J. B. Lee and K. B. Lee
Nagoya Math. J.
Vol. 196 (2009), 117–134

AVERAGING FORMULA FOR NIELSEN NUMBERS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R)

JONG BUM LEE AND KYUNG BAI LEE

Abstract. We prove that the averaging formula for Nielsen numbers holds for continuous maps on infra-solvmanifolds of type (R): Let M be such a manifold with holonomy group Ψ and let $f : M \to M$ be a continuous map. The averaging formula for Nielsen numbers

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|$$

is proved. This is a workable formula for the difficult number $N(f)$.

§1. Introduction

Let M be a closed manifold and let $f : M \to M$ be a continuous map. In order to study the fixed points of f, the Lefschetz number $L(f)$ and Nielsen number $N(f)$ are associated to f. These numbers are homotopy invariants. The Nielsen number gives more precise information concerning the existence of fixed points than the Lefschetz number, but its computation when compared with that of the Lefschetz number is in general much more difficult.

Therefore, there have been attempts to find some relations between these two numbers. In [3], Brooks, Brown, Pak and Taylor show that for a continuous map f on a torus, $|L(f)| = N(f)$. Anosov [1] extended this to nilmanifolds. However, such an equality does not hold on infra-nilmanifolds as shown in [1]: there is a continuous map f on the Klein bottle for which $N(f) \neq |L(f)|$. If M is an infra-nilmanifold, and f is homotopically periodic or more generally virtually unipotent, then it is known in [13], [18] that

Received June 2, 2008.
Revised March 9, 2009.
Accepted May 10, 2009.
2000 Mathematics Subject Classification: 55M20, 57S30.
The first author was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MOST) (No. R01-2007-000-10097-0).

$L(f) = N(f)$. Other interesting investigations on the relation $|L(f)| = N(f)$ were made for flat Riemannian manifolds and Anosov diffeomorphisms f in [4] and for infra-nilmanifolds in [5].

The following averaging formula for Lefschetz numbers is well-known [9, Theorem III.2.12]:

$$L(f) = \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} L(\bar{\alpha}\bar{f}).$$

For Nielsen numbers, the averaging formula does not always hold. Instead, the following inequality holds [11, Theorem 3.1]:

$$N(f) \geq \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} N(\bar{\alpha}\bar{f}).$$

It is proved in [11] that if X is an infra-nilmanifold, the above inequality becomes the equality. Furthermore, in [14], we offered algebraic and practical computation formulas for the Nielsen and Lefschetz numbers of a continuous map on an infra-nilmanifold in terms of the holonomy of the manifold. The purpose of this work is to extend these results from infra-nilmanifolds to infra-solvmanifolds of type (R), thereby showing that the deviation from the equality $N(f) = |L(f)|$ can be measured by the holonomy of the manifold.

§2. Infra-solvmanifolds

Let G be a Lie group, let $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G and let $\mathrm{Endo}(G)$ be the group of continuous endomorphisms of G. The affine group of G is the semi-direct product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with the multiplication $(a, A)(b, B) = (aA(b), AB)$. It has a Lie group structure and acts on G by $(a, A) \cdot x = aA(x)$ for all $x \in G$. Suppose that G is connected and has a linear connection defined by left-invariant vector fields. It turns out that $\mathrm{Aff}(G)$ is the group of connection preserving diffeomorphisms of G.

We recall from [6], [8], [12] some definitions about solvable Lie groups and give some basic properties which are necessary for our discussion. A connected solvable Lie group S is called of *type* (NR) (for "no roots") [12] if the eigenvalues of $\mathrm{Ad}(x) : \mathfrak{S} \to \mathfrak{S}$ are always either equal to 1 or else they are not roots of unity. Solvable Lie groups of type (NR) were considered first in [12]. A connected solvable Lie group S is called of *type* (R) (or *completely solvable*) if $\mathrm{ad}(X) : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for each $X \in \mathfrak{S}$.

A connected solvable Lie group S is called of *type* (E) (or *exponential*) if $\exp : \mathfrak{S} \to S$ is surjective. Some important properties of such groups related to our paper are listed below. See [6], [8] for more details.

(1) Abelian $\Longrightarrow$ Nilpotent $\Longrightarrow$ type (R) $\Longrightarrow$ type (E) $\Longrightarrow$ type (NR).

(2) (Rigidity of Lattices) Let S and S' be connected and simply connected solvable Lie groups of type (R), and let Γ be a lattice of S. Then any homomorphism from Γ to S' extends uniquely to a homomorphism of S to S'.

Let S be a connected and simply connected solvable Lie group and H be a closed subgroup of S. The coset space $H\backslash S$ is called a *solvmanifold*. We shall deal with *compact* solvmanifolds only. Let π be the fundamental group of the solvmanifold $M = H\backslash S$. Then $\pi = H/H_0$ where H_0 is the identity component of H. Such a group is known to be a *Mostow-Wang* group or a *strongly torsion-free solvable* group, i.e., π contains a finitely generated torsion-free, nilpotent normal subgroup with torsion-free abelian quotient group of finite rank.

A discrete subgroup Γ of S is a *lattice* of S if $\Gamma\backslash S$ is compact, and in this case, we say that $\Gamma\backslash S$ is a *special* solvmanifold. Let $\pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of Γ. Then π acts freely on S, and the manifold $\pi\backslash S$ is called an *infra-solvmanifold*. The group $\Psi = \pi/\Gamma$ is the holonomy group of π or $\pi\backslash S$. It sits naturally in $\mathrm{Aut}(S)$. Thus *every infra-solvmanifold is finitely covered by a special solvmanifold.* An infra-solvmanifold $M = \pi\backslash S$ is of type (R) if S is of type (R).

First we generalize Lemma 3.1 of [14], in which the existence of a fully invariant subgroup of finite index in an almost Bieberbach group is proved. The proof consists merely of straightforward adaptation of that of [14, Lemma 3.1] to this more general, but very analogous situation.

Lemma 2.1. *Let S, S' be connected and simply connected solvable Lie groups, and let $\pi, \pi' \subset \mathrm{Aff}(S)$ be finite extensions of lattices Γ, Γ' of S, S', respectively. Then there exist fully invariant subgroups $\Lambda \subset \Gamma$, $\Lambda' \subset \Gamma'$ of π, π', respectively, which are of finite index, so that any homomorphism $\theta : \pi \to \pi'$ maps Λ into Λ'.*

Next we state the following, which generalizes [16, Theorem 1.1] from almost crystallographic groups to finite extensions of lattices of a simply connected solvable Lie group of type (R), and [17, Theorem 3.1] from isomorphisms to homomorphisms. The crucial point is our Lemma 2.1 and the rigidity of lattices of simply connected solvable Lie groups of type (R). Then we just follow the argument of [16, Theorem 1.1] or [17, Theorem 3.1].

THEOREM 2.2. *Let S be a connected and simply connected solvable Lie group of type* (R). *Let $\pi, \pi' \subset \mathrm{Aff}(S)$ be finite extensions of lattices of S. Then any homomorphism $\theta : \pi \to \pi'$ is semi-conjugate by an "affine map". That is, for any homomorphism $\theta : \pi \to \pi'$, there exist $d \in S$ and a homomorphism $D : S \to S$ such that $\theta(\alpha) \circ (d, D) = (d, D) \circ \alpha$, or the following diagram is commutative*

$$\begin{array}{ccc} S & \xrightarrow{(d,D)} & S \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \theta(\alpha)} \\ S & \xrightarrow{(d,D)} & S \end{array}$$

for all $\alpha \in \pi$.

Proof. By Lemma 2.1, there exist lattices Λ, Λ' of S so that the homomorphism $\theta : \pi \to \pi'$ restricts to a homomorphism $\theta : \Lambda \to \Lambda'$. Consider the homomorphism $\Lambda \to \Lambda' \hookrightarrow S$, where the first map is the restriction of θ. Since Λ is a lattice of S, by the rigidity of lattices ([20], [21], [23], cf. [8, Theorem 2.2]), any such a homomorphism extends uniquely to a continuous homomorphism $C : S \to S$. Thus $\theta|_\Lambda = C|_\Lambda$; and hence $\theta(z, I) = (Cz, I)$ for all $z \in \Lambda$ (more precisely, $(z, I) \in \Lambda \subset \mathrm{Aff}(S)$). Let us denote the composite homomorphism $\pi \to \pi' \to S \rtimes \mathrm{Aut}(S) \to \mathrm{Aut}(S)$ by $\bar{\theta}$; and define a map $f : \pi \to S$ by

$$\theta(w, K) = (Cw \cdot f(w, K), \bar{\theta}(w, K)).$$

Using exactly the same method employed in the proof of [17, Theorem 3.1], we can show that f and $\bar{\theta}$ factor through $Q = \pi/\Lambda$. We still use the notation f and $\bar{\theta}$ to denote the induced maps $Q \to S$ and $Q \to \mathrm{Aut}(S)$, respectively. Furthermore, we can show that, with the Q-structure on S via $\bar{\theta} : Q \to \mathrm{Aut}(S)$, $f : Q \to S$ is a "principal" crossed homomorphism. In other words, there exists $d \in S$ such that

$$f(w, K) = d \cdot \bar{\theta}(w, K)(d^{-1}).$$

Denoting by $\tau_{d^{-1}}$ the conjugation by d^{-1}, we write $D = \tau_{d^{-1}} \circ C$, and we check that θ is "semi-conjugation" by $(d, D) = (d, \tau_{d^{-1}} \circ C)$. ∎

§3. Linearization of maps on solvmanifolds

In this section we will restate the main result of [12]. This restatement, Theorem 3.1, makes possible to prove our crucial Corollary 3.2 and Lemma 3.3, and their proofs become simple.

For the restatement, first we need to understand the solvmanifolds. In [15] (cf. [18] also), we exploited the fundamental group structure of solvmanifolds and produced, using Seifert fiber space constructions, a fibration structure on the solvmanifold over a torus, with a nilmanifold as a base. Let $M = H\backslash S$ be a solvmanifold. Then there is an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \mathbb{Z}^s \longrightarrow 1,$$

where

- Γ is a fully invariant subgroup of π, and
- Γ is a finitely generated torsion-free nilpotent group.

Let G be the Malćev completion of Γ. Doing the Seifert construction, one obtains a Seifert fibering with typical fiber the nilmanifold $\Gamma\backslash G$. The π action on $G \times \mathbb{R}^s$ is properly discontinuous, and free since π is torsion-free. Hence the Seifert fiber space $\pi\backslash(G \times \mathbb{R}^s)$ is a closed smooth manifold. This manifold has a bundle structure over the torus $\mathbb{Z}^s\backslash\mathbb{R}^s$ with fiber the nilmanifold $\Gamma\backslash G$:

$$\Gamma\backslash G \longrightarrow M' = \pi\backslash(G \times \mathbb{R}^s) \xrightarrow{q} \mathbb{Z}^s\backslash\mathbb{R}^s = T^s.$$

Note also that the given solvmanifold M is homotopic to the Seifert fiber space M'. Let $\alpha : M \to M'$ be a homotopy equivalence with a homotopy inverse β. Let $f : M \to M$. Consider $f' = \alpha \circ f \circ \beta : M' \to M'$. The homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha} \\ M' & \xrightarrow{f'} & M' \end{array}$$

implies that $N(f) = N(f')$. Thus we may assume that M is the total space of the bundle $\Gamma\backslash G \to M \xrightarrow{q} T^s$. That is, $M = \pi\backslash(G \times \mathbb{R}^s)$, and $f : M \to M$.

Consider a lifting $\tilde{f} : G \times \mathbb{R}^s \to G \times \mathbb{R}^s$ of $f : M \to M$. Then it induces a homomorphism $\varphi : \pi \to \pi$ defined by $\varphi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$ for all $\alpha \in \pi$.

Since Γ is a fully invariant subgroup of π, φ induces a homomorphism $\varphi' = \varphi|_\Gamma : \Gamma \to \Gamma$, and in turn induces a homomorphism $\bar{\varphi} : \mathbb{Z}^s \to \mathbb{Z}^s$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \xrightarrow{\psi} & \mathbb{Z}^s & \longrightarrow & 1 \\ & & \Big\downarrow \varphi' & & \Big\downarrow \varphi & & \Big\downarrow \bar{\varphi} & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \xrightarrow{\psi} & \mathbb{Z}^s & \longrightarrow & 1 \end{array}$$

Now $\bar{\varphi} : \mathbb{Z}^s \to \mathbb{Z}^s$ extends uniquely to a homomorphism $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$, which induces a map $\phi_{\bar{F}} : T^s \to T^s$ so that the induced homomorphism on the group of covering transformations, $\mathbb{Z}^s$, is exactly $\bar{\varphi}$. Since $\bar{\varphi}\psi = \psi\varphi$, we have a homotopy $h_t : q \circ f \simeq \phi_{\bar{F}} \circ q$. By the Homotopy Lifting Property of the fibration $M \to T^s$, there exists a lifting homotopy $H_t : M \to M$ of h_t such that $H_0 = f$. Let $f' = H_1$. Then $q \circ f' = \phi_{\bar{F}} \circ q$, and f' is fiber-preserving and homotopic to f. Moreover, $f' : M \to M$ induces the map $\phi_{\bar{F}} : T^s \to T^s$ which is a homomorphism.

We recall the definition of the linearization of the solvmanifold [12]. There exists a finite descending central series of Γ:

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_c \supset \{1\}$$

such that

- each $\Gamma_i := \Gamma \cap \gamma_i(G)$ is a fully invariant subgroup of Γ,
- each $\Lambda_i := \Gamma_{i-1}/\Gamma_i$ is torsion-free abelian,
- Γ acts trivially on each Λ_i, and
- there is a well-defined action of $\mathbb{Z}^s$ on each Λ_i. We denote its action homomorphism by $A_i : \mathbb{Z}^s \to \mathrm{Aut}(\Lambda_i)$.

Then the collection $\{\Lambda_i, A_i\}$ is called the linearization of the solvmanifold M.

Recall that $\varphi' : \Gamma \to \Gamma$ extends uniquely to a homomorphism $F' : G \to G$. Then F' induces homomorphisms $\gamma_i(G) \to \gamma_i(G)$ and then in turn induces homomorphisms

$$\varphi'_i = \varphi'|_\Gamma : \Gamma_i = \Gamma \cap \gamma_i(G) \longrightarrow \Gamma_i.$$

Therefore, there are induced homomorphisms

$$\hat{\varphi}'_i : \Lambda_i \longrightarrow \Lambda_i.$$

Furthermore the above commutative diagram produces the following equality: for each i and $\lambda \in \mathbb{Z}^s$,

$$\hat{\varphi}'_i \circ A_i(\lambda) = A_i(\bar{\varphi}(\lambda)) \circ \hat{\varphi}'_i.$$

We shall say that the *linearization* of the map $f : M \to M$ is the collection of homomorphisms $\{\hat{\varphi}'_i, \bar{\varphi}\}$, or simply, the pair $(F', \bar{F})$ of the homomorphisms $F' : G \to G$ and $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$. Notice that the differential $F'_* : \mathfrak{G} \to \mathfrak{G}$ can be expressed as a matrix of the form

$$\begin{bmatrix} \hat{\varphi}'_1 & * & \cdots & * \\ 0 & \hat{\varphi}'_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{\varphi}'_c \end{bmatrix}$$

by choosing a suitable basis of $\mathfrak{G}$.

Therefore, the main result of [12] can be restated as follows:

THEOREM 3.1. ([12, Theorem 3.1]) *If $f : M \to M$ is a self-map on the solvmanifold of type* (NR) *with linearization $(F', \bar{F})$, then*

$$L(f) = \det(I - \bar{F}_*) \det(I - F'_*), \quad N(f) = |L(f)|.$$

COROLLARY 3.2. *Let $M = \Gamma \backslash S$ be a special solvmanifold of type* (NR)*. If a homomorphism $D : S \to S$ induces a map $\phi_D : M \to M$, then $L(\phi_D) = \det(I - D_*)$ and $N(\phi_D) = |L(\phi_D)|$.*

Proof. It is easy to see that the homomorphism $\varphi : \Gamma \to \Gamma$ induced by the lifting D of ϕ_D is exactly $\varphi = D|_\Gamma$. Thus we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [\Gamma,\Gamma] & \longrightarrow & \Gamma & \longrightarrow & \Gamma/[\Gamma,\Gamma] = \mathbb{Z}^s & \longrightarrow & 1 \\ & & \downarrow D|_{[\Gamma,\Gamma]} & & \downarrow D|_\Gamma & & \downarrow \bar{D}|_{\mathbb{Z}^s} & & \\ 1 & \longrightarrow & [\Gamma,\Gamma] & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^s & \longrightarrow & 1 \end{array}$$

Thus a linearization of ϕ_D is $(D|_{[\Gamma,\Gamma]}, \bar{D}|_{\mathbb{Z}^s})$, and hence

$$L(\phi_D) = \det(I - D|_{[\Gamma,\Gamma]*}) \det(I - \bar{D}|_{\mathbb{Z}^s *}) = \det(I - D_*).$$

This proves our claim. ∎

The following slightly generalizes [14, Lemma 3.2] from simply connected nilpotent Lie groups to simply connected solvable Lie groups.

LEMMA 3.3. *Let S be a connected and simply connected solvable Lie group. For any $g \in S$ and $D \in \mathrm{Endo}(S)$,*

$$\det(\mathrm{Ad}(g) - D_*) = \det(I - D_*).$$

Proof. For $g \in S$, τ_g denotes the conjugation by g so that $\tau_g(x) = gxg^{-1}$. Let $\hat{S} = [S,S]$, $\bar{S} = S/[S,S]$. Consider the exact sequence $1 \to \hat{S} \to S \to \bar{S} \to 1$. Then the endomorphisms D and τ_g on S induce endomorphisms $\hat{D}$ and $\hat{\tau}_g$ on $\hat{S}$ and hence, induce endomorphisms $\bar{D}$ and $\bar{\tau}_g$ on $\bar{S}$. Note that $\bar{\tau}_g = \mathrm{id}_{\bar{S}}$. Thus the left hand side diagram commutes, which implies that the right hand side diagram commutes.

$$\begin{array}{ccccc} \hat{S} & \longrightarrow & S & \longrightarrow & \bar{S} \\ {\scriptstyle \hat{D}}\downarrow{\scriptstyle \hat{\tau}_g} & & {\scriptstyle D}\downarrow{\scriptstyle \tau_g} & & {\scriptstyle \bar{D}}\downarrow{\scriptstyle \mathrm{id}} \\ \hat{S} & \longrightarrow & S & \longrightarrow & \bar{S} \end{array} \qquad \begin{array}{ccccc} \hat{\mathfrak{S}} & \longrightarrow & \mathfrak{S} & \longrightarrow & \bar{\mathfrak{S}} \\ {\scriptstyle \hat{D}_*}\downarrow{\scriptstyle \widehat{\mathrm{Ad}}(g)} & & {\scriptstyle D_*}\downarrow{\scriptstyle \mathrm{Ad}(g)} & & {\scriptstyle \bar{D}_*}\downarrow{\scriptstyle \mathrm{id}} \\ \hat{\mathfrak{S}} & \longrightarrow & \mathfrak{S} & \longrightarrow & \bar{\mathfrak{S}} \end{array}$$

Thus we can find a basis of $\mathfrak{S}$ so that D_* and $\mathrm{Ad}(g)$ are of the form

$$D_* = \begin{bmatrix} \hat{D}_* & * \\ 0 & \bar{D}_* \end{bmatrix}, \quad \mathrm{Ad}(g) = \begin{bmatrix} \widehat{\mathrm{Ad}}(g) & * \\ 0 & I \end{bmatrix}.$$

Now since $[S,S]$ is nilpotent, by [14, Lemma 3.2], we have

$$\det(\widehat{\mathrm{Ad}}(g) - \hat{D}_*) = \det(I - \hat{D}_*).$$

Hence

$$\begin{aligned} \det(\mathrm{Ad}(g) - D_*) &= \det(\widehat{\mathrm{Ad}}(g) - \hat{D}_*) \det(I - \bar{D}_*) \\ &= \det(I - \hat{D}_*) \det(I - \bar{D}_*) \\ &= \det(I - D_*). \end{aligned}$$

This finishes the proof of our claim. ∎

§4. Averaging formula for Nielsen numbers

To state and prove the main result, we shall briefly explain some necessary facts about the mod K Nielsen fixed point theory. Let $f : X \to X$ be a self-map on a compact connected space X, and let K be a normal subgroup of $\pi = \pi_1(X)$ of finite index. Suppose $f_*(K) \subset K$. For the fixed liftings $\tilde{f} : \tilde{X} \to \tilde{X}$ and $\bar{f} : K\backslash\tilde{X} \to K\backslash\tilde{X}$ of f, we have homomorphisms

$$\bar{\varphi} : \pi/K \to \pi/K \quad \text{defined by } \bar{f}\bar{\alpha} = \bar{\varphi}(\bar{\alpha})\bar{f},$$
$$\varphi : \pi \to \pi \quad \text{defined by } \tilde{f}\alpha = \varphi(\alpha)\tilde{f},$$

so that $\varphi' = \varphi|_K : K \to K$ and the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\
1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1
\end{array}$$

Let $p : \tilde{X} \to X$, $p' : \tilde{X} \to K\backslash X$ be covering maps. The fixed point classes of f are the subsets $p(\mathrm{Fix}(\alpha\tilde{f}))$ $(\alpha \in \pi)$ of the fixed point set $\mathrm{Fix}(f)$ of f. For each $\alpha \in \pi$, the fixed point classes of $\bar{\alpha}\bar{f}$ are the subsets $p'(\mathrm{Fix}(k\alpha\tilde{f}))$ $(k \in K)$ of the fixed point set $\mathrm{Fix}(\bar{\alpha}\bar{f})$ of $\bar{\alpha}\bar{f}$.

We denote the subgroup of π fixed by a homomorphism $\psi : \pi \to \pi$ by

$$\mathrm{fix}(\psi) = \{\alpha \in \pi \mid \psi(\alpha) = \alpha\}.$$

Then the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tau_\alpha\varphi'} & & \downarrow{\scriptstyle \tau_\alpha\varphi} & & \downarrow{\scriptstyle \tau_{\bar{\alpha}}\bar{\varphi}} & & \\
1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1
\end{array}$$

is commutative, and the following sequence of groups

$$1 \longrightarrow \mathrm{fix}(\tau_\alpha\varphi') \xrightarrow{i} \mathrm{fix}(\tau_\alpha\varphi) \xrightarrow{q} \mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})$$

is exact. With the above notation, we have

THEOREM 4.1. ([11, Theorem 3.1]) *Let $f : X \to X$ be a self-map on a compact connected space X, and let K be a normal subgroup of the fundamental group π of X with finite index such that $f_*(K) \subset K$. Then*

$$N(f) \geq \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} N(\bar{\alpha}\bar{f}),$$

and equality holds if and only if for each $k \in K$ and $\alpha \in \Pi$ with $p(\mathrm{Fix}(k\alpha\tilde{f}))$ an essential fixed point class, $|q(\mathrm{fix}(\tau_{k\alpha}\varphi))| = 1$.

We will show that the equality in the above theorem holds for infra-solvmanifolds of type (R). Thus we generalize the averaging formula for Nielsen numbers for continuous maps on infra-nilmanifolds [11, Theorem 3.5] to infra-solvmanifolds of type (R). With Theorem 2.2, our proof requires a straightforward adaptation of the proof of [11, Theorem 3.5].

THEOREM 4.2. *Let M be an infra-solvmanifold of type* (R) *and $f : M \to M$ be any self map. Suppose that N is a regular covering of M which is a solvmanifold with fundamental group K. Assume that $f_*(K) \subset K$. Then*

$$N(f) = \frac{1}{[\pi : K]} \sum N(\bar{f}),$$

where the sum ranges over all the liftings $\bar{f}$ of f onto N. In particular, $N(f) \geq |L(f)|$.

Proof. Let $M = \pi\backslash S$ be the infra-solvmanifold. Then $K \subset \pi \cap S$ and $N = K\backslash S$. Now we fix a lifting $\tilde{f} : S \to S$ of $f : M \to M$. Then $\tilde{f}$ induces a homomorphism $\varphi = f_* : \pi \to \pi$ on the group π of covering transformations of $p : S \to M$. Namely, for any $\alpha \in \pi$, $\varphi(\alpha)\tilde{f} = \tilde{f}\alpha$. Since $\varphi : \pi \to \pi$ induces $\varphi' : K \to K$, there is a continuous map $\bar{f} : N \to N$ which makes the following diagram

$$\begin{array}{ccc}
S & \xrightarrow{\tilde{f}} & S \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p'} \\
N & \xrightarrow{\bar{f}} & N \\
{\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\
M & \xrightarrow{f} & M
\end{array}$$

commutative.

By Theorem 2.2, there exists an affine map (d, D) on S such that

$$\varphi(\alpha)\cdot(d, D) = (d, D)\cdot\alpha$$

for all $\alpha\in\pi$. This implies that the map $(d, D) : S\to S$ induces a map $\Phi_{(d,D)} : M\to M$, and furthermore, $\Phi_{(d,D)}$ and f induce exactly the same homomorphism φ on π. Since M is a $K(\pi, 1)$-manifold, $\Phi_{(d,D)}$ and f are homotopic. Similarly, (d, D) induces a map $\phi_{(d,D)} : N\to N$ so that $\phi_{(d,D)}$ and $\bar{f}$ are homotopic. By the homotopy invariance of the Nielsen number, we can replace $\tilde{f}$, $\bar{f}$ and f in the diagram by (d, D), $\phi_{(d,D)}$ and $\Phi_{(d,D)}$, respectively, in the following discussion.

The formula above reads $\varphi(k)\cdot(d, D) = (d, D)\cdot k$ for $\alpha = k\in K$. Noting that $k\in K$ acts on S as a left translation, this yields $\varphi(k) = d\cdot D(k)\cdot d^{-1}$. That is, $\varphi = \tau_d\circ D$ on K. Recall that for any lattice K in a solvable Lie group S of type (R), any endomorphism of K extends uniquely to an endomorphism of S. This means that the equality $\varphi = \tau_d\circ D$ extends to S.

Write $E = \tau_d\circ D\in\mathrm{Endo}(S)$. Then we have

$$\varphi'(k) = \varphi(k) = E(k)\quad\text{for all } k\in K \text{ (and also } k\in S).$$

Thus the map $E : S\to S$ yields a map $\phi_E : N\to N$ on the solvmanifold N of type (R). Moreover, for any $k\in K$,

$$\varphi(k)\circ E = E\circ k,$$

where k and $\varphi(k)$ are left translations. This implies $\bar{f}\simeq\phi_E$. By Theorem 3.2, $N(\bar{f}) = |\det(I-E_*)|$, where E_* is the linear map on the Lie algebra of the Lie group S induced by E. Since $\det(I-E_*) = \det(I-\mathrm{Ad}(d)D_*) = \det(I-D_*)$ (Lemma 3.3), $N(\bar{f}) = |\det(I-D_*)|$, and hence $N(\bar{f})\neq 0$ if and only if $\mathrm{fix}(D_*) = \{0\}$ if and only if $\mathrm{Fix}(D) = \{e\}$.

Now we observe the following: Let $F : S\to S$ be any homomorphism on S and $g\in S$. For $(g, F)\in S\rtimes\mathrm{Endo}(S)$, if $\mathrm{Fix}((g, F))\neq\emptyset$, then there is a one-to-one correspondence between $\mathrm{Fix}((g, F))$ and $\mathrm{Fix}(F)$. Let $h\in\mathrm{Fix}((g, F))$. Then $(h, I)^{-1}(g, F)(h, I) = (e, F) = F$ and thus $k\in\mathrm{Fix}((g, F))\mapsto h^{-1}k\in\mathrm{Fix}(F)$ is the required correspondence. Therefore, $\mathrm{Fix}((g, F)) = h\cdot\mathrm{Fix}(F)$.

Let $\alpha = (a, A)$. Since $\tilde{f} = (d, D)$, we have $k\alpha\tilde{f} = (k, I)(a, A)(d, D) = (k\cdot a\cdot A(d), AD)$, $\bar{\alpha}\bar{f} = \phi_{(k\cdot a\cdot A(d),AD)}$ and $N(\bar{\alpha}\bar{f}) = |\det(I-A_*D_*)|$. Therefore if $N(\bar{\alpha}\bar{f})\neq 0$, the above observation implies that for all $k\in K$, $\mathrm{Fix}(k\alpha\tilde{f})$ has only one point.

Let $g \in \mathrm{fix}(\tau_{k\alpha}\varphi) \subset \pi$ and $\tilde{x} \in \mathrm{Fix}(k\alpha\tilde{f})$. Then

$$\begin{aligned} k\alpha\tilde{f}(g\cdot\tilde{x}) &= k\alpha(\varphi(g)\tilde{f})(\tilde{x}) \\ &= ((k\alpha)\varphi(g)(k\alpha)^{-1})k\alpha\tilde{f}(\tilde{x}) \\ &= (\tau_{k\alpha}\varphi)(g)(\tilde{x}) \quad (\text{since } \tilde{x} \in \mathrm{Fix}(k\alpha\tilde{f})) \\ &= g\cdot\tilde{x} \quad (\text{since } g \in \mathrm{fix}(\tau_{k\alpha}\varphi)) \end{aligned}$$

so that $g \cdot \tilde{x} \in \mathrm{Fix}(k\alpha\tilde{f})$. This implies $g\cdot\tilde{x} = \tilde{x}$. Since $g \in \pi$, $g = 1$. It follows that $\mathrm{fix}(\tau_{k\alpha}\varphi) = \{1\}$ and so $|q(\mathrm{fix}(\tau_{k\alpha}\varphi))| = 1$ for all $k \in K$.

On the other hand, if $N(\bar{\alpha}\bar{f}) = 0$, then the fixed point class $p'(\mathrm{Fix}(k\alpha\tilde{f}))$ of $\bar{\alpha}\bar{f}$ is inessential. By [11, Remark 2.7], the fixed point class $p(\mathrm{Fix}(k\alpha\tilde{f}))$ of f is also inessential. Hence our result follows from Theorem 4.1. ∎

We now explain how a continuous map $f : M \to M$ on the infra-solvmanifold $M = \pi\backslash S$ of type (R) induces an endomorphism of the Lie algebra, $f_* : \mathfrak{S} \to \mathfrak{S}$, naturally. The continuous map $f : M \to M$ induces a homomorphism $\varphi : \pi \to \pi$. Let Λ be a fully invariant subgroup of π in Lemma 2.1. Note that Λ is a lattice of S. Then, the induced homomorphism $\varphi : \pi \to \pi$ restricts to a homomorphism $\varphi' = \varphi|_\Lambda : \Lambda \to \Lambda$, which extends to an endomorphism of the Lie group S in a unique way. See [8, Theorem 2.2]. The differential of this map is an endomorphism of the Lie algebra, $f_* : \mathfrak{S} \to \mathfrak{S}$.

In all, we can prove our main result which computes the Lefschetz number $L(f)$ and the Nielsen number $N(f)$ of any continuous map f on an infra-solvmanifold M of type (R) in terms of the *holonomy* of the manifold. Thus we generalize the algebraic computation formula for Nielsen numbers for continuous maps on infra-nilmanifolds [14, Theorem 3.4] to infra-solvmanifolds of type (R). Our proof requires a straightforward adaptation of the proof of [14, Theorem 3.4]. Namely,

THEOREM 4.3. *Let $f : M \to M$ be any continuous map on an infra-solvmanifold M of type* (R) *with the holonomy group Ψ. Then*

$$L(f) = \frac{1}{|\Psi|}\sum_{A\in\Psi}\frac{\det(A_* - f_*)}{\det A_*},$$

$$N(f) = \frac{1}{|\Psi|}\sum_{A\in\Psi}|\det(A_* - f_*)|.$$

Proof. Let Λ be a fully invariant subgroup of π as in Lemma 2.1. Let $N = \Lambda\backslash S$ be the special solvmanifold. With $K = \Lambda$ as in the proof of Theorem 4.2, we have the following *homotopy commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{(d,D)} & S \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p'} \\ N & \xrightarrow{\phi_E} & N \\ {\scriptstyle \bar p}\downarrow & & \downarrow{\scriptstyle \bar p} \\ M & \xrightarrow{\Phi_{(d,D)}} & M \end{array}$$

where $E = \tau_d \circ D$. Here, we recall that $(d, D) : S \to S$ stands for the affine map defined by $(d, D)(x) = d \cdot D(x)$ for $x \in S$. Recall also that, as mentioned in the proof of Theorem 4.2, (d, D), ϕ_E and $\Phi_{(d,D)}$ are homotopic to $\tilde{f}$, $\bar{f}$ and f, respectively. By Corollary 3.2, we have

$$\begin{aligned} L(\bar f) &= L(\phi_E) = \det(I - E_*), \\ N(\bar f) &= N(\phi_E) = |\det(I - E_*)|. \end{aligned}$$

Let $\alpha \in \pi/\Lambda$. Choose a preimage $(a, A) \in \pi$ of α under the quotient map $\pi \to \pi/\Lambda$. Then α becomes a covering transformation of $\Lambda\backslash S$ via the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{(a,A)} & S \\ \downarrow & & \downarrow \\ \Lambda\backslash S & \xrightarrow{\alpha} & \Lambda\backslash S \end{array}$$

This means that α is the map induced by $(a, A) : S \to S$, i.e., $\alpha = \phi_{(a,A)}$. Moreover, $\alpha_* = (\tau_a \circ A)_* = \mathrm{ad}(a) \circ A_*$ on $\mathfrak{S}$. Consequently,

$$\begin{aligned} L(\alpha^{-1} \circ \bar f) &= \det(I - (\alpha_*^{-1} \circ E_*)) \\ &= \det(I - A_*^{-1}\,\mathrm{ad}(a^{-1})E_*) \\ &= \det(I - \mathrm{ad}(a^{-1})E_*A_*^{-1}) \\ &= \det(I - E_*A_*^{-1}) \quad \text{by Lemma 3.3} \\ &= \frac{\det(A_* - f_*)}{\det A_*}. \end{aligned}$$

Recalling that $\phi_{(a,A)}$ induces an isomorphism of the lattice Γ, we have $|\det A_*| = 1$, and $N(\alpha^{-1}\circ \bar f) = |L(\alpha^{-1}\circ \bar f)| = \frac{|\det(A_*-f_*)|}{|\det A_*|} = |\det(A_*-f_*)|$. Therefore, by [9, Theorem III.2.12, (p. 52)] and by Theorem 4.2, we have

$$\begin{aligned} L(f) &= \frac{1}{[\pi:\Lambda]}\sum_{\alpha\in\pi/\Lambda} L(\alpha^{-1}\circ \bar f)\\ &= \frac{1}{|\Psi|}\sum_{A\in\Psi}\frac{\det(A_*-f_*)}{\det A_*},\\ N(f) &= \frac{1}{[\pi:\Lambda]}\sum_{\alpha\in\pi/\Lambda} |\det(A_*-f_*)|\\ &= \frac{1}{|\Psi|}\sum_{A\in\Psi}|\det(A_*-f_*)|. \end{aligned}$$

This finishes the proof of Theorem. ∎

Example 4.4. The solvable Lie group Sol is one of the eight geometries that one considers in the study of 3-manifolds [22]. One can describe Sol as a semi-direct product $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $t\in\mathbb{R}$ acts on $\mathbb{R}^2$ via the map

$$\varphi(t) = \begin{bmatrix} e^t & 0\\ 0 & e^{-t}\end{bmatrix}.$$

Its Lie algebra $\mathfrak{sol}$ is given as $\mathfrak{sol} = \mathbb{R}^2 \rtimes_\sigma \mathbb{R}$ where

$$\sigma(t) = \begin{bmatrix} t & 0\\ 0 & -t\end{bmatrix}.$$

The Lie group Sol can be embedded into Aff(3) as

$$\begin{bmatrix} e^t & 0 & 0 & x\\ 0 & e^{-t} & 0 & y\\ 0 & 0 & 1 & t\\ 0 & 0 & 0 & 1\end{bmatrix}$$

where x, y and t are real numbers, and hence its Lie algebra $\mathfrak{sol}$ is isomorphic to the algebra of matrices

$$\begin{bmatrix} t & 0 & 0 & a\\ 0 & -t & 0 & b\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since Sol is of type (R), it is of type (NR). We denote the general element of Sol by $\{x, y, t\}$. Let Γ be the subgroup of Sol which is generated by

$$\left\{\tfrac{1}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}, 0\right\}, \quad \left\{\tfrac{\sqrt{5}-1}{2\sqrt{5}}, \tfrac{\sqrt{5}+1}{2\sqrt{5}}, 0\right\}, \quad \left\{0, 0, \ln\tfrac{3+\sqrt{5}}{2}\right\}.$$

Then Γ is isomorphic to the group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ where

$$\phi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is an element of $\mathrm{SL}(2, \mathbb{Z})$ with eigenvalues $\frac{3\pm\sqrt{5}}{2}$, and in fact Γ is a lattice of Sol.

Let $a = \left\{0, 0, \frac{1}{2}\ln\frac{3+\sqrt{5}}{2}\right\} \in \mathrm{Sol}$ and $A : \mathrm{Sol} \to \mathrm{Sol}$ be the automorphism of Sol given by

$$A(\{x, y, t\}) = \{-x, -y, t\}.$$

Then A has period 2, and $(a, A)^2 = \left(\left\{0, 0, \ln\frac{3+\sqrt{5}}{2}\right\}, I\right) \in \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$, where I is the identity automorphism of Sol. The subgroup

$$\pi = \langle \Gamma, (a, A) \rangle \subset \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$$

generated by the lattice Γ and the element (a, A) is discrete and torsion free, and Γ is a normal subgroup of π of index 2. Thus π is a torsion-free finite extension of the lattice Γ, and $\pi\backslash\mathrm{Sol}$ is an infra-solvmanifold, which has a double covering $\Gamma\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ by its holonomy group, $\Psi = \pi/\Gamma = \{1, A\} \cong \mathbb{Z}_2$.

Let $D : \mathrm{Sol} \to \mathrm{Sol}$ be the automorphism of Sol given by

$$D(\{x, y, t\}) = \{my, mx, -t\}$$

where m is any nonzero integer (cf. [7, Proposition 2.3]). Then $DA = AD$ and the conjugation by $(\{0, 0, 0\}, D) \in \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$ maps π into π (and Γ into Γ). Thus, the affine map $(\{0, 0, 0\}, D) : \mathrm{Sol} \to \mathrm{Sol}$ induces $\phi_D : \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ and $\Phi_D : \pi\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ so that the following diagram is commutative:

$$\begin{array}{ccc}
(\pi, \mathrm{Sol}) & \xrightarrow{(\{0,0,0\}, D)} & (\pi, \mathrm{Sol}) \\
\downarrow & & \downarrow \\
(\pi/\Gamma, \Gamma\backslash\mathrm{Sol}) & \xrightarrow{\phi_D} & (\pi/\Gamma, \Gamma\backslash\mathrm{Sol}) \\
\downarrow & & \downarrow \\
\pi\backslash\mathrm{Sol} & \xrightarrow{\Phi_D} & \pi\backslash\mathrm{Sol}
\end{array}$$

We take an ordered (linear) basis for the Lie algebra of Sol as follows:

$$\mathbf{e}_1 = \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}.$$

With respect to this basis, the differentials of A and D are

$$A_* = \begin{bmatrix} -1&0&0\\ 0&-1&0\\ 0&0&1 \end{bmatrix}, \quad D_* = \begin{bmatrix} 0&m&0\\ m&0&0\\ 0&0&-1 \end{bmatrix}.$$

Therefore by Theorem 4.3, the Lefschetz number and the Nielsen number of the map $\Phi_D : \pi\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ are:

$$\begin{aligned} L(\Phi_D) &= \frac{1}{2}\left(\frac{\det(I-D_*)}{\det(I)} + \frac{\det(A_*-D_*)}{\det(A_*)}\right) \\ &= \frac{1}{2}\big(2(1-m^2) + 2(1-m^2)\big) = 2(1-m^2), \\ N(\Phi_D) &= \frac{1}{2}(|\det(I-D_*)| + |\det(A_*-D_*)|) \\ &= \frac{1}{2}\big(|2(1-m^2)| + |2(1-m^2)|\big) = 2|1-m^2|. \end{aligned}$$

Acknowledgement. The authors would like to thank the referee for pointing out some errors and making careful corrections to a few expressions in the original version of the paper.

References

[1] D. V. Anosov, *The Nielsen numbers of maps of nil-manifolds*, Uspehi Mat. Nauk, **40** (1985), 133–134; Russian Math. Survey, **40** (1985), 149–150.

[2] O. Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology, **43** (2004), 903–924.

[3] R. B. S. Brooks, R. F. Brown, J. Pak and D. H. Taylor, *Nielsen numbers of maps of tori*, Proc. Amer. Math. Soc., **52** (1975), 398–400.

[4] K. Dekimpe, B. De Rock and W. Malfait, *The Nielsen numbers of Anosov diffeomorphisms on flat Riemannian manifolds*, Forum Math., **17** (2005), 325–341.

[5] K. Dekimpe, B. De Rock and W. Malfait, *The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds*, Monatsh. Math., **150** (2007), 1–10.

[6] K. Dekimpe, K. B. Lee and F. Raymond, *Bieberbach theorems for solvable Lie groups*, Asian J. Math., **5** (2001), 499–508.

[7] K. Y. Ha and J. B. Lee, *Left invariant metrics and curvatures on simply connected three dimensional Lie groups*, Math. Nachr., **282** (2009), 868–898.

[8] J. Jezierski, J. Kędra and W. Marzantowicz, *Homotopy minimal periods for NR-solvmanifolds maps*, Topology Appl., **144** (2004), 29–49.

[9] B. Jiang, Lectures on Nielsen fixed point theory, Contemporary Mathematics Amer. Math. Soc., **14**, 1983.

[10] Y. Kamishima, K. B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, Quart. J. Math. Oxford Ser. (2), **34** (1983), 433–452.

[11] S. W. Kim, J. B. Lee and K. B. Lee, *Averaging formula for Nielsen numbers*, Nagoya J. Math., **178** (2005), 37–53.

[12] E. Keppelmann and C. McCord, *The Anosov theorem for exponential solvmanifolds*, Pacific J. Math., **170** (1995), 143–159.

[13] S. Kwasik and K. B. Lee, *The Nielsen numbers of homotopically periodic maps of infranilmanifolds*, J. London Math. Soc. (2), **38** (1988), 544–554.

[14] J. B. Lee and K. B. Lee, *Lefschetz numbers for continuous maps, and periods for expanding maps on infra-nilmanifolds*, J. Geom. Phys., **56** (2006), 2011–2023.

[15] K. B. Lee, *Nielsen numbers of periodic maps on solvmanifolds*, Proc. Amer. Math. Soc., **116** (1991), 575–579.

[16] K. B. Lee, *Maps on infra-nilmanifolds*, Pacific J. Math., **168** (1995), 157–166.

[17] K. B. Lee, *Infra-solvmanifolds of type* (R), Quart. J. Math. Oxford Ser. (2), **46** (1995), 185–195.

[18] C. McCord, *Nielsen numbers and Lefschetz numbers on solvmanifolds*, Pacific J. Math., **147** (1991), 153–164.

[19] M. S. Raghunathan, Discrete subgroups of Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete **68**, Springer-Verlag, 1972.

[20] M. Saito, *Sur certains groupes de Lie resolubles*, Sci. Papers Coll. Gen. Ed. Univ. Tokyo, **7** (1957) 1–11.

[21] M. Saito, *Sur certains groupes de Lie resolubles II*, Sci. Papers Coll. Gen. Ed. Univ. Tokyo, **7** (1957) 157–168.

[22] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc., **15** (1983), 401–487.

[23] E. B. Vinberg, V. V. Gorbatsevich and O. V. Shvartsman, *Discrete subgroups of Lie groups*, Current Problems in Mathematics. Fundamental Directions, vol. 21, Itogi Nauki i Tekhniki, vol. 215, Akad. Nauk SSSR, VINITI, Moscow, 1988, pp. 5–120 (in Russian).

Jong Bum Lee
Department of Mathematics
Sogang University
Seoul 121-742
Korea
jlee@sogang.ac.kr

Kyung Bai Lee
Department of Mathematics
University of Oklahoma
Norman, OK 73019
U.S.A.
kblee@math.ou.edu