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PROJECTIVITY OF MOMENT MAP QUOTIENTS

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Let G be a complex reductive group acting algebraically on a complex projective variety X. Given a polarization of X, i.e., an ample G-line bundle L over X, Mumford (see [16]) defined the notion of stability: A point $x \in X$ is said to be semistable with respect to L if and only if there exists $m \in \mathbb{N}$ and an invariant section $s : X \to L^m$ such that $s(x) \neq 0$. Let $X(L)$ denote the set of semistable points in X, then there is a projective variety $X(L)/\!/G$ and a G-invariant surjective algebraic map $\pi : X(L) \to X(L)/\!/G$ such that
(i) π is an affine map and
(ii) $\mathcal{O}_{X(L)/\!/G} = (\pi_* \mathcal{O}_{X(L)})^G$.

In particular, for an open affine subset U of $X(L)/\!/G$, it follows that $\pi^{-1}(U) = \operatorname{Spec} \mathbb{C}[U]^G$ where $\mathbb{C}[U]$ denotes the coordinate ring of $\pi^{-1}(U)$ and $\mathbb{C}[U]^G$ is the algebra of invariant functions.

There is a completely analogous picture for a holomorphic action of a complex reductive group G on a Kählerian space X. The role of a polarization is taken over by a Hamiltonian action of a maximal compact subgroup K of G. Here one considers a maximal compact subgroup K of G, assumes the Kähler structure to be K-invariant and that there is an equivariant moment map $\mu : X \to \mathfrak{k}^*$ with respect to ω. In this situation $X(\mu) = \{x \in X;\ \overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}$ is called the set of semistable points of X with respect to μ. Here $\overline{G \cdot x}$ denotes the topological closure of the G-orbit through x. The following result has been proved in [11] (c.f. [18]).

The set $X(\mu)$ is open in X and there is a complex space $X(\mu)/\!/G$ and a G-invariant surjective holomorphic map $\pi : X(\mu) \to X(\mu)/\!/G$ such that
(i) π is a Stein map and
(ii) $\mathcal{O}_{X(\mu)/\!/G} = (\pi_* \mathcal{O}_{X(\mu)})^G$.

In fact there is one more analogy between these two constructions. In the case where X is projective, the line bundle L induces a line bundle $\overline{L}$ on $X(L)/\!/G$ which turns out to be ample. In the Kähler case ω induces a Kählerian structure $\overline{\omega}$ on $X(\mu)/\!/G$.

A very ample G-line bundle L over X induces a G-equivariant holomorphic embedding of X into $\mathbb{P}(V)$ where V is the dual vector space of the space of sections $\Gamma(X, L)$ and the G-action on $\mathbb{P}(V)$ is induced by the natural linear G-action on $\Gamma(X, L)$. Now one may assume the K-representation to be unitary and therefore the pull back of the Fubini-Study form $\omega_{\mathbb{P}(V)}$ to X is a K-invariant Kähler form ω and the pull back of the natural moment map $\mu_{\mathbb{P}(V)}$ to X gives a moment map $\mu : X \to \mathfrak{k}^*$. In this case, using a result of Kempf-Ness (see [12]), one checks that $X(\mu) = X(L)$, i.e., the set of Mumford-semistable subsets of X is a subset of the set of momentum-semistable sets (see [13], [17] or Sec. 3.).

Of course in general a given invariant Kähler form ω on a projective G-manifold X may not be integral. Therefore associated moment maps are not in an obvious way related to G-line bundles. Nevertheless, our goal here is to prove the following

Semistability Theorem. *Let X be a smooth projective variety endowed with a holomorphic action of a complex reductive group $G = K^{\mathbb{C}}$, ω a K-invariant Kähler form and $\mu : X \to \mathfrak{k}^*$ a K-equivariant moment map. Then there is a very ample G-line bundle L over X such that*

$$X(\mu) = X(L).$$

Recently there has been some interest in the question of how $X(L)$ and $X(L)/\!/G$ vary in dependence of L (see e.g. [4], [19]). The above obviously implies that these results extend to the case where μ is moving.

1. Mumford quotients

Let G be a complex reductive group and V a G-representation, i.e., there is given a holomorphic homomorphism $\rho : G \to \mathrm{GL}(V)$. Since G is reductive, it is in fact a linear algebraic group and ρ is an algebraic map (see e.g. [3]). Moreover the algebra $\mathbb{C}[V]^G$ of G-invariant polynomials is finitely generated. The corresponding affine variety is denoted by $V/\!/G$. The inclusion $\mathbb{C}[V]^G \hookrightarrow \mathbb{C}[V]$ induces a polynomial map $\pi : V \to V/\!/G$ which turns out to be surjective. Explicitly $\pi : V \to V/\!/G$ can be realized as follows. Let $q_1, \ldots, q_k$ be a set of generators of the algebra $\mathbb{C}[V]^G$ and $q := (q_1, \ldots, q_k)$. Then $Y := q(V)$ is a Zariski-closed subset of $\mathbb{C}^k$ which is isomorphic with $V/\!/G$. Under this isomorphism $\pi : V \to V/\!/G$ is given by q.

Since the group G and the action $G \times V \to V$, $(g, v) \to g \cdot v$, are algebraic, every G-orbit is Zariski-open in its closure. In particular, for every $x \in \overline{G \cdot v} \setminus G \cdot v$ we have $\dim G \cdot x < \dim G \cdot v$. This implies that the closure of every G-orbit contains a closed G-orbit which may be defined as a G-orbit of smallest dimension in $\overline{G \cdot v}$. Now G-invariant polynomials separate G-invariant Zariski-closed subsets. This can be seen by using integration over a maximal compact subgroup K of G. Thus the closed G-orbit in $\overline{G \cdot v}$ is unique. Moreover for $v, w \in V$ we have $\pi(v) = \pi(w)$ if and only if $\overline{G \cdot v} \cap \overline{G \cdot w} \neq \emptyset$ and this is the case if and only if $\overline{G \cdot v}$ and $\overline{G \cdot w}$ contain the same closed orbit. Consequently, if $G \cdot v_0$ is the closed orbit in $\overline{G \cdot v}$, then $\pi^{-1}(\pi(v)) = \{w \in V; G \cdot v_0 \subset \overline{G \cdot w}\}$. This is often expressed by the phrase that the quotient $V/\!/G$ parametrises the closed G-orbits in V.

Assume now that X is a projective G-variety which is realized as a G-stable Zariski-closed subset of $\mathbb{P}(V)$. In general there is no way to associate to X a quotient $X/\!/G$ which has reasonable properties. For example if V is irreducible, then $\mathbb{P}(V)$ contains a unique G-orbit which is compact. This orbit is the image of a G-orbit through a maximal weight-vector in V. Since every G-orbit in $\mathbb{P}(V)$ contains a closed G-orbit in its closure, the unique compact orbit is contained in the closure of every other G-orbit in $\mathbb{P}(V)$. If one were to try to define a Hausdorff quotient, then every point would have to be identified with the points in the unique compact orbit. The resulting quotient would be a point.

In order to resolve this difficulty Mumford introduced the following procedure (see [16]). Let N be the Null-cone in V, i.e., the fiber through the origin of the quotient map $\pi : V \to V/\!/G$ and let $p : V \setminus \{0\} \to \mathbb{P}(V)$ denote the $\mathbb{C}^*$-principal bundle which defines the projective space $\mathbb{P}(V)$. For a subset Y of $\mathbb{P}(V)$ let $\hat{Y} := p^{-1}(Y)$. A point $x \in X$ is said to be semistable with respect to V if $\hat{x} = p^{-1}(x) \subset \hat{X} \setminus N$. Let $X(V) := p(\hat{X} \setminus N)$ denote the set of semistable points in X with respect to the representation V. Thus $X(V)$ is obtained by removing the image of the Null-cone from X.

The cone $C(X) := \hat{X} \cup \{0\}$ in V over X is a G-stable closed affine subset of V and N is saturated with respect to $\pi_V : V \to V/\!/G$. Thus $\hat{X}(V) := \hat{X} \setminus N = C(X) \setminus N$ is saturated with respect to $\pi_{\hat{X}} : \hat{X} \to \hat{X}/\!/G$. In particular there is a quotient $\hat{\pi} : \hat{X}(V) \to \hat{X}(V)/\!/G$ which is given by restricting $\pi_V : V \to V/\!/G$ to $\hat{X}(V)$. The $\mathbb{C}^*$-action on V defined by multiplication commutes with the G-action and stabilizes $\hat{X}(V)$. Thus there is an induced $\mathbb{C}^*$-action on $\hat{X}(V)/\!/G$ which can be described explicitly as follows. Let $q_1, \dots, q_k$ be a set of homogeneous generators of $\mathbb{C}[V]^G$ with $\deg q_j = d_j$. The map $q : V \to \mathbb{C}^k$ is equivariant with respect to $\mathbb{C}^*$. More precisely we have $q(t \cdot v) = (t^{d_1} q_1(v), \dots, t^{d_k} q_k(v))$. Moreover $q(V \setminus N) = q(V) \setminus \{0\} \subset \mathbb{C}^k \setminus \{0\}$. Note that $\mathbb{C}^*$ acts properly on $\mathbb{C}^k \setminus \{0\}$. In particular there is a geometrical quotient $\mathbb{C}^k \setminus \{0\}/\mathbb{C}^* =: \mathbb{P}(d_1, \dots, d_k)$ which is a projective variety. This implies that $X(V)/\!/G := (\hat{X}(V)/\!/G)/\mathbb{C}^*$ is also a projective variety, since it is a Zariski-closed subspace of $\mathbb{P}(d_1, \dots, d_k)$. The map $\hat{X}(V) \to X(V)/\!/G$ is $\mathbb{C}^*$-invariant and induces therefore an algebraic map $\pi : X(V) \to X(V)/\!/G$ which is the quotient map for the G-action on $X(V)$.

There is a standard procedure to realize a given G-variety X as a G-stable subvariety of some projective space $\mathbb{P}(V)$ where V is a G-representation. For this assume that L is a very ample line bundle over X and let $\Gamma(X, L)$ denote the space of holomorphic sections of L. Thus the natural map $\iota_L : X \to \mathbb{P}(V)$ which is given by evaluation where $V := \Gamma(X, L)^*$ is the dual of $\Gamma(X, L)$ is an embedding. Now if the G-action on X lifts to a G-action on L, then V is a G-representation in a natural way and ι_L is G-equivariant. The set $X(L) := \{x \in X; s(x) \neq 0$ for some invariant section $s \in \Gamma(X, L^m),\ m \in \mathbb{N}\}$ coincides with $X(V)$ after identifying X with $\iota_L(X) \subset \mathbb{P}(V)$ and is called the set of semistable points of X with respect to the G-line bundle L. Note that $X(L)$ depends on L and on the lifting of the G-action to L. The following two elementary facts concerning G-actions on line bundles are often useful (see [16]).

Lemma. *Let X be a smooth projective G-variety.*
(i) *If L is ample, then there is a lifting of the G-action to some positive power L^m of L.*
(ii) *Two liftings of the G-action to L differ by a character of G.*

Proof. The last statement follows since X is compact and therefore a G-action on the trivial bundle $X \times \mathbb{C} = L \otimes L^{-1}$ is given by $g \cdot (x, z) = (g \cdot x, \chi(g)z)$ where $\chi : G \to \mathbb{C}^*$ is a character of G.

For the first statement one may assume that G acts effectively. Since G is connected the induced action on $\mathrm{Pic}(X)$ is trivial. This implies that there is a subgroup $\tilde{G}$ of the automorphism group of L and an exact sequence of the form

$$1 \to \mathbb{C}^* \to \tilde{G} \to G \overset{\alpha}{\to} 1$$

where α is given by restricting $\tilde{g} \in \tilde{G}$ to the zero section $X \hookrightarrow L$. This sequence splits after replacing G by a finite covering. Hence the G-action on X lifts to L^m for some positive m. □

2. Moment map quotients

Let G be a complex reductive group which acts holomorphically on a complex manifold X. Now choose a maximal compact subgroup K of G and let ω be a K-invariant Kähler form on X. By definition the K-action on X is said to be Hamiltonian with moment map μ if there is given an equivariant smooth map μ from X into the dual $\mathfrak{k}^*$ of the Lie algebra $\mathfrak{k}$ of K such that

$$(*) \qquad d\mu_\xi = \imath_{\xi_X}\omega$$

for all $\xi \in \mathfrak{k}$. Here ξ_X denotes the vector field on X associated with ξ, $\mu_\xi = \mu(\xi)$ and $\imath_{\xi_X}\omega$ is the one form $\eta \to \omega(\xi_X, \eta)$. Note that for a connected manifold X an equivariant moment map is uniquely defined by $(*)$ up to a constant in $\mathfrak{k}^*$ which lies in the set of fixed points. In particular, if the group K is semisimple then an equivariant moment map is unique. Moreover in the semisimple case it can be shown that μ exists for a given K-invariant Kähler form ω (see e.g. [6])

EXAMPLE. Let $\rho : X \to \mathbb{R}$ be a smooth K-invariant function, $\omega := 2i\partial\overline{\partial}\rho$ and let $\mu : X \to \mathfrak{k}^*$ be the associated K-equivariant map which is defined by $\mu_\xi = d\rho(J\xi_X)$. Here J denotes the complex structure tensor on X. A direct calculation shows that $d\mu_\xi = \imath_{\xi_X}\omega$ holds for every $\xi \in \mathfrak{k}$. In particular, if ρ is strictly plurisubharmonic, i.e.,

ω is Kähler, then μ is a moment map. We refer to $\mu =: \mu^\rho$ as the moment map given by ρ.

Similar to the case of an ample G-line bundle there is a notion of semistability with respect to μ. A point $x \in X$ is said to be semistable with respect to μ if $\overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset$. Let $X(\mu)$ denote the set of semistable points with respect to μ.

The following is proved in [11] (see also [18]).

Theorem 1. *The set of semistable points $X(\mu)$ is open in X and the semistable quotient $\pi : X(\mu) \to X(\mu)//G$ exists. The inclusion $\mu^{-1}(0) \hookrightarrow X(\mu)$ induces a homeomorphism $\mu^{-1}(0)/K \cong X(\mu)//G$.*

By a semistable quotient of a complex space Z (see [10] for more details) endowed with a holomorphic action of G we mean a complex space $Z//G$ together with a G-invariant surjective map $\pi : Z \to Z//G$ such that:
(i) The structure sheaf $\mathcal{O}_{Z//G}$ is given by $(\pi_* \mathcal{O}_Z)^G$, i.e., the holomorphic functions on an open subset of $Z//G$ are exactly the invariant holomorphic functions on its inverse image in Z.
(ii) The map $\pi : Z \to Z//G$ is a Stein map, i.e., the inverse image of a Stein subspace of $Z//G$ is a Stein subspace of Z.
In [9] it is shown that each point $q \in X(\mu)//G$ has an open neighborhood Q such that $\omega = 2i\partial\bar{\partial}\rho$ on $\pi^{-1}(Q)$ for some K-invariant smooth function ρ. Furthermore, the moment map μ restricted to $\pi^{-1}(Q)$ is given by ρ, i.e., $\mu = \mu^\rho$. A result of Azad and Loeb (see [2]) asserts that, if $x \in \mu^{-1}(0)$, then ρ is an exhaustion on $G \cdot x$ i.e., is bounded from below and proper. In particular $G \cdot x$ is closed in $X(\mu)$ for every $x \in \mu^{-1}(0)$. The converse is also true in the following sense. If $G \cdot x$ is closed in $X(\mu)$, then $\mu(g \cdot x) = 0$ for some $g \in G$. Furthermore in [8] it is shown that the restriction of ρ to each fiber over Q is an exhaustion, i.e., is bounded from below and proper. This Exhaustion Lemma and also a refinement of it (see Sec. 6) will be used several times in the remainder of this paper. For example, it implies the following (see [8]).

Theorem 2. *Let X be a compact complex manifold with a holomorphic G-action and let $\mu : X \to \mathfrak{k}^*$ be a moment map with respect to a K-invariant Kähler form ω. Let $\tilde{\omega}$ be a K-invariant Kähler form on X which lies in the cohomology class of ω. Then there exists a moment map $\tilde{\mu} : X \to \mathfrak{k}^*$ with respect to $\tilde{\omega}$ such that*

$$X(\mu) = X(\tilde{\mu}).$$

Proof. We recall the argument given in [8]. Since $\tilde{\omega}$ is cohomologous to ω and X is a compact Kähler manifold, there exists a differentiable K-invariant function $f : X \to \mathbb{R}$ so that $\tilde{\omega} = \omega + 2i\partial\bar{\partial} f$. Define $\mu^f : X \to \mathfrak{k}^*$ by $\mu^f_\xi = J\xi_X(f)$ for $\xi \in \operatorname{Lie} K$

and set $\tilde{\mu} = \mu + \mu^f$. Then $\tilde{\mu}$ is a moment map with respect to $\tilde{\omega}$. For every $x \in X(\mu)$ there exists a strictly plurisubharmonic K-invariant function $\rho : Z \to \mathbb{R}$, where $Z := \overline{G \cdot x} \cap X(\mu)$, so that $\mu | Z = \mu^\rho$, where μ^ρ is the moment map associated to ρ (see [9]). Since $Z \cap \mu^{-1}(0) \neq \emptyset$, the above mentioned Exhaustion Lemma implies that $\rho : Z \to \mathbb{R}$ is an exhaustion. Now f attains its minimum and maximum on X and ρ is an exhaustion. Hence the strictly plurisubharmonic K-invariant function $\tilde{\rho} := \rho + f$ is also an exhaustion on Z. This shows that $Z \subset X(\tilde{\mu})$, i.e., $X(\mu) \subset X(\tilde{\mu})$. By symmetry we have $X(\mu) = X(\tilde{\mu})$. □

If G is a connected semisimple Lie group, then a moment map with respect to a K-invariant Kähler form ω always exists and is unique. Thus in this case Theorem 2 shows that $X(\mu)$ depends only on the cohomology class of ω.

3. Moment maps associated to representations

Let V be a G-representation whose restriction to the maximal compact subgroup K of G is unitary with Hermitian inner product $\langle\ ,\ \rangle$. Then $\rho : V \to \mathbb{R}$, $\rho(z) = (1/2)\|z\|^2 = (1/2)\langle z, z\rangle$, is a K-invariant strictly plurisubharmonic exhaustion function on V and consequently $V = V(\mu)$ where the moment map $\mu : V \to \mathfrak{k}^*$ is given by $\mu_\xi(z) = d\rho(J\xi z) = (1/2)(\langle J\xi z, z\rangle + \langle z, J\xi z\rangle) = (1/i)\langle \xi z, z\rangle$. The Kähler form $\omega_V = 2i\partial\bar{\partial}\rho$ is given by $\omega_V(v, w) = -\operatorname{Im}\langle v, w\rangle$. Since in this case the restriction of ρ to every π-fibre is an exhaustion, we have $V(\mu) = V$ and the inclusion $\mu^{-1}(0) \hookrightarrow V$ induces a homeomorphism $\mu^{-1}(0) \cong V/\!/G$ (see Sec. 2 Theorem 1). The essential part of this statement has already been proved in [12].

Let $S := S(V) := \{z \in V; \|z\| = 1\}$ denote the unit sphere in V. Note that S is a co-isotropic submanifold of V with respect to ω_V, i.e., $(T_z S)^{\perp_{\omega_V}} = T_z(S^1 \cdot z) \subset T_z S$ where the circle group $S^1 = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$ acts on V by multiplication. This is easily seen by using the orthogonal decomposition $T_z V = T_z(\mathbb{C}^* z) \oplus W$ where $W := T_z S \cap i T_z S$ denotes the complex tangent space of S at z. The complex structure on W induces the standard complex structure on $\mathbb{P}(V) = S(V)/S^1$. Moreover since S is co-isotropic, there is a unique symplectic structure $\omega_{\mathbb{P}(V)}$ on $\mathbb{P}(V)$ such that $\iota_S^* p^* \omega_{\mathbb{P}(V)} = \iota_S^* \omega_V$. Here $p : (V \setminus \{0\}) \to (V \setminus \{0\})/\mathbb{C}^* = \mathbb{P}(V)$ denotes the quotient map and $\iota_S : S \hookrightarrow V$ is the inclusion. Furthermore, the definition of the complex structure and of $\omega_{\mathbb{P}(V)}$ are compatible so that $\omega_{\mathbb{P}(V)}$ is in fact a Kähler form on $\mathbb{P}(V)$. Up to a positive constant it is the unique Kähler form on $\mathbb{P}(V)$ which is invariant with respect to the unitary group $\mathrm{U}(V)$. Note that $\omega_{\mathbb{P}(V)}$ is determined by

$$p^*\omega_{\mathbb{P}(V)} = 2i\partial\bar{\partial}\log\rho = 2i\left(-\frac{1}{\rho^2}\partial\rho \wedge \bar{\partial}\rho + \frac{1}{\rho}\partial\bar{\partial}\rho\right).$$

The induced K-action on $\mathbb{P}(V)$ is again Hamiltonian. The moment map is given by $(\mu_{\mathbb{P}(V)})_\xi([z]) = (2/i)(\langle \xi z, z\rangle/\|z\|^2) = d\log\rho(z)(J\xi z)$. In particular we have $\overline{G \cdot [z]} \cap \mu_{\mathbb{P}(V)}^{-1}(0) \neq \emptyset$ if and only if $\overline{G \cdot z} \cap \mu^{-1}(0) \neq \emptyset$ and this is the case if and only if

$f(z) \neq 0$ for some G-invariant homogeneous polynomial f on V.

Now let X be a G-stable subvariety of $\mathbb{P}(V)$. The pull back of $\omega_{\mathbb{P}(V)}$ to X induces a Kählerian structure ω on X and the K-action is Hamiltonian with moment map $\mu : X \to \mathfrak{k}^*$, $\mu = \mu_{\mathbb{P}(V)}|X$. We call μ the standard moment map induced by the embedding into $\mathbb{P}(V)$. The above construction shows the following well known

Lemma. *Let L be a very ample G-line bundle over X and consider X as a G-stable subvariety on $\mathbb{P}(V)$ where the embedding is given by $\Gamma(X, L)$ and $V = \Gamma(X, L)^*$. Then*

$$X(\mu) = X(L),$$

i.e., the semistable points with respect to the standard moment map on $\mathbb{P}(V)$ are the semistable points with respect to L.

4. The main result

Let G be a connected complex reductive group and K a maximal compact subgroup of G, i.e., $G = K^{\mathbb{C}}$. By a G-variety we mean in the following an algebraic variety together with an algebraic action of G.

Let X be a smooth projective G-variety and ω a K-invariant Kähler form on X. Assume that the K-action is Hamiltonian with respect to ω, i.e., there is a K-equivariant moment map $\mu : X \to \mathfrak{k}^*$, and denote by $X(\mu) := \{x \in X; \overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}$ the set of semistable points with respect to μ.

Semistability Theorem. There is a very ample G-line bundle L over X such that

$$X(\mu) = X(L).$$

Here $X(L)$ denotes the set of semistable points in X in the sense of Mumford, i.e., $X(L) = \{x \in X; s(x) \neq 0$ for some G-invariant holomorphic section s of L^m, $m \in \mathbb{N}\}$.

The case where ω is assumed to be integral is well known and follows rather directly from the definitions using standard Kempf-Ness type arguments. In fact it is a consequence of Theorem 2 of Sec. 2 and the Lemma in Sec. 3.

The proof in the general case is divided into two steps. In the first part we consider forms ω whose cohomology class $[\omega]$ is contained in the $\mathbb{R}$-linear span of the ample cone in $H^{1,1}(X)$. The second part of the proof is more involved. It is a reduction procedure to the first case.

At least implicitly (see e.g. [4], [13], [17]) the ample cone case seems to be known. In order to be complete we include a proof in the next paragraph.

5. The ample cone case

In this section G is a connected complex reductive group with a fixed maximal compact subgroup K and X is a smooth projective G-variety. Let ω be a K-invariant Kähler form and let $\mu : X \to \mathfrak{k}^*$ be a K-equivariant moment map. In this section we prove the following

Proposition. *Assume that the cohomology class of ω lies in the real linear span of the ample cone in $H^{1,1}(X)$. Then there exists a very ample G-line bundle L over X such that*

$$X(\mu) = X(L).$$

Proof. Since $X(\mu)$ essentially depends only on the cohomology class of ω (see Sec. 2 Theorem 2), we may assume that there are equivariant holomorphic embeddings $\iota_k : X \to \mathbb{P}(V_k)$, $k = 1, \ldots, m$, so that

$$\omega = \sum a_k \iota_k^* \omega_{\mathbb{P}(V_k)}$$

where a_k are positive real numbers.

Let $\iota : X \to \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ be the diagonal embedding. Then

$$\omega = \iota^* \left(\sum a_k \pi_k^* \omega_{\mathbb{P}(V_k)} \right),$$

where $\pi_k : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) \to \mathbb{P}(V_k)$ denotes the projection. Hence the moment map μ is the restriction of a moment map on $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ with respect to $\sum a_k \pi_k^* \omega_{\mathbb{P}(V_k)}$ which also will be denoted by μ. Since X is closed in $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$, we have

$$X(\mu) = (\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m))(\mu) \cap X.$$

Thus for the proof of the proposition we may assume that $X = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$, $\omega = \sum a_k \pi_k^* \omega_{\mathbb{P}(V_k)}$ and the G-action is given by a representation $G \to \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_m)$.

Let T be a maximal compact torus in K. Then $T^{\mathbb{C}}$ is a maximal algebraic torus in G. We now reduce the proof of the proposition to the case where $G = T^{\mathbb{C}}$ as follows.

Let $\mu_T : X \to \mathfrak{t}^*$ be the moment map for the T-action which is induced by μ and the embedding $\mathfrak{t} \hookrightarrow \mathfrak{k}$. Then it follows that

$$X(\mu) = \bigcap_{k \in K} k \cdot X(\mu_T)$$

by the Hilbert Lemma version in [13] (Sec. 8.8.). Thus it is sufficient to show the following

CLAIM. There exists a very ample G-line bundle L over $X = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ such that

$$X(\mu_T) = X(L_T)$$

where $X(L_T)$ denotes the set of semistable points with respect to L if one considers L as a $T^{\mathbb{C}}$-bundle.

The proposition follows from the above claim, since

$$X(\mu) = \bigcap_{k \in K} k \cdot X(\mu_T) = \bigcap_{k \in K} k \cdot X(L_T) = X(L).$$

In order to prove the claim one may proceed as follows.

Let $S = S_1 \times \cdots \times S_m$ be the maximal torus in $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_m)$ which contains the image of T and $\mu_k : \mathbb{P}(V_k) \to \mathfrak{s}_k^*$ the standard moment map on $\mathbb{P}(V_k)$. We will consider μ_k as a moment map with respect to $S = S_1 \times \cdots \times S_m$ where the factors of S different from S_k act trivially on $X = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$. Since $\omega = \sum a_k \omega_{\mathbb{P}(V_k)}$, the moment map $\mu : X \to \mathfrak{t}^*$ is given by

$$\mu = a_1 \mu_1 + \cdots + a_m \mu_m + c$$

where $c \in \mathfrak{t}^*$ and μ_k now denotes the map from X to $\mathfrak{t}^*$ which is given by $\mu_k : X \to \mathfrak{s}^*$ composed with the dual of $\mathfrak{t} \to \mathfrak{s}$. Now if $\tilde{a}_k$ are positive rational numbers and $\tilde{c}$ is rational, then $\tilde{\mu} := \tilde{a}_1 \mu_1 + \cdots + \tilde{a}_m \mu_k + \tilde{c}$ is a moment map with respect to $\tilde{\omega} := \sum \tilde{a}_k \pi_k^* \omega_{\mathbb{P}(V_k)}$. Since $\tilde{a}_k$ and $\tilde{c}$ are rational, it follows that there is a very ample G-line bundle L over X such that $X(L) = X(\tilde{\mu})$. Thus we have to show the following

There exists $\tilde{a}_k$ and $\tilde{c}$ such that $X(\mu) = X(\tilde{\mu})$.

This statement follows from convexity properties of μ as follows. Since T is compact, the set X^T of T-fixed points in X is smooth. Let $X^T = \cup_{j \in J} F_j$ be the decomposition into connected components. Note that μ is constant on every F_j, $j \in J$. For the set J let $\mathcal{P}(J)$ be the set of subsets of J. We say that $\mathcal{L} \in \mathcal{P}(J)$ is μ-semistable if $0 \in \mathrm{Conv}\{\mu(F_j); j \in \mathcal{L}\}$ where Conv denotes the convex hull operation in $\mathfrak{t}^*$. Let $X_{\mathcal{L}} := \{x \in X; \overline{T^{\mathbb{C}} \cdot x} \cap F_j \neq \emptyset \text{ for all } j \in \mathcal{L}\}$. Since $\mu(\overline{T^{\mathbb{C}} \cdot x}) = \mathrm{Conv}\{\mu(F_j); \overline{T^{\mathbb{C}} \cdot x} \cap F_j \neq \emptyset\}$ (see [1]), it follows that

$$X(\mu) = \bigcup X_{\mathcal{L}}.$$

Here the union is taken over the elements $\mathcal{L}$ of $\mathcal{P}(J)$ which are μ-semistable. For a given μ denote by $I(\mu)$ the set of μ-semistable subsets of J. We show now that if a collection of subsets is of the form $I(\mu)$, then $I(\mu) = I(\tilde{\mu})$ for some positive rational $\tilde{a}_k$ and rational $\tilde{c}$.

In order to see this, let $\Lambda_{kj} := \mu_k(F_j) \in \mathfrak{t}^*$. Note that Λ_{kj} are integral points in $\mathfrak{t}^*$. A subset $I \subset \mathcal{P}(J)$ is of the form $I(\mu)$ if and only if there exist positive real numbers

a_k and $c \in \mathfrak{t}^*$ such that for all $\mathcal{L} \in \mathcal{P}(J)$ the following holds.

$$0 \in \operatorname{Conv}\left\{\sum_k a_k \Lambda_{kj} + c;\, j \in \mathcal{L}\right\} \text{ if and only if } \mathcal{L} \in I.$$

This condition is equivalent to a collection of linear inequalities with integral coefficients in the unknowns a_k's and c which have a real solution if and only if they have a rational one. □

6. Cohomologous Kähler forms on orbits

In this section let G be a connected complex reductive group with maximal compact subgroup K and let $X = G \cdot x_0$ be a G-homogeneous manifold. We assume that there are given K-invariant Kähler forms ω^j, $j = 0, 1$, on X which are cohomologous and set

$$\omega^t = (1-t)\omega^0 + t\omega^1, \quad t \in [0,1].$$

Moreover, assume that there are K-equivariant moment maps

$$\mu^t : X \to \mathfrak{k}^*, \quad t \in [0,1]$$

with respect to ω^t such that the dependence on t is continuous.

REMARK. We have $\mu^t = (1-t)\mu^0 + t\mu^1 + c^t$ where $c^t \in \mathfrak{z}^*$. Here $\mathfrak{z}$ is the Lie algebra of the center of K. The goal of this section is to obtain some control about the semistable set $M_K^t := (\mu^t)^{-1}(0)$ if t varies.

Lemma. *If $M_K^{t_0} \neq \emptyset$ for some $t_0 \in [0,1]$, then $\omega^t = 2i\partial\bar{\partial}\rho^t$ where $\rho^t = (1-t)\rho^0 + t\rho^1$ and $\rho^j : X \to \mathbb{R}$, $j = 0, 1$, are K-invariant smooth functions.*

Proof. Since $M_K^{t_0} \neq \emptyset$, the orbit $X = G \cdot x_0$ is a Stein manifold (see e.g. [7] or [9]). Now ω^0 and ω^1 are assumed to be cohomologous. Thus there is a K-invariant smooth function $u : X \to \mathbb{R}$ such that $\omega^1 - \omega^0 = 2i\partial\bar{\partial}u$. On the other hand $\omega^{t_0} = 2i\partial\bar{\partial}f$ for some K-invariant smooth function $f : X \to \mathbb{R}$ (see Sec. 2 and [9]). Thus $\omega^t = 2i\partial\bar{\partial}\rho^t$ where $\rho^0 := f - t_0 u$ and $\rho^1 := f + (1-t_0)u$. □

Now let Z denote the connected component of the identity of the center of K and let S be a semisimple factor of K. Thus $K = S \cdot Z$ and $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{z}$ on the level of Lie algebras. Let μ_S^t (resp. μ_Z^t) be the moment map with respect to the S-action (resp. Z-action), i.e., the composition of μ^t with the dual of the inclusion $\mathfrak{s} \hookrightarrow \mathfrak{k}$ (resp. $\mathfrak{z} \hookrightarrow \mathfrak{k}$). We also set $M_K^t := (\mu^t)^{-1}(0)$, $M_S^t := (\mu_S^t)^{-1}(0)$ and $M_Z^t = (\mu_Z^t)^{-1}(0)$.

Proposition 3. *If $M_K^{t_0} \neq \emptyset$ and if the set $X(\mu_Z^t)$ of $Z^{\mathbb{C}}$-semistable points is independent of t, then there are pluriharmonic K-invariant functions $h^t : X \to \mathbb{R}$ which depend continuously on t such that*

$$\mu^t = \mu^{\rho^t + h^t}.$$

Proof. It follows from the definition of a moment map that it is unique up to a constant. Thus $\mu^{\rho^t} = \mu^t + c^t$ where c^t is a K-invariant constant, i.e., $c^t \in \mathfrak{z}^*$.

The proof of the Proposition will be reduced to the case of a compact Abelian group $T \cong (S^1)^k$. In this situation we have $T^{\mathbb{C}} \cong (\mathbb{C}^*)^k$ and $\mathfrak{t} = \operatorname{Lie} T \cong \mathbb{R}^k$. Moreover, for any $c^t \in \mathfrak{t}^*$, $c^t = (c_1^t, \ldots, c_k^t)$, the function $\tilde{h}^t(z_1, \ldots, z_k) = c_1^t \log|z_1| + \cdots + c_k^t \log|z_k|$ is pluriharmonic on $T^{\mathbb{C}}$ and satisfies $\mu^{\tilde{h}^t} = c^t$.

Let $x_0 \in M_K^{t_0}$ and set $L := K_{x_0}$. Then we have the following orthogonal decomposition of the Lie algebra $\mathfrak{k}$.

$$\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{z}_L \oplus \mathfrak{s}$$

where $\mathfrak{z}_L := \mathfrak{z} \cap (\mathfrak{s} + \mathfrak{l})$, $\mathfrak{z} = \mathfrak{t} \oplus \mathfrak{z}_L$ and $\mathfrak{s} + \mathfrak{l} = \mathfrak{s} \oplus \mathfrak{z}_L$.

Note that $\mathfrak{z}$ is the Lie algebra of the group K/S and $\mathfrak{s} + \mathfrak{l}$ is the Lie algebra of the subgroup $S \cdot L$ of K. Since K is connected $K/SL = (K/S)/(SL/S) =: T$ is a compact connected Abelian group. Hence we have $T \cong (S^1)^k$ and $\operatorname{Lie} T \cong (\mathfrak{k}/\mathfrak{s})/((\mathfrak{s} + \mathfrak{l})/\mathfrak{s}) \cong \mathfrak{z}/\mathfrak{z}_L = \mathfrak{t}$. Now identify $\mathfrak{k} \cong \mathfrak{k}^*$, i.e., we have the orthogonal splitting

$$\mathfrak{k}^* = \mathfrak{t}^* \oplus \mathfrak{z}_L^* \oplus \mathfrak{s}^*.$$

CLAIM. $c^t \in \mathfrak{t}^*$.

For the proof let $x_0 \in M_K^{t_0}$ be given and note that $Z^{\mathbb{C}} \cdot x_0$ is closed in $X(\mu_Z^{t_0}) = X(\mu_Z^t)$. Thus there are $x_t \in Z^{\mathbb{C}} \cdot x_0$ such that $\mu_Z^t(x_t) = 0$. In particular we have $c^t = \mu^{\rho^t}(x_t)$. Now let $\xi = \tau + \lambda + \sigma$, where $\tau \in \mathfrak{t}$, $\lambda \in \mathfrak{z}_L$ and $\sigma \in \mathfrak{s}$. Then, since the moment map is unique for a semisimple Lie group, it follows that

$$0 = \mu_\sigma^t(x_t) = \mu_\sigma^{\rho^t}(x_t).$$

For $\lambda \in \mathfrak{z}_L$ we have $\lambda = \lambda_S + \lambda_L$ for some $\lambda_S \in \mathfrak{s}$ and $\lambda_L \in \mathfrak{l}$ and $[\lambda, \lambda_L] = 0$. Thus, using the fact that x_t is an $L^{\mathbb{C}}$-fixed point, we have

$$\exp is\lambda \cdot x_t = \exp is\lambda_S \cdot \exp is\lambda_L \cdot x_t = \exp is\lambda_S \cdot x_t.$$

This implies

$$\begin{aligned} 0 &= \mu_{\lambda_S}^t(x_t) \\ &= \mu_{\lambda_S}^{\rho^t}(x_t) \end{aligned}$$

$$= \left(\frac{d}{ds}\right)_{s=0} \rho^t(\exp is\lambda_S \cdot x_t)$$
$$= \left(\frac{d}{ds}\right)_{s=0} \rho^t(\exp is\lambda \cdot x_t)$$
$$= \mu_\lambda^{\rho^t}(x_t).$$

Since

$$\mu_\xi^{\rho^t}(x_t) = \mu_\tau^{\rho^t}(x_t) + \mu_\lambda^{\rho^t}(x_t) + \mu_\sigma^{\rho^t}(x_t) = \mu_\tau^{\rho^t}(x_t),$$

this implies the claim.

Now, as we already observed, on $T^{\mathbb{C}}$ there exists a pluriharmonic function $\tilde{h}^t : T^{\mathbb{C}} \to \mathbb{R}$ such that $\mu^{\tilde{h}^t} = c^t := \mu^{\rho^t}(x_t) \in \mathfrak{t}^*$. Since $T^{\mathbb{C}} = (K^{\mathbb{C}}/S^{\mathbb{C}})/(S^{\mathbb{C}}L^{\mathbb{C}}/S^{\mathbb{C}}) = K^{\mathbb{C}}/S^{\mathbb{C}}L^{\mathbb{C}}$, the natural map $q : K^{\mathbb{C}}/L^{\mathbb{C}} \to T^{\mathbb{C}}$ is $K^{\mathbb{C}}$-equivariant. Thus $h^t := \tilde{h}^t \circ q$ is a K-invariant pluriharmonic function on $X = K^{\mathbb{C}}/L^{\mathbb{C}}$ such that $\mu^{h^t}(x_t) = c^t$. Therefore $\rho^t - h^t$ is a smooth K-invariant function such that $\omega^t = 2i\partial\bar{\partial}(\rho^t - h^t)$ and, since $\mu^t(x_t) = \mu^{\rho^t - h^t}(x_t) = 0$ and X is connected, $\mu^t = \mu^{\rho^t - h^t}$. □

7. Action of a torus

Let $T \cong (S^1)^m$ be a torus and X a complex projective manifold with an algebraic action of the complexified torus $T^{\mathbb{C}} \cong (\mathbb{C}^*)^m$. Let ω^j be T-invariant Kähler forms on X with moment maps $\mu^j : X \to \mathfrak{t}$.

We say that ω^0 and ω^1 are cohomologous on the closure Y of a $T^{\mathbb{C}}$-orbit in X if there is a $T^{\mathbb{C}}$-equivariant projective desingularization $p : \tilde{Y} \to Y$ such that the pull back of the forms to $\tilde{Y}$ are cohomologous, i.e., such that $p^*\omega^1 - p^*\omega^0 = 2i\partial\bar{\partial}f$ for some T-invariant smooth function $f : X \to \mathbb{R}$.

For $t \in [0, 1]$ we set $\omega^t := (1 - t)\omega^0 + t\omega^1$ and $\mu^t := (1 - t)\mu^0 + t\mu^1$. Note that μ^t is a moment map with respect to ω^t and that ω^t and ω^0 are cohomologous on the closure of every $T^{\mathbb{C}}$-orbit in X if this is the case for ω^0 and ω^1.

Proposition. *If ω^0 and ω^1 are cohomologous on the closure of every $T^{\mathbb{C}}$-orbit in X, then there is a constant $c^t \in \mathfrak{t}^*$ depending continuously on t such that*

$$X(\mu^0) = X(\mu^t + c^t).$$

For the proof of the Proposition we consider first the case where $T \cong S^1$, i.e., we fix a one dimensional subtorus $S^1 = \{\exp z\xi; z \in \mathbb{R}\}$ where ξ is chosen to be a generator of the kernel of the one-parameter group $z \to \exp z\xi$. With respect to this S^1-action let $X^{S^1} = \cup F_\alpha$ be the decomposition of the set of S^1-fixed points of X into connected components. The set of these components is endowed with a partial order relation which is generated by $F_\alpha < F_\beta$. Here we set $F_\alpha < F_\beta$ if and only there is a point $x \in X$ such that $\lim_{z\to 0} z \cdot x \in F_\alpha$ and $\lim_{z\to\infty} z \cdot x \in F_\beta$ where $z \in \mathbb{C}^* = (S^1)^{\mathbb{C}}$.

Let $\mu^t_\xi : X \to \mathbb{R}$ where $\mu^t_\xi = \langle \mu^t, \xi \rangle$ denote the moment map with respect to the given S^1-action. Since $d\mu^t_\xi = \imath_{\xi_X}\omega^t$, the moment map μ^t_ξ is constant on every F_α.

Lemma. *If $F_\alpha < F_\beta$, then $\mu^0_\xi(F_\alpha) - \mu^0_\xi(F_\beta) = \mu^t_\xi(F_\alpha) - \mu^t_\xi(F_\beta)$.*

Proof. Let $x_0 \in X$ be such that $\lim_{z\to 0} z \cdot x_0 \in F_\alpha$ and $\lim_{z\to\infty} z \cdot x_0 \in F_\beta$. We may assume that the map $\mathbb{C}^* \to \mathbb{C}^* \cdot x_0$, $z \to z \cdot x_0$ is an isomorphism and extends to a holomorphic map $b : \mathbb{P}_1(\mathbb{C}) \to X$ with $b(0) = x_\alpha$ and $b(\infty) = x_\beta$.

Now since by assumption the pull back of $\eta := \omega^t - \omega^0$ to the desingularization $\mathbb{P}_1(\mathbb{C})$ of $\overline{\mathbb{C}^* \cdot x_0}$ is cohomologous to zero we have

$$\begin{aligned} 0 &= \int_{\overline{\mathbb{C}^*\cdot x_0}} \eta \\ &= \int_{\mathbb{C}^*\cdot x_0} \eta \\ &= \int_{\mathbb{R}^+\cdot x_0} \imath_{\xi_X}\eta \\ &= \int_{\mathbb{R}^+\cdot x_0} d(\mu^t_\xi - \mu^0_\xi) \\ &= \mu^t_\xi(x_\beta) - \mu^0_\xi(x_\beta) - (\mu^t_\xi(x_\alpha) - \mu^0_\xi(x_\alpha)). \end{aligned}$$

Here $\mathbb{R}^+ \cdot x_0$ denotes the $\mathbb{R}^+ := \{z \in \mathbb{R}; z > 0\}$-orbit through x_0. □

REMARK. Implicitly we used that under the above assumption ω^0 and ω^1 are cohomologous on the normalization of $\overline{\mathbb{C}^* \cdot x_0}$.

Proof of the Proposition. The above Lemma implies that there is a constant $c^t \in \mathfrak{t}^*$ depending continuously on t such that μ^0 and $\tilde{\mu}^t := \mu^t + c^t$ assume the same values on every component of the set X^T of T-fixed points in X. Since $\tilde{\mu}^t(\overline{T^{\mathbb{C}} \cdot x})$ is the convex hull of the images of $\tilde{\mu}^t(F_\alpha)$ where $F_\alpha \cap \overline{T^{\mathbb{C}} \cdot x} \neq \emptyset$ (see [1]) it follows that $X(\mu^0) = \{x \in X; 0 \in \mu^0(\overline{T^{\mathbb{C}} \cdot x})\} = \{x \in X; 0 \in \tilde{\mu}^t(\overline{T^{\mathbb{C}} \cdot x})\} = X(\tilde{\mu}^t)$. □

8. Action of a semisimple group

Let G be a connected complex semisimple Lie group with maximal compact subgroup K and X a projective manifold with an algebraic G-action. As in the last section we say that two given closed forms ω^0 and ω^1 are cohomologous on the closure Y of a G-orbit in X if there is a G-equivariant desingularization $p : \tilde{Y} \to Y$ such that $p^*\omega^0 - p^*\omega^1 = 2i\partial\bar{\partial} f$ for some smooth function $f : \tilde{Y} \to \mathbb{R}$.

Proposition. *Let $\omega^j : X \to \mathfrak{k}^*$, $j = 0, 1$, be two K-invariant Kähler forms on X which are cohomologous on every G-orbit closure and let μ^j be the unique moment map with respect to ω^j. Then*

$$X(\mu^0) = X(\mu^1).$$

Proof. For $x \in (\mu^0)^{-1}(0)$ set $Y := \overline{G \cdot x}$ and let $p : \tilde{Y} \to Y$ an equivariant resolution of singularities such that $p^*\omega^1 - p^*\omega^0 = 2i\partial\bar{\partial} f$ for a smooth K-invariant function f. In particular f is bounded on $G \cdot x$.

Since $G \cdot x$ is closed in $X(\mu^0)$ it follows from the Exhaustion Lemma that $\mu^0|G \cdot x = \mu^\rho$ for some K-invariant plurisubharmonic exhaustion function $\rho : G \cdot x \to \mathbb{R}$.

Therefore $\rho + f$ is likewise an exhaustion and in particular has a minimum on $G \cdot x$. Since μ^1 is unique, we have $\mu^1 = \mu^0 + \mu^f = \mu^{\rho+f}$. Thus $X(\mu^0) \subset X(\mu^1)$ and the reverse inclusion follows by symmetry. □

9. Reduction to Levi factors

Let G be a connected complex reductive group with maximal compact subgroup K and let X be a compact connected manifold endowed with a holomorphic action of G. We assume that there are given K-invariant Kähler forms ω^j, $j = 0, 1$, on X which are cohomologous on any G-orbit and set

$$\omega^t = (1 - t)\omega^0 + t\omega^1, \quad t \in [0, 1].$$

Moreover, assume that there are K-equivariant moment maps

$$\mu^t : X \to \mathfrak{k}^*, \quad t \in [0, 1]$$

with respect to ω^t which depend continuously on t. We set $M_K^t := (\mu^t)^{-1}(0)$.

Let Z be the center of K and S the semisimple part of K, i.e., $K = Z \cdot S$ where $Z \cap S$ is a finite group and assume the following condition:

(∗) $\quad X(\mu_Z^t)$ is independent of $t \in [0, 1]$.

Lemma 4. *Assume the condition* (∗) *and for $x_0 \in X$ let $\Omega := G \cdot x_0$. Then, for $t \in [0, 1]$,*

$$M_K^t \cap \Omega \neq \emptyset$$

is an open condition.

Proof. Let $t_0 \in [0, 1]$ be such that $M_K^{t_0} \cap \Omega \neq \emptyset$. It follows that there exists a smooth curve ρ^t of K-invariant smooth functions so that $\omega^t = 2i\partial\bar{\partial}\rho^t$ and $\mu^t = \mu^{\rho^t}$ on

Ω. Furthermore, since $M_K^{t_0} \cap \Omega \neq \emptyset$, it follows that ρ^{t_0} is an exhaustion of Ω. For t near t_0 the function ρ^t has the same convexity properties as ρ^{t_0} and is therefore likewise an exhaustion (see [8], proof of Lemma 2 in Sec. 2). The points where it has its minimum are those in $M_K^t \cap \Omega$. □

Lemma 5. *Assume the condition* $(*)$ *and for* $x_0 \in X$ *let* $\Omega := G \cdot x_0$. *Then, for* $t \in [0, 1]$,

$$M_K^t \cap \Omega \neq \emptyset$$

is a closed condition.

Proof. We have to show that $M_K^t \cap \Omega \neq \emptyset$ for $t < t_0$ implies $M_K^{t_0} \cap \Omega \neq \emptyset$.

Since μ_K^t depends continuously on t and X is compact, it follows that $M_K^{t_0} \cap \overline{G \cdot x_0} \neq \emptyset$. Let $y_0 \in M_K^{t_0} \cap \overline{G \cdot x_0}$. If $G \cdot y_0 \neq \Omega$, then by Lemma 1 for t near t_0 we have that $M_K^t \cap \overline{G \cdot x_0} \subset G \cdot y_0$. However the intersection of M_k^t with $\overline{G \cdot x_0}$ consist of precisely one K-orbit, which would be contrary to $M_K^t \cap \Omega$ also being non-empty. □

Proposition. *Assume that condition* $(*)$ *is fulfilled. Then* $X(\mu_K^t)$ *does not depend on* $t \in [0, 1]$.

Proof. Let $x \in M_K^{t_0}$ and $\Omega := G \cdot x$. From the above two Lemma it follows that $M_K^t \cap \Omega \neq \emptyset$ for all t. Thus, the condition that Ω is a closed G-orbit in $X(\mu_K^t)$ is satisfied for some t if and only if this is the case for all t. □

10. Proof of the Semistability Theorem

For the proof of the Semistability Theorem we need to associate to a given Kähler form one whose cohomology class lies in the real span of the ample cone. (see [15], §3). Let X be a smooth projective variety and denote by $H \in H^2(X, \mathbb{R})$ the cohomology class of a hyperplane section.

Let $\mathcal{C}_1$ be the subspace of the second homology group $H_2(X, \mathbb{R})$ which is spanned by the images of closed analytic curves and $\mathcal{C}_{n-1}$ the subspace of $H^2(X, \mathbb{R})$ spanned by divisors, or, what is the same, Chern classes of holomorphic line bundles.

The following lemma is well known (see e.g. [15]).

Lemma 1. *The pairing* $\mathcal{C}_1 \times \mathcal{C}_{n-1} \longrightarrow \mathbb{R}$ *which is induced by associating to a line bundle* L *and a curve* C *the intersection number* $L \cdot C := \deg L_C$ *is perfect.*

Lemma 2. *Let ω be a Kähler form on X. Then there exist a Kähler form $\widetilde{\omega}$ whose cohomology class $[\widetilde{\omega}]$ lies in the span of the ample cone such that*

$$\int_C \widetilde{\omega} = \int_C \omega$$

holds for all one-dimensional analytic cycles C.

Proof. Consider the linear map $\lambda : \mathcal{C}_1 \to \mathbb{R}$, $\lambda(C) = \int_C \omega$, given by $\lambda(C) = \int_C \omega$. By Lemma 1 there is a class $\tilde{D}$ in $\mathcal{C}_{n-1}$ such that $\lambda(C) = \tilde{D} \cdot C$ for all 1-cycles C. Since $\tilde{D}$ is a divisor, the cohomology class of $\tilde{D}$ lies in the span of the ample cone. Moreover, it follows that the cohomology class of $\tilde{D}$ contains a Kähler form $\widetilde{\omega}$ ([15] §3, see also [14]). □

We need the following elementary observation.

Lemma 3. *Let Y be a connected smooth projective variety and assume that G has an open orbit on Y. Then there are no non-zero holomorphic p-forms on Y for $p \geq 1$.*

As a consequence we obtain the following

Corollary. *If α is a smooth closed $(1,1)$-form on Y such that $\int_C \alpha = 0$ for every one-dimensional analytic cycle C, then $\alpha = 2i\partial\bar{\partial} f$ for some smooth function $f : Y \to \mathbb{R}$.*

For the proof of the semistability Theorem we also need

Lemma 4. *If the K-action on X is Hamiltonian with respect to the K-invariant Kähler form ω, then it is also Hamiltonian with respect to any other K-invariant Kähler form $\tilde{\omega}$.*

Proof of the Semistability Theorem. Given a smooth K-invariant Kähler form ω on a smooth projective G-variety X we already know from Lemma 2 that there is a Kähler form $\tilde{\omega}$ on X which lies in the $\mathbb{R}$-span of the ample cone of X such that

$$(*) \qquad \int_C \omega = \int_C \tilde{\omega}$$

on every analytic curve C in X. Since K is assumed to be connected the cohomology class of $\tilde{\omega}$ is K-invariant. Hence, after integration over the compact group K, we may assume that $\tilde{\omega}$ is K invariant and still satisfies $(*)$.

Now it follows from Lemma 4 just above, the Proposition in Sec. 7 and the existence of a moment map in the semisimple case that there is a moment map $\mu^t : X \to$

$\mathfrak{k}^*$ with respect to ω^t where $\omega^t = (1-t)\tilde{\omega} + t\omega$ such that the μ^t depends continuously on t and such that $X(\mu^t_Z) = X(\mu^0_Z)$ for all $t \in [0, 1]$. Here Z denotes the connected component of the center of K.

Moreover (∗) implies

$$(**) \qquad \int_C \omega^t = \int_C \omega.$$

Since the closure of every G-orbit in X has an equivariant algebraic desingularization, it follows from the above Corollary that the forms are cohomologous on the closure of every G-orbit. The statement of the theorem now follows from the Proposition in Sec. 9 and the Proposition in Sec. 5. □

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