M. Artebani
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A COMPACTIFICATION OF $\mathcal{M}_3$ VIA $K3$ SURFACES

MICHELA ARTEBANI

Abstract. S. Kondō defined a birational period map between the moduli space of genus three curves and a moduli space of polarized $K3$ surfaces. In this paper we give a resolution of the period map, providing a surjective morphism from a suitable compactification of $\mathcal{M}_3$ to the Baily-Borel compactification of a six dimensional ball quotient.

Introduction

The theory of hypergeometric differential equations developed by Deligne and Mostow in the eighties implies that certain moduli spaces of weighted points in $\mathbb{P}^1$ are isomorphic to quotients of complex balls by the action of a discrete group ([6]). More recently several authors have constructed birational maps between moduli spaces of algebraic varieties such as curves of low genus, Del Pezzo surfaces, cubic threefolds, and complex ball quotients (e.g. [1], [10], [20], [21]). These kinds of correspondences are interesting both because they give a new insight in the compactification problem and because they naturally give an interplay between geometry and arithmetic. In this paper we provide an explicit resolution of the birational map between the moduli space of genus three curves and a six dimensional ball quotient constructed by S. Kondō in [18].

Let $V = |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the space of plane quartics and V_0 be the open subset parametrizing smooth curves. The degree four cyclic cover of the plane branched along a curve in V_0 is a $K3$ surface X equipped with an order four non-symplectic automorphism σ. This construction defines a holomorphic period map:

$$\mathcal{P}_0 : \mathcal{Q}_0 \longrightarrow \mathcal{M},$$

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where $\mathcal{Q}_0$ is the geometric quotient of V_0 by the action of PGL_3 and $\mathcal{M}$ is the moduli space of pairs (X, σ), isomorphic to a six dimensional ball quotient. In [18] S. Kondō shows that $\mathcal{P}_0$ gives an isomorphism between $\mathcal{Q}_0$ and the complement of two irreducible divisors $\mathcal{D}_n$, $\mathcal{D}_h$ in $\mathcal{M}$, corresponding to plane quartics with a node and to smooth hyperelliptic genus three curves respectively. A compactification for the moduli space $\mathcal{M}$ is given by the Baily-Borel compactification $\mathcal{M}^*$, which in this case is the union of $\mathcal{M}$ and one point (see [2]). On the other hand, the moduli space $\mathcal{Q}_0$ is known to be birational to the coarse moduli space $\mathcal{M}_3$ of genus three curves. Natural compactifications for this moduli space are the Deligne-Mumford compactification $\overline{\mathcal{M}}_3$ and the GIT compactification $\mathcal{Q}$, obtained by taking the PGL_3-categorical quotient of V.

In this paper we construct a distinct compactification $\widetilde{\mathcal{Q}}$ of $\mathcal{M}_3$ by blowing up the orbit of double conics in $\mathcal{Q}$ such that the exceptional divisor is a GIT moduli space of hyperelliptic genus three curves. This compactification parametrizes genus three curves with nodes and cusps and the boundary is a rational curve, parametrizing tacnodal curves. The main theorem is the following

THEOREM. *The map $\mathcal{P}_0$ can be extended to a holomorphic surjective map*

$$\mathcal{P} : \widetilde{\mathcal{Q}} \longrightarrow \mathcal{M}^*$$

with the following properties:

a) *it induces an isomorphism between the locus of stable curves and $\mathcal{M}$*;
b) *the exceptional divisor in $\widetilde{\mathcal{Q}}$ is mapped isomorphically onto the Baily-Borel compactification of $\mathcal{D}_h$*;
c) *the rational curve at the boundary is contracted to the boundary point of $\mathcal{M}^*$.*

In [15] Hyeon and Lee describe the birational geometry of $\overline{\mathcal{M}}_3$. One of its log minimal models, called moduli space of h-stable curves, is an intermediate compactification between the Deligne-Mumford and the GIT compactification. We report here a proof, due to D. Hyeon, of the isomorphism between this compactification and $\widetilde{\mathcal{Q}}$ (see 4.3).

Here is the plan of the paper.

In the first section we review the geometric construction by Kondō, which defines the period map $\mathcal{P}_0$, and show that this can be easily extended to GIT-stable quartics (i.e. having at most ordinary nodes and cusps), defining a holomorphic map

$$\mathcal{P}_1 : \mathcal{Q}_s \longrightarrow \mathcal{M},$$

where $\mathcal{Q}_s$ is the geometric quotient of the locus of GIT-stable quartics for the action of PGL_3. The analogous construction for strictly GIT-semistable quartics gives either surfaces with elliptic singularities or significant limit singularities (see [29]). In Section 2 we prove that, taking the Baily-Borel compactification $\mathcal{M}^*$ of $\mathcal{M}$, the period map can be extended holomorphically to give

$$\mathcal{P}_2 : \mathcal{Q}\backslash\{q\} \longrightarrow \mathcal{M}^*,$$

where the point q represents the orbit of double conics i.e. the canonical models of hyperelliptic genus three curves. In Section 3 we describe a correspondence between hyperelliptic genus three curves and certain hyperelliptic polarized $K3$ surfaces parametrized by $\mathcal{D}_h$. In [19] Kondō defines a period map

$$\mathcal{P}^h : \mathcal{Q}^h \longrightarrow \mathcal{D}_h^*$$

from the GIT moduli space $\mathcal{Q}^h$ of sets of eight points in $\mathbb{P}^1$ to the Baily-Borel compactification of $\mathcal{D}_h$ and proves that it is an isomorphism. We recall his results giving a different approach. In particular, we show that $\mathcal{D}_h$ is a moduli space for degree four cyclic covers of a cone over a rational normal quartic in $\mathbb{P}^5$ branched along a quadratic section.

The last section contains the main theorem. We first construct the blowing up $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ in q such that the exceptional divisor is isomorphic to the moduli space $\mathcal{Q}^h$. We then prove that the period maps $\mathcal{P}_2$ and $\mathcal{P}^h$ glue together to define a global period map

$$\mathcal{P} : \widetilde{\mathcal{Q}} \longrightarrow \mathcal{M}^*.$$

Similar descriptions for sextic double planes have been given by E. Horikawa in [13] and by J. Shah in [30]. These two papers and [32] by H. J. M. Sterk are all important references for this work.

In Subsection 4.3 we explain the relation between $\widetilde{\mathcal{Q}}$ and other known birational models of $\overline{\mathcal{M}}_3$.

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§1. GIT-stable quartics

1.1. Smooth quartics

We start recalling the geometric construction introduced by Kondō in [18]. This can be resumed by the following commutative diagram:

$$\begin{tikzcd} X_C \arrow[rr, "\pi"] \arrow[dr, "\pi_2"'] & & \mathbb{P}^2 \\ & S_C \arrow[ur, "\pi_1"'] & \end{tikzcd},$$

where C is a smooth quartic curve, π_1 is the double cover of the plane branched along C and π is the degree four cyclic cover of the plane branched along C. Note that S_C is a Del Pezzo surface of degree two and π_1 is the morphism associated to the anti-canonical linear system. Its double cover X_C is a $K3$ surface. In coordinates, if the quartic C is defined by the equation $f_4(x,y,z)=0$:

$$X_C = \{(x,y,z,t) \in \mathbb{P}^3 : t^4 = f(x,y,z)\}.$$

The $K3$ surface X_C has a natural degree four polarization induced by the embedding in $\mathbb{P}^3$ and given by $\pi^*(l)$, where l is the class of a line in the plane.

Let G be the covering transformation group of π. A generator σ for G can be chosen such that the space of holomorphic two-forms of X_C lies in the i-eigenspace W of the isometry $\rho = \sigma^*$ on $H^2(X_C, \mathbb{C})$. In particular the Picard lattice of X_C contains the pull-back L_+ of the Picard lattice of S_C as a sublattice. Hence the period point of X_C lies in the six dimensional complex ball

$$B = \{x \in \mathbb{P}(W) : (x, \bar{x}) > 0\} \subset \mathbb{P}(L_- \otimes \mathbb{C}),$$

where L_- is the orthogonal complement of L_+ in $H^2(X_C, \mathbb{Z})$.

Proposition 1.1. $L_+ \cong \langle 2\rangle \oplus A_1^{\oplus 7}$, $L_- \cong \langle 2\rangle^{\oplus 2} \oplus D_4^{\oplus 3}$.

Proof. Since L_+ and L_- are the eigenspaces of an involution, then both are 2-elementary lattices i.e. $(L_+)^*/L_+ \cong (L_-)^*/L_- \cong \mathbb{Z}_2^{\ell}$ (see [25]). Let δ

be the invariant defined in [26]. By [25], Theorem 4.2.2 and Theorem 4.3.2, we get that $\ell = 8$, $\delta = 1$ and it follows that L_+ and L_- are isomorphic to the above lattices. $\square$

Remark 1.2. The isometry ρ acts on L_- without non zero fixed vectors. It can be easily proved that, for a suitable choice of bases, it acts on $\langle 2\rangle^{\oplus 2}$ and each copy of D_4 as J and $J^{\oplus 2}$ respectively, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A $K3$ surface X is (L_+, ρ)-*polarized* if it is L_+-polarized and has period point in B, up to the choice of an isometry $H^2(X, \mathbb{Z}) \cong H^2(X_C, \mathbb{Z})$ ([10]). As proved in [18], the moduli space $\mathcal{M}$ of these $K3$ surfaces is the quotient of the six dimensional complex ball B by the action of an arithmetic group Γ of automorphisms:

$$\mathcal{M} = B/\Gamma \qquad \Gamma = \{\gamma \in \mathrm{O}(L_-) : \gamma \circ \rho = \rho \circ \gamma\}.$$

Since projectively equivalent plane quartics give isomorphic polarized $K3$ surfaces, then the above construction defines a holomorphic period map:

$$\mathcal{P}_0 : \mathcal{Q}_0 \longrightarrow \mathcal{M},$$

where $\mathcal{Q}_0$ is the geometric PGL_3-quotient of the moduli space of smooth quartics.

Note that all (L_+, ρ)-polarized $K3$ surfaces have a degree four polarization given by the ρ-invariant lattice $L^\rho = \langle h \rangle$ in L_+. The polarization is *ample* if there are no (-2)-curves orthogonal to h, i.e. a $K3$ surface is not ample polarized iff its class belongs to the *discriminant locus* $\mathcal{D}$ in $\mathcal{M}$:

$$\mathcal{D} = \left(\bigcup_{r \in \Delta} H_r \right) / \Gamma,$$

where $\Delta = \{r \in L_- : r^2 = -2\}$ is the set of *roots* of L_- and $H_r = B \cap r^\perp$. In fact, ample (L_+, ρ)-polarized $K3$ surfaces correspond to smooth plane quartics.

THEOREM 1.3. (Theorem 2.5, [18]) *The period map gives an isomorphism*:

$$\mathcal{P}_0 : \mathcal{Q}_0 \cong \mathcal{M}_3 \backslash \mathcal{M}_3^h \longrightarrow \mathcal{M} \backslash \mathcal{D}.$$

If $\Lambda_r = \langle r, \rho(r)\rangle$ and $\Lambda_r^\perp$ is its orthogonal lattice in L_-, then the roots can be divided in two classes (Lemma 3.3, [18]):

$$\Delta_n = \{r \in \Delta :\ \Lambda_r^\perp \cong U^{\oplus 2} \oplus A_1^{\oplus 8}\},\ \ \Delta_h = \{r \in \Delta :\ \Lambda_r^\perp \cong U(2)^{\oplus 2} \oplus D_8\}.$$

This leads to a natural decomposition of $\mathcal{D}$ in the union of two divisors, called *mirrors*:

$$\mathcal{D}_n = \left(\bigcup_{r\in\Delta_n} H_r\right)/\Gamma, \quad \mathcal{D}_h = \left(\bigcup_{r\in\Delta_h} H_r\right)/\Gamma.$$

In the following sections we will see that these two divisors have a clear interpretation in terms of genus three curves.

1.2. GIT-stable quartics

Let $V = |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the space of plane quartic curves. We now consider the categorical quotient of V with respect to the canonical action of PGL_3. We adopt the following notation:

$$V_{ss} = \{\text{GIT-semistable points in } V\},\ \ V_s = \{\text{GIT-stable points in } V\},$$
$$V_0 = \{\text{smooth curves in } V\}.$$

These loci can be described as follows (see [24]):

LEMMA 1.4. *Let $C \in V$ be a plane quartic, then*:

i) $C \in V \backslash V_{ss}$ *if and only if it has a triple point or is the union of a cubic with an inflectional tangent line.*

ii) $C \in V_s$ *if and only if it has at most ordinary nodes and cusps.*

iii) $C \in V_{ss}\backslash V_s$ *if it has a tacnode. Moreover, C belongs to the minimal orbit if and only if it is either a double conic or the union of two tangent conics* (*where at least one is smooth*).

A double point p on a strictly semistable plane quartic C is *inadmissible* if there is a projective transformation which carries p to the point $(0,0,1)$ such that C has an affine equation of the form:

$$f = (y + \alpha x^2)^2 + \sum_{i+2j>4} a_{ij} x^i y^j.$$

Notice that any point on a double conic is inadmissible and not all tacnodes are inadmissible. By [24] there exists a universal categorical quotient

$$\phi : V_{ss} \longrightarrow \mathcal{Q} = V_{ss} /\!\!/ \mathrm{PGL}_3,$$

where $\mathcal{Q}$ is a projective variety and ϕ is a PGL_3-invariant surjective morphism such that its restriction to V_s is a geometric quotient. We define $\mathcal{Q}_0 = \phi(V_0)$ and $\mathcal{Q}_s = \phi(V_s)$. Then there is a natural isomorphism ([33]):

$$\mathcal{Q}_0 \cong \mathcal{M}_3 \backslash \mathcal{M}_3^h,$$

where $\mathcal{M}_3^h$ denotes the hyperelliptic locus. We now show that the period map has a good behaviour on the stable locus.

LEMMA 1.5. *The minimal resolution of the degree four cyclic cover of $\mathbb{P}^2$ branched along a GIT-stable quartic is a (L_+, ρ)-polarized $K3$ surface.*

Proof. Let $\pi : Y \to \mathbb{P}^2$ be the degree four cyclic cover branched along a GIT-stable plane quartic C. Local computations show that $\pi^{-1}(p)$ is a rational double point of type A_3 if p is a node and of type E_6 if p is a cusp of C (see Ch.III, [3]). The minimal resolution $r : \tilde{Y} \to Y$ of Y is a $K3$ surface since $r^*(K_Y) = K_{\tilde{Y}} = 0$ and $H^1(\mathcal{O}_{\tilde{Y}}) = H^1(\mathcal{O}_Y) = 0$. Since these $K3$ surfaces are degenerations of ample (L_+, ρ)-polarized $K3$ surfaces, then their period points belong to B. $\square$

By the previous Lemma a polarized $K3$ surface X_C can be associated to any GIT-stable plane quartic C. This defines a natural extension $\mathcal{P}_1$ of $\mathcal{P}_0$ to the locus $\mathcal{Q}_s$ representing GIT-stable quartics in $\mathcal{Q}$:

$$\mathcal{P}_1 : \mathcal{Q}_s \longrightarrow \mathcal{M}.$$

PROPOSITION 1.6. *The period map $\mathcal{P}_1$ is a holomorphic extension of $\mathcal{P}_0$ and maps $\mathcal{Q}_s \backslash \mathcal{Q}_0$ to $\mathcal{D}_n$. Moreover, the map is generically surjective onto $\mathcal{D}_n$.*

Proof. By a result of E. Brieskorn on simultaneous resolution of singularities (see [5]) the period map $\mathcal{P}_0 \circ \phi$ can be extended holomorphically to the open subset of GIT-stable curves in V. By taking the quotient for the action of PGL_3 we get the first assertion. The generic point in $\mathcal{Q}_s \backslash \mathcal{Q}_0$ corresponds to a plane quartic with one node. This case is analyzed in detail in [18], in particular it is proved that the polarization on X_C is not ample since, for general $C \in \mathcal{Q}_s \backslash \mathcal{Q}_0$:

$$\mathrm{Pic}(X_C) \cap L_- = \langle r, \rho(r) \rangle, \quad r \in \Delta_n.$$

This implies the first assertion, the second one is proved in [18], §4. ∎

Let C be a GIT-stable quartic having at least n nodes and c cusps. The $K3$ surface X_C carries a natural involution $\tau_{n,c}$ in the covering group of the 4:1 map $X_C \to \mathbb{P}^2$. Let $\tau_{n,c}^*$ be the induced isometry on $H^2(X_C, \mathbb{Z})$.

Proposition 1.7. *Let C be the generic quartic with n nodes and c cusps. Then* $\mathrm{Pic}(X_C)$ *equals the fixed lattice of $\tau_{n,c}^*$ and its isomorphism class, in case C is irreducible, is given in the following table.*

(n, c)	$\mathrm{Pic}(X_C)$
$(1, 0)$	$U \oplus A_1^{\oplus 8}$
$(2, 0)$	$U \oplus A_1^{\oplus 6} \oplus D_4$
$(3, 0)$	$U \oplus A_1^{\oplus 6} \oplus D_6$
$(1, 1)$	$U \oplus A_1^{\oplus 2} \oplus D_4 \oplus D_6$
$(1, 2)$	$U \oplus A_1^{\oplus 2} \oplus D_6 \oplus E_8$
$(2, 1)$	$U \oplus A_1^{\oplus 2} \oplus D_4 \oplus D_8$
$(0, 1)$	$U \oplus A_1^{\oplus 4} \oplus D_6$
$(0, 2)$	$U \oplus A_1^{\oplus 2} \oplus D_4 \oplus E_8$
$(0, 3)$	$U \oplus A_1^{\oplus 2} \oplus E_8^{\oplus 2}$

Proof. If C is a plane quartic with n nodes and c cusps, then it can be easily seen, by performing the minimal resolution of the degree four cyclic cover, that

$$\mathrm{Pic}(X_C) \cap L_- \cong A_1^{\oplus 2n} \oplus D_4^{\oplus c},$$

where $A_1^{\oplus 2}$ and D_4 are intersection lattices of exceptional divisors over a node and a cusp respectively. Hence the rank of the fixed lattice equals $\mathrm{rank}(L_+) + 2n + 4c$ and, for generic C, it coincides with the Picard lattice by dimension reasons.

Assume C to be an irreducible plane quartic and $0 \leq n, c \leq 3$. In this case the fixed locus of $\tau_{n,c}$ is given by an irreducible curve of genus $3 - n - c$ and $n + 3c$ smooth rational curves. The lattices in the table can be thus determined as in the proof of Lemma 1.1. Note that, if $(n, c) \neq (1, 1), (1, 2)$, then $\delta_{n,c} = 1$ by [25], Theorem 4.3.2. Otherwise, let x be the class of a

-2-curve in L_+ corresponding to a bitangent of C not passing through the singular points. Then it is easy to see that $x/2$ belongs to the dual of the fixed lattice of $\tau_{n,c}$ and $(x/2)^2 = -1/2 \notin \mathbb{Z}$. It follows that $\delta_{n,c} = 1$ also in these two cases. ∎

Remark 1.8. In [34] È. B. Vinberg proves that the $K3$ surface with transcendental lattice isomorphic to $\langle 2\rangle \oplus \langle 2\rangle$ can be obtained as the 4:1 cover of the plane branched along either a quartic with 6 nodes or a quartic with 3 cusps. This can be also proved with the same method in the proof of Proposition 1.7.

§2. GIT-semistable quartics

Let C be a plane quartic with an ordinary tacnode at p and Y be the 4:1 cyclic cover of the plane branched along C. The local equation of Y over the point p is analytically isomorphic to:

$$t^4 + y^2 + x^4 = 0,$$

so Y has an elliptic singularity of type $\tilde{E}_7$. This suggests that there will be no $K3$ surfaces corresponding to these curves. However, we prove that the period map can be still extended holomorphically to strictly semistable admissible quartics if we consider the Baily-Borel compactification of the moduli space $\mathcal{M}$. In particular, we show that these quartics are mapped to the boundary and we describe the corresponding degenerations of $K3$ surfaces.

Let $\mathcal{M}^*$ be the Baily-Borel compactification of $\mathcal{M}$. It easy to prove that the i-eigenspace W of ρ in $L_- \otimes \mathbb{C}$ is given by

$$W = \{x - i\rho(x) : x \in L_- \otimes_{\mathbb{Z}} \mathbb{R}\} \cong L_- \otimes_{\mathbb{Z}} \mathbb{R}.$$

We say that $w \in W$ is *defined over* $\mathbb{Z}$ if $w = x - i\rho(x)$ with $x \in L_- \subset L_- \otimes_{\mathbb{Z}} \mathbb{R}$. The rational boundary components of the Baily-Borel compactification are given by points and curves corresponding respectively to isotropic lines and planes with generators in the lattice (see [2]). In our case this implies the following.

LEMMA 2.1. *The boundary of $\mathcal{M}^*$ is a disjoint union of points corresponding to isotropic lines in W defined over $\mathbb{Z}$.*

Let $\phi : V_{ss} \to \mathcal{Q}$ be the quotient morphism to the categorical quotient of V_{ss} by the action of PGL_3 and $V'_{ss} \subset V$ be the set of admissible semistable plane quartics. The following result and its proof are similar to those of Theorem 4.1, [13] and Theorem 5, [12]. The main tool is an extension theorem by Borel, which implies that the map $p_1 = \mathcal{P}_1 \circ \phi$ can be extended to a holomorphic map of V'_{ss} into $\mathcal{M}^*$ if $V'_{ss} \backslash V_s$ is locally contained in a divisor with normal crossing singularities. The following lemma gives this property for p_1.

Lemma 2.2. *Let C be a point in $V'_{ss} \backslash V_s$ i.e. C has an admissible tacnode. Then $V'_{ss} \backslash V_s$ is smooth at C of codimension* 3.

Proof. We first assume that C has only one tacnode in p. Then, after a projective transformation, the curve C can be defined by an equation of degree four of the form:

$$f = \sum_{i+2j \geq 4} a_{ij} x^i y^j, \quad a_{02} = 1,$$

in affine coordinates (x, y) centered at p. The quartic curves in a neighbourhood U of C in V can be defined by the equations:

$$f(t) = \sum_{i+2j \geq 4} a_{ij} x^i y^j + \sum t_{ij} x^i y^j = 0, \tag{1}$$

where $t = \{t_{ij}\}$ is a system of parameters for U. We call $C(t)$ the quartic curve defined by $f(t)$. If U is sufficiently small, then a curve belongs to $U \cap (V'_{ss} \backslash V_s)$ if and only if it has a tacnode. We can assume that $C(t)$ has the tacnode $p(t)$ in the intersection point of the lines:

$$x + s = 0, \quad y + ux + v = 0,$$

where s, u, v depend on t and $y + ux + v = 0$ is the tangent line at $p(t)$. Then $f(t)$ can be written in the form:

$$\sum_{i+2j \geq 4} (a_{ij} + b_{ij})(x + s)^i (y + ux + v)^j \tag{2}$$

for suitable coefficients b_{ij}. We now compare the equations 1 and 2 to get relations of the following types:

a) for indices (i, j) such that $i + 2j \geq 4$:

$$t_{ij} = b_{ij} + \text{terms divisible by } s,\ u \text{ or } v;$$

b) for indices (i, j) such that $i + 2j < 4$:

$$t_{ij} = \text{polynomial in } s,\ u,\ v,\ b_{ij}.$$

Notice that equations of type a) can be solved in b_{ij} as functions of s, u, v and t_{ij}, $i + 2j \geq 4$. Three equations of type b), by forgetting higher terms in u, s, v and b_{ij}, $i + 2j \geq 4$ are given by:

b1) $t_{01} = 2v + \cdots$,

b2) $t_{30} = a_{31}v + 4a_{40}s + a_{21}u + \cdots$,

b3) $t_{11} = 2a_{12}v + 2a_{21}s + 2u + \cdots$.

We prove that these equations are independent. Otherwise we would get:

$$a_{21}^2 - 4a_{40} = 0.$$

Let $a_{21} = 2\alpha$, then $a_{40} = \alpha^2$. This would imply that C has an equation of the form:

$$f = (y + \alpha x^2)^2 + \sum_{i+2j \geq 5} a_{ij} x^i y^j,$$

i.e. p would be an inadmissible tacnode. Thus we can solve b1), b2), b3) in u, s, v as functions of t_{01}, t_{30}, t_{11} and t_{ij}, $i + 2j \geq 4$. Notice that the cardinality of the set of indices (i, j) with $i + 2j < 4$ is equal to 6. Hence we get 3 independent equations and it can be easily seen that they define a variety which is smooth at C. In case of a quartic curve C with two tacnodes, similar computations give that $V'_{ss} \backslash V_s$ has two smooth components of codimension 3 in C. By Lemma 6 in [12], we can find two transversal hyperplanes L_i, $i = 1, 2$ such that $V'_{ss} \backslash V_s \subset L_1 \cup L_2$ in a neighbourhood of C. ∎

Proposition 2.3. *The period map $p_1 = \mathcal{P}_1 \circ \phi$ extends to a holomorphic map*:

$$\mathcal{P}_2 : V'_{ss} \longrightarrow \mathcal{M}^*.$$

Proof. It follows from Lemma 2.2 and Borel's extension theorem ([4], Theorem A and Remark 3.8). ∎

In fact, the locus of inadmissible plane quartics is mapped to one point in $\mathcal{Q}$.

LEMMA 2.4. *The quotient morphism $\phi : V_{ss} \to \mathcal{Q}$ maps $V_{ss} \backslash V'_{ss}$ to the point q representing the orbit of double conics.*

Proof. The proof is similar to that of Lemma 11, [13]. ∎

THEOREM 2.5. *The period map $\mathcal{P}_0$ can be extended to a holomorphic map*:

$$\mathcal{P}_2 : \mathcal{Q} \backslash \{q\} \longrightarrow \mathcal{M}^*.$$

The subvariety $\mathcal{Q} \backslash (\mathcal{Q}_s \cup \{q\})$ is a smooth rational curve parametrizing type II degenerations of $K3$ surfaces and it is mapped to the boundary of $\mathcal{M}^*$.

Proof. The first assertion follows from Proposition 2.3 and Lemma 2.4. Let C be a quartic curve in $V'_{ss} \backslash V_s$ and Δ be the open unit disc in $\mathbb{C}$. Consider a one dimensional family of plane quartic curves $F : \mathcal{C} \to \Delta$ with smooth general fiber C_t, $t \neq 0$ and central fiber $C_0 = C$. By [24], after passing to a ramified covering of Δ, we can assume that C_0 is a quartic curve in a minimal orbit. Hence, by Lemma 2.4 we can assume C_0 to be the union of two conics tangent in two points. Notice that the set of plane quartics which are the union of two tangent conics maps onto a smooth rational curve in $\mathcal{Q}$ (with q in its closure). Taking the 4:1 cyclic coverings of $\mathbb{P}^2$ branched along the curves C_t we get a family $G : \mathcal{X} \to \Delta$ of quartic surfaces X_t in $\mathbb{P}^3$. The central fiber X_0 has two singular points of type $\tilde{E}_7$ and it is birational to a ruled variety with base curve of genus 1. In fact, by Theorem 2.4 in [31], this degeneration is a surface of Type II i.e. the monodromy transformation N satisfies $N^2 = 0$, $N \neq 0$. In particular, the monodromy transformation has infinite order. This implies that the class of C in $\mathcal{Q}$ is mapped to the boundary of $\mathcal{M}^*$. ∎

Remark 2.6. In [31], Theorem 2.4, J. Shah gives a classification of GIT-semistable quartic surfaces. It can be easily proved that the 4:1 cyclic cover of $\mathbb{P}^2$ branched along a plane quartic is a stable quartic surface if and only if the plane quartic is stable. In this case, the quartic surface has at most rational double points. Hence, we only have surfaces of Type I (case A, Type I in Shah's theorem). In the case of strictly semistable and reduced plane quartics, the corresponding quartic surface is strictly semistable with isolated singularities. In particular, we get quartic surfaces of Type II (case B, Type II, (i) in Shah's theorem). In the case of a double conic, the 4:1 cyclic cover of the plane branched along this curve is the union of two quadrics tangent to each other along the ramification curve. In particular, this surface X has significant limit singularities (case B, Surfaces with significant limit singularities, (ii) in Shah's theorem), this means that the order of the monodromy transformation depends on the family of surfaces specializing to X.

§3. Hyperelliptic genus three curves

In Remark 2.6 we observed that the degree four cyclic cover of the plane branched along a double conic has significant limit singularities, hence the period map can not be extended to the point q. This is connected to the existence of *hyperelliptic* (L_+, ρ)-polarized $K3$ surfaces i.e. such that the curves in the linear system defined by the degree four polarization introduced in Section 1.1. are hyperelliptic (see [8]).

3.1. Smooth hyperelliptic curves

Let $C \subset \mathbb{P}^5$ be a smooth hyperelliptic genus three curve embedded by the bicanonical map and i be the hyperelliptic involution on C. Consider the surface Σ in $\mathbb{P}^5$ defined as follows

$$\Sigma = \overline{\bigcup_{p \in C} \langle p, i(p) \rangle},$$

where $\langle\ ,\ \rangle$ denotes the line spanned by two points.

Proposition 3.1. *The surface Σ is a cone over a rational normal quartic and C is a quadratic section of Σ not passing through the vertex. Let $\widetilde{\Sigma}$ be surface obtained by blowing up of the vertex of Σ. This is a 4-th Hirzebruch surface and the class of the curve C in $Pic(\widetilde{\Sigma})$ is given by*:

$$C = 2S_\infty + 8F,$$

where S_∞ is the class of the section with $S_\infty^2 = -4$ and F is the class of a fiber.

Proof. A hyperelliptic genus three curve C can be given by an equation of the form

$$y^2 = \prod_{i=1}^{8} (x - \lambda_i),$$

for some complex numbers $\lambda_1, \dots, \lambda_8$. In these coordinates the hyperelliptic involution i is given by $(x, y) \mapsto (x, -y)$ and the bicanonical map is

$$\phi_{|2K_C|} : C \longrightarrow \mathbb{P}^5, \quad (x, y) \longmapsto (1, x, x^2, x^3, x^4, y).$$

Hence, if $(z_0, \dots, z_5)$ are coordinates for $\mathbb{P}^5$, the surface Σ is a cone with vertex $(0, \dots, 0, 1)$ over the rational normal quartic obtained by projecting C on the hyperplane $z_5 = 0$. Moreover, since the curve C is a quadratic section of Σ, its inverse image in $\widetilde{\Sigma}$ has the intersection numbers: $(C, S_\infty) = 0$, $(C, F) = 2$. This determines the class of C in $\mathrm{Pic}(\widetilde{\Sigma})$. □

Let X_C be the 4:1 cyclic cover of the rational ruled surface $\widetilde{\Sigma}$ branched along the reducible curve:

$$C \cup 2S_\infty \in |4S_\infty + 8F|.$$

It can be easily proved that X_C is a $K3$ surface.

Remark 3.2. In [18] and [19] S. Kondō associates a $K3$ surface to a smooth hyperelliptic genus three curve by embedding the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ as a divisor of bidegree $(4, 2)$ and then taking the minimal resolution of the double cover of the quadric branched along C and two lines L_1, L_2. It is easy to realize that the double cover of $\widetilde{\Sigma}$ branched along C is isomorphic to the double cover of the quadric branched along $C \cup L_1 \cup L_2$. Hence the construction by S. Kondō leads to the same $K3$ surface X_C defined above.

The previous construction associates a polarized $K3$ surface X_C to any smooth hyperelliptic genus three curve C. In [18] and [19] Kondō proved that this defines a holomorphic period map

$$\mathcal{P}_0^h : \mathcal{M}_3^h \longrightarrow \mathcal{D}_h \subset \mathcal{M}.$$

Moreover, if we denote with $\mathcal{D}'$ the discriminant locus in $\mathcal{D}^h$ (i.e. the union of hyperplane sections of $\mathcal{D}_h$ defined by roots), then:

Theorem 3.3. (Theorem 5.3, [18], Theorem 3.8, [19]) *The period map induces an isomorphism*:

$$\mathcal{P}_0^h : \mathcal{M}_3^h \longrightarrow \mathcal{D}_h \backslash \mathcal{D}'.$$

3.2. GIT-stable hyperelliptic curves

The canonical morphism gives a natural correspondence between hyperelliptic genus three curves and collections of eight (unordered) points in $V^h = |\mathcal{O}_{\mathbb{P}^1}(8)|$ up to projectivities. We now consider the categorical quotient of V^h with respect to the canonical action of PGL_2. We adopt the following notation:

$$V^h_{ss} = \{\text{GIT-semistable points in } V^h\}, \;\; V^h_s = \{\text{GIT-stable points in } V^h\},$$
$$V_0 = \{\text{collections of 8 distinct points in } \mathbb{P}^1\}.$$

These loci can be described as follows (see [24]):

LEMMA 3.4. *Let $f \in V^h$. Then*:

i) $f \in V^h_s$ *iff it has no point of multiplicity* ≥ 4.
ii) $f \in V^h_{ss} \backslash V^h_s$ *if it has points of multiplicity* 4. *Moreover, it belongs to a minimal orbit iff it has two distinct points of multiplicity* 4.

A natural compactification for $\mathcal{M}^h_3$ is the categorical quotient:

$$\mathcal{Q}^h = V^h_{ss} /\!\!/ \, \mathrm{PGL}_2 \, .$$

We will denote by $\mathcal{Q}^h_s$ the image of V^h_s in $\mathcal{Q}^h$. It is known that there is an isomorphism

$$\mathcal{M}^h_3 \cong \mathcal{Q}^h_0 = V^h_0 /\!\!/ \, \mathrm{PGL}_2 \, .$$

Let $\Sigma \subset \mathbb{P}^5$ be a cone over a rational normal quartic.

PROPOSITION 3.5. *The space $\mathcal{Q}^h$ is isomorphic to a GIT quotient of the space of quadratic sections of Σ not passing through the vertex with respect to the action of the automorphism group of Σ.*

Proof. We omit the proof since it completely analogous to that in [30, §2]. $\square$

We now give a characterization of stability as follows:

PROPOSITION 3.6. *A quadratic section of Σ not passing through the vertex is GIT-stable if and only if it has at most ordinary nodes and cusps.*

Proof. Let C be a quadratic section not passing through the vertex, $p \in C$ and F be the class of a fiber in $\widetilde{\Sigma}$. We choose a basis $\{u, v\}$ of $H^0(\widetilde{\Sigma}, F)$ such that u vanishes at p. Let $l_0 = \{u = 0\}$, $l_\infty = \{v = 0\}$, $q_\infty \in H^0(\widetilde{\Sigma}, S_\infty)$ and $q_0 \in H^0(\widetilde{\Sigma}, 4F + S_\infty)$. We consider the affine coordinates $x = u/v$ and $y = q_0/q_\infty$. The divisor C is defined by an equation of the form $f = q_0^2 + b = 0$, where $b \in H^0(\widetilde{\Sigma}, F)$ and has multiplicity ≤ 4 at p. In the affine set $\widetilde{\Sigma} - S_\infty - l_\infty$ an equation for C is of the form $y^2 + p_8(x) = 0$, where p_8 is a polynomial of degree 8 which is not divisible by x^5. Notice that C is reduced and has at most double points. If C is GIT-stable then x^4 doesn't divide p_8, hence C has at most a node or an ordinary cusp in p. ∎

The minimal resolution of the degree four cyclic cover of $\widetilde{\Sigma}$ branched along the curve $C \cup 2S_\infty$, where C is a stable point in V^h, is an (L_+, ρ)-polarized $K3$ surface. An application of Brieskorn's result as in the proof of Proposition 1.7 gives

Lemma 3.7. *The period map $\mathcal{P}_0^h$ extends to a holomorphic map*:

$$\mathcal{P}_1^h : \mathcal{Q}_s^h \longrightarrow \mathcal{D}_h.$$

As in the case of stable plane quartics, a natural stratification for $\mathcal{D}_h$ is induced by the number of nodes and cusps of a stable hyperelliptic curve. The Picard lattice of the generic $K3$ surface in each stratum is given in [19], §4.5.

3.3. GIT-semistable hyperelliptic curves

Let C be a stricly GIT-semistable hyperelliptic genus three curve in a minimal orbit. With the notation of the proof of Proposition 3.6, a local equation for the embedding of the curve C in $\widetilde{\Sigma}$ is given by:

$$y^2 - p_8(x) = 0,$$

where $p_8(x) = (x-a)^4(x-b)^4$, $a, b \in \mathbb{C}$. In fact, note that the locus $\mathcal{Q}^h \backslash \mathcal{Q}_s^h$ is just one point. The curve C in $\widetilde{\Sigma}$ has two tacnodes, this implies that the 4:1 cyclic cover X_C of $\widetilde{\Sigma}$ branched along $C \cup 2S_\infty$ has two elliptic double points. As in the case of semistable plane quartics, it can be proved that the period map can be extended to $\mathcal{Q}^h$ by considering the Baily-Borel compactification $\mathcal{D}_h^*$ of the ball quotient $\mathcal{D}_h$. In fact, the boundary of $\mathcal{D}_h^*$ is given by one point (the *cusp*) and we have the following result.

THEOREM 3.8. (Theorem 4.7, [19]) *The period map can be extended to an isomorphism*:

$$\mathcal{P}^h : \mathcal{Q}^h \longrightarrow \mathcal{D}_h^*.$$

The point $\mathcal{Q}^h \backslash \mathcal{Q}_s^h$ *is mapped to the cusp.*

§4. A period map for genus three curves

In this section we construct a blow-up of $\mathcal{Q}$ in q such that the exceptional divisor is isomorphic to $\mathcal{Q}^h$ and we prove that the period map $\mathcal{P}_2$ can be extended to this variety. In fact, we prove that this extension coincides with the period map $\mathcal{P}^h$ on the exceptional divisor.

4.1. Blow-up

Let T be a conic in $\mathbb{P}^2$ and $2T$ be the corresponding double conic. We denote by $G_3(2T)$ the orbit of $2T$ by the action of PGL_3.

PROPOSITION 4.1. *Let* $\widetilde{V}$ *be the blow-up of* V_{ss} *along* $G_3(2T)$*. For a proper choice of the* PGL_3*-linearization on* $\widetilde{V}$*, the fibre of*

$$b : \widetilde{V} \longrightarrow V_{ss}$$

over $2T$ *is the semistable locus* V_{ss}^h *(with respect to the action of* PGL_2*). The exceptional divisor* $\mathcal{E}$ *of the induced blowing-up*

$$\widetilde{V} /\!\!/ \mathrm{PGL}_3 \longrightarrow \mathcal{Q} = V_{ss} /\!\!/ \mathrm{PGL}_3$$

is isomorphic to $\mathcal{Q}^h$.

Proof. The result follows from a theorem of F. Kirwan (§7, [17]) if we prove that the normal space at $G_3(2T)$ in a point $2T$ is isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^1}(8))$ and that, under this isomorphism, the isotropy group of $2T$ acts on it as PGL_2.

Let $q \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ be a section vanishing on T. The isotropy group of T in PGL_3 can be easily identified with PGL_2. There exists a unique GL_2-invariant decomposition:

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \cong \mathbb{C}q \oplus \Theta,$$

where $\Theta \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. Moreover:

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \cong \mathbb{C}q^2 \oplus q\Theta \oplus \Psi,$$

where $\Psi \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$. The normal space at $G_3(2T)$ in $2T$ is isomorphic to Ψ. Besides, up to the isomorphism of Ψ with $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$, the action of PGL_2 is the canonical one. Hence, by the result of F. Kirwan, the action of PGL_3 can be lifted to $\widetilde{V}$ in such a way that the exceptional divisor over q is isomorphic to the universal categorical quotient of V_{ss}^h by the action of PGL_2. ∎

We denote by $\widetilde{\mathcal{Q}}$ the blowing-up of $\mathcal{Q}$ in q given in Proposition 4.1:

$$\widetilde{\mathcal{Q}} = \widetilde{V} /\!\!/ \mathrm{PGL}_3 \,.$$

Let $\widetilde{\mathcal{Q}}_s$ be the subvariety corresponding to the stable locus in $\widetilde{V}$.

4.2. Final extension

We can now associate the isomorphism class of a genus three curve to each point of $\widetilde{\mathcal{Q}}$. If the point is not on the exceptional divisor, then the curve can be embedded in $\mathbb{P}^2$ as a GIT-semistable plane quartic. Otherwise, it represents a GIT-semistable hyperelliptic genus three curve and can be embedded in the cone Σ as a quadratic section not passing through the vertex. Moreover, we defined two period maps:

$$\mathcal{P}_2 : \widetilde{\mathcal{Q}} \backslash \mathcal{E} \longrightarrow \mathcal{M}^* \backslash \mathcal{D}_h,$$
$$\mathcal{P}^h : \mathcal{E} \longrightarrow \mathcal{D}_h^*.$$

We now prove that $\mathcal{P}_2$ and $\mathcal{P}^h$ give a global holomorphic period map on $\widetilde{\mathcal{Q}}$. For similar results and methods see [32] and [30].

We start considering the case of a one parameter family of smooth quartic curves degenerating to a double conic $2T$. Let X be the degree four cyclic cover of the Hirzebruch surface $\widetilde{\Sigma}$ branched along $C + 2S_\infty$, where C is a quadratic section of the cone Σ not passing through the vertex. We define the *degree four cyclic cover of* Σ *branched along* C *and the vertex* to be the surface obtained by contracting the inverse images of S_∞ in X.

Proposition 4.2. *Let $\mathcal{C} \to \Delta$ be a family of plane quartics over the unit complex disk Δ intersecting transversally the orbit of double conics in a smooth double conic over $0 \in \Delta$.*
Let $F : \mathcal{X} \to \Delta$ be the family of quartic surfaces in $\mathbb{P}^3$ giving the 4:1 *cyclic cover of $\mathbb{P}^2$ branched along $\mathcal{C}$.*
Then there exist a cover $\xi : \Delta \to \Delta$ ramified over $0 \in \Delta$ and a family $F' : \mathcal{Y} \to \Delta$ such that:

i) *the fibers of* F' *and* ξ^*F *are isomorphic over* $t \neq 0$;

ii) *the central fiber* Y_0 *of* F' *is the* 4:1 *cover of a cone* $\Sigma \subset \mathbb{P}^5$ *over a rational normal quartic, branched along the vertex and a quadratic section not passing through the vertex.*

Proof. Let (x_0, x_1, x_2) be coordinates for $\mathbb{P}^2$ and $t \in \Delta$. The equation of the family $\mathcal{C}$ can be written in the form:

$$\mathcal{C}: \ q^2(x_i) + t\phi(t, x_i) = 0,$$

where $\phi(t) \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$, $\phi(0)$ is not divisible by q (from the transversality condition) and $q \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ defines a smooth conic T. Let ξ be the base change of order two $t \mapsto t^2$ on Δ and call $\mathcal{C}'$ the family of curves over Δ obtained as pull-back of $\mathcal{C}$ by ξ. The 4:1 cyclic cover $\mathcal{X}'$ of $\Delta \times \mathbb{P}^2$ branched along the family $\mathcal{C}'$ has a natural embedding in $\Delta \times \mathbb{P}^3$, in coordinates $(t; x_0, x_1, x_2, w)$:

$$\mathcal{X}': \ w^4 = q^2(x_i) + t^2\phi(t^2, x_i).$$

Consider the embedding:

$$\Delta \times \mathbb{P}^3 \longrightarrow \Delta \times \mathbb{P}^9$$

given by the identity on the first factor and by the 2-nd Veronese embedding on the second one. We consider the coordinates in $\mathbb{P}^9$:

$$z_{ij} = x_i x_j, \ \ s = w^2, \ \ y_k = t x_k, \quad 0 \le i, j, k \le 2.$$

The Veronese embedding $V(\mathbb{P}^3)$ of $\mathbb{P}^3$ is given by the equations:

$$\begin{cases} z_{ij}z_{kl} - z_{ik}z_{jl} = 0, \\ s z_{ij} - y_i y_j = 0, \qquad\quad 0 \le i, j \le 2. \end{cases}$$

Then the embedding of $\mathcal{X}'$ in $\Delta \times \mathbb{P}^9$ is the intersection of $V(\mathbb{P}^3)$ with the quadratic section:

$$s^2 = q^2(z_{ij}) + t^2\phi(z_{ij}, t^2),$$

where we think $q \in H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(1))$ and $\phi(t^2) \in H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(2))$. We still denote by T the hyperplane section defined by q in $\mathbb{P}^9$ and consider the blowing-up of $\Delta \times \mathbb{P}^9$ along $\{0\} \times T$. This gives a family with general fiber isomorphic to $\mathbb{P}^9$ and central fiber given by the union of a copy of $\mathbb{P}^9$ and the exceptional divisor:

$$E \cong \mathbb{P}(\mathcal{O}_T \oplus \mathcal{O}_T(1)).$$

By Grauert's contraction criterion (see [11]), the $\mathbb{P}^9$ component in the central fiber can be contracted to a point. The proper transform $\mathcal{Y}$ of $\mathcal{X}'$ in this new variety is the intersection of the cone over $V(\mathbb{P}^3)$ with the quadratic sections:

$$\begin{cases} s^2 = \epsilon^2 + t^2\phi, \\ q - \epsilon t = 0. \end{cases}$$

Let $\mathcal{S}$ be the projection of $\mathcal{Y}$ on the linear subspace defined by:

$$y_i = 0, \quad 0 \leq i \leq 2.$$

The surface S_t is defined by:

$$\begin{cases} s^2 = \epsilon^2 + t^2\phi, \\ z_{ij}z_{kl} - z_{ik}z_{jl} = 0, \quad 0 \leq i, j \leq 2, \\ q - \epsilon t = 0. \end{cases}$$

We denote by $\pi_{2,t} : \mathcal{Y} \longrightarrow \mathcal{S}$ the projection.
Consider now the projection $\mathcal{W}$ of $\mathcal{S}$ on the subspace defined by:

$$s = y_i = 0, \quad 0 \leq i \leq 2.$$

The surface W_t is defined by:

$$\begin{cases} z_{ij}z_{kl} - z_{ik}z_{jl} = 0, \quad 0 \leq i, j \leq 2, \\ q - \epsilon t = 0. \end{cases}$$

We denote by $\pi_{1,t} : \mathcal{S} \longrightarrow \mathcal{W}$ the projection and by $\pi_t = \pi_{2,t} \circ \pi_{1,t}$. Then we have the following two cases:

i) $t \neq 0$:
The surface W_t is a linear section of the cone over the Veronese surface not passing through the vertex, hence it is isomorphic to the Veronese surface. The projection $\pi_{1,t}$ is the double cover of W_t branched along the section B_t defined by:

$$\epsilon^2 + \phi(z_{ij}, t^2) = 0.$$

In particular S_t is a Del Pezzo surface of degree two. The projection $\pi_{2,t}$ is the double cover of S_t branched along $\pi_{1,t}^{-1}(B_t)$. Hence π_t is the 4:1 cyclic cover of W_t branched along B_t. The surface Y_t is isomorphic to the quartic surface X_t.

ii) $t = 0$:

The central fiber W_0 is a cone over the rational normal quartic given by the intersection of the Veronese surface with the hyperplane $q = 0$. Let $\tilde{W}_0$ be the 4-th Hirzebruch surface obtained by blowing up the vertex of W_0, S_∞ be the exceptional divisor and $\tilde{B}_0$ be the proper transform of B_0. The projection $\pi_{1,0}$ is the double cover of W_0 branched along the section B_0:

$$\epsilon^2 + \phi(z_{ij}, 0) = 0.$$

In particular S_0 has two singular points $P_\pm$ over the vertex of W_0. The projection $\pi_{2,0}$ is the double cover of S_0 branched along $\pi_{1,0}^{-1}(B_0)$ and the points $P_\pm$. Hence π_0 is the 4:1 cyclic cover of W_0 branched along B_0 and the vertex of the cone i.e. the blowing up $\tilde{Y}_0$ of Y_0 in the points $\pi_{2,0}^{-1}(P_\pm)$ is the 4:1 cyclic cover of $\tilde{W}_0$ branched along $\tilde{B}_0 + 2S_\infty$. ∎

Corollary 4.3. *Let $\mathcal{F}$ be a one-parameter family in $\widetilde{\mathcal{Q}}$ intersecting transversally the exceptional divisor $\mathcal{E}$. Then the period maps $\mathcal{P}_2$ and $\mathcal{P}^h$ glue holomorphically on $\mathcal{F}$.*

Proof. By [30], Proposition 2.1, up to base change, there exists a family $\widetilde{\mathcal{F}}$ in $\widetilde{V}$ representing $\mathcal{F}$ (such that the central fiber belongs to a minimal orbit). Let w be the intersection of $\widetilde{\mathcal{F}}$ with the exceptional divisor. The projection of $\widetilde{\mathcal{F}}$ to V_{ss} is a family of plane quartics $\mathcal{C}$ with central fiber equal to a double conic $2T$. By Proposition 4.2, after a base change of order two, we can associate to $\mathcal{C}$ a family of surfaces such that the general fiber is isomorphic to the 4:1 cyclic cover of $\mathbb{P}^2$ branched along C_t and the central fiber is the 4:1 cyclic cover of a cone Σ branched along a quadratic section B_0 and the vertex. In fact, it follows from the proof of Proposition 4.2 that the quadratic section B_0 is the section corresponding to w as given in Proposition 3.5. Since the period map is invariant by the action of the Galois group of the double cover, by taking the minimal resolution of surfaces in the family, we get a period map $\mathcal{P}_{\mathcal{F}} : \mathcal{F} \to \mathcal{M}^*$ which is the gluing of $\mathcal{P}_2$ and $\mathcal{P}^h$. ∎

The existence of a global extension follows from the following version of Hartogs' Theorem (see [14]):

Lemma 4.4. (Theorem 2.2.8, [14]) *Let $f : U \to \mathbb{C}$ be a function defined in the open set $U \subset \mathbb{C}^n$. Assume that f is holomorphic in each variable z_j when the other variables z_k, $k \neq j$ are fixed. Then f is holomorphic in U.*

Theorem 4.5. *The period map $\mathcal{P}_2$ can be extended holomorphically to a period map:*

$$\mathcal{P} : \widetilde{\mathcal{Q}} \longrightarrow \mathcal{M}^*$$

such that its restriction to the exceptional divisor $\mathcal{E}$ coincides with the period map $\mathcal{P}^h$. Moreover:

i) the locus $\widetilde{\mathcal{Q}} \backslash \widetilde{\mathcal{Q}}_s$ is a smooth rational curve mapped to the boundary of $\mathcal{M}^*$;

ii) $\mathcal{P}$ induces an isomorphism $\mathcal{P}_{|\widetilde{\mathcal{Q}}_s} : \widetilde{\mathcal{Q}}_s \to \mathcal{M}$.

Proof. Let $\mathcal{P}$ be the gluing of $\mathcal{P}_2$ and $\mathcal{P}^h$ on $\widetilde{\mathcal{Q}}$. The map $\mathcal{P}$ is holomorphic in $\widetilde{\mathcal{V}} \backslash \mathcal{E}$ (by Theorem 2.3) and in $\mathcal{E}$ (by Theorem 3.8). Moreover, by Proposition 4.2, it is holomorphic on the generic one-dimensional family intersecting $\mathcal{E}$ transversally. Hence, by Lemma 4.4, the period map $\mathcal{P}$ is holomorphic. Since $\widetilde{\mathcal{Q}}$ is a compact variety (see [24]) we also have that $\mathcal{P}$ is surjective.

Let B be the rational curve given by the image of strictly GIT-semistable quartics in $\mathcal{Q} - \{q\}$ and b be the point in $\mathcal{E}$ corresponding to effective divisors of the form $4p_1 + 4p_2$ in $|\mathcal{O}_{\mathbb{P}^1}(8)|_{ss}$. We prove that the point b lies in the closure of the curve B. In fact, a point in B can be given by:

$$(q + tz^2)(q - tz^2) = q^2 - t^2 z^4$$

where $q = xy - z^2$ and $t \in \mathbb{C}$. From the proof of Proposition 4.2, it follows that the corresponding quadratic section on the cone is given by the union of two hyperplane sections:

$$q^2 - z^4 = (q + z^2)(q - z^2) = 0.$$

The equation $z^4 = 0$ intersects the conic $q = 0$ in two points with multiplicity 4, hence this quadratic section corresponds to the point b.

Moreover, note that the injectivity of the period map on $\mathcal{Q}_s$ follows as in the generic case. Hence assertions *i*) and *ii*) follow from the surjectivity of the period map and Theorems 2.5 and 3.8. ∎

Two easy corollaries are the following:

COROLLARY 4.6. *The boundary of the Baily-Borel compactification $\mathcal{M}^*$ is given by a unique point (the cusp).*

Proof. This follows from Lemma 2.1 and Theorem 4.5. ∎

COROLLARY 4.7. *Every (L^+, ρ)-polarized $K3$ surface X carries an order four automorphism σ such that $\sigma^* = \pm iI$ on $H^{2,0}(X)$.*

Proof. By Theorem 4.5 an (L_+, ρ)-polarized $K3$ surface is either the 4:1 cyclic cover of $\mathbb{P}^2$ branched along a stable quartic or the 4:1 cyclic cover of the cone Σ branched along a quadratic section. In the first case, the $K3$ surface carries an order four covering automorphism σ_q such that the quotient of X by the involution $\tau_q = \sigma_q^2$ is a rational surface (the blow up of a Del Pezzo surface of degree two). Hence, τ_q^* acts as minus the identity on the transcendental lattice $T(X)$, as in the generic case. Similarly, in the second case we have an order four covering automorphism σ_h and the quotient of X by the action of the involution $\tau_h = \sigma_h^2$ is a rational surface (a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$). ∎

4.3. On the relation with other birational models of $\overline{\mathcal{M}}_3$

In [15] Hyeon and Lee determined all log minimal models of the Deligne-Mumford compactification $\overline{\mathcal{M}}_3$ and describe the birational maps between them. The general picture is described by the following diagram (§1, [15])

$$\overline{\mathcal{M}}_3 \xrightarrow[T]{} \overline{\mathcal{M}}_3^{ps} \xrightarrow[\Psi]{} \overline{\mathcal{M}}_3^{cs} \xleftarrow[\Psi^+]{} \overline{\mathcal{M}}_3^{hs} \xrightarrow[\Theta]{} \mathcal{Q}\,,$$

where $\overline{\mathcal{M}}_3^{ps}$, $\overline{\mathcal{M}}_3^{cs}$ and $\overline{\mathcal{M}}_3^{hs}$ are the moduli spaces of pseudo-stable, c-stable and h-stable curves respectively; T is a birational contraction, Ψ is a small contraction, Ψ^+ is the flip of Ψ contracting tacnodal curves and Θ is the contraction of the hyperelliptic locus.

The space $\overline{\mathcal{M}}_3^{ps}$ is the moduli space introduced by Schubert in [28]. In [15, Proposition 21] the authors prove that the compactification $\overline{\mathcal{M}}_3^{cs}$ is isomorphic to the Baily-Borel compactification $\mathcal{M}^*$.

The moduli space of h-stable curves is actually isomorphic to $\widetilde{\mathcal{Q}}$. The following proof is due to D. Hyeon. Let $Z, H \subset \overline{\mathcal{M}}_3^{hs}$ be respectively the locus of tacnodal curves and the hyperelliptic locus. There are natural isomorphisms

$$\alpha : \overline{\mathcal{M}}_3^{hs} \backslash Z \longrightarrow \widetilde{\mathcal{Q}}_s, \quad \beta : \overline{\mathcal{M}}_3^{hs} \backslash H \longrightarrow \widetilde{\mathcal{Q}} \backslash \mathcal{E}.$$

In fact, note that the first two spaces are isomorphic to $\mathcal{M}$ and the second ones to $\mathcal{Q}_s$ (Theorem 4.5 and [15, Theorem 1]). Since isomorphisms between coarse moduli spaces are unique up to unique isomorphism, the isomorphisms α and β coincide away from $Z \cup H$. Thus they glue together to give an isomorphism of $\overline{\mathcal{M}}_3^{hs}$ and $\widetilde{\mathcal{Q}}$ away from the points $\widetilde{\mathcal{Q}} \backslash \widetilde{\mathcal{Q}}_s \cap \mathcal{E}$ and $Z \cap H$. Since the two spaces are normal, then this map extends to an isomorphism by Hartogs' Lemma.

In [21] E. Looijenga defined a compactification of $\mathcal{M}$ which is a small modification of the Baily-Borel compactification, essentially replacing the cusp by a curve. Let $\mathcal{W}$ be this moduli space and $Z', H' \subset \mathcal{W}$ be the exceptional curve over the cusp and the proper transform of the hyperelliptic divisor $\mathcal{D}_h$. Then $\mathcal{W} \backslash Z'$ is again isomorphic to $\widetilde{\mathcal{Q}}_s$ and $\mathcal{W} \backslash H'$ to $\widetilde{\mathcal{Q}}_s$ (see [21, Theorem 7.1]). Thus it can be proved as before that $\widetilde{\mathcal{Q}}$ and Looijenga's compactification are isomorphic.

The relationship between $\mathcal{Q}$, $\overline{\mathcal{M}}_3$ and $\overline{\mathcal{M}}_3^{ps}$ are also described by D. Mumford in [23] and by A. Yukie in [35]. In particular A. Yukie constructs a blow-up of $\mathcal{Q}$ which maps to $\overline{\mathcal{M}}_3$.

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Departamento de Matemática
Universidad de Concepción
Casilla 160-C, Concepción
Chile

martebani@udec.cl