic coordinate system. If we retrivialize $(TX)_x$ respectively $(T\mathbf{C}^r)_{K(x)}$ with the forms dz_i ($i = 1, \ldots, n$ respectively $i \in A$) instead, then the given vector is expressed by $(z_1 Z^1_1, \ldots, z_l Z^l_1, Z^{l+1}_1, \ldots, Z^n_1)$ (see equations (1.9) and (1.10)). Furthermore, in the latter trivialization, the map $(K_*)_x$ is just the projection to the components given by A. So equation (1.18) follows. Hence, the assertion follows from equation (1.17). □

If we trivialize $\overline{J}_k X$ over U by $\omega^1 = dz_1/z_1, \dots, \omega^l = dz_l/z_l$, $\omega^{l+1} = dz_{l+1}, \dots, \omega^n = dz_n$ as in equations (1.9) and (1.10), we obtain the following extension of Proposition 1.2.

Proposition 1.4. *Let the setup be as above.*
a) *There are polynomials Q^i_h in the variables Z^i_j with coefficients which are holomorphic functions on U such that*

$$\begin{aligned}\overline{J}_k V_x = \{(Z^i_j)_{i=1,\dots,n;\,j=1,\dots,k} \mid\ & Z^i_h = Q^i_h(x, Z^1_1, \dots, Z^n_1, \dots, Z^1_{h-1}, \dots \\ & \dots, Z^n_{h-1}, Z^1_h, \dots, Z^r_h) \text{ for } h = 1, \dots, k,\ \ i \in B\}.\end{aligned} \tag{1.19}$$

b) $\overline{J}_k V \subset \overline{J}_k X$ *is a submanifold and the projection map K defined in* Proposition 1.3 *induces a bundle isomorphism*

$$K_k : \overline{J}_k V|_U \to K^{-1}(\overline{J}_k(\mathbf{C}^r \setminus E)).$$

Proof for a). By using the coordinate system $(z_1, \dots, z_n)$ on U, the map

$$\Psi : \mathbf{C}^n \to \mathbf{C}^n;\ (w_1, \dots, w_n) \to (e^{w_1}, \dots, e^{w_l}, w_{l+1}, \dots, w_n) = (z_1, \dots, z_n)$$

induces a locally biholomorphic map $\Psi : \Psi^{-1}(U) \to U \setminus D$. Let $\hat{V} = \Psi_*^{-1}(V) \subset T(\Psi^{-1}(U))$ and let W^i_j, $i = 1, \dots, n;\ j = 1, \dots, k$ be the components of the first part of the trivialization map

$$(d\tilde{w}_1, \dots, d\tilde{w}_n) \times \pi : J_k(\Psi^{-1}(U)) \to (\mathbf{C}^k)^n \times \Psi^{-1}(U).$$

Lemma 1.5. *On $J_k(\Psi^{-1}(U))$, we have*

$$W^i_j = Z^i_j \circ \Psi_k \ \ i = 1, \dots, n;\ j = 1, \dots, k.$$

Proof of Lemma 1.5. Let $j_k(f) \in J_k(\Psi^{-1}(U))$ and let $f = (f_1, \dots, f_n) : (\mathbf{C}, 0) \to \Psi^{-1}(U)$ represent it. We put $f^* dw_i = df_i(t) = C_i(t)dt$. Then we have $W^i_j(j_k(f)) = \{\partial^{j-1} C_i(t)/\partial t^{j-1}\}|_{t=0}$. On the other hand,

$$(\Psi \circ f)^* \omega^i = df_i(t) = C_i(t)dt$$

independently of i, and hence,

$$Z^i_j \circ \Psi_k(j_k(f)) = \frac{\partial^{j-1} C_i(t)}{\partial t^{j-1}}\bigg|_{t=0} = W^i_j(j_k(f)). \qquad \square$$

Since $\hat{V} \subset T(\Psi^{-1}(U))$ is the inverse image of $V \subset T(U \setminus D)$, we have

$$\hat{V}_w = \left\{ (W_1^1, \ldots, W_1^n) \;\middle|\; W_1^i = \sum_{m \in A} a_{im} \circ \Psi(w) \cdot W_1^m \text{ for } i \in B \right\}, \tag{1.20}$$

for $w \in \Psi^{-1}(U)$. Using Proposition 1.2, we get that

$$\begin{aligned}(J_k \hat{V})_w \;=\; \{(W_j^i)_{i=1,\ldots,n;j=1,\ldots,k} \mid \; & W_h^i = P_h^i(w, W_1^1, \ldots, W_1^n, \ldots, W_{h-1}^1, \ldots \\ & \ldots, W_{h-1}^n, W_h^1, \ldots, W_h^r) \text{ for } h = 1, \ldots, k, \quad i = r+1, \ldots, n\},\end{aligned}$$

where the P_h^i are the polynomials in the variables W_j^i with coefficients depending holomorphically on $w \in \Psi^{-1}(U)$ obtained by formally differentiating

$$f_i'(t) = \sum_{1 \le m \le r} a_{im} \circ \Psi(f(t)) \cdot f_m'(t) \tag{1.21}$$

$h-1$ times. The important point is now that the coefficient functions factor through Ψ by holomorphic functions which are still holomorphic for all $x \in U$:

Main Lemma 1.6. *The coefficients of the polynomials P_h^i factor through Ψ by holomorphic functions which are defined on all of U. Namely,*

$$\begin{aligned}& P_h^i(w, W_1^1, \ldots, W_1^n, \ldots, W_{h-1}^1, \ldots, W_{h-1}^n, W_h^1, \ldots, W_h^r) \\ =\; & Q_h^i(x, W_1^1, \ldots, W_1^n, \ldots, W_{h-1}^1, \ldots, W_{h-1}^n, W_h^1, \ldots, W_h^r)\end{aligned}$$

for $x = \Psi(w) \in U \setminus D$, where the Q_h^i are polynomials in the variables W_j^i, with coefficients which are holomorphic in x on all of U.

Proof of the Main Lemma. If $\alpha : U \to \mathbf{C}$ is holomorphic, we have:

$$\begin{aligned}&\frac{\partial}{\partial t}\alpha \circ \Psi(f(t)) = \sum_{\mu=1}^{n} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot \frac{\partial}{\partial t}(\Psi_\mu \circ f)(t) \\ &= \sum_{\mu=1}^{l} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot \frac{\partial}{\partial t} e^{f_\mu}(t) + \sum_{\mu=l+1}^{n} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot \frac{\partial}{\partial t} f_\mu(t) \\ &= \sum_{\mu=1}^{l} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot (z_\mu \circ \Psi)(f(t)) \cdot f_\mu'(t) + \sum_{\mu=l+1}^{n} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot f_\mu'(t) \\ &= \sum_{\mu=1}^{l} \left(\left(z_\mu \frac{\partial \alpha}{\partial z_\mu}\right) \circ \Psi\right)(f(t)) \cdot f_\mu'(t) + \sum_{\mu=l+1}^{n} \left(\frac{\partial \alpha}{\partial z_\mu} \circ \Psi\right)(f(t)) \cdot f_\mu'(t).\end{aligned}$$

Now the assertion follows by induction on h as the coefficients of the polynomials P_h^i

are obtained by formally differentiating the equation in equation (1.21) $h-1$ times and that the functions a_{im} are holomorphic on all of U. $\square$

Finally we patch together these results and obtain the proof of Proposition 1.4 (a): Using Lemma 1.5, the Main Lemma and the local isomorphisms Ψ^{-1} we see that this assertion holds for all $x \in U \setminus D$, with equations $Z_h^i = Q_h^i(\Psi(\Psi^{-1}(x)), \ldots W_j^i \ldots) = Q_h^i(x, \ldots Z_j^i \ldots)$ which are independent of the choice of the local isomorphism Ψ^{-1}. Moreover, their coefficient functions are still holomorphic on U. Since $\overline{J}_k V$ is defined as the closure of $J_k V$ in $\overline{J}_k X$, the structure of the equations $Z_h^i = Q_h^i$ implies the assertion for all $x \in U$.

Proof for Proposition 1.4 b). It is verbatim that of Proposition 1.2 b). $\square$

1.4. Regular jets Let (X, V) be a directed manifold. The subset $J_k V^{\text{sing}} \subset J_k V$ of *singular k-jets* is defined to be the subset of k-jets $j_k(f) \in J_k V$ of germs $f : (\mathbf{C}, 0) \to (X, x)$ such that $f'(0) = 0$. Its complement $J_k V^{\text{reg}} = J_k V \setminus J_k V^{\text{sing}}$ defines the *regular k-jets*.

Let now $(\overline{X}, D, \overline{V})$ be a log-directed manifold. Define $\overline{J}_k V^{\text{sing}} \subset \overline{J}_k V$ to be the closure $\overline{J_k V^{\text{sing}}} \subset \overline{J}_k V$ of $J_k V^{\text{sing}}$ in $\overline{J}_k V$ and set $\overline{J}_k V^{\text{reg}} = \overline{J}_k V \setminus \overline{J}_k V^{\text{sing}}$.

Proposition 1.7. a) $\overline{J}_k V^{\text{sing}} \subset \overline{J}_k V$ *is a smooth submanifold of codimension* $r = \operatorname{rank} \overline{V}$. *In terms of the local coordinates of* $\overline{J}_k V \subset \overline{J}_k X$ (*see* Proposition 1.4), *this submanifold is given by the equations*

$$Z_1^i = 0, i \in A.$$

b) *The bundle isomorphism* K_k *given in* Proposition 1.4 b) *respects the singular and regular jets.*

Proof for a). Using equations (1.9), (1.10) and (1.17), we see that $J_k V^{\text{sing}}$ in $\overline{J}_k V$ is given locally by the equations $Z_1^i = 0, i \in A$. So the assertion follows.

Proof for b). This follows directly from the proof of Proposition 1.4 b). $\square$

2. Log-Demailly-Semple jet bundles

A natural notion of higher order contact structures was introduced on a firm setting by Demailly ([2]) in the holomorphic category for the study of complex hyperbolic geometry. These structures realize natural "quotient" spaces of directed jet bundles. Demailly called them Semple Jet bundles after Semple ([19]), who constructed and worked with these bundles over $\mathbf{P}^2$. In this section, we generalize these bundles to the logarithmic case and prove some important properties, like functoriality and local triviality. Their connection with log-directed jet bundles will be discussed in Section 3.

2.1. Definition of log-Demailly-Semple jet bundles We begin with a log-directed manifold $X_0 = (\overline{X}_0, D_0, \overline{V}_0)$. We inductively define $(\overline{X}_k, D_k, \overline{V}_k)$ as follows. Let $\overline{X}_k = \mathbf{P}(\overline{V}_{k-1})$ with its natural projection π_k to $\overline{X}_{k-1}$. Set $D_k = \pi_k^{-1}(D_{k-1})$ and $X_k = \overline{X}_k \setminus D_k$. Let $\mathcal{O}_{\overline{X}_k}(-1)$ be the tautological subbundle of $\pi_k^{-1}\overline{V}_{k-1} \subseteq \pi_k^{-1}\overline{T}X_{k-1}$, and set

$$\overline{V}_k = (\pi_k)_\star^{-1}\Big(\mathcal{O}_{\overline{X}_k}(-1)\Big). \tag{2.1}$$

Equivalently, $\overline{V}_k \subset \overline{T}X_k$ is defined, for every point $(x, [v]) \in \overline{X}_k = \mathbf{P}(\overline{V}_{k-1})$ associated with a vector $v \in (\overline{V}_{k-1})_x$ for $x \in \overline{X}_{k-1}$, by

$$(\overline{V}_k)_{(x,[v])} = \big\{\xi \in (\overline{T}X_k)_{(x,[v])} : \ (\pi_k)_*\xi \in \mathbf{C}v\big\}, \quad \mathbf{C}v \subset (\overline{V}_{k-1})_x \subset (\overline{T}X_{k-1})_x.$$

Since $(\pi_k)_\star : \overline{T}X_k \to \pi_k^{-1}\overline{T}X_{k-1}$ has maximal rank everywhere as it is a bundle projection, we see that $\overline{V}_k$ is a subbundle of $\overline{T}X_k$ giving a log-directed structure for X_k and also for π_k, thus completing our inductive definition.

We set $\overline{P}_kV = \overline{X}_k$, $P_kV = X_k$, $\overline{P}_kX = \overline{P}_kTX$ and $P_kX = P_kTX$. Let

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : \overline{P}_kV \to \overline{P}_jV$$

for $j < k$. We also put $(\overline{P}_kV)_x = (\pi_{0,k})^{-1}(x)$ and $(\overline{V}_k)_x = \overline{V}_k|_{(\overline{P}_kV)_x}$ for $x \in \overline{X}$.

Note that $\ker(\pi_k)_\star = T_{\overline{P}_kV/\overline{P}_{k-1}V}$ by definition. This gives the following short exact sequence of vector bundles over $\overline{P}_kV$:

$$0 \longrightarrow T_{\overline{P}_kV/\overline{P}_{k-1}V} \longrightarrow \overline{V}_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{\overline{P}_kV}(-1) \longrightarrow 0. \tag{2.2}$$

Furthermore, we have the Euler exact sequence for projectivized bundles (applied to the bundle $\mathbf{P}(\overline{V}_{k-1}) \to \overline{P}_{k-1}V = \overline{X}_{k-1}$)

$$0 \longrightarrow \mathcal{O}_{\overline{P}_kV} \longrightarrow \pi_k^{-1}\overline{V}_{k-1} \otimes \mathcal{O}_{\overline{P}_kV}(1) \longrightarrow T_{\overline{P}_kV/\overline{P}_{k-1}V} \longrightarrow 0. \tag{2.3}$$

The composition of vector bundle morphisms over $\overline{P}_kV$

$$\mathcal{O}_{\overline{P}_kV}(-1) \hookrightarrow \pi_k^{-1}\overline{V}_{k-1} \xrightarrow{(\pi_k)^{-1}(\pi_{k-1})_\star} \pi_k^{-1}\mathcal{O}_{\overline{P}_{k-1}V}(-1)$$

yields an effective divisor Γ_k corresponding to a section of

$$\mathcal{O}_{\overline{P}_kV}(1) \otimes \pi_k^{-1}\mathcal{O}_{\overline{P}_{k-1}V}(-1) = \mathcal{O}(\Gamma_k). \tag{2.4}$$

There is a canonical divisor on $\overline{P}_kV$ given by

$$\overline{P}_kV^{\text{sing}} = \bigcup_{2 \le j \le k} \pi_{j,k}^{-1}(\Gamma_j) \subset \overline{P}_kV.$$

Finally, we set

$$\overline{P}_k V^{\mathrm{reg}} = \overline{P}_k V \setminus \overline{P}_k V^{\mathrm{sing}} \quad \text{and} \quad \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}} = \left(\mathcal{O}_{\overline{P}_k V}(-1)|_{\overline{P}_k V^{\mathrm{reg}}}\right) \setminus \overline{P}_k V,$$

where the last $\overline{P}_k V$ denotes the zero section.

2.2. Properties of log-Demailly-Semple jet bundles

Proposition 2.1. *Let $F : (\overline{X}', D', \overline{V}') \to (\overline{X}, D, \overline{V})$ be a log-directed morphism.*
a) *For all $k \geq 0$ there exist log-directed meromorphic maps* (*log-directed morphisms outside the locus of indeterminacy*)

$$F_k : (\overline{P}_k V', D'_k, \overline{V}'_k) \cdots \to (\overline{P}_k V, D_k, \overline{V}_k)$$

which commute with the projections, more specifically for all $0 \leq l \leq k-1$ one has

$$\pi_{l,k} \circ F_k = F_l \circ \pi'_{l,k}.$$

These maps in turn induce meromorphic maps

$$(F_k)_* : \mathcal{O}_{\overline{P}_{k+1} V'}(-1) \cdots \to \mathcal{O}_{\overline{P}_{k+1} V}(-1)$$

(*holomorphic where F_k is*) *which also commute with the projections.*
b) *If the differential map $F_* : \overline{V}' \to F^{-1}(\overline{V})$ is injective over a point $x_0 \in \overline{X}'$, then there exists a neighborhood U of x_0 in $\overline{X}'$ over which the maps F_k are log-directed morphisms and the induced maps*

$$F_k : \overline{P}_k V' \to F^{-1}(\overline{P}_k V)$$

are holomorphic embeddings and the induced maps between line bundles

$$(F_k)_* : \mathcal{O}_{\overline{P}_{k+1} V'}(-1) \to F^{-1}(\mathcal{O}_{\overline{P}_{k+1} V}(-1))$$

over these embeddings are injective.
c) *If the differential map $F_* : \overline{V}' \to F^{-1}(\overline{V})$ is bijective in a point $x_0 \in \overline{X}'$, then there exists a neighborhood U of x_0 in $\overline{X}'$ over which the maps F_k are log-directed morphisms and the induced maps*

$$F_k : \overline{P}_k V' \to F^{-1}(\overline{P}_k V)$$
$$(F_k)_* : \mathcal{O}_{\overline{P}_{k+1} V'}(-1) \to F^{-1}(\mathcal{O}_{\overline{P}_{k+1} V}(-1))$$

are all bundle isomorphisms over U.

Combining Proposition 2.1 with Proposition 1.3 yields the following, which we will use in the next section to study $\overline{P}_k V$ by studying $K^{-1}(\overline{P}_k(\mathbf{C}^r \setminus E))$:

Proposition 2.2. *Let $(\overline{X}, D, \overline{V})$ be a log-directed manifold. There exists a neighborhood U of x_0 in $\overline{X}$ and a log-directed projection*

$$K : (\overline{X}, D, \overline{V})|_U \to (\mathbf{C}^r, E, \overline{T}\mathbf{C}^r)\ ; \quad (z_1, \dots, z_n) \to (z_1, \dots z_a, z_{l+1}, \dots, z_{l+b}),$$

with $E = \{z_1 \dots z_a = 0\}$ *and* $a + b = r = \operatorname{rank} V$, *which induces*

$$\begin{aligned}(K_k)_* &: \mathcal{O}_{\overline{P}_{k+1}V}(-1)|_U \to K^{-1}(\mathcal{O}_{\overline{P}_{k+1}(\mathbf{C}^r \setminus E)}(-1)),\\ K_k &: \overline{P}_k V|_U \to K^{-1}(\overline{P}_k(\mathbf{C}^r \setminus E))\end{aligned}$$

as bundle isomorphisms. □

Proof for a). We proceed by induction on k. The case $k = 0$ holds by assumption. Assume the case $k - 1$ holds. If $(F_{k-1})_* : \overline{V}'_{k-1} \to \overline{V}_{k-1}$, define

$$F_k := \mathbf{P}((F_{k-1})_*) : \overline{P}_k V' \to \overline{P}_k V.$$

Then by definition of $D'_k = (\pi'_k)^{-1} D'_{k-1}$ and $D_k = \pi_k^{-1} D_{k-1}$ the map F_k is a log-meromorphic morphism which commutes with projections. Let $(F_k)_* : \overline{T} P_k V' \to \overline{T} P_k V$ be the log-differential map defined as in equation (1.8). If $\xi \in (\overline{V}'_k)_{(w,[v])}$ with $w \in \overline{P}_{k-1} V'$ and $v \in (\overline{V}'_{k-1})_w$, then

$$\begin{aligned}(\pi_k)_*((F_k)_*\xi) &= (\pi_k \circ F_k)_*\xi = (F_{k-1} \circ \pi'_{k-1})_*\xi\\ &= (F_{k-1})_*(\pi'_{k-1})_*\xi \in (F_{k-1})_*\mathbf{C}v = \mathbf{C}((F_{k-1})_*v),\end{aligned}$$

hence, $(F_k)_*\xi \in (\overline{V}_k)_{(F_{k-1}(w),[(F_{k-1})_*v])}$, so F_k is a log-directed meromorphic morphism. The second part of the assertion is clear.

Proof for b and c). $(F_*)_x : (\overline{V}')_x \to (\overline{V})_{F(x)}$ remains injective (respectively bijective) for all x in a neighborhood $U(x_0)$. By the bundle structures it suffices to prove that for all $x \in U(x_0)$, the maps $F_k : (\overline{P}_k V')_x \to (\overline{P}_k V)_{F(x)}$ are holomorphic embeddings (respectively biholomorphic maps) and the maps $(F_k)_* : (V'_k)_x \to (V_k)_{F(x)}$ are injective (respectively bijective) bundle maps over them. We prove this by induction on k. The case $k = 0$ holds by assumption. Assume the case $k - 1$ holds. By projectivizing the injective (respectively bijective) bundle map $(F_{k-1})_*$ and by a), we get that $F_k : (\overline{P}_k V')_x \to (\overline{P}_k V)_{F(x)}$ is an injective (respectively bijective) log-directed morphism. Furthermore, since $F_{k-1} : (\overline{P}_{k-1} V')_x \to (\overline{P}_{k-1} V)_{F(x)}$ is a holomorphic embedding (resp biholomorphic) and since $(F_{k-1})_* \overline{V}'_k \subset \overline{V}_k|_{F_{k-1}(\overline{P}_{k-1}V')}$ is a holomorphic subbundle (respectively the same bundle) the map F_k is an embedding (respectively biholomorphic). It remains to show that $(F_k)_*$ is again injective (respectively bijective).

Let $\xi \in (\overline{V}'_k)_w$ for $w \in \overline{P}_k V'$ such that $(F_k)_*\xi = 0$. Then

$$0 = (\pi_k)_*(F_k)_*\xi = (F_{k-1})_*(\pi'_k)_*\xi.$$

Since the map $(F_{k-1})_*$ is injective we get $\xi \in \ker(\pi_k)_*$. But the subbundle $\ker(\pi_k)_* \subset \overline{V}'_k \subset \overline{T} P_k V'$ can be canonically identified with the relative tangent bundle $T_{\overline{P}_k V'/\overline{P}_{k-1} V'}$, which is a subbundle of $T\overline{P}_k V'$. Since we have shown that F_k is a holomorphic embedding, $(F_k)_*$ is injective on $(T\overline{P}_k V')_w$, which contains ξ. As $(F_k)_*\xi = 0$ this forces $\xi = 0$. So $(F_k)_*$ is injective on $(\overline{V}'_k)_w$. Moreover, if the assumption in c) holds, then $\operatorname{rank} V'_k = \operatorname{rank} V' = \operatorname{rank} V = \operatorname{rank} V_k$, and so $(F_k)_*$ is bijective. □

Corollary 2.3. *We have $F_k((\overline{P}_k V')^{\mathrm{sing}}) \subset \overline{P}_k V^{\mathrm{sing}}$. Moreover, if $F_* : V' \to F^{-1}(V)$ is injective at a point $x_0 \in \overline{X}'$, then there is a neighborhood of x_0 over which $(\overline{P}_k V')^{\mathrm{sing}}$ is isomorphic to $F_k^{-1}(\overline{P}_k V^{\mathrm{sing}})$.*

Proof. By the definition of the singular locus and Proposition 2.1 a) it suffices to show $F_j(\Gamma'_j) \subset \Gamma_j$ (respectively $F_j^{-1}(\Gamma_j) = \Gamma'_j$) for $2 \le j \le k$. Moreover, since the Γ'_j and Γ_j are divisors without vertical components, it suffices to prove the assertions where all maps F_i, $i \le j$ are holomorphic. The first assertion follows immediately from the definitions of Γ'_j and Γ_j and the equation

$$(\pi_{l-1})_* \circ (F_{l-1})_* = (F_{l-2})_* \circ (\pi'_{l-1})_*. \tag{2.5}$$

The second follows from this equation and the injectivity of $(F_{l-2})_*$. □

Corollary 2.4. *Let $(\overline{X}, D, \overline{V})$ be a log-directed manifold. If $\overline{V} \subset \overline{W} \subset \overline{T}X$ are holomorphic subbundles, then we have natural inclusions of submanifolds*

$$\overline{P}_k V \subset \overline{P}_k W \subset \overline{P}_k X$$

and the associated maps over these inclusions of the line bundles

$$\mathcal{O}_{\overline{P}_k V}(-1) \subset \mathcal{O}_{\overline{P}_k W}(-1) \subset \mathcal{O}_{\overline{P}_k X}(-1)$$

are line bundle restrictions.

Proof. We apply Proposition 2.1 to the log-directed morphism $i : (\overline{X}, D, \overline{V}) \to (\overline{X}, D, \overline{W})$, where $i : \overline{X} \to \overline{X}$ is the identity map and induces the bundle inclusion $i_* : \overline{V} \to \overline{W}$. By Proposition 2.1 a) and b) we obtain a log-directed morphism $i_k : \overline{P}_k V \to \overline{P}_k W$ which locally over $\overline{X}$ is, moreover, a holomorphic embedding $i_k : \overline{P}_k V \to i^{-1}(\overline{P}_k W) = \overline{P}_k W$. Hence, $i_k : \overline{P}_k V \to \overline{P}_k W$ is a holomorphic embedding. The other statements follow in a similar way. □

3. Log-directed jet differentials

3.1. Demailly-Semple jet bundles and jet differentials In this subsection we recall parts of some basic results of the work [2] of Demailly on his construction of the Demailly-Semple jet bundles, which we generalize to the logarithmic setting in the next subsections.

Let (X, V) be a directed manifold. Without loss of generality we assume $r = \operatorname{rank} V \geq 2$ in Section 3, for the situation is trivial otherwise. Let

$$G_k = J_k\mathbf{C}_0^{\mathrm{reg}} = \left\{ t \to \phi(t) = \sum_{i=1}^{k} a_i t^i \ , \quad a_1 \in \mathbf{C}^*,\ a_i \in \mathbf{C},\ i \geq 2 \right\}$$

be the *group of reparametrizations*. Elements $\phi \in G_k$ act on J_kV as holomorphic automorphisms by

$$\phi : J_kV \to J_kV;\ j_k(f) \to j_k(f \circ \phi).$$

In particular, $\mathbf{C}^*$ acts on J_kV.

Every nonconstant germ $f : (\mathbf{C}, 0) \to X$ tangent to V lifts to a unique germ $f_{[k]} : (\mathbf{C}, 0) \to P_kV$ tangent to V_k. This $f_{[k]}$ can be defined inductively to be the projectivization of $f'_{[k-1]} : (\mathbf{C}, 0) \to V_{k-1}$. As such we also have a germ

$$f'_{[k-1]} : (\mathbf{C}, 0) \to \mathcal{O}_{P_kV}(-1).$$

This construction is actually a special case of our construction in Proposition 2.1, since $P_k\mathbf{C} = \mathbf{C}$. We get, moreover:

Proposition 3.1. *Let $F : (X', V') \to (X, V)$ be a directed morphism. Let $f : (\mathbf{C}, 0) \to X'$ be a germ tangent to V' such that the germ $F \circ f : (\mathbf{C}, 0) \to X$ is nonconstant. Then there exists a neighborhood U of $0 \in \mathbf{C}$ such that for all $t \in U$, $t \neq 0$, and for all $k \geq 0$, the map $F_k : P_kV' \to P_kV$ (see* Proposition 2.1) *is a morphism around $f_{[k]}(t)$, and we have on U:*

$$(F \circ f)_{[k]} = F_k \circ (f_{[k]}). \tag{3.1}$$

Proof. Since the germ $F \circ f : (\mathbf{C}, 0) \to X$ is nonconstant, we can find a neighborhood U of $0 \in \mathbf{C}$ such that $(F \circ f)'(t) \neq 0$ for all $t \in U$, $t \neq 0$. From the equation $(F \circ f)'(t) = (\pi_{0,k})_* \circ (F \circ f)'_{[k]}(t)$, we get that $(F \circ f)'_{[k]}(t) \neq 0$ for all $k \geq 0$. We now proceed by induction on k. The case $k = 0$ is trivial. Assume the case $k - 1$. This means that for all $t \in U$, $t \neq 0$, the map $F_{k-1} : P_{k-1}V' \to P_{k-1}V$ is a morphism at $f_{[k-1]}(t)$, and we have on U:

$$(F \circ f)_{[k-1]} = F_{k-1}(f_{[k-1]}).$$

Taking the derivative, we obtain

$$(F \circ f)'_{[k-1]}(t) = (F_{k-1})_*(f_{[k-1]})'(t).$$

Now the left hand side is nonzero for $t \neq 0$, so the right hand side is nonzero, too, and we just can projectivize and obtain the assertion for $t \neq 0$. Finally equation (3.1) still holds for $t = 0$ by analytic continuation. □

The *bundle of directed invariant jet differentials of order k and degree m*, denoted by $E_{k,m}V^*$, is defined as follows: Its sheaf of sections $\mathcal{O}(E_{k,m}V^*)$ over X consists of holomorphic functions on $J_kV|_O$ which satisfy

$$Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \ \ \forall\, j_k(f) \in J_kV^{\mathrm{reg}}|_O \text{ and } \phi \in G_k \tag{3.2}$$

as O varies over the open subsets of X. We remark that equation (3.2) implies that the functions Q, restricted to the fibers of J_kV, are polynomials of weighted degree m with respect to the $\mathbf{C}^*$-action, so that our definition coincides with the usual one.

Theorem 3.2 (Demailly ([2])). *Let (X, V) be a directed manifold.*
a) *The maps*

$$\begin{aligned}\tilde{\alpha}_k : J_kV^{\mathrm{reg}} &\to \mathcal{O}_{P_kV}(-1)^{\mathrm{reg}}, \quad j_k(f) \to f'_{[k-1]}(0),\\ \alpha_k : J_kV^{\mathrm{reg}} &\to P_kV^{\mathrm{reg}}, \quad j_k(f) \to f_{[k]}(0)\end{aligned}$$

are well defined, holomorphic and surjective.
b) *If $\phi \in G_k$ is a reparametrization, one has*

$$\begin{aligned}(f \circ \phi)'_{[k-1]}(0) &= f'_{[k-1]}(0) \cdot \phi'(0),\\ (f \circ \phi)_{[k]}(0) &= f_{[k]}(0).\end{aligned}$$

c) *The quotient J_kV^{reg}/G_k of J_kV^{reg} by G_k has the structure of a locally trivial fiber bundle over X, and the map*

$$\alpha_k/G_k : J_kV^{\mathrm{reg}}/G_k \to P_kV$$

is a holomorphic embedding which identifies J_kV^{reg}/G_k with P_kV^{reg}.
d) *The direct image sheaf*

$$(\pi_{0,k})_*\mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$

can be identified with the sheaf $\mathcal{O}(E_{k,m}V^*)$.

Corollary 3.3. 1) *The group $G_k^o = \{t \to \phi(t) = \sum_{i=1}^k a_i t^i\ ,\ \ a_1 = 1\}$ acts transitively on the fibers of $\tilde{\alpha}_k$.*

2) *The maps $\tilde{\alpha}_k$ and α_k are holomorphic submersions.*

Proof for 1). By Theorem 3.2 b) and c), the group G_k acts transitively on the fibers of α_k. So for any two points p and q of a fixed fiber of $\tilde{\alpha}_k$ we find $\phi \in G_k$ such that $\phi(p) = q$. Again by b) we have $\phi'(0) = 1$, so $\phi \in G_k^o$.

Proof for 2). It suffices to prove the statement for α_k, since by the action of $\mathbf{C}^* = G_k/G_k^o$ on $J_k V^{\mathrm{reg}}$ and Theorem 3.2 b), it also follows for $\tilde{\alpha}_k$. The assertion is equivalent with the existence of local sections for α_k through every point $j_k(f) \in J_k V^{\mathrm{reg}}$.
Let $j_k(f) \in J_k V^{\mathrm{reg}}$ be given, and let $w_0 = \alpha(j_k(f))$. Since $f'(0) \not\equiv 0$, we get that $f'_{[k-1]}(0) \not\equiv 0$. Then by Corollary 5.12 in Demailly's paper [2] and its proof we find a neighborhood $U(w_0) \subset P_k V^{\mathrm{reg}}$ and a holomorphic family of germs (f_w), $w \in U(w_0)$, such that $(f_w)_{[k]}(0) = w$ and $f_{w_0} = f$. After possibly shrinking $U(w_0)$, we may assume that $f'_w(0) \not\equiv 0$ for all $w \in U(w_0)$. Thus $w \mapsto j_k(f_w)$ defines a local holomorphic section $s : U(w_0) \to J_k V^{\mathrm{reg}}$; $w \mapsto j_k(f_w)$ with $s(\alpha_k(j_k(f))) = s(w_0) = j_k(f_{w_0}) = j_k(f)$. □

3.2. Local trivializations

Proposition 3.4. *Let $z_1, \ldots, z_r$ be the standard coordinates of $\mathbf{C}^r$, let $a \leq r$, let $E = \{z_1 \ldots z_a = 0\}$ and $P = (\overbrace{1, \ldots, 1}^{a}, \overbrace{0, \ldots, 0}^{r-a}) \in \mathbf{C}^r$.*
a) *The trivialization of $\overline{J}_k(\mathbf{C}^r \setminus E)$ by the forms $\omega^1 = dz_1/z_1, \ldots, \omega^a = dz_a/z_a$, $\omega^{a+1} = dz_{a+1}, \ldots, \omega^r = dz_r$, induce an isomorphism*

$$\overline{J}_k(\mathbf{C}^r \setminus E) \to J_k(\mathbf{C}^r)_P \times \mathbf{C}^r \tag{3.3}$$

which respects regular and singular jets and commutes (outside E) with G_k.
b) *For the log-manifold $(\mathbf{C}^r, E)$ there exists a line bundle isomorphism*

$$\mathcal{O}_{\overline{P}_k(\mathbf{C}^r \setminus E)}(-1) \to \mathcal{O}_{P_k \mathbf{C}^r}(-1)_P \times \mathbf{C}^r \tag{3.4}$$

which respects regular and singular jets and such that the diagram

$$\begin{tikzcd} \overline{J}_k(\mathbf{C}^r \setminus E)^{\mathrm{reg}}|_{\mathbf{C}^r \setminus E} \arrow[r] \arrow[d, "\tilde{\alpha}_k"] & J_k(\mathbf{C}^r)_P^{\mathrm{reg}} \times (\mathbf{C}^r \setminus E) \arrow[d, "(\tilde{\alpha}_k)_P \times id"] \\ \mathcal{O}_{\overline{P}_k(\mathbf{C}^r \setminus E)}(-1)^{\mathrm{reg}}|_{\mathbf{C}^r \setminus E} \arrow[r] & \mathcal{O}_{P_k(\mathbf{C}^r)}(-1)_P^{\mathrm{reg}} \times (\mathbf{C}^r \setminus E) \end{tikzcd} \tag{3.5}$$

commutes.

Proof for a). The composition

$$(J_k\mathbf{C}^r)_P = J_k(\mathbf{C}^r \setminus E)_P \hookrightarrow \overline{J}_k(\mathbf{C}^r \setminus E) \to (\mathbf{C}^k)^r$$

is an isomorphism, where the last morphism is given by the first factor of the trivialization map in equation (1.9). We compose the isomorphism of equation (1.9) with the inverse of the above to obtain the isomorphism

$$\begin{aligned} \overline{J}_k(\mathbf{C}^r \setminus E) &\to J_k(\mathbf{C}^r)_P \times \mathbf{C}^r\,; \\ ((Z^i_j)_{i=1,\dots,r;j=1,\dots,k}; x) &\to (((Z^i_j)_{i=1,\dots,r;j=1,\dots,k}; P); x)\,. \end{aligned}$$

This isomorphism respects regular and singular jets, since the subset of the singular jets is given in every fiber by $\{Z^i_1 = 0\,,\ i = 1, \dots, r\}$ by Proposition 1.7.

Let us understand this isomorphism, restricted to $\mathbf{C}^r \setminus E$, in a more geometric way. As in the proof of Proposition 1.4, let

$$\Psi : \mathbf{C}^r \to \mathbf{C}^r;\ (w_1, \dots, w_r) \to (e^{w_1}, \dots, e^{w_a}, w_{a+1}, \dots, w_r) = (z_1, \dots, z_r),$$

and let W^i_j, $i = 1, \dots, r;\ j = 1, \dots, k$ be the components of the first part of

$$(d\tilde{w}_1, \dots, d\tilde{w}_r) \times \pi : J_k(\mathbf{C}^r) \to (\mathbf{C}^k)^r \times \mathbf{C}^r.$$

Then we claim that the above isomorphism, restricted to $\mathbf{C}^r \setminus E$, is given by

$$J_k(\mathbf{C}^r \setminus E) \to J_k(\mathbf{C}^r)_P \times \mathbf{C}^r \setminus E\,;\ j_k(f) \to (j_k(\Psi(\Psi^{-1} \circ f - \Psi^{-1} \circ f(0))), f(0)),$$

where subtraction means subtraction in $\mathbf{C}^r$. Note that Ψ^{-1} is only defined up to addition of summands $2\pi i m$, $m \in \mathbb{Z}$ for the first a components, but the germ $\Psi^{-1} \circ f - \Psi^{-1} \circ f(0)$ is well defined. In fact, by Lemma 1.5 the above isomorphism is given by trivial shift with respect to the coordinates W^i_j, but this is, by the definition of these coordinates, just subtraction of the value of the germ for $t = 0$. It follows that the above isomorphism commutes with the action of G_k. In fact, reparametrization does not depend on the coordinates and so it commutes with Ψ and Ψ^{-1}. Furthermore, it commutes with subtraction of constants in $\mathbf{C}^r$. This proves a).

Proof for b). We use the following strategy: Using some results of Demailly's paper [2] we first define the isomorphism of equation (3.4) on $\mathbf{C}^r \setminus E$ similarly to our geometric way above. With this isomorphism it is easy to verify the diagram of equation (3.5). We then extend the isomorphism over the complement of a divisor in $\overline{P}_k(\mathbf{C}^r \setminus E)$ which does not contain any entire fiber over $\mathbf{C}^r$. For this, we introduce an explicit local coordinate chart in $\overline{P}_k(\mathbf{C}^r \setminus E)$ the complement of which is such a divisor. In order to extend over the remaining codimension two locus we use the fact that our objects are defined inductively by projectivizing vector bundles, and that for vector bundle maps, the Riemann Extension Theorem holds. This way is not the shortest possible (see Lemma 5.10, which gives a shorter and intrinsic proof of this extension over the divisor E), but it explains well the geometric contents of the isomorphism in equation (3.4) via explicit local coordinates (see also Corollary 3.7), which is useful for applications.

By Corollary 5.12 of [2], for all points $w \in P_k(\mathbf{C}^r \setminus E)$, there exists a germ $f : (\mathbf{C}, 0) \to \mathbf{C}^r \setminus E$ such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$. We claim that by composing this germ with the map $t \to at + t^2$, $a \in \mathbf{C}$, the vector $f'_{[k-1]}(0)$ can be made equal to an arbitrary vector in the complex line $\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1)$ over w. This follows from Theorem 3.2 b) for $a \neq 0$. Since the image of the germ $f_{[k]}$ does not change, we get, after an easy computation, that $f'_{[k-1]}(0) = 0$ for $a = 0$. So every vector of $\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1)$ is obtained this way.

As above, the map

$$\begin{aligned} &\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1) \to (\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1))_P \times (\mathbf{C}^r \setminus E)\,; \\ &f'_{[k-1]}(0) \to (\Psi(\Psi^{-1}\circ f - \Psi^{-1}\circ f(0))'_{[k-1]}(0); f(0)) \end{aligned}$$

is a well defined isomorphism, its inverse being given by the well defined map

$$\begin{aligned} &(\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1))_P \times \mathbf{C}^r \setminus E \to \mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1)\,; \\ &(f'_{[k-1]}(0), p) \to \Psi(\Psi^{-1}\circ f + \Psi^{-1}(p))'_{[k-1]}(0)\,. \end{aligned}$$

Now, by Proposition 5.11 of [2], the singular locus of $\mathcal{O}_{P_k(\mathbf{C}^r\setminus E)}(-1)$ can be characterized by $f'_{[k-1]} = 0$ along with the multiplicities of $f_{[j]}$, $j = 0, \dots, k-1$, which remain invariant under changes of coordinates or additions by constants. So this isomorphism respects the regular and singular jet loci. So we can restrict it to the regular loci. If now $j_k(f) \in J_k(\mathbf{C}^r\setminus E)^{\mathrm{reg}}$, then this is mapped to $(\Psi(\Psi^{-1}\circ f - \Psi^{-1}\circ f(0))'_{[k-1]}(0), f(0))$ by both compositions of the maps in the diagram in equation (3.5), so this diagram commutes.

We now carry out the above strategy via the following lemma. Let $\overline{V}_k^{\mathrm{reg}} = \overline{V}_k \setminus \overline{P}_k V|_{\overline{P}_k V^{\mathrm{reg}}}$, where the $\overline{P}_k V$ denotes the zero section of $\overline{V}_k$. Then we have a canonical identification

$$\mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}} \xrightarrow{\sim} \overline{V}_{k-1}^{\mathrm{reg}}\,.$$

We now introduce r coordinate charts on $\overline{V}_{k-1}^{\mathrm{reg}}$ in the same way as Demailly did for $P_k V^{\mathrm{reg}}$ in equations (4.9) and (5.7) and Theorem 6.8 of [2].

Lemma 3.5. *Let* $((Z^i_j)_{i=1,\dots,r;j=1,\dots,k};(z_1,\dots,z_r))$ *be the coordinates of* $\overline{J}_k(\mathbf{C}\setminus E)$, *and let*

$$\overline{A}_{k,r}=\left(\bigcap_{j=2}^{k}\{Z^r_j=0\}\right)\cap(\overline{J}_k(\mathbf{C}\setminus E)\setminus\{Z^r_1=0\}).$$

Then the map

$$\tilde{\alpha}_k:\overline{A}_{k,r}|_{\mathbf{C}^r\setminus E}\to\overline{V}_{k-1}|_{\mathbf{C}^r\setminus E} \tag{3.6}$$

extends over E *to a map which is biholomorphic onto its image* $\tilde{\alpha}_k(A_{k,r})$, *such that this image contains the complement of a divisor in* $\overline{V}_{k-1}$ *which is nowhere dense in* $(\overline{V}_{k-1})_x$ *for all* $x\in\mathbf{C}^r$. *More precisely*:

CLAIM S(k). $\overline{V}_{k-1}\to\overline{P}_{k-1}(\mathbf{C}^r\setminus E)$ is a vector bundle of rank r over a $(k-1)$-stage tower of $\mathbf{P}^{r-1}$-bundles, and we can introduce inhomogenous coordinates on these bundles corresponding to the coordinates $(z_1,\dots,z_r)$ of $\mathbf{C}^r$, in which the map $\tilde{\alpha}_k$ of equation (3.6) is given by

$$\begin{aligned}&((Z^i_j)_{i=1,\dots,r-1;j=1,\dots,k},Z^r_1;(z_1,\dots,z_r))\to\\&\quad\left(\left(\frac{Z^1_1}{Z^r_1},\dots,\frac{Z^{r-1}_1}{Z^r_1}\right),\left(\frac{Z^1_2}{(Z^r_1)^2},\dots,\frac{Z^{r-1}_2}{(Z^r_1)^2}\right),\dots,\right.\\&\quad\left.\left(\frac{Z^1_{k-1}}{(Z^r_1)^{k-1}},\dots,\frac{Z^{r-1}_{k-1}}{(Z^r_1)^{k-1}}\right),\left(\frac{Z^1_k}{(Z^r_1)^{k-1}},\dots,\frac{Z^{r-1}_k}{(Z^r_1)^{k-1}},Z^r_1\right);(z_1,\dots,z_r)\right).\end{aligned}$$

Proof of Lemma 3.5. It suffices to prove S(k). We prove this by induction. The statement S(1) is trivial. Assume by induction that S($k-1$) holds. Then the corresponding inhomogenous coordinates of $\overline{P}_{k-1}(\mathbf{C}^r\setminus E)$ are given by

$$\begin{aligned}&\left(\left(\frac{Z^1_1}{Z^r_1},\dots,\frac{Z^{r-1}_1}{Z^r_1}\right),\left(\frac{Z^1_2}{(Z^r_1)^2},\dots,\frac{Z^{r-1}_2}{(Z^r_1)^2}\right),\right.\\&\quad\left.\dots,\left(\frac{Z^1_{k-1}}{(Z^r_1)^{k-1}},\dots,\frac{Z^{r-1}_{k-1}}{(Z^r_1)^{k-1}}\right);(z_1,\dots,z_r)\right).\end{aligned} \tag{3.7}$$

In order to find coordinates over this affine chart, we proceed in two steps:

The first step is to find the coordinates of the logarithmic tangent bundle

$$\overline{T}P_{k-1}(\mathbf{C}^r\setminus E)\to\overline{P}_{k-1}(\mathbf{C}^r\setminus E)$$

over our affine chart. Note that, in the coordinates of equation (3.7), the divisor $E_{k-1}=\pi^{-1}_{0,k-1}(E)$ is given by $\{z_1\dots z_a=0\}$ and, hence, is independent of any of the fiber coordinates Z^i_j or of their quotients $Z^i_j/(Z^r_1)^j$. So the coordinates of $\overline{T}P_{k-1}(\mathbf{C}^r \setminus E)$ are given by those of equation (3.7) and their differentials, except for $z_1, \dots, z_a$, where the log-differentials are needed.

The second step is to restrict this coordinate system to the subbundle $\overline{V}_{k-1} \subset \overline{T}P_{k-1}(\mathbf{C}^r \setminus E)$. By the definition of this subbundle in equation (2.1) (see also Demailly's equation (5.7) in [2]) we choose the differentials of the $r-1$ coordinate functions of equation (3.7) which describe the fibers of the map $\pi_{k-1} : \overline{P}_{k-1}(\mathbf{C}^r \setminus E) \to \overline{P}_{k-2}(\mathbf{C}^r \setminus E)$ and of an extra component which corresponds to how we have chosen the inhomogenous coordinates: These are the (nonlog-) differentials $d(Z^i_{k-1}/(Z^r_1)^{k-1})$, $i = 1, \dots, r$ plus the extra component, corresponding to the log- (in case $a = r$) or nonlog- (in case $a < r$) differential of z_r, which is Z^r_1 (see equations (1.3) and (1.9)).

It remains to express these coordinates without the differential as in Claim S(k). It suffices to do this outside E, since these expressions then extend over E by the identity theorem. We have for $1 \le i \le r-1$:

$$d\left(\frac{Z^i_{k-1}}{(Z^r_1)^{k-1}}\right) = \frac{dZ^i_{k-1}}{(Z^r_1)^{k-1}} - \frac{Z^i_{k-1}}{(Z^r_1)^{k-1}} \cdot (k-1)\frac{dZ^r_1}{Z^r_1}\,.$$

By equations (1.3) and (1.9) we get

$$dZ^i_{k-1}(j_k(f)) = \left(\frac{d}{dt}\frac{d^{k-2}}{dt^{k-2}}\frac{f^*\omega^i}{dt}\right)\Big|_{t=0} = \left(\frac{d^{k-1}}{dt^{k-1}}\frac{f^*\omega^i}{dt}\right)\Big|_{t=0} = Z^i_k(j_k(f))\,.$$

As we only work on the submanifold $\overline{A}_{k,r}$, we have $j_k(z_r \circ f) = a_0(x) + a_1(x)t$. We now again use equations (1.3) and (1.9), and distinguish two cases:
If $a < r$, then

$$dZ^1_r(j_k(f)) = \left(\frac{d}{dt}\frac{f^*dz_r}{dt}\right)\Big|_{t=0} = \left(\frac{d}{dt}a_1(x)\right)\Big|_{t=0} = 0\,.$$

If $a = r$, then

$$\begin{aligned} dZ^1_r(j_k(f)) &= \left(\frac{d}{dt}\frac{f^*dz_r}{(z_r \circ f)dt}\right)\Big|_{t=0} = \left(\frac{d}{dt}\frac{a_1(x)}{a_0(x)+a_1(x)t}\right)\Big|_{t=0} \\ &= -\left(\frac{a_1(x)}{a_0(x)}\right)^2 = -\left(\frac{f^*dz_r}{(z_r \circ f)dt}\Big|_{t=0}\right)^2 = -\left(Z^r_1\right)^2(j_k(f))\,. \end{aligned}$$

So we have

$$d\left(\frac{Z^i_{k-1}}{(Z^r_1)^{k-1}}\right) = \frac{Z^i_k}{(Z^r_1)^{k-1}} + (k-1)\cdot\begin{cases} 0 & : a < r \\ Z^r_1 & : a = r \end{cases}\,.$$

These coordinates can be expressed in those of claim S(k) and vice versa. □

Now we use Lemma 3.5 to extend the diagram in equation (3.4), and, thus, the isomorphism of equation (3.4): The diagram becomes

$$\begin{tikzcd}
\overline{A}_{k,r}|_{\mathbf{C}^r\setminus E} \arrow[r] \arrow[d, "\tilde{\alpha}_k"] & (A_{k,r})_P \times (\mathbf{C}^r \setminus E) \arrow[d, "(\tilde{\alpha}_k)_P \times id"] \\
\overline{V}_{k-1}|_{\mathbf{C}^r\setminus E} \arrow[r] & (V_{k-1})_P \times (\mathbf{C}^r \setminus E)
\end{tikzcd}$$

where the two vertical arrows are now biholomorphic onto their images. By Lemma 3.5 these isomorphisms extend to isomorphisms over $\mathbf{C}^r$. So the isomorphism of equation (3.4) extends over E outside a horizontal divisor which is nowhere dense in all fibers, giving an isomorphism outside an analytic set of codimension at least two.

We finally prove, by induction over k, that these isomorphisms extend to

$$\begin{tikzcd}
\overline{V}_{k-1} \arrow[r] \arrow[d] & (V_{k-1})_P \times \mathbf{C}^r \arrow[d] \\
\overline{P}_{k-1}V \arrow[r] & (P_{k-1}V)_P \times \mathbf{C}^r
\end{tikzcd}$$

which induce the desired isomorphisms of equation (3.4). The case S(1) is trivial (there is nothing to extend any more in this case). Assume by induction that S$(k-1)$ is true. Then by projectivizing we have an isomorphism $\overline{P}_{k-1}V \to (P_{k-1}V)_P \times \mathbf{C}^r$, and over this we have an isomorphism $\overline{V}_{k-1} \to (V_{k-1})_P \times \mathbf{C}^r$ up to a subvariety of codimension two. Now this isomorphism extends, since for vector bundle maps, the Riemann Extension Theorem holds. (For any point $w \in \overline{P}_{k-1}V$, take a dual basis of $\overline{V}_{k-1}$ around w. Then the extension of the maps, in both directions, is reduced to extension of holomorphic functions once we compose these maps with the dual vectors.) The fact that the extended maps are still inverses to each other follows from the Identity Theorem. This ends the proof of Proposition 3.4. $\square$

Important Remark 1. The local isomorphisms of equations (3.3) and (3.4) are fiber bundle isomorphisms. But they are *not* induced by (directed) morphisms. As a result, these local isomorphisms have a priori no functoriality, and every compatibility which one needs has to be proved explicitly, which we proceed to do.

Proposition 3.6. *Let $(\overline{X}, D, \overline{V})$ be a log-directed manifold. Let $x_0 \in \overline{X}$ and let U and the log-directed projection*

$$K : (\overline{X}, D, \overline{V})|_U \to (\mathbf{C}^r, E, \overline{T}\mathbf{C}^r)\ ;\ \ (z_1, \ldots, z_n) \to (z_1, \ldots z_a, z_{l+1}, \ldots, z_{l+b}),$$

with $E = \{z_1 \ldots z_a = 0\}$ and $a + b = r = \operatorname{rank} V$, be as in Proposition 2.2. *Without loss of generality, let* $P = (\overbrace{1, \ldots, 1}^{l}, \overbrace{0, \ldots, 0}^{n-l}) \in U$.

a) *The isomorphisms of* Proposition 3.4, Proposition 1.4 *and* Proposition 2.2 *induce isomorphisms*

$$\overline{J}_k V|_U \to J_k V_P \times U$$

and

$$\mathcal{O}_{\overline{P}_k V}(-1)|_U \to \mathcal{O}_{P_k V}(-1)_P \times U$$

respecting regular and singular jets such that the first isomorphism commutes (*outside* D) *with the action of* G_k *and such that the diagrams*

$$\begin{tikzcd}
K^{-1}(\overline{J}_k(\mathbf{C}^r \setminus E))|_U \arrow[rr] & & (K^{-1}(J_k\mathbf{C}^r))_P \times U \\
& \overline{J}_k V|_U \arrow[ul] \arrow[rr] & & J_k V_P \times U \arrow[ul]
\end{tikzcd} \tag{3.8}$$

and

$$\begin{tikzcd}
K^{-1}(\mathcal{O}_{\overline{P}_k(\mathbf{C}^r\setminus E)}(-1))|_U \arrow[rr] & & (K^{-1}(\mathcal{O}_{P_k\mathbf{C}^r}(-1)))_P \times U \\
& \mathcal{O}_{\overline{P}_k V}(-1)|_U \arrow[ul] \arrow[rr] & & \mathcal{O}_{P_k V}(-1)_P \times U \arrow[ul]
\end{tikzcd} \tag{3.9}$$

commute.

b) *Moreover, outside the divisor* D, *they induce the following cubic diagram*

(3.10)

$$\begin{tikzcd}
K^{-1}(\overline{J}_k(\mathbf{C}^r \setminus E)^{\mathrm{reg}})|_{U\setminus D} \arrow[rr] \arrow[dd, "\tilde{\alpha}_k"] & & (K^{-1}(J_k\mathbf{C}^r)^{\mathrm{reg}})_P \times (U\setminus D) \arrow[dd, "(\tilde{\alpha}_k)_P\times id"] & \\
& \overline{J}_k V^{\mathrm{reg}}|_{U\setminus D} \arrow[ul] \arrow[rr] \arrow[dd, "\tilde{\alpha}_k"] & & J_k V_P^{\mathrm{reg}} \times (U\setminus D) \arrow[ul] \arrow[dd, "(\tilde{\alpha}_k)_P\times id"] \\
K^{-1}(\mathcal{O}_{\overline{P}_k(\mathbf{C}^r\setminus E)}(-1)^{\mathrm{reg}})|_{U\setminus D} \arrow[rr] & & \begin{array}{c}(K^{-1}(\mathcal{O}_{P_k\mathbf{C}^r}(-1))^{\mathrm{reg}})_P \\ \times (U\setminus D)\end{array} & \\
& \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_{U\setminus D} \arrow[ul] \arrow[rr] & & \mathcal{O}_{P_k V}(-1)_P^{\mathrm{reg}} \times (U\setminus D) \arrow[ul]
\end{tikzcd}$$

c) *By combining with the canonical line bundle projections we get the same isomorphisms and diagrams with* $\overline{P}_k(\mathbf{C}^r \setminus E)$, $\overline{P}_k V$ *and* α_k *instead of* $\mathcal{O}_{\overline{P}_k(\mathbf{C}^r\setminus E)}(-1)$,

$\mathcal{O}_{\overline{P}_k V}(-1)$ *and* $\tilde{\alpha}_k$.

Proof for a). We *define* the isomorphisms $\overline{J}_k V|_U \to J_k V_P \times U$ respectively $\mathcal{O}_{\overline{P}_k V}(-1)|_U \to \mathcal{O}_{P_k V}(-1)_P \times U$ by the other three arrows of the respective diagrams. In this way we obtain trivializations which, by definition, make the diagrams commutative. By Proposition 3.4, Proposition 1.7, Proposition 1.4 and Corollary 2.3, the regular and singular loci are preserved. The first isomorphism commutes with the action of G_k. In fact, by Proposition 3.4 a) this is true for the upper line of the diagram in equation (3.8). Furthermore, the isomorphism K_k in the vertical arrows is, outside D, just given by $K_k(j_k(f)) = j_k(K \circ f)$, and this trivially commutes with the action of G_k.

Proof for b). The back side of this cubic diagram (the side with the K^{-1}) is just the pull back the diagram in equation (3.5). The upper and the lower sides of the cubic diagram are the restrictions of the diagrams in equations (3.8) respectively (3.9) to the regular locus over $U \setminus D$. The left hand side respectively the right hand side of the cubic diagram commute by the functoriality of the map K, see also the equations (1.12) and (3.1). It is an easy exercise to see that then the front side of the cubic diagram commutes also and, furthermore, that the whole diagram commutes.

Proof for c). This is clear from the diagrams. $\square$

Important Remark 2. For all local isomorphisms given by the horizontal left-to-right arrows in the above diagrams, our Important Remark 1 also applies. However, the local isomorphisms induced by K are functorial.

Corollary 3.7. a) *The fiber bundles* $\overline{P}_k V$, $\mathcal{O}_{\overline{P}_k V}(-1)$ *and* $\overline{V}_k$ *and their regular and singular jet loci are all locally trivialized over* $\overline{X}$ *in a way which is compatible, through the maps* α_k *respectively* $\tilde{\alpha}_k$, *with the trivialization of* $\overline{J}_k V$ *by using local logarithmic coordinates.*
b) *Let* $U \subset \overline{X}$ *and* K *be like in* Proposition 2.2. *Let* $\overline{A}_{k,i} \subset \overline{J}_k(\mathbf{C}^r \setminus E)$, $i = 1, \ldots, r$, *be like in* Lemma 3.5, *and let* $\overline{B}_{k,i} = \overline{A}_{k,i} \cap \{Z_1^i = 1\}$. *Then there exist* r *coordinate charts*

$$K^{-1}(\overline{A}_{k,i}) \to \overline{V}_{k-1} \text{ (respectively } K^{-1}(\overline{A}_{k,i}) \to \mathcal{O}_{\overline{P}_k V}(-1));$$

$$\left((Z_j^i)_{i=1,\ldots,r-1;j=1,\ldots,k}, Z_1^r; (z_1, \ldots, z_n)\right) \to$$

$$\left(\left(\frac{Z_1^1}{Z_1^r}, \ldots, \frac{Z_1^{r-1}}{Z_1^r}\right), \left(\frac{Z_2^1}{(Z_1^r)^2}, \ldots, \frac{Z_2^{r-1}}{(Z_1^r)^2}\right), \ldots, \right.$$
$$\left. \left(\frac{Z_{k-1}^1}{(Z_1^r)^{k-1}}, \ldots, \frac{Z_{k-1}^{r-1}}{(Z_1^r)^{k-1}}\right), \left(\frac{Z_k^1}{(Z_1^r)^{k-1}}, \ldots, \frac{Z_k^{r-1}}{(Z_1^r)^{k-1}}, Z_1^r\right); (z_1, \ldots, z_n)\right)$$

which cover $\overline{V}^{\mathrm{reg}}_{k-1}$ *(respectively* $\mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}$*), and* r *coordinate charts*

$$
\begin{aligned}
&K^{-1}(\overline{B}_{k,i}) \to \overline{P}_{k-1}V;\\
&\left((Z^i_j)_{i=1,\dots,r-1;j=1,\dots,k};(z_1,\dots,z_n)\right) \to\\
&\qquad \left(\left(Z^1_1,\dots,Z^{r-1}_1\right),\left(Z^1_2,\dots,Z^{r-1}_2\right),\dots,\left(Z^1_k,\dots,Z^{r-1}_k\right);(z_1,\dots,z_n)\right)
\end{aligned}
$$

which cover $\overline{P}_{k-1}V^{\mathrm{reg}}$.

Proof. a) is contained in Proposition 3.6. The existence of the coordinate charts in b) follows from Proposition 3.6 and Lemma 3.5. These charts cover the locus of regular jets by Theorem 3.2, a) outside D. Thus by our local trivializations which are compatible with the charts and with the locus of regular jets, these charts cover the locus of regular jets everywhere. □

REMARK. The coordinates can also be obtained directly without Lemma 3.5.

3.3. Log-directed jets and log-Demailly-Semple jets This subsection extends Theorem 3.2 a), b) and c) to the log-directed case.

Proposition 3.8. *Let* $(\overline{X}, D, \overline{V})$ *be a log-directed manifold.*
a) *The maps* $\tilde{\alpha}_k$ *and* α_k *of* Theorem 3.2 a) *extend to holomorphic and surjective maps*

$$
\begin{aligned}
\tilde{\alpha}_k &: \overline{J}_k V^{\mathrm{reg}} \to \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}},\\
\alpha_k &: \overline{J}_k V^{\mathrm{reg}} \to \overline{P}_k V^{\mathrm{reg}}.
\end{aligned}
$$

b) *The action of* $\phi \in G_k$ *extends to an automorphism of* $\overline{J}_k V$ *leaving* $\overline{J}_k V^{\mathrm{reg}}$ *and* $\overline{J}_k V^{\mathrm{sing}}$ *invariant and*

$$\tilde{\alpha}_k \circ \phi = \tilde{\alpha} \cdot \phi'(0)\,,\ \ \alpha_k \circ \phi = \alpha_k\,.$$

c) *The quotient* $\overline{J}_k V^{\mathrm{reg}}/G_k$ *has the structure of a locally trivial fiber bundle over* $\overline{X}$*, and the map*

$$\alpha_k/G_k\ : \overline{J}_k V^{\mathrm{reg}}/G_k \to \overline{P}_k V$$

is a holomorphic embedding which identifies $\overline{J}_k V^{\mathrm{reg}}/G_k$ *with* $\overline{P}_k V^{\mathrm{reg}}$.

Proof for a). By Theorem 3.2, the map $\tilde{\alpha}_k$ is defined outside D:

$$\tilde{\alpha}_k : \overline{J}_k V^{\mathrm{reg}}|_{\overline{X}\setminus D} \to \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_{\overline{X}\setminus D}. \tag{3.11}$$

Let $x \in D$. By Proposition 3.6 b), there exists a neighborhood U of x with

$$
\begin{tikzcd}
\overline{J}_k V^{\mathrm{reg}}|_{U \setminus D} \arrow[r] \arrow[d, "\tilde{\alpha}_k"] & J_k V_P^{\mathrm{reg}} \times (U \setminus D) \arrow[d, "(\tilde{\alpha}_k)_P \times id"] \\
\mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_{U \setminus D} \arrow[r] & \mathcal{O}_{P_k V}(-1)_P^{\mathrm{reg}} \times (U \setminus D)
\end{tikzcd}
\tag{3.12}
$$

Here the horizontal arrows are isomorphisms, which, by Proposition 3.6 a), extend as isomorphisms over U, and $(\tilde{\alpha}_k)_P \times id$ is clearly extendable to U to a surjective holomorphic map on the right hand side. So $\tilde{\alpha}_k$ is also extendable to a surjective holomorphic map over U on the left hand side. Since $x \in D$ is arbitrary, and since by equation (3.11) the extension of $\tilde{\alpha}_k$ is unique if it exists, we obtain a well defined surjective holomorphic map

$$\tilde{\alpha}_k : \overline{J}_k V^{\mathrm{reg}} \to \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}. \tag{3.13}$$

By combining with the canonical line bundle projections we get in the same way a surjective holomorphic map

$$\alpha_k : \overline{J}_k V^{\mathrm{reg}} \to \overline{P}_k V^{\mathrm{reg}} \tag{3.14}$$

which extends the corresponding map α_k of Theorem 3.2 from $\overline{X} \setminus D$ to $\overline{X}$.

Proof for b). If $\phi \in G_k$ is a reparametrization, one has on $\overline{J}_k V^{\mathrm{reg}}|_{\overline{X} \setminus D}$ by Theorem 3.2:

$$\tilde{\alpha}_k \circ \phi = \tilde{\alpha}_k \cdot \phi'(0)\,, \quad \alpha_k \circ \phi = \alpha_k, \tag{3.15}$$

where in the first equation the multiplication $\tilde{\alpha}_k \cdot \phi'(0)$ denotes the multiplication with scalars in the line bundle $\mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_{\overline{X} \setminus D}$. By Proposition 3.6 a), we have the diagram

$$
\begin{tikzcd}
\overline{J}_k V|_{U \setminus D} \arrow[r] \arrow[d, "\phi"] & J_k V_P \times (U \setminus D) \arrow[d, "\phi_P \times id"] \\
\overline{J}_k V|_{U \setminus D} \arrow[r] & J_k V_P \times (U \setminus D).
\end{tikzcd}
\tag{3.16}
$$

By a similar argument as in a), the map ϕ extends to a holomorphic automorphism on $\overline{J}_k V$. From this diagram, it also follows that ϕ maps $\overline{J}_k V^{\mathrm{reg}}$ onto itself, since all the arrows of this diagram preserve regular and singular jets (even over D by Proposition 1.7). Finally, equation (3.15) extends from $\overline{J}_k V^{\mathrm{reg}}|_{\overline{X} \setminus D}$ to $\overline{J}_k V^{\mathrm{reg}}$ by the Identity Theorem.

Proof for c). By b), the quotient $\overline{J}_k V^{\mathrm{reg}}/G_k$ is well defined (as set). By the diagrams of equations (3.15) and (3.16), we obtain from Proposition 3.6 c):

$$\begin{tikzcd} \overline{J}_k V^{\mathrm{reg}}/G_k|_{U\setminus D} \arrow[r] \arrow[d, "\alpha_k/G_k"] & (J_k V^{\mathrm{reg}}/G_k)_P \times (U \setminus D) \arrow[d, "(\alpha_k/G_k)_P \times id"] \\ \overline{P}_k V^{\mathrm{reg}}|_{U\setminus D} \arrow[r] & P_k V_P^{\mathrm{reg}} \times (U \setminus D) \end{tikzcd} \tag{3.17}$$

By Demailly ([2]), the vertical arrows in this diagram are isomorphisms. By a similar argument as in a), one obtains a holomorphic isomorphism

$$\alpha_k/G_k : \overline{J}_k V^{\mathrm{reg}}/G_k \to \overline{P}_k V^{\mathrm{reg}} \tag{3.18}$$

over $\overline{X}$. Equation (3.17) shows that this isomorphism makes $\overline{J}_k V^{\mathrm{reg}}/G_k$ into a holomorphic fiber bundle over $\overline{X}$. ☐

3.4. Characterization of log-directed jet differentials In this subsection we generalize Theorem 3.2 d). More precisely we prove:

Proposition 3.9. *A holomorphic (respectively meromorphic) function Q on $\overline{J}_k V|_O$ for some connected open subset $O \subset \overline{X}$ which satisfies*

$$Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \ \ \forall\, j_k(f) \in J_k V^{\mathrm{reg}} \text{ and } \forall \phi \in G_k \tag{3.19}$$

over some open subset of O' of $O \setminus D$ defines a holomorphic (respectively meromorphic) section of $\mathcal{O}_{\overline{P}_k V}(m)$ over O, and vice versa.

Proof. Let $Q : \overline{J}_k V|_O \to \mathbf{C}$ be a meromorphic function which satisfies

$$Q \circ \phi = \phi'(0)^m Q \ \ \forall \phi \in G_k \tag{3.20}$$

over O'. Since $\overline{J}_k V^{\mathrm{reg}}|_O$ is connected, equation (3.20) holds over O by the Identity Theorem. Since $\tilde{\alpha}_k$ and α_k were obtained over D by trivial extensions in the diagrams of Proposition 3.6, the results of Corollary 3.3 extend also over D. In particular, G_k^o of Corollary 3.3 acts transitively on the fibers of $\tilde{\alpha}_k$. Since the function Q is invariant under the action of G_k^o by equation (3.20), there exists a Zariski-densely defined function $\tilde{Q} : \mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_O \to \mathbf{C}$ such that $Q = \tilde{Q} \circ \tilde{\alpha}_k$. Again by Corollary 3.3, $\tilde{\alpha}_k$ has local holomorphic sections everywhere. So $\tilde{Q}$ is a meromorphic function on $\mathcal{O}_{\overline{P}_k V}(-1)^{\mathrm{reg}}|_O$. By equation (3.20), this function is m-linear with respect to the $\mathbf{C}^*$-action and so corresponds to a meromorphic section s of $\mathcal{O}_{\overline{P}_k V}(m)^{\mathrm{reg}}|_O$. In order to extend this section to the singular locus, we have to redo an argument of Demailly ([2]): In a neighborhood W of any point $w_0 \in \overline{P}_k V|_{O\setminus D}$ we can find a holomorphic family of germs f_w such that $(f_w)_{[k]}(0) = w$ and $(f_w)'_{[k-1]}(0) \not\equiv 0$. Then we get

$s(w) = Q(f'_w, \dots, f_w^{(k)})(0)((f_w)'_{[k-1]})^m$ on $W \cap \overline{P}_k V|_{O\setminus D}$. Now the right hand side extends to a section of $\mathcal{O}_{\overline{P}_k V}(m)|_{O\setminus D}$, so the left hand side does, too. So s is a meromorphic section of $\mathcal{O}_{\overline{P}_k V}(m)|_{\overline{P}_k V^{\mathrm{reg}} \cup \overline{P}_k V|_{O\setminus D}}$. The complement of the latter set is of codimension two in $\overline{P}_k V|_O$, so s extends to a section of $\mathcal{O}_{\overline{P}_k V}(m)|_O$.

Conversely let $\tilde{Q}$ be an m-linear meromorphic function on $\mathcal{O}_{\overline{P}_k V}(-1)|_O$ corresponding to a meromorphic section of $\mathcal{O}_{\overline{P}_k V}(m)|_O$. Then $Q := \tilde{Q} \circ \tilde{\alpha}_k$ is a meromorphic function on $\overline{J}_k V^{\mathrm{reg}}|_O$. By the Riemann Extension Theorem, it extends to a meromorphic function on $\overline{J}_k V|_O$. It satisfies equation (3.20) on $\overline{J}_k V^{\mathrm{reg}}|_O$ since $\tilde{Q}$ corresponds to a section of $\mathcal{O}_{\overline{P}_k V}(m)|_O$ and since the fibers of $\tilde{\alpha}_k$ are invariant under the action of G_k^o. Hence, it satisfies equation (3.20) over O by the Identity Theorem.

Finally we remark that if we start with a holomorphic rather than a meromorphic function (respectively, section) in the arguments above, we would obtain a holomorphic section (respectively, function) as a result. □

4. Log-directed jet metrics

4.1. The case of 1-jets This case was already treated in the second named author's thesis. We recall the basic results after some definitions.

For a line bundle L over a complex variety X, let E_L be the union of the base locus

$$\mathrm{Bs}|L| := \{x \in X : \ s(x) = 0 \text{ for all } s \in H^0(X, L)\},$$

of L and the restricted exceptional locus

$$\{x \in X \setminus \mathrm{Bs}|L| \ : \ \dim_x \varphi_L^{-1}(\varphi_L(x)) > 0\}$$

of the rational map

$$\varphi_L := [s_1 : \ldots : s_n] : X \ \cdots \to \mathbf{P}^{n-1},$$

where $\{s_1, \ldots, s_n\}$ is a basis of $H^0(X, L)$. We will call E_L the *basic locus* of L. Define the *stable basic locus* of L to be

$$S_L := \bigcap_{m>0} E_{mL}.$$

A standard argument (worked out in details in the appendix) shows that for any line bundle H on a normal variety X, we have

$$\mathrm{Bs}|mL - H| \subset S_L$$

for some sufficiently large m. Let X_{reg} be the smooth part of X. Let L^* be the dual bundle of L. Recall that a continuous function $g : L^* \to [0, \infty]$ such that

$$g(cv) = |c|^2 g(v) \tag{4.1}$$

for all $c \in \mathbf{C}$ and $v \in L^*$ is called a *singular metric* on L^*. By equation (4.1), the set $g^{-1}(0) \cup g^{-1}(\infty)$ consists of the zero section of L^* and the inverse image in L^* of a closed subset Σ_g of X. For our purpose, we will always assume that the open set $U = X_{\text{reg}} \setminus \Sigma_g$ is dense in X and that g is twice differentiable on $L^*|_U$. Then $dd^c \log g$ is a real $(1,1)$ form outside the zero section of $L^*|_U$ invariant under the C^* action given by equation (4.1) and is thus the pull back of a real $(1,1)$ form on U denoted by

$$\operatorname{Ric}(g) = \Theta_{g^{-1}} = \Theta_{g^{-1}}(L)\,,$$

which is known as the curvature form of g. By convention, g is called a *pseudometric* if $g^{-1}(\infty) = \emptyset$ and g is called a *metric* if also $\Sigma_g = \emptyset$.

Let now $(\overline{X}, D)$ be a log-manifold, and set again $X = \overline{X} \setminus D$. A Kähler metric ω on X gives a metric on TX which in turn gives a metric g_ω on $\mathcal{O}_{\overline{P}_1X}(-1)|_{P_1X}$. If g_ω extends to a metric on $\mathcal{O}_{\overline{P}_1X}(-1)$, then we can use it to dominate a scalar multiple of any pseudometic on $\mathcal{O}_{\overline{P}_1X}(-1)$ by appealing to the compactness of $\overline{X}$. This is the basic strategy used to obtain the following result (Proposition 1 of [12]). We remark that Noguchi ([14]) had already similar results in the case X is compact under the assumption that $\mathcal{O}_{P_1X}(m)$ is spanned by global sections everywhere on $\mathcal{O}_{P_1X}(m)$ for m large enough.

Proposition 4.1. *Assume $(\overline{X}, D)$ is a log-manifold and that $\overline{X}$ is projective. Let $\pi_1 : \overline{P}_1X \to \overline{X}$ be the natural projection, let $\overline{\Xi}$ be a subvariety of $\overline{P}_1X$ and let $\sigma : \overline{Z} \to \overline{\Xi} \subseteq \overline{P}_1X$ be the normalization of $\overline{\Xi}$. Let $\overline{L}_\sigma = \sigma^{-1}\mathcal{O}_{\overline{P}_1X}(1)$, $Z = \sigma^{-1}(P_1X)$ and $L_\sigma = \overline{L}_\sigma|_Z$. Then there is a pseudometric g on $L^*_\sigma = \sigma^{-1}(\mathcal{O}_{P_1X}(-1))$ with $\Sigma_g \subset S_{\overline{L}_\sigma}$, such that* $\operatorname{Ric}(g)$ *is the pullback of a Kähler metric ω on X, specifically*

$$\operatorname{Ric}(g) = (\pi_1 \circ \sigma)^* \omega\,,$$

such that this Kähler metric ω dominates g, in the sense that

$$(\sigma^* g_\omega)(\xi) \geq g(\xi) \tag{4.2}$$

for all $\xi \in L^*_\sigma$ outside $S_{\overline{L}_\sigma}$ and $\sigma^{-1}\big(\operatorname{Sing}(\overline{\Xi})\big)$.

By the usual definition of holomorphic sectional curvature, we see that equation (4.2) says precisely that g, as a "length" function on X in the tangent directions defined by $\overline{\Xi}$, has holomorphic sectional curvature bounded from above by -1, and g is nonvanishing outside $S_{\overline{L}_\sigma}$.

Hence, the usual Ahlfors' Lemma applies to show that if $f : \Delta \to X$ is any holomorphic map from the unit disk $\Delta \subset \mathbf{C}$ whose lifting $f_{[1]}$ has image in $\overline{\Xi}$ but not completely in $\sigma(S_{\overline{L}_\sigma}) \cup \operatorname{Sing}(\overline{\Xi})$, then f must satisfy the distance decreasing property.

From this, the following result is derived by elementary arguments in [12] (see also Noguchi ([14], [15])), which we quote.

Theorem 4.2. *With the same setup as in* Proposition 4.1, *we let* $\Delta^* = \Delta \setminus \{0\}$ *be the punctured unit disk and set* $\overline{L}_0 = \mathcal{O}_{\overline{P}_1 X}(1)|_{\overline{\Xi}}$.
(a) (*Distance decreasing property*) *If* $f : \Delta \to X$ *is a holomorphic map whose lift* $f_{[1]}$ *has values in* Ξ *but not entirely in* $S_{\overline{L}_0} \cup \mathrm{Sing}(\overline{\Xi})$, *then* $f^*g \leq \rho$ (*where* ρ *is the Poincaré metric on* Δ).
(b) (*Degeneracy of Holomorphic Curve*) *If* $f : \mathbf{C} \to X$ *is holomorphic such that* $f_{[1]}$ *has values in* $\overline{\Xi}$, *then* $f_{[1]}(\mathbf{C}) \subset S_{\overline{L}_0} \cup \mathrm{Sing}(\overline{\Xi})$.
(c) (*Big Picard Theorem*) *If* $f : \Delta^* \to X$ *is holomorphic such that* $f_{[1]}$ *has values in* $\overline{\Xi}$ *but not entirely in* $S_{\overline{L}_0} \cup \mathrm{Sing}(\overline{\Xi})$, *then* f *extends to a holomorphic map* $\overline{f} : \Delta \to \overline{X}$.

4.2. The general case We call a pseudometric h on $\mathcal{O}_{P_k V}(-1)$ a *k-jet pseudometric* on (X, V), and define $B_k = S_{\mathcal{O}_{\overline{P}_k V}(1)}$, the stable basic locus of $\mathcal{O}_{\overline{P}_k V}(1)$.

Theorem 4.3. *With the notations as in* Theorem 4.2, *assume that* $(\overline{X}, D, \overline{V})$ *is a log-directed manifold and that* $\overline{X}$ *is projective.*
a) *If* $B_k \neq \overline{P}_k V$, *then there exists a k-jet pseudometric* h *on* (X, V) *with* $\Sigma_h \subseteq B_k$ *such that* h *has curvature bounded from above by* -1 *in the sense that* $\mathrm{Ric}(h) = \pi_k^* \omega$ *is the pullback of a Kähler metric* ω *on* $P_{k-1}V$ *such that* g_ω *dominates* h. *In particular, we have*

$$\langle \Theta_{h^{-1}}, |\xi|^2 \rangle \;=\; \langle \mathrm{Ric}(h), |\xi|^2 \rangle \;\geq\; h((\pi_k)_* \xi) \quad \text{for} \quad \xi \in V_k.$$

b) *If* $f : \mathbf{C} \to \overline{X} \setminus D$ *is holomorphic with* $f_*(T\mathbf{C}) \subset \overline{V}$, *then* $f_{[k]}(\mathbf{C}) \subset B_k$.
c) *If* $f : \Delta^* \to \overline{X} \setminus D$ *is holomorphic with* $f_*(T\Delta^*) \subset \overline{V}$, *then*:
Either f *extends to a holomorphic map* $\overline{f} : \Delta \to \overline{X}$ *or* $f_{[k]}(\Delta^*) \subset B_k$.
Moreover, let $Y \subset \overline{P}_k V$ *be any subvariety. We define* $B_k(Y) = S_{\mathcal{O}_{\overline{P}_k V}(1)|_Y}$. *If* f *lifts to a map with values in* Y, *then* b) *and* c) *hold with* $B_k(Y) \cup \mathrm{Sing}(Y)$ *instead of* B_k.

Proof. Apply Proposition 4.1 and Theorem 4.2 to the log-manifold $(\overline{P}_{k-1}V, D_{k-1})$ and the subvariety $\overline{\Xi} = \overline{P}_k V$ (or $\overline{\Xi} = Y \subset \overline{P}_k V$). Note that, since $\overline{V}_k \subset \overline{T}(P_{k-1}V)$ is a holomorphic subbundle, $\overline{P}_1 V_k = \overline{P}_k V$ is a submanifold in $\overline{P}_1(P_{k-1}V)$ and $\mathcal{O}_{P_k V}(-1) = \mathcal{O}_{\overline{P}_1(P_{k-1}V)}(-1)|_{P_k V}$ by Proposition 2.4. □

5. Logarithmic Bloch's and Lang's Conjecture

In this section we apply our method to the special case of semi-abelian varieties where our Ahlfors-Schwarz Lemma (Theorem 4.3) gives a logarithmic version of Bloch's Theorem and our big Picard Theorem yields a big Picard version of Bloch's Theorem. By using the Wronskian associated to the theta function of an effective divisor in a semi-abelian variety ([21], [17]), we affirm furthermore a logarithmic version of Lang's Conjecture and a big Picard analogue of it, all via metric geometry on logarithmic Demailly-Semple jets.

5.1. Statement of the results We first recall the definition and some basic facts on semi-abelian varieties (see [17], [8], [9]) needed to state our results.

A quasiprojective variety G is called a semi-abelian variety if it is a commutative group which admits an exact sequence of groups

$$0 \to (\mathbf{C}^*)^\ell \to G \to A \to 0,$$

where A is an abelian variety of dimension m.

Taking the pushforward of $(\mathbf{C}^*)^\ell \subset G$ with the natural embedding $(\mathbf{C}^*)^\ell \subset (\mathbf{P}^1)^\ell$, we obtain a smooth completion

$$\overline{G} = (\mathbf{P}^1)^\ell \times_{(\mathbf{C}^*)^\ell} G$$

of G with boundary divisor S, which has only normal crossing singularities. We denote the natural action of G on $\overline{G}$ on the right as addition. It follows that the exponential map from the Lie algebra $\mathbf{C}^n$ is a group homomorphism and, hence, it is also the universal covering map of $G = \mathbf{C}^n/\Lambda$, where $\Lambda = \Pi_1(G)$ is a discrete subgroup of $\mathbf{C}^n$ and $n = \mathsf{m} + \ell$.

Following Iitaka ([9]), we have the following trivialization of the logarithmic tangent and cotangent bundles of $\overline{G}$: Let $z_1, \ldots, z_n$ be the standard coordinates of $\mathbf{C}^n$. Since $dz_1, \ldots, dz_n$ are invariant under the group action of translation on $\mathbf{C}^n$, they descend to forms on G. There they extend to logarithmic forms on $\overline{G}$ along S, which are elements of $H^0(\overline{G}, \overline{\Omega}G)$. These logarithmic 1-forms are everywhere linearly independent on $\overline{G}$. Thus, they globally trivialize the vector bundle $\overline{\Omega}G$. Finally, we note that these logarithmic forms are invariant under the group action of G on $\overline{G}$, and, hence, the associated trivialization of $\overline{\Omega}G$ over $\overline{G}$ is also invariant.

We now state the main theorems of this section. With the above setup, let $f : \Gamma \to G$ be a holomorphic map, where Γ is either $\mathbf{C}$ or the punctured disk Δ^*. Denote by $\overline{X}(f)$ the Zariski closure of $f(\Gamma)$ in $\overline{G}$ and let $X(f) = \overline{X}(f) \cap G$. Furthermore, let $D \subset G$ be a reduced algebraic divisor in G, which we regard as a union of 1-codimensional algebraic subvarieties of G. We note that an algebraic subvariety of G which is also a subgroup is necessarily a semi-abelian variety as well, see [9].

Theorem 5.1. (a) *Let $f : \mathbf{C} \to G$ be a holomorphic curve. Then $X(f)$ is a translate of an algebraic subgroup of G.*
(b) *Let $f : \mathbf{C} \to (G \setminus D)$ be a holomorphic curve. Then $X(f) \cap D = \emptyset$.*

Corollary 5.2. *If D has nonempty intersection with any translate of an algebraic subgroup of G (of positive dimension), then $G \setminus D$ is Brody hyperbolic.*
In particular, this holds if $G = A$ is an abelian variety and D is ample. □

Theorem 5.1 (a) is a logarithmic version of Bloch's Theorem, first proved by Noguchi ([15]), (b) is a logarithmic version of Lang's Conjecture. Both Theorem 5.1 and Corollary 5.2 were obtained by Noguchi ([17]), and were, in the nonlogarithmic case, first proved by Siu-Yeung ([21]).

Theorem 5.3. (a) *Let $f : \Delta^* \to G$ be a holomorphic map. Then either it extends to a holomorphic map $\overline{f} : \Delta \to \overline{G}$, or there exists a maximal algebraic subgroup G' of G of positive dimension such that $X(f)$ is foliated by translates of G'.*
(b) *Let $f : \Delta^* \to (G \setminus D)$ be holomorphic. Then one of the following holds*:
(i) *f extends to $\overline{f} : \Delta \to \overline{G}$.*
(ii) *$X(f) \cap D = \emptyset$.*
(iii) *There is an algebraic subgroup $G'' \subset G'$ of positive dimension such that $X(f) \cap D$ is foliated by translates of G''.*
(c) *Let now $f : \Delta^* \to (A \setminus D)$, where G is the special case of an abelian variety A. Then one of the following holds*:
(i) *f extends to $\overline{f} : \Delta \to A$.*
(ii) *There exists an algebraic subgroup $G'' \subset G'$ of positive dimension such that D is foliated by translates of G''.*

Corollary 5.4. *If $G = A$ is an abelian variety and D is ample, then $f : \Delta^* \to A \setminus D$ extends to a holomorphic map $\overline{f} : \Delta \to A$.*

We remark that Theorem 5.3 and Corollary 5.4 are big Picard type Theorems. Aside from (a), which can be found in Noguchi ([15]), these are, to our knowledge, new to the literature.

Proof of Corollary 5.4. Corollary 5.4 follows from Theorem 5.3 (c) and the fact that an ample divisor D in an abelian variety A cannot be foliated by translates of an algebraic subgroup A'' of A of positive dimension. For assume it were. Then $D = q^{-1}(\overline{D})$, where $\overline{D}$ is a divisor in A/A'' and $q : A \to A/A''$ is the quotient map. But then $\mathcal{O}(D) = q^{-1}\mathcal{O}(\overline{D})$ is trivial along A'', since A'' is a fiber of the map q. This is a contradiction. $\square$

REMARK. The last part of Corollary 5.2 follows from Corollary 5.4 as follows. The $\mathbf{C} \setminus \Delta$ is biholomorphic to Δ^*. So we can conclude from Corollary 5.4 that any entire curve $f : \mathbf{C} \to A$ extends to a holomorphic map $\overline{f} : \mathbf{P}^1 \to A$. Hence, $\overline{f}$ must be constant, since all coordinate 1-forms on A must pull back to the zero form on $\mathbf{P}^1$ as $\mathbf{P}^1$ has no nontrivial 1-forms.

However, Corollary 5.4 does not follow from Corollary 5.2. It would if $A \setminus D$ were hyperbolically embedded in A. This would be the case, for example, if D were hyperbolic (see for example [11]). But even a very ample divisor in A is not hyperbolic in general. To see this, choose any translate T of an algebraic subvariety which is of codimension at least 3 in A. Then there always exists an irreducible ample divisor in A which contains T, as can be deduced by applying the Bertini's Theorem 7.19 in [7]. Note that a hyperbolic open subset V in a projective variety $\overline{V}$ need not be hyperbolically embedded in general, as one can easily see by blowing up a point of $\overline{V} \setminus V$.

REMARK. Let $G = (\mathbf{C}^*)^n \subset \mathbf{P}^n$, $n \geq 4$. Let $\overline{D}$ be a generic hyperplane in $\mathbf{P}^n$ and $\overline{H}$ be another hyperplane with $G \cap \overline{H} \neq \emptyset$ and $\overline{H} \cap \overline{D} \cap G = \emptyset$. Then $\overline{H} \cap (G \setminus \overline{D})$ is equal to $\mathbf{P}^{n-1}$ minus at most $n+2$ hyperplanes, which contains nontrivial images of $\mathbf{C}$ and hence, admits maps f from Δ^* which do not extend to Δ. This is because the complement of $n+2$ hyperplanes in $\mathbf{P}^{n-1}$ contains nontrivial diagonals for $n \geq 4$, which are nonhyperbolic. So we get examples of f for Theorems 5.1 (b) and 5.3 (b) (ii) with nontrivial $X(f)$.

REMARK. Let A be an abelian variety, $D \subset A$ a divisor and $f_1 : \Delta \to A \setminus D$ a holomorphic map. Let $X(f_1)$ be the Zariski closure of $f_1(\Delta) \subset A$. Let E be an elliptic curve and $q : \mathbf{C} \to E$ be the universal cover. Then

$$f(z) = \left(f_1(z), q \circ \exp\left(\frac{1}{z}\right)\right) : \Delta^* \to A \times E$$

does not extend.

This easy construction provides examples which are relevant to Theorem 5.3:
(1) It makes Theorem 5.3 (b) (iii) and (c) (ii) sharp.
(2) Choose f_1 in such a way that $X(f_1)$ is not a translate of an algebraic subgroup in A. Then we have an example for (a) where $X(f)$ is not itself a translate of an algebraic subgroup of $A \times E$.

5.2. Some results on semi-abelian varieties We first summarize some elementary properties of semi-abelian varieties.

Lemma 5.5. (a) *The quotient of a semi-abelian variety G by an algebraic subgroup G' is again a semi-abelian variety, and the quotient map $q : G \to G/G'$ is an algebraic morphism.*
(b) *If $X \subset G$ is an algebraic variety foliated by translates of G', then*

$$X/G' \subset G/G'$$

is again an algebraic variety.
(c) *If X is an algebraic subvariety of G, and $h : X \to \mathbf{P}^1$ is a rational function, then the closed subgroup*

$$G' = \{a \in G : X = (X + a)\} \cap \{a \in G : h(x) = h(x + a) \text{ for all } x \in X\}$$

is again an algebraic subvariety.

Proof. Lemma 5.5 should be well known, but since we do not know a precise reference we indicate a proof. From the fact that connected algebraic subgroups of a semi-abelian variety are again semi-abelian, it is easy to see that one can consider quotients of G by G' by taking the quotient on the abelian and the $(\mathbf{C}^*)^\ell$ factors separately. Now the quotient of the abelian factor by an algebraic subgroup is abelian by isogeny, and the quotient of $(\mathbf{C}^*)^\ell$ by a connected algebraic subgroup is likewise a product of $\mathbf{C}^*$, see [9]. Hence, we obtain a $(\mathbf{C}^*)^l$ bundle over an abelian variety for some l, which projectivizes to a $\mathbf{P}^l$ bundle over a projective variety, and, therefore, must be projective. From this the entire Lemma 5.5 follows. □

Lemma 5.6. *Let A be an abelian variety and $D \subset A$ a reduced algebraic divisor. Let A' be an algebraic subgroup of A and T a translate of A' in A. Assume $T \cap D = \emptyset$. Then D is foliated by translates of A'.*

Proof. Without loss of generality we may assume that D is irreducible. Let $q : A \to A/A'$ denote the quotient map. Since q is a proper map, $q(D)$ is a projective subvariety in A/A'. Since D is irreducible and $T \cap D = \emptyset$, $q(D)$ is an irreducible divisor. So $\tilde{D} = q^{-1}(q(D)) \subset A$ is also an irreducible divisor containing D as q is smooth. This forces $D = \tilde{D}$. □

REMARK. Lemma 5.6 is false for semi-abelian varieties. For we may take $G = (\mathbf{C}^*)^2$, $T = \{(z_1, z_2) \in G : z_1 = 1\}$, $D = \{(z_1, z_2) \in G : (z_1 - 1)z_2 = 1\}$.

5.3. Jet bundles on semi-abelian varieties To simplify notation, we work exclusively with a semi-abelian variety G and its associated log-manifold $(\overline{G}, S)$ defined as before. We remark, however, that many of the definitions and results hold for an arbitrary log-manifold.

Recall from Subsections 1.2 and 2.1 that $\overline{P}_k V$ denotes the logarithmic k-jet bundle of $(\overline{G}, S, \overline{V})$ and that $\overline{P}_k G$ denotes the logarithmic jet bundle of $(\overline{G}, S) = (\overline{G}, S, \overline{T}G)$. Note that the log-directed morphism $i : (\overline{G}, S, \overline{V}) \to (\overline{G}, S, \overline{T}G)$ induces a canonical realization of $\overline{P}_k V$ as a submanifold of $\overline{P}_k G$ as $\overline{V}$ is a subbundle of $\overline{T}G$ over $\overline{G}$.

Let $D \subset G$ be a reduced algebraic divisor. Then, by Hironaka ([6]), there exist a log-manifold $(\overline{Y}, E)$ and a log-morphism $p : (\overline{Y}, E) \to (\overline{G}, S)$ with

1) $p^{-1}(S \cup D) = E$,
2) $p : p^{-1}(\overline{G} \setminus D) \to \overline{G} \setminus D$ is biholomorphic.

Given a subbundle $\overline{V}$ of $\overline{T}G$, we have the following commutative diagram:

$$\tag{5.1}
\begin{tikzcd}
\mathcal{O}_{\overline{P}_k Y}(-1) \arrow[r, "(p_{[k-1]})_*"] \arrow[d] & \mathcal{O}_{\overline{P}_k G}(-1) \arrow[d] & \mathcal{O}_{\overline{P}_k (V)}(-1) \arrow[l, hook', "(i_{[k-1]})_*"'] \arrow[d] \\
\overline{P}_k Y \arrow[r, "p_{[k]}"] \arrow[d] & \overline{P}_k G \arrow[d] & \overline{P}_k (V) \arrow[l, hook', "i_{[k]}"'] \arrow[d] \\
(\overline{Y}, E) \arrow[r, "p"] & (\overline{G}, S) & (\overline{G}, S, \overline{V}) \arrow[l, hook', "i"']
\end{tikzcd}$$

Here, $i_{[k]}$ realizes $\overline{P}_k V$ as a submanifold of $\overline{P}_k G$ by Proposition 2.1. Outside $\pi_{0,k}^{-1}(D) \subset \overline{P}_k G$, the maps $p_{[k]}$ (and hence, $p_{[k-1]_*}$) are isomorphisms. All other maps are holomorphic. We define

$$\overline{Z}_k := \overline{p_{[k]}^{-1}(\overline{P}_k V \setminus \pi_{0,k}^{-1}(D))}^{\text{Zariski}} \subset \overline{P}_k Y.$$

DEFINITION 5.7. A meromorphic section s of $\mathcal{O}_{\overline{P}_k V}(m)$ is said to have at most log-poles along D if it pulls back, via the map $(p_{[k-1]})_*|_{\overline{Z}_k} = (p_{[k-1]}|_{\overline{Z}_k})_*$, to a holomorphic section[2] of $\mathcal{O}_{\overline{P}_k Y}(m)|_{\overline{Z}_k}$.

Suppose that a meromorphic section s of $\mathcal{O}_{\overline{P}_k V}(m)$ over an open subset $U \subseteq \overline{G}$ is defined by a meromorphic function Q on $J_k G|_U$ satisfying equation (3.2), see Proposition 3.9. Suppose more precisely that Q is given by a polynomial in the differentials up to order $k-1$ of sections of $\overline{\Omega} G|_U$ as well as the differentials up to order $k-1$ of $d \log \theta$, where θ is a meromorphic function on U nonvanishing and holomorphic on $U_0 = U \setminus \{D \cup S\}$. Then, after composing with p given above, these differentials, and so also the polynomial in them, become holomorphic functions on $\overline{J}_k Y|_{p^{-1}(U)}$ by Proposition 1.1 (c). Furthermore, the resulting polynomial still satisfies equation (3.2) on $p^{-1}(U)$. Hence, by Proposition 3.9, we obtain a holomorphic section of $\mathcal{O}_{\overline{P}_k Y}(m)|_{\tilde{U}}$ that matches with the pullback of s on an open set of $\overline{Z}_k$. Therefore, $s|_U$ is meromorphic with at most log-poles along D. If such a description is possible on a neighborhood U of each point in $\overline{G}$, then s is meromorphic with at most log-poles along D. We will consider examples of such an s in Subsection 5.5.

[2]With this we mean that, after pulling back the section s over the part of $\overline{Z}_k$ where the meromorphic map $(p_{[k-1]})_*$ is holomorphic, it extends to a holomorphic section of $\mathcal{O}_{\overline{P}_k Y}(m)|_{\overline{Z}_k}$.

Lemma 5.8. *Let $(\overline{G}, S, \overline{V})$ be as above.*
(1) *There exist injective maps*

$$\overline{P}_{k+l}V \to \overline{P}_l(P_kV) \quad \textit{and} \quad \mathcal{O}_{\overline{P}_{k+l}V}(-1) \to \mathcal{O}_{\overline{P}_l(P_kV)}(-1)$$

which are given outside S by $f_{[k+l]} \mapsto (f_{[k]})_{[l]}$ and by $f'_{[k+l-1]} \mapsto (f_{[k]})'_{[l-1]}$, respectively, and which realize $\overline{P}_{k+l}V \subset \overline{P}_l(P_kV)$ and $\mathcal{O}_{\overline{P}_{k+l}V}(-1) \subset \mathcal{O}_{\overline{P}_l(P_kV)}(-1)$, respectively, as submanifolds.
(2) *Furthermore, let Γ be a curve and $f : \Gamma \to X$ be a holomorphic map which is tangent to V. As before, let $\overline{X}_k(f) \subset \overline{P}_kV$ be the Zariski closure of the image of the k-th lift $f_{[k]} : \Gamma \to \overline{P}_kV$ of the map f. We denote $\pi_{0,k}^{-1}(S)$ again by S and $\pi_{0,k}^{-1}(G) \cap \overline{X}_k(f)$ by $X_k(f)$. We recall that by Hironaka there exists a log-morphism*

$$\Psi : (\overline{\tilde{X}_k}(f), \tilde{S}) \to (\overline{P}_kV, S) \qquad \textit{such that}:$$

(a) $\Psi(\overline{\tilde{X}_k}(f)) = \overline{X}_k(f)$.
(b) $\Psi^{-1}(S) = \tilde{S}$.
(c) Ψ *is biholomorphic outside* $\Psi^{-1}\Big(\mathrm{Sing}(\overline{X}_k(f))\Big)$.

We set $\tilde{X}_k(f) = \overline{\tilde{X}_k}(f) \backslash \tilde{S}$. With this setup, we have the following commutative diagram:

(5.2)
$$\begin{array}{ccccc}
\overline{P}_l(\tilde{X}_k(f)) & & \xrightarrow{\quad\Psi_{[l]}\quad} & & \overline{P}_l(P_kV) \\
\Big\downarrow & & & & \Big\uparrow{\scriptstyle i} \\
 & & \overline{X}_{k+l}(f) & \subset & \overline{P}_{k+l}(V) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\overline{\tilde{X}_k}(f) & \xrightarrow{\Psi} & \overline{X}_k(f) & \subset & \overline{P}_k(V)
\end{array}$$

where $\Psi_{[l]}$ may be meromorphic, all other maps in the diagram are holomorphic and the following holds:

$$i(\overline{X}_{k+l}(f)) \subset \Psi_{[l]}(\overline{P}_l(\tilde{X}_k(f))).$$

(3) *Let s, t be meromorphic sections of the bundle $\mathcal{O}_{\overline{P}_kV}(m)$ with at most log-poles along D, and assume t is not the zero section. Then $t^2 \cdot d(s/t)$ can be considered as a meromorphic section of the bundle $\mathcal{O}_{\overline{P}_{k+1}V}(2m+1)$ with at most log-poles along D.*

Proof for (1). This follows directly from Corollary 2.4 and Proposition 2.1 applied to the subbundle inclusion $\overline{V}_k \subset \overline{T}P_{k-1}V$. The presentation of the maps outside S follows from Proposition 3.1.

Proof for (2). $\Psi_{[l]} : \overline{P}_l(\tilde{X}_k(f)) \to \overline{P}_l(P_kV)$ is a proper rational map. Hence,

$\Psi_{[l]}(\overline{P}_l(\tilde{X}_k(f))$ is an algebraic subset containing $i(f_{[k+l]}(\Gamma)) = (f_{[k]})_{[l]}(\Gamma)$. Therefore, we also have $i(\overline{X}_{k+l}(f)) \subset \Psi_{[l]}(\overline{P}_l(\tilde{X}_k(f))$.

Proof for (3). s/t is a rational function on $\overline{P}_k V$. Hence, $d(s/t)$ is a rational section of $\mathcal{O}_{\overline{P}_1(P_k V)}(1)$, which lifts back, via the inclusion map $\mathcal{O}_{\overline{P}_{k+1} V}(-1) \to \mathcal{O}_{\overline{P}_1(P_k V)}(-1)$, to a rational section of $\mathcal{O}_{\overline{P}_{k+1} V}(1)$. Hence, $t^2 \cdot d(s/t)$ is a rational section of $\mathcal{O}_{\overline{P}_{k+1} V}(2m+1)$. To prove that $t^2 \cdot d(s/t)$ again is a meromorphic section with at most log-poles along D, we pull back the sections s, t and $t^2 \cdot d(s/t)$ to $\overline{Z}_k$ by the map $(p_{[k-1]})_*$. Then by definition the sections s and t become holomorphic sections on $\overline{Z}_k$. It suffices to show that the rational section $t^2 \cdot d(s/t)$ also is holomorphic on $\overline{Z}_k$. It suffices to prove this locally on $\overline{Z}_k$. Given any point $w \in \overline{Z}_k \subset \overline{P}_k Y$ there exists an open neighborhood U of w in $\overline{P}_k Y$ such that the holomorphic sections s and t over $\overline{Z}_k \cap U$ extend to holomorphic sections of the bundle $\mathcal{O}_{\overline{P}_k Y}(m)$ over U, and that this bundle is trivial over U. After choosing such a trivialization, one has, by the product rule for the holomorphic functions s and t,

$$t^2 \cdot d\left(\frac{s}{t}\right) = t\,ds - s\,dt\,,$$

the latter being a holomorphic section of $\mathcal{O}_{\overline{P}_1(U\setminus E)}(1)$. □

Next, we want to prove that $\overline{P}_k G$, and even $\overline{P}_k V$, are trivial over $\overline{G}$ for certain subbundles $\overline{V} \subset \overline{T}G$, which we will call special. Let $z_1, \ldots, z_n$ be any linear coordinates of the universal cover $\mathbf{C}^n \to G$. We have observed that $\overline{T}G = \mathbf{C}^n \times \overline{G}$, where the trivialization is given by $dz_1, \ldots dz_n$.

DEFINITION 5.9. $\overline{V}$ is said to be special if $\overline{V} = \mathbf{C}^r \times \overline{G}$ in this trivialization.

From now on, all subbundles $\overline{V} \subset \overline{T}G$ we use are assumed to be special.

Lemma 5.10. (a) *The map*

$$\mathcal{O}_{P_k V}(-1) \to (\mathcal{O}_{P_k V}(-1))_0 \times G;\;\; f'_{[k-1]}(0) \mapsto ((f - f(0))'_{[k-1]}(0), f(0)) \tag{5.3}$$

gives an isomorphism $\mathcal{O}_{P_k V}(-1) \to \mathcal{O}_{\overline{P}_k V}(-1)|_G$, and this isomorphism is invariant under the action of G.
(b) *This isomorphism can be extended to a G-invariant isomorphism*

$$\overline{\Psi}_k : \mathcal{O}_{\overline{P}_k V}(-1) \to (\mathcal{O}_{P_k V}(-1))_0 \times \overline{G}$$

respecting the fibers of the line bundles.
(c) *By combining with the canonical line bundle projections we get the same isomorphisms with $P_k V$ and $\overline{P}_k V$ instead of $\mathcal{O}_{P_k V}(-1)$ and $\mathcal{O}_{\overline{P}_k V}(-1)$.*

Proof. (a) is immediate, and (c) follows immediately from (b). To prove (b), we use that by Corollary 3.7 (a) the trivialization of (a) extends locally, so by the Identity Theorem it extends globally. The invariance under the action of G of this trivialization extends from G to $\overline{G}$ by continuity.

We would like, however, also to indicate a direct proof. It is obtained by proving the following more precise statement by induction over k.

CLAIM S(k). There exist trivialization maps $\overline{\psi}_k$, induced canonically by the trivialization of V by $dz_1, \ldots, dz_n$, such that

$$(+)_k \qquad \begin{array}{ccc} \overline{V}_k & \xrightarrow{\overline{\psi}_k} & (V_k)_0 \times \overline{G} \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle (\pi_k)_0 \times id_{\overline{G}}} \\ \overline{P}_{k-1}V & \xrightarrow{\overline{\Psi}_{k-1}} & (P_{k-1}V)_0 \times \overline{G} \end{array}$$

commutes and the upper line projectivizes to

$$(++)_k \qquad \begin{array}{ccc} \overline{P}_k V & \xrightarrow{\overline{\Psi}_k} & (P_k V)_0 \times \overline{G} \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle (\pi_k)_0 \times id_{\overline{G}}} \\ \overline{P}_{k-1}V & \xrightarrow{\overline{\Psi}_{k-1}} & (P_{k-1}V)_0 \times \overline{G} \end{array}$$

where Ψ_k extends the isomorphism in equation (5.3) in a G-invariant way.

Now S(1) is clear from the trivialization

$$V \xrightarrow{\ \sim\ } \mathbf{C}^r \times \overline{G},$$

since we are given that V is special. Assuming, by induction, that S(k) is true. Then we get $(++)_{k+1}$ by projectivizing $(+)_{k+1}$. It remains to show $(+)_{k+1}$. By $(++)_k$ we get induced trivializations of the logarithmic tangent bundles

$$\begin{array}{ccccc} \overline{T}(P_k V) & \xrightarrow{(\overline{\Psi}_k)_*} & T((P_k V)_0) \times \overline{T}G & \longrightarrow & T((P_k V)_0) \times \mathbf{C}^r \times G \\ \Big\downarrow{\scriptstyle (\pi_k)_*} & & \Big\downarrow{\scriptstyle ((\pi_k)_0)_* \times (id_{\overline{G}})_*} & & \\ \overline{T}(P_{k-1}V) & \xrightarrow{(\overline{\Psi}_{k-1})_*} & T((P_{k-1}V)_0) \times \overline{T}G & \longrightarrow & T((P_{k-1}V)_0) \times \mathbf{C}^r \times G \end{array}$$

where the isomorphisms on the right hand side are obtained by trivializing $\overline{T}G$ by the forms $dz_1, \ldots, dz_n$. We want to show it we restrict the isomorphism in the upper line of this diagram from $\overline{T}(P_k V)$ to $\overline{V}_k$, we also get a trivialization of $\overline{V}_k$ over $\overline{G}$. Then we can denote this trivialization by ψ_{k+1} and the rest follows easily. The key point of the proof is now that by equation (2.1), namely

$$\overline{V}_k(G) := (\pi_k)_*^{-1}(\mathcal{O}_{\overline{P}_k V}(-1)) \subset \overline{T}(P_k V),$$

the subbundle $\overline{V}_k \subset \overline{T} P_k V$ is defined in an intrinsic way which is compatible with the isomorphisms of the diagram above. $\square$

Lemma 5.11. *Let Y be a complex manifold, $Z \subset Y$ a complex submanifold and denote by $i : (Z, TZ) \to (Y, TY)$ the directed inclusion map. Let $g : \Delta \to Y$ be holomorphic with $dg(0) \not\equiv 0$, where Δ denotes again the unit disk in* $\mathbf{C}$. *Assume that $g_{[l]}(0)$ is in the image of the* (*composed*) *morphism*

$$P_l Z \xrightarrow{i_{[l]}} i^{-1} P_l Y \to P_l Y \tag{5.4}$$

for all $l \geq 0$. Then $g(\Delta) \subset Z$.

Proof. By Proposition 2.1 a) and b) the map in equation (5.4) is a morphism. Let $U \subset Y$ be a neighborhood of $g(0)$ and $F : U \to \mathbf{C}$ be a holomorphic function with $F|_{Z \cap U} \equiv 0$. It suffices to show $F \circ g \equiv 0$. There exist a small disk Δ_ϵ and a map $h_l : \Delta_\epsilon \to Z$ with $i_{[l]}((h_l)_{[l]}(0)) = g_{[l]}(0)$ and by Corollary 2.3 we may assume $dh_l(0) \not\equiv 0$. By Proposition 3.1 we have $i_{[l]}(h_l)_{[l]} = (i \circ h_l)_{[l]}$ and hence, $(i \circ h_l)_{[l]}(0) = g_{[l]}(0)$ and $d(i \circ h_l)(0) \not\equiv 0$. Hence, we can reparametrize $i \circ h_l$ in a way that it has the same Taylor expansion as g up to order l. We assume that this has been done, and note that h_l still maps a neighborhood of the origin to Z. Hence,

$$\left(\frac{\partial^j}{\partial t^j} F \circ g\right)(0) = \left(\frac{\partial^j}{\partial t^j} F \circ i \circ h_l\right)(0) = 0 \quad \text{for} \quad j \leq l.$$

Since l is arbitrary, we get $F \circ g \equiv 0$. $\square$

5.4. The Main Lemma The following Main Lemma is the key step in proving Theorem 5.1 and Theorem 5.3. In the case $\Gamma = \mathbf{C}$, it is a generalization of a lemma contained in [17], and for $G = A$, it generalizes a lemma in [21].

Main Lemma 5.12. *With the same setup as that for* Theorem 5.1 (*or* Theorem 5.3), *let $\overline{V} \subset \overline{T} G$ be a special subbundle. Assume that $f : \Gamma \to G \setminus D$ is tangent to V and, in the case $\Gamma = \Delta^*$, that f does not extend to Δ as a map to $\overline{G}$. Let, for $k \geq 0$, $\overline{X}_k(f)$ denote the Zariski closure of $f_{[k]}(\Gamma)$ in $\overline{P}_k V$. Let, for $k, m \geq 1$, Θ be meromorphic section of the line bundle $\mathcal{O}_{\overline{P}_k V}(m)$ with at most log-poles along D. Then there exists an algebraic subgroup $G' \subset G$, of positive dimension, which leaves $\overline{X}_k(f)$ and, for $k \geq 1$, also $\Theta|_{\overline{X}_k(f)}$ invariant.*

REMARK. The same is true for finitely many different sections Θ.

The rest of this subsection will be devoted to the proof of the Main Lemma. It suffices to consider the case $k \geq 1$. In fact, to prove the case $k = 0$ we apply the Main Lemma for $k = 1$ and for Θ being the zero section in $\mathcal{O}_{\overline{P}_1 V}(1)$. Since the map $\pi_1 : \overline{P}_1 V \to \overline{X}$ is equivariant under the action of G' and maps $\overline{X}_1(f)$ surjectively $\overline{X}_0(f)$, the subgroup G' also leaves $\overline{X}_0(f)$ invariant. So for the rest of the proof of the Main Lemma we assume $k \geq 1$.

We fix $u \in \Gamma$ to be any point for which $df(u) \not\equiv 0$. Then all $f_{[k+l]}(u)$, $l \geq 0$, are regular jets. Let $s_0 \in H^0(\overline{P}_1 V, \mathcal{O}_{\overline{P}_1 V}(1))$ be a global section, which is invariant under the action of G, and which is nonvanishing at $f_{[1]}(u)$. It exists because $df(u) \not\equiv 0$ and $\mathcal{O}_{\overline{P}_1 V}(1) = \mathcal{O}_{P_1 V_0}(1) \times \overline{G}$ (see Lemma 5.10). Choose an infinite sequence $\{n_0, n_1, n_2, n_3, \ldots\}$ of natural numbers such that the following two conditions hold:

$$(2(k+l)-1) | n_l \text{ for } l \geq 0,$$
$$n_l \geq 2(n_{l-1}+1) \text{ for } l \geq 1.$$

For example, $n_l = 2^l(2(k+l)-1)m$, where $m = \deg \Theta$ will work. Let

$$\Theta_0 = \Theta \cdot (s_0^{n_0 - m})$$

and, for $l \geq 1$, define inductively:

$$\Theta_l = d\left(\frac{\Theta_{l-1}}{s_0^{n_{l-1}}}\right) \cdot \left(s_0^{n_l - 1}\right).$$

Then, by Lemma 5.8 (3), Θ_l is a meromorphic section of $\mathcal{O}_{\overline{P}_{k+l} V}(n_l)$ with at most log-poles along D (here $s_0 = s_0 \circ (\pi_{0,k+l-1})_*$).

By Lemma 5.10 we may identify $\overline{P}_k V$ as $P_k V_0 \times \overline{G}$. Then we have:

$$\begin{tikzcd}
 & \mathcal{O}_{\overline{P}_{k+l}(V)}(-1)|_{\overline{X}_{k+l}(f)} \arrow[d] & \subset & \mathcal{O}_{\overline{P}_{k+l}(V)}(-1) \arrow[rr, "\alpha"] \arrow[d] & & & \mathcal{O}_{P_{k+l}V_0}(-1) \arrow[d] \\
 & \overline{X}_{k+l}(f) \arrow[d] & \subset & \overline{P}_{k+l}(V) \arrow[d] & = & P_{k+l}V_0 \times \overline{G} \arrow[r, "p_1"] \arrow[d] & P_{k+l}V_0 \arrow[d] \\
 & \overline{X}_k(f) \arrow[d] & \subset & \overline{P}_k(V) \arrow[d] & = & P_k V_0 \times \overline{G} \arrow[r, "p_1"] \arrow[d] & P_k V_0 \arrow[d] \\
\Gamma \arrow[r, "f"] & \overline{X}(f) & \subset & \overline{G} & = & \overline{G} \arrow[r] & 0
\end{tikzcd}$$

where $0 \in G \setminus D$ and p_1 is projection to the first factor. Define, for $l \geq 0$:

$$W_l = \left\{ a \in G : (f+a)_{[k+l]}(u) \in \overline{X}_{k+l}(f) \text{ and } \frac{\Theta_i}{s_0^{n_i}} \Big|_{f_{[k+i]}(u)} = \frac{\Theta_i}{s_0^{n_i}} \Big|_{(f+a)_{[k+i]}(u)},\ i = 0, \dots, l \right\}.$$

Lemma 5.13. *With the hypothesis and the setup as in the Main Lemma, $W := \bigcap_{l=0}^{\infty} W_l$ is an algebraic subvariety of G and* $\dim_0 W \geq 1$.

Proof of Lemma 5.13. W_l is an algebraic subvariety of G as the group action of G on itself is algebraic. Hence, W is also algebraic. Let $l_1 > l_2$ and $\pi_{k+l_2,k+l_1} : \overline{P}_{k+l_1}V \to \overline{P}_{k+l_2}V$. If $(f+a)_{[k+l_1]}(u) \in \overline{X}_{k+l_1}(f)$, then

$$(f+a)_{[k+l_2]}(u) = \pi_{k+l_2,k+l_1} \circ (f+a)_{[k+l_1]}(u) \in \pi_{k+l_2,k+l_1}(\overline{X}_{k+l_1}(f)) = \overline{X}_{k+l_2}(f).$$

Hence, W_l, $l \geq 0$, is a decreasing sequence of algebraic subvarieties of G. So the proof of Lemma 5.13 is complete if we show:

$$\dim_0 W_l \geq 1 \text{ for } l \geq 0.$$

By the beginning of part iii) of the proof of Theorem 6.8 of Demailly in [2], the rational map, obtained by a basis of holomorphic sections of the line bundle $\mathcal{O}_{P_{k+l}V_0}(2(k+l)-1)$, is a morphism on the subset $P_{k+l}(V)_0^{\mathrm{reg}}$ of regular jets in $P_{k+l}V_0$ and separates all points there. Denote by $L_{(2(k+l)-1)}$ the linear system obtained by the pull backs of these sections by the map α (see the last diagram). Then we have

$$(L_{(2(k+l)-1)})^{n_l/(2(k+l)-1)} \subset H^0(\overline{P}_{k+l}V, \mathcal{O}_{\overline{P}_{k+l}V}(n_l)).$$

Therefore, the sections of $H^0(\overline{P}_{k+l}V, \mathcal{O}_{\overline{P}_{k+l}V}(n_l))$ still separate points in the subset of regular jets of each fiber of the map $\overline{P}_{k+l}V \to \overline{G}$. Let a map $\Phi_l : \overline{P}_{k+l}V \to \mathbf{P}^{N_l}$ be defined by a basis of these sections. Then the fiber of the map Φ_l through a regular jet $\xi \in \overline{P}_{k+l}V$ is necessarily of the form $\{\xi + a,\ a \in R\}$, where $R \subset \overline{G}$ is algebraic.

To the basis of holomorphic sections which define the map Φ_l, we now add some extra sections which we allow in addition to have log-poles along the divisor D, namely the sections $\Theta_i \cdot s_0^{n_l - n_i}$, $i = 0, \dots, l$. So we get a map

$$\tilde{\Phi}_l : \overline{P}_{k+l}V \to \mathbf{P}^{N_l+l+1}.$$

This map will in general only separate the subset of regular jets of those fibers of the map $\pi_{0,k+l} : \overline{P}_{k+l}V \to \overline{G}$ which are not over the divisor D. But the fibers of the map $\tilde{\Phi}_l : \overline{P}_{k+l}V \to \mathbf{P}^{N_l+l+1}$ through a regular jet $\xi \in \overline{P}_{k+l}V \setminus \pi_{0,k+l}^{-1}(D)$ must still be of the form $\{\xi + a,\ a \in R_\xi\}$, where $R_\xi \subset R \subset \overline{G}$ is an algebraic subset. So, Lemma 5.13 is proved if we show that $\tilde{\Phi}_l : \overline{X}_{k+l}(f) \to \mathbf{P}^{N_l+l+1}$ has positive dimensional fiber through $\xi = f_{[k+l]}(u)$.

For proving this, we want to use our Ahlfors Lemma 4.3. But this only applies for holomorphic sections. We first extend the diagram in equation (5.1) to

$$\begin{array}{ccccccc}
\overline{Z}_{k+l}(f) & \subset & \overline{P}_{k+l}Y \xrightarrow{p_{[k+l]}} \overline{P}_{k+l}G & \supset & \overline{P}_{k+l}(V) & \supset & X_{k+l}(f) \\
 & & \downarrow \qquad\qquad \downarrow & & \downarrow & & \\
\overline{Z}_{k}(f) & \subset & \overline{P}_{k}Y \xrightarrow{p_{[k]}} \overline{P}_{k}G & \supset & \overline{P}_{k}(V) & \supset & X_{k}(f) \\
 & & \downarrow \qquad\qquad \downarrow & & \downarrow & & \\
 & & (\overline{Y}, E) \xrightarrow{p} (\overline{G}, S) & \supset & (\overline{G}, S, \overline{V}) & &
\end{array}$$

where

$$\begin{aligned}
\overline{Z}_k(f) &= \overline{p_{[k]}^{-1}(X_k(f) \setminus \pi_{0,k}^{-1}(D))}^{\text{Zariski}} \subset \overline{Z}_k \subset \overline{P}_k Y, \\
\overline{Z}_{k+l}(f) &= \overline{p_{[k+l]}^{-1}(X_{k+l}(f) \setminus \pi_{0,k+l}^{-1}(D))}^{\text{Zariski}} \subset \overline{Z}_{k+l} \subset \overline{P}_{k+l} Y.
\end{aligned}$$

By functoriality of the jet bundles and Definition 5.7, the sections which define $\tilde{\Phi}_l$ pull back to holomorphic sections to span a linear system

$$\tilde{L}_{k+l} \subset H^0\Big(\overline{Z}_{k+l}(f), \mathcal{O}_{\overline{P}_{k+l}Y}(n_l)\Big).$$

The elements of $\tilde{L}_{k+l}$ define the pull back of $\tilde{\Phi}_l$ to $\overline{Z}_{k+l(f)}$ (which we denote again by $\tilde{\Phi}_l$).

So we can apply our Ahlfors Lemma 4.3 to the map

$$\tilde{\Phi}_l : \overline{Z}_{k+l}(f) \to \mathbf{P}^{N_l+l+1}$$

to conclude that $\tilde{f}_{[k+l]}(u) \in B_{k+l}(\overline{Z}_{k+l}(f))$, where $\tilde{f} = p^{-1} \circ f$ and, without loss of generality, $u \notin \mathrm{Sing}\tilde{X}_{k+l}$. For otherwise, the map $\tilde{f}$ would be constant or (in the case of $\Gamma = \Delta^*$) at least extendable, which would imply that f has this property, or the image of $\tilde{f}_{[k+l]}$ would be contained in $\mathrm{Sing}\overline{Z}_{k+l}(f)$, which is impossible, since $\overline{Z}_{k+l}(f)$ is the proper transform of the Zariski closure of the image of $f_{[k+l]}$. Hence, $\tilde{\Phi}_l : \overline{Z}_{k+l}(f) \to \mathbf{P}^{N_l+l+1}$ has positive dimensional fiber through $\tilde{f}_{[k+l]}(u)$. Since $f_{[k+l]}(u) \notin \pi_{0,k+l}^{-1}(S \cup D)$, the map $p_{[k+l]}^{-1}$ is an isomorphism around $f_{[k+l]}(0)$. Hence, $\tilde{\Phi}_l : X_{k+l}(f) \to \mathbf{P}^{N_l+l+1}$ also has positive dimensional fiber. This ends the proof of Lemma 5.13. □

Let us now continue with the proof of the Main Lemma. Without loss of generality we may assume that $f_{[k]}(u) \notin \mathrm{Sing}(X_k(f))$. Then there exists a neighborhood $U = U(0) \subset G$ such that, for all $a \in U$, we have $(f+a)_{[k]}(u) \notin \mathrm{Sing}(X_k(f))$. By Lemma 5.8 (2), we get

$$((f+a)_{[k]})_{[l]}(u) \in \overline{P}_l(\tilde{X}_k(f)) \subset \overline{P}_l(P_k(V))$$

for $a \in W \cap U$ and all $l \geq 0$, where we have omitted to write the map Ψ. This is justified by the fact that around $(f + a)_{[k]}(u)$, for $a \in U$, the variety $X_k(f)$ is smooth, and so Ψ is an isomorphism there. Applying Lemma 5.11, we get that $(f + a)_{[k]}(\Gamma) \subset \tilde{X}_k(f)$. Hence,

$$(f + a)_{[k]}(\Gamma) \subset X_k(f)$$

for $a \in W \cap U$. But this means

$$f(\Gamma) \subset X_k(f) \cap (X_k(f) + a).$$

Since $X_k(f)$ was the Zariski closure of $f(\Gamma)$, we get

$$X_k(f) = X_k(f) + a$$

for all $a \in W \cap U$. We next want to show:

Lemma 5.14.

$$\left.\frac{\Theta}{s_0^{\deg \Theta}}\right|_{X_k(f)}$$

is invariant under the action of all $a \in W \cap U$

Proof of Lemma 5.14. Let $a \in W \cap U$ be fixed. In order to simplify notation, we denote by $F_{(i)}$, $i \in \mathbb{N}_0$, the following rational function on $\overline{P}_{k+i}V$:

$$F_{(i)}(y) := \frac{\Theta_i}{s_0^{n_i}}(y) - \frac{\Theta_i}{s_0^{n_i}}(y + a).$$

Set $F = F_{(0)}$. It suffices to show that, for all $i \in \mathbb{N}_0$, we have

$$\frac{\partial^i}{\partial t^i}\big(F \circ f_{[k]}\big)(u) = 0. \tag{5.5}$$

For then, by analytic continuation applied to $f_{[k]} : \Delta \to \overline{P}_k V$, we will have $F \circ f_{[k]}(\Gamma) \equiv 0$, so that $F \equiv 0$ on $X_k(f)$ as required by Lemma 5.14.

By abuse of notation, we have identified $s_0 \circ (\pi_{0,k+l-1})_*$ with s_0. Since these sections are maps from $\mathcal{O}_{\overline{P}_{k+l}V}(1)$ respectively $\mathcal{O}_{\overline{P}_1 V}(1)$, this means we have

$$s_0\big(f_{[k+l]}(t)\big) \cdot f'_{[k+l-1]}(t) = s_0\big(f_{[1]}(t)\big) \cdot f'(t)\,. \tag{5.6}$$

Recall that the section s_0 is nonvanishing at $f_{[1]}(u)$, and that $f'(u) \not\equiv 0$. So, after possibly shrinking Δ, we may reparametrize f such that

$$s_0\big(f_{[k+l]}(t)\big) \cdot f'_{[k+l-1]}(t) = s_0\big(f_{[1]}(t)\big) \cdot f'(t) \equiv 1\,. \tag{5.7}$$

For the rest of this proof we fix the parameter t in such a way that equation (5.7) is satisfied. We claim that for any $i \in \mathbb{N}_0$ we have:

$$\frac{\partial^i}{\partial t^i}\big(F \circ f_{[k]}\big)(t) = F_{(i)} \circ f_{[k+i]}(t)\,. \tag{5.8}$$

We prove this by induction over $i \in \mathbb{N}_0$. The case S(0) is clear by definition. Assume that S(i), $i < l$ is true. Then we have

$$\begin{aligned}
\frac{\partial^l}{\partial t^l}\big(F \circ f_{[k]}\big)(t) &= \frac{\partial}{\partial t}\left(\frac{\partial^{l-1}}{\partial t^{l-1}}\big(F \circ f_{[k]}\big)\right)(t)\\
&= \frac{\partial}{\partial t}\big(F_{(l-1)} \circ f_{[k+l-1]}\big)(t)\\
&= (dF_{(l-1)})((f_{[k+l-1]})(t)) \cdot f'_{[k+l-1]}(t)\\
&= \frac{(dF_{(l-1)})((f_{[k+l-1]})(t)) \cdot f'_{[k+l-1]}(t)}{s_0((f_{[k+l]})(t)) \cdot f'_{[k+l-1]}(t)}\\
&= \frac{(dF_{(l-1)})((f_{[k+l]})(t)) \cdot f'_{[k+l-1]}(t)}{s_0((f_{[k+l]})(t)) \cdot f'_{[k+l-1]}(t)}\\
&= \frac{dF_{(l-1)}}{s_0}((f_{[k+l]})(t)) = F_{(l)}((f_{[k+l]}(t)).
\end{aligned}$$

Here, we regard the differentials as linear maps on $\mathcal{O}(-1)$, more specifically, sections of $\mathcal{O}(1)$. To see the second equality from below, recall that the section $dF_{(l-1)}$ of $\mathcal{O}_{\overline{P}_1(P_{k+l-1}(V))}(1)$ naturally restricts to a section of $\mathcal{O}_{\overline{P}_{k+l}V}(1)$. This proves equation (5.8). But from equation (5.8), equation (5.5) follows immediately. This is because, by the definition of W, $F_{(i)} \circ f_{[k+i]}(u) = 0$ for all $i \in \mathbb{N}_0$. □

Now the proof of the Main Lemma is immediate. Let us define

$$\tilde{W} = \left\{a \in G : X_k(f) = X_k(f) + a,\ \frac{\Theta}{s_0^{\deg \Theta}} \text{ is invariant under } a\right\}.$$

$\tilde{W}$ is clearly a group, which is algebraic by Lemma 5.5. It is of positive dimension, since $(W \cap U) \subset \tilde{W}$ and $\dim_0(W \cap U) \geq 1$. This finishes the proof of the Main Lemma 5.12.

5.5. Proof of Theorems 5.1 and 5.3

Proof of Theorems 5.1 (a) and 5.3 (a). We apply the Main Lemma 5.12 in the special case where $k = 0$ and $V = \overline{T}G$. Then Theorem 5.3 (a) is immediate. Theorem 5.1 (a) is obtained by dividing out by the biggest algebraic subgroup of G under which $X(f)$ is invariant. □

In order to prove the remaining parts of Theorems 5.1 and 5.3, we first have to choose the section Θ appropriately. We do this in the same way as Siu-Yeung ([21]) or Noguchi ([17]).

By Noguchi ([17], Lemma 2.1), there exists a theta function for $D \subset G$. This means the following. Let $\pi : \mathbf{C}^n \to G$ be the universal covering with a 'semi-lattice' $\Pi_1(G)$. There exists an entire function $\theta : \mathbf{C}^n \to \mathbf{C}$ such that

$$(\theta) = \pi^* D.$$

Moreover, for any $\gamma \in \Pi_1(G)$, there is an affine linear function L_γ in x with

$$\theta(x + \gamma) = e^{L_\gamma(x)}\theta(x), \;\; x \in \mathbf{C}^n. \tag{5.9}$$

But the proof which Noguchi gives actually yields more. Consider G as a $(\mathbf{C}^*)^\ell$-principal fiber bundle over an abelian variety A, and denote the projection map by $p : G \to A$. Let $\pi : \mathbf{C}^{\mathrm{m}} \to A$ be the universal covering. Then the fibered products of π with G respectively $\overline{G}$ over A are $(\mathbf{C}^*)^\ell \times \mathbf{C}^{\mathrm{m}}$ respectively $(\mathbf{P}^1)^\ell \times \mathbf{C}^{\mathrm{m}}$. Let $\tilde{\pi} : \mathbf{C}^n \to (\mathbf{C}^*)^\ell \times \mathbf{C}^{\mathrm{m}}$ be the universal covering. Then Noguchi's proof yields that there exists a holomorphic function $\tilde{\theta} : (\mathbf{C}^*)^\ell \times \mathbf{C}^{\mathrm{m}} \to \mathbf{C}$ which extends to a meromorphic function on $(\mathbf{P}^1)^\ell \times \mathbf{C}^{\mathrm{m}}$ such that $\theta = \tilde{\theta} \circ \tilde{\pi}$ is a theta function for $D \subset G$ as above. More precisely, if $(w_1, \ldots, w_t)$ is a multiplicative coordinate system of $(\mathbf{C}^*)^\ell$ and $U \subset \mathbf{C}^{\mathrm{m}}$ is a small neighborhood of a point in $\mathbf{C}^{\mathrm{m}}$, then $\tilde{\theta}|_{(\mathbf{C}^*)^\ell \times U}$ can be written as

$$\sum_{\text{finite}} a_{l_1 \ldots l_t}(y) w_1^{l_1} \ldots w_t^{l_t}, \tag{5.10}$$

where the coefficients $a_{l_1 \ldots l_t}(y)$ are holomorphic functions on U. As $(\mathbf{P}^1)^\ell \times \mathbf{C}^{\mathrm{m}}$ is the universal covering space of $\overline{G}$, $\tilde{\theta}$ gives a multivalued defining function for D on $\overline{G}$ locally given by equation (5.10). It follows that, whenever an algebraic expression in θ descends to G, it extends to $\overline{G}$ to a meromorphic object, and, hence, to a rational object.

Let $\mu : \tilde{\Gamma} \to \Gamma$ be the universal cover, and let $\tilde{f} : \tilde{\Gamma} \to \mathbf{C}^n$ be any lift of the map $f \circ \mu : \tilde{\Gamma} \to G \setminus D$. Then we can choose a linear coordinate system $(z_1, \ldots, z_n)$ of $\mathbf{C}^n$ such that in these coordinates, $\tilde{f}$ is expressed as

$$\tilde{f}(z) = (f_1(z), \ldots, f_{n'}(z), 0, \ldots, 0)$$

with holomorphic functions $f_1(z), \ldots, f_{n'}(z)$, for which the functions

$$1, f_1, f_2, \ldots, f_{n'}$$

are linearly independent. The set of differential equations $dz_{n'+1} = 0, \ldots, dz_n = 0$ obviously defines a subbundle $\tilde{V} \subset T\mathbf{C}^n$, which is invariant under translation and, hence, descends to G. It is important to remark that this subbundle extends to a special subbundle $V \subset \overline{T}G$. This is true, since by definition, V is just of the form $V = \mathbf{C}^{n'} \times G$ with respect to the trivialization of $\overline{T}G$ given by the standard logarithmic forms $dz_1, \ldots, dz_n$.

Siu-Yeung ([21]) defined the following logarithmic jet differential:

$$\Theta = \begin{vmatrix} d\log\theta & dz_1 & \ldots & dz_{n'} \\ d^2\log\theta & d^2 z_1 & \ldots & d^2 z_{n'} \\ \vdots & \vdots & \vdots & \vdots \\ d^{n'+1}\log\theta & d^{n'+1} z_1 & \ldots & d^{n'+1} z_{n'} \end{vmatrix}$$

We want to use this Θ as the Θ in our Main Lemma 5.12. So we need:

Lemma 5.15. *Θ can be considered as a meromorphic section in the line bundle*

$$\mathcal{O}_{\overline{P}_{n'+1}V}\left(\frac{(n'+1)(n'+2)}{2}\right)$$

with at most log-poles along D.

Proof of Lemma 5.15. We first show (part (a)) that Θ is a meromorphic function on $J_k\mathbf{C}^n$, and, hence, also on $J_k\tilde{V}$, which satisfies equation (3.2) with order $k = n'+1$ and degree $m = (n'+1)(n'+2)/2 = k(k+1)/2$ (recall that $\tilde{V} \subset T\mathbf{C}^n$ is defined by $dz_{n'+1} = \cdots = dz_n = 0$). By using equation (5.10) it follows actually that Θ defines such a function on $J_k(\mathbf{C}^{*\ell} \times \mathbf{C}^{\mathrm{m}})$, which extends, again by equation (5.10), to a meromorphic function on $\overline{J}_k(\mathbf{C}^{*\ell} \times \mathbf{C}^{\mathrm{m}})$. This meromorphic function descends to a multivalued function on $\overline{J}_kG$. Then we show (part (b)) that it is actually singlevalued on $\overline{J}_kV \subset \overline{J}_kG$. So it corresponds, by Proposition 3.9, to a meromorphic section of $\mathcal{O}_{\overline{P}_kV}(m)$. That it is a meromorphic section of $\mathcal{O}_{\overline{P}_kV}(m)$ with at most log-poles along D then follows from the local description of $\tilde{\theta}$ in equation (5.10) and from the local description of meromorphic sections with at most log-poles along D right after Definition 5.7: In fact, by this description, applied to the multivalued meromorphic function on $\overline{J}_kG$, it gives rise to a (possibly) multivalued holomorphic function on $\overline{J}_kY$. But, as we saw above, it is singlevalued over $\overline{J}_kV$, and hence, it is also singlevalued over $\overline{J}_kY$. Thus, it yields a meromorphic section with at most log-poles along D.

(a) We show more generally: Let $h_1, \ldots, h_r : (\Delta, 0) \to \mathbf{C}$ be nonvanishing germs of holomorphic functions. Let $g_i = \log h_i$, $i = 1, \ldots, r$. Then

$$\begin{vmatrix} dg_1 & dg_2 & \ldots & dg_r \\ d^2 g_1 & d^2 g_2 & \ldots & d^2 g_r \\ \vdots & \vdots & \vdots & \vdots \\ d^r g_1 & d^r g_2 & \ldots & d^r g_r \end{vmatrix}$$

gives a jet differential on $J_r\Delta_0$ which is equivariant under the full reparametrization group $J_r\Delta_0$ in the sense of equation (3.2). (We will apply this for the case where the germs h_j are obtained by composing θ and the exponentials of the z_i's with the germ $f:(\mathbf{C},0)\to\mathbf{C}^n$ representing the jet in our case.)

Only the equivariance is nontrivial. Let $g=(g_i)$. Let $\phi\in J_r\mathbf{C}_0$. Then, by using the identity

$$(g\circ\phi)^{(j)}(0)=g^{(j)}(0)(\phi'(0))^j+\sum_{s=1}^{j-1}\sum_{i_1+\cdots+i_s=j}c_{i_1\dots i_s}g^{(s)}(0)\phi^{(i_1)}(0)\dots\phi^{(i_s)}(0),$$

we get by induction on r that

$$(g\circ\phi)'\wedge\cdots\wedge(g\circ\phi)^{(r)}(0)=g'\wedge\cdots\wedge g^{(r)}\cdot(\phi'(0))^{r(r+1)/2}.$$

This gives the desired equivariance.

(b) From equation (5.9) we have, for $\gamma\in\Pi_1(G)\subset\mathbf{C}^n$:

$$d^i\log\theta(x+\gamma)=d^i\log\theta(x)+d^iL_\gamma(x)=d^i\log\theta(x)+\sum_{j=1}^{n'}a_jd^ix_j+\sum_{j=n'+1}^{n}a_jd^ix_j,$$

where $a_i\in\mathbf{C}$ are constants. Then, from the properties of the determinant and the fact that we restrict Θ to $J_{n'+1}\tilde{V}$, it follows that this jet differential is invariant under the action of $\Pi_1(G)$. Hence, it descends to G. $\square$

Lemma 5.16. *Under the assumptions of* Theorem 5.1 (b), *or* Theorem 5.3 (b) *and* (c) *and the additional assumption that f does not extend, the following holds*: *If $X(f)\cap D\neq\emptyset$, then $X(f)\cap D$ is foliated by translates of an algebraic subgroup $G''\subset G'$ of positive dimension, where G' is the maximal subgroup whose translates foliate $X(f)$.*

Proof of Lemma 5.16. We may assume that f is nonconstant and, for the case that $\Gamma=\Delta^*$, that the map f does not extend. We apply the Main Lemma 5.12 to get the existence of an algebraic subgroup $G''\subset G$ of positive dimension which leaves $X_{n'+1}(f)$ and $\Theta|_{X_{n'+1}(f)}$ invariant. As $X_{n'+1}(f)$ is invariant under the action of G'' and the projection $\pi_{1,n'+1}$ (respectively $\pi_{0,n'+1}$) maps $X_{n'+1}(f)$ surjectively onto $X_1(f)$ (respectively $X(f)$), we see that $X_1(f)$ and $X(f)$ are also invariant.

Take any $a \in G''$. Since $\Theta|_{X_{n'+1}(f)}$ is invariant under translation by a,

$$\begin{vmatrix} d\log\frac{\theta(f)}{\theta(f+a)} & df_1 & \dots & df_{n'} \\ d^2\log\frac{\theta(f)}{\theta(f+a)} & d^2 f_1 & \dots & d^2 f_{n'} \\ \vdots & \vdots & \vdots & \vdots \\ d^{n'+1}\log\frac{\theta(f)}{\theta(f+a)} & d^{n'+1} f_1 & \dots & d^{n'+1} f_{n'} \end{vmatrix} \equiv 0 \text{ on } \Gamma.$$

Now $d\log\{\theta(x)/\theta(x+a)\}$ is a rational differential on G. Since $f+a$ cannot map entirely into the zero set of θ, because $X(f)$ is the Zariski closure of $f(\Gamma)$,

$$\frac{\partial}{\partial z}\left(\log\frac{\theta(f)(z)}{\theta(f+a)(z)}\right), \frac{\partial}{\partial z} f_1(z), \dots, \frac{\partial}{\partial z} f_{n'}(z)$$

are well defined meromorphic functions on Γ. The functions $(\partial/\partial z)f_1(z), \dots,$ $(\partial/\partial z)f_{n'}(z)$ are linearly independent as $1, f_1, \dots, f_{n'}$ were so. Hence, we get, by the classical Lemma of the Wronskian [1], that there exist complex numbers $c_1, \dots, c_{n'}$ (which may depend on $a \in G''$) such that

$$d\log\frac{\theta(f)(z)}{\theta(f+a)(z)} + c_1 df_1(z) + \dots + c_{n'} df_{n'}(z) \equiv 0 \text{ on } \Gamma.$$

So we have

$$d\log\frac{\theta(x)}{\theta(x+a)} + c_1 dx_1 + \dots + c_{n'} dx_{n'} \equiv 0 \tag{5.11}$$

on $f_{[1]}(\Gamma)$. Moreover, since $d\log\{\theta(x)/\theta(x+a)\}$ is a rational differential on G, this equation holds on $X_1(f)$.

Assume now that Lemma 5.16 does not hold. Then there exists $x_0 \in X(f) \cap D$ and $a_0 \in G''$ such that $x_0 + a_0 \notin D$. We want to show that this assumption leads to a contradiction. From equation (5.11) we get that

$$d\log\frac{\theta(x)}{\theta(x+a_0)} = d\log\frac{\theta(x+b)}{\theta(x+a_0+b)} \tag{5.12}$$

on $X_1(f)$ for $b \in G''$. This means that

$$d\log\left(\frac{\theta(f)}{\theta(f+b)}\frac{\theta(f+a_0+b)}{\theta(f+a_0)}\right) \equiv 0 \text{ on } \Gamma.$$

Hence,

$$\frac{\theta(f)}{\theta(f+b)}\frac{\theta(f+a_0+b)}{\theta(f+a_0)} = c_{a_0,b} \text{ on } \Gamma,$$

where $c_{a_0,b} \in \mathbf{C}$ is a constant, which may depend on a_0 and b. Since $\{\theta(x)/\theta(x+b)\} \times \{\theta(x+a_0+b)/\theta(x+a_0)\}$ is a well defined rational function on G, we have

$$\frac{\theta(x)}{\theta(x+b)} \frac{\theta(x+a_0+b)}{\theta(x+a_0)} = c_{a_0,b} \text{ on } X(f), \tag{5.13}$$

where $b \in G''$. Now $x_0 + a_0 \notin D$, but $x_0 \in D$. So we get, for $b = a_0$ and $x = x_0$, that $c_{a_0,a_0} = 0$. This means, as $X(f)$ is irreducible, that either $\theta(x) \equiv 0$ or $\theta(x+2a_0) \equiv 0$ on $X(f)$. But both is not true, as one sees by taking $x = x_0 + a_0$ respectively $x = x_0 + a_0 - 2a_0$ (remark that the latter is still in $X(f)$, since $X(f)$ is invariant under the action of G''). So our assumption was wrong, and we have proved Lemma 5.16. $\square$

Proof of Theorem 5.1 (b). Assume that $X(f) \cap D \neq \emptyset$. We want to show that this assumption leads to a contradiction. After applying a translation, we may assume, by Theorem 5.1 (a), that $X(f)$ again is a semi abelian variety G' with nonempty divisor D' in G', where D' is the reduction of $X(f) \cap D$. Now we divide through the maximal algebraic subgroup $\tilde{G}$ of G' which foliates D'. Then, by applying Lemma 5.16 to the quotient $G'/\tilde{G}$ and by taking the inverse image under the quotient map, we get $X(f) \cap D = \emptyset$, which contradicts our assumption. $\square$

Proof of Theorem 5.3 (b) and (c). Part (b) follows immediately from Lemma 5.16. For (c), let G be an abelian variety. Let G' again be the maximal algebraic subgroup the translates of which foliate $X(f)$. We may assume that all translates of G' which foliate $X(f)$ intersect D (in particular $X(f) \cap D \neq \emptyset$), for otherwise we finish the proof by using Lemma 5.6. Then there must be such a translate T_0 of G' such that $T_0 \not\subset D$. Now by Lemma 5.16 we find a subgroup $G'' \subset G'$ of positive dimension which foliates $X(f) \cap D$. Hence, $T \cap D$ is foliated by translates of G''. But since T is also foliated by translates of G'', there must be such a translate not hitting D at all. This finishes the proof again by using Lemma 5.6. $\square$

6. Appendix

We use the notations of Subsection 4.1. We now give a key result via which pseudometrics of negative curvature are usually constructed (see (2.7) of [5]). We point out that our version is sharper than the ones in [2, 5] for the basic locus in our definition is smaller than theirs.

Lemma 6.1. *Let the setup be as in* Subsection 4.1, *and assume further that X is normal. Then given line bundles L and H over X, there is an integer $l_1 \geq 1$ such that $x \notin E_L$ implies that $x \notin \mathrm{Bs}|lL - H|$ for all positive multiples l of l_1, more specifically,*

$$E_L \supseteq \mathrm{Bs}|lL - H|.$$

Proof. Observe that we may always write $H + H' = H''$, where H' and H'' are very ample divisors. Then $\mathrm{Bs}|lL - H''| \supseteq \mathrm{Bs}|lL - H|$, as one can see from the fact that $\mathrm{Bs}|G - G'| \cup \mathrm{Bs}|G'| \supseteq \mathrm{Bs}|G|$ for arbitrary line bundles G and G'. Hence, we will assume without loss of generality that H is very ample.

Let $x \in X$ be outside E_L. Then, we may assume that φ_L is birational onto its image after replacing L by a suitable multiple of L (see 1.10 and 5.7 of [23]). It will be sufficient to show that x is outside $\mathrm{Bs}|lL - H|$ for some l, and hence, for all multiples thereof, as Lemma 6.1 would then follow by the quasi-compactness of $X \setminus E_L$.

Consider the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ generated by the global sections of L. By blowing up this ideal sheaf, we obtain a modification $\sigma : \tilde{X} \to X$ so that $\mathcal{J} = \sigma^\star \mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is an invertible ideal sheaf of $\mathcal{O}_{\tilde{X}}$ generated by a global section s of the line bundle $F = \mathcal{O}_\sigma(-1)$, namely, $\mathcal{J} = \mathrm{Im}\{\mathcal{O}(F^*) \xrightarrow{\otimes s} \mathcal{O}\}$. Then $\overline{L} := \sigma^{-1}L - F$ is spanned by the sections $(\sigma^{-1}t)/s$ as t ranges over $H^0(L)$. Note that $E_{\overline{L}} = \sigma^{-1}(E_L)$, that σ^{-1} is an isomorphism on the neighborhood $X \setminus E_L$ of x and that any section of $l\overline{L} - \sigma^{-1}H$ not vanishing on the point $\sigma^{-1}(x)$ gives rise to a section of $lL - H$ not vanishing on x by tensoring with s^l (Zariski's Main Theorem and $s(x) \neq 0$). Hence, replacing (X, L) by $(\tilde{X}, \overline{L})$ we may assume that $\varphi_L : X \to \mathbf{P}^n$ is a birational morphism onto its image $W = \varphi_L(X)$. Let $\sigma_0 : W_0 \to W \subseteq \mathbf{P}^n$ be the normalization of W. Then $H_0 := \sigma_0^{-1}\mathcal{O}_{\mathbf{P}^n}(1)$ is ample so that there is a positive integer d such that dH_0 is very ample on W_0. As X is normal, there is a canonical morphism $\varphi : X \to W_0$ such that $\varphi_L = \sigma_0 \circ \varphi$. Noting $\varphi^{-1}H_0 = L$, we see that the image W_1 of the morphism φ_{dL} admits a birational morphism r to W_0 and that $\varphi = r \circ \varphi_{dL}$. As φ is connected by Zariski's Main Theorem, $\varphi^{-1}(\varphi(x)) = \{x\}$. Hence, replacing L by dL we may assume that

$$\varphi_L^{-1}(\varphi_L(x)) = \{x\}. \tag{6.1}$$

As H is very ample, $|H|$ has an element D such that $x \notin D \not\subseteq E_L$ by general positioning. We may now choose, thanks to equation (6.1), a hypersurface of sufficiently high degree l in $\mathbf{P}^n$ containing $\varphi_L(D)$ but not $\varphi_L(x)$. This gives a divisor in $|lL - H|$ not containing x as desired. □

In practice, Lemma 6.1 is all that one uses. But one can easily deduce the following strengthened version in order to complete the picture.

Lemma 6.2. *Let X be a normal complex projective variety with any line bundle H. For any line bundle L over X, there is an integer m_0 such that*

$$E_{m_0 L} \supseteq S_L \supseteq \mathrm{Bs}|m_0 L - H|\,.$$

Moreover, if H is very ample, the both inclusions are equalities.

Proof. Clearly there is an integer N such that $S_L = \cap_{m=1}^N E_{mL}$. For each m, there is an integer $l_m > 0$, such that $E_{mL} \supseteq \mathrm{Bs}|lL - H|$ for all positive multiple l of l_m by

Lemma 6.1. Letting m_0 be a common multiple of $l_1, \ldots, l_N$, we see that $E_{m_0 L} \supseteq S_L \supseteq \mathrm{Bs}|m_0 L - H|$. If, furthermore, H is very ample, one easily verifies that $\mathrm{Bs}|mL - H| \supseteq E_{mL}$ for all m. $\square$

REMARK. As $\mathrm{Bs}|lL - H| \supseteq E_{lL} \supseteq S_L$, it follows that $S_L = \cap_{l>0} \mathrm{Bs}|lL - H|$ for any very ample H.

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