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THE COMMON LIMIT OF A QUADRUPLE SEQUENCE AND THE HYPERGEOMETRIC FUNCTION F_D OF THREE VARIABLES

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Dedicated to Professor Masaaki Yoshida on his sixtieth birthday

Abstract. We study a quadruple sequence and express its common limit by Lauricella's hypergeometric function $F_D\big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; z_1, z_2, z_3\big)$ of three variables. We give a functional equation of F_D, which is the key to get our expression of the common limit.

§1. Introduction

For two positive real numbers a_0 and b_0 with $a_0 \geq b_0$, the double sequence $\{a_n\}$ and $\{b_n\}$ given as

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

has a common limit, which is called the arithmetic-geometric mean $M(a_0, b_0)$ of a_0 and b_0. It is shown by C. F. Gauss that $M(a_0, b_0)$ can be expressed by the hypergeometric function:

$$\frac{a_0}{M(a_0, b_0)} = \frac{1}{M(1, x)} = F\Big(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2\Big),$$

where $x = b_0/a_0$.

In this paper, we study a quadruple sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$

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given by

$$(a_0, b_0, c_0, d_0) = (a, b, c, d), \quad a \geq b \geq c \geq d \geq 0,$$

$$(1) \qquad a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4}, \qquad b_{n+1} = \frac{\sqrt{(a_n + d_n)(b_n + c_n)}}{2},$$
$$c_{n+1} = \frac{\sqrt{(a_n + c_n)(b_n + d_n)}}{2}, \qquad d_{n+1} = \frac{\sqrt{(a_n + b_n)(c_n + d_n)}}{2}.$$

We can easily see that it has a common limit $\mu(a, b, c, d)$. Our main theorem is the expression of $\mu(a, b, c, d)$ by Lauricella's hypergeometric function F_D of three variables:

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D\Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - x_1^2, 1 - x_2^2, 1 - x_3^2\Big)^2.$$

The key for our main theorem is Proposition 1, which is the functional equation of the hypergeometric function F_D corresponding to the property

$$\frac{a_0}{a_1}\mu\Big(1, \frac{b_0}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{a_0}\Big) = \mu\Big(1, \frac{b_1}{a_1}, \frac{c_1}{a_1}, \frac{d_1}{a_1}\Big).$$

It turns out that

$$\mu\Big(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n}\Big) F_D\Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - \Big(\frac{b_n}{a_n}\Big)^2, 1 - \Big(\frac{c_n}{a_n}\Big)^2, 1 - \Big(\frac{d_n}{a_n}\Big)^2\Big)^2$$

is independent of n. This fact implies our main theorem. In order to show Proposition 1, we prepare some essential facts for integrable Pfaffian systems in Section 3 and give the integrable Pfaffian system with respect to $F_D(\alpha, \beta, \gamma; z)$ of three variables in Fact 4.

C. W. Borchardt considers in [B] the quadruple sequence

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4}, \qquad b_{n+1} = \frac{\sqrt{a_n b_n} + \sqrt{c_n d_n}}{2},$$
$$c_{n+1} = \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2}, \qquad d_{n+1} = \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2},$$

with positive initial terms a_0, b_0, c_0, d_0. Its common limit $B(a_0, b_0, c_0, d_0)$ is expressed in terms of period integrals of a hyperelliptic curve C of genus 2. J. F. Mestre studies the expression of fixed points under the hyperelliptic involution on C by the initial terms and shows that $B(a_0, b_0, c_0, d_0)$ can be expressed by the arithmetic-geometric mean $M(a_0, c_0)$ when $a_0 = b_0$ and $c_0 = d_0$, refer to [M].

J. M. Borwein and P. B. Borwein consider in [BB] two double sequences

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \sqrt[3]{b_n \frac{a_n^2 + a_n b_n + b_n^2}{3}},$$

and

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \quad b_{n+1} = \sqrt{b_n \frac{a_n + b_n}{2}};$$

they express their common limits $M_3(a_0, b_0)$ and $M_4(a_0, b_0)$ by $F\left(\frac{1}{3}, \frac{2}{3}, 1; z\right)$ and $F\left(\frac{1}{4}, \frac{3}{4}, 1; z\right)$, respectively. We remark that the expression of $M_4(a_0, b_0)$ can be obtained by our main theorem as a special case $b_0 = c_0 = d_0$.

As a generalization of $M_3(a_0, b_0)$, K. Koike and H. Shiga give a triple sequence and express its common limit by Appell's hypergeometric function $F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; z_1, z_2\right)$ of two variables z_1, z_2, refer to [KS1]. They study an extension of the arithmetic-geometric mean and give its expression by Appell's hypergeometric function F_1 with different parameters in [KS2].

For other studies related to the arithmetic-geometric mean, refer to [MM].

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§2. The quadruple sequence

LEMMA 1. *The quadruple sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ given as* (1) *satisfies*

$$a \geq a_{n-1} \geq a_n \geq b_n \geq c_n \geq d_n \geq d_{n-1} \geq d$$

for any $n \in \mathbb{N}$. It has a common limit, which is denoted by $\mu(a, b, c, d)$.

Proof. We assume $a_n \geq b_n \geq c_n \geq d_n \geq 0$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(4a_{n+1})^2 - (4b_{n+1})^2 &= (a_n - b_n - c_n + d_n)^2 \geq 0, \\
(2b_{n+1})^2 - (2c_{n+1})^2 &= (a_n - b_n)(c_n - d_n) \geq 0, \\
(2c_{n+1})^2 - (2d_{n+1})^2 &= (a_n - d_n)(b_n - c_n) \geq 0, \\
a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4} &\leq \frac{a_n + a_n + a_n + a_n}{4} = a_n, \\
d_{n+1} = \frac{\sqrt{(a_n + b_n)(c_n + d_n)}}{2} &\geq \frac{\sqrt{(d_n + d_n)(d_n + d_n)}}{2} = d_n,
\end{aligned}$$

which imply $a_n \geq a_{n+1} \geq b_{n+1} \geq c_{n+1} \geq d_{n+1} \geq d_n \geq 0$. Since the sequences $\{a_n\}$ and $\{d_n\}$ are monotonous and bounded, they converge. Since

$$\begin{aligned} a_{n+1} - d_{n+1} &= \frac{1}{4}(\sqrt{a_n + b_n} - \sqrt{c_n + d_n})^2 \leq \frac{1}{4}(\sqrt{2a_n} - \sqrt{2d_n})^2 \\ &= \frac{1}{2}(a_n + d_n - 2\sqrt{a_n d_n}) \leq \frac{a_n - d_n}{2} \leq \frac{a-d}{2^{n+1}}, \end{aligned}$$

we have $\lim_{n\to\infty}(a_n - d_n) = 0$. Thus the quadruple sequence (1) has a common limit. □

Remark 1. 1. We have

$$\begin{aligned} (4a_{n+1})^2 - (4b_{n+1})^2 &= (a_n - b_n - c_n + d_n)^2, \\ (4a_{n+1})^2 - (4c_{n+1})^2 &= (a_n - b_n + c_n - d_n)^2, \\ (4a_{n+1})^2 - (4d_{n+1})^2 &= (a_n + b_n - c_n - d_n)^2. \end{aligned}$$

2. The quadruple sequence (1) quadratically converges, since

$$a_{n+1} - d_{n+1} \leq \frac{1}{4}(\sqrt{2a_n} - \sqrt{2d_n})^2 = \frac{1}{2}\frac{(a_n - d_n)^2}{(\sqrt{a_n} + \sqrt{d_n})^2}.$$

It is easy to see that

$$\mu(a,b,c,d) = a\mu\Big(1, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}\Big),$$

$$\begin{aligned} &\mu(a,b,c,d) \\ &= \mu\left(\frac{a+b+c+d}{4}, \frac{\sqrt{(a+d)(b+c)}}{2}, \frac{\sqrt{(a+c)(b+d)}}{2}, \frac{\sqrt{(a+b)(c+d)}}{2}\right). \end{aligned}$$

By putting $x_1 = b/a$, $x_2 = c/a$, $x_3 = d/a$ for these equalities, we have the following lemma.

Lemma 2. *Let (y_1, y_2, y_3) be the image of (x_1, x_2, x_3) by the map φ*

$$\begin{aligned} &\varphi(x_1, x_2, x_3) \\ &= \left(\frac{2\sqrt{(1+x_3)(x_1+x_2)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_2)(x_1+x_3)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_1)(x_2+x_3)}}{1+x_1+x_2+x_3}\right). \end{aligned}$$

Then μ satisfies the relation

$$\frac{4}{1+x_1+x_2+x_3}\mu(1, x_1, x_2, x_3) = \mu(1, y_1, y_2, y_3) \tag{2}$$

for $0 < x_3 \leq x_2 \leq x_1 \leq 1$.

§3. Integrable Pfaffian systems

In this section, we prepare some facts of integrable Pfaffian systems. We consider a system of first-order partial differential equations with r unknowns $f_1, \dots, f_r$ and n variables $x_1, \dots, x_n$ in the following form

$$df(x) = \Omega(x) f(x), \tag{3}$$

where $x = (x_1, \dots, x_n)$ is in an open set U, $f(x) = {}^t(f_1(x), \dots, f_r(x))$ and $\Omega(x)$ is an $r \times r$ matrix whose entries are 1-forms on U. The system (3) is called a Pfaffian system on U and $\Omega(x)$ is called the connection matrix of (3). If $\Omega(x)$ satisfies the integrability condition

$$d\Omega(x) = \Omega(x) \wedge \Omega(x),$$

then the system (3) is integrable.

FACT 1. 1. *The system* (3) *has exactly* r *linearly independent vector valued solutions if and only if it is integrable.*

2. *If the system* (3) *is integrable, then there exists a unique solution* f *around* $u \in U$ *such that* $f(u) = p$ *for a given initial vector* $p \in \mathbb{C}^r$.

FACT 2. *For an integrable Pfaffian system* (3) *and an invertible* $r \times r$ *functional matrix* $P(x)$ *on* U*, the vector valued function* $g(x) = P(x)f(x)$ *satisfies the Pfaffian system*

$$dg(x) = [P(x)\Omega(x)P(x)^{-1} + dP(x)P(x)^{-1}]g(x).$$

Let f_0 be a function of n-variables $(y_1, \dots, y_n)$ on an open set V. We assume that the vector valued function

$$f(y) = {}^t\left(f_0(y), \frac{\partial f_0}{\partial y_1}(y), \dots, \frac{\partial f_0}{\partial y_n}(y)\right)$$

satisfies an integrable Pfaffian system

$$df(y) = \Omega(y) f(y)$$

on V. Let η be a map from an open set U to V given as

$$\eta : U \ni x = (x_1, \dots, x_n) \longmapsto y = (\eta_1(x), \dots, \eta_n(x)) \in V,$$

and let J be the Jacobi matrix of η:

$$J = \left(\frac{\partial \eta_i}{\partial x_j}\right)_{ij} = \begin{pmatrix} \frac{\partial \eta_1}{\partial x_1} & \cdots & \frac{\partial \eta_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial \eta_n}{\partial x_1} & \cdots & \frac{\partial \eta_n}{\partial x_n} \end{pmatrix}.$$

FACT 3. *If* $\det(J) \neq 0$ *on* U *then the function* $h_0(x) = f_0(\eta(x))$ *satisfies*

$$dh(x) = [\tilde{J}\Omega(x)\tilde{J}^{-1} + d\tilde{J}\tilde{J}^{-1}]h(x),$$

where

$$h(x) = {}^t\left(h_0(x), \frac{\partial h_0}{\partial x_1}(x), \dots, \frac{\partial h_0}{\partial x_n}(x)\right), \quad \tilde{J} = \begin{pmatrix} 1 & \\ & {}^tJ \end{pmatrix},$$

and $\Omega(x)$ *is the pull-back of* $\Omega(y)$ *under the map* η.

§4. Lauricella's hypergeometric function F_D

Lauricella's hypergeometric function F_D of m-variables $z_1, \dots, z_m$ with parameters $\alpha, \beta_1, \dots, \beta_m, \gamma$ is defined as

$$F_D(\alpha, \beta, \gamma; z) = \sum_{n_1,\dots,n_m \geq 0}^{\infty} \frac{(\alpha, \sum_{j=1}^m n_j) \prod_{j=1}^m (\beta_j, n_j)}{(\gamma, \sum_{j=1}^m n_j) \prod_{j=1}^m (1, n_j)} \prod_{j=1}^m z_j^{n_j},$$

where $z = (z_1, \dots, z_m)$ satisfies $|z_j| < 1$ $(j = 1, \dots, m)$, $\beta = (\beta_1, \dots, \beta_m)$, $\gamma \neq 0, -1, -2, \dots$ and $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This function admits the integral representation:

$$F_D(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\gamma-\alpha} \prod_{j=1}^m (1 - z_j t)^{-\beta_j} \frac{dt}{t(1-t)}.$$

When $m = 1$, $F_D(\alpha, \beta, \gamma; z)$ coincides with the Gauss hypergeometric function $F(\alpha, \beta, \gamma; z)$. We consider $F_D(\alpha, \beta_1, \beta_2, \beta_3, \gamma; z_1, z_2, z_3)$ of three variables.

FACT 4. *The function* $F_D(\alpha, \beta, \gamma; z)$ *of three variables satisfies the integrable Pfaffian system given as*

$$df = \sum_{1 \leq i < j \leq 5} A_{ij} d\log(z_i - z_j) f,$$

where $f = {}^t(f_0, f_1, f_2, f_3)$, $f_0 = F_D(\alpha, \beta, \gamma; z)$, $f_i = z_i \frac{\partial f_0}{\partial z_i}$ $(i = 1, 2, 3)$, $z_4 = 0$, $z_5 = 1$ *and*

$$A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_2 & \beta_1 & 0 \\ 0 & \beta_2 & -\beta_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1+\beta_2+\beta_3-\gamma & 0 & 0 \\ 0 & -\beta_2 & 0 & 0 \\ 0 & -\beta_3 & 0 & 0 \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_3 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & \beta_3 & 0 & -\beta_1 \end{pmatrix}, \quad A_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 1+\beta_1+\beta_3-\gamma & 0 \\ 0 & 0 & -\beta_3 & 0 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_3 & \beta_2 \\ 0 & 0 & \beta_3 & -\beta_2 \end{pmatrix}, \quad A_{34} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 1+\beta_1+\beta_2-\gamma \end{pmatrix},$$

$$A_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta_1 & \gamma-\alpha-\beta_1-1 & -\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_2 & -\beta_2 & \gamma-\alpha-\beta_2-1 & -\beta_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_3 & -\beta_3 & -\beta_3 & \gamma-\alpha-\beta_3-1 \end{pmatrix}.$$

Remark 2. The A_{ij} and $A_{i,n+1}$ in the proof of Proposition 9.1.4 in [IKSY] are wrong. Professor K. Ohara informed us of the correct Pfaffian system given by the system [O].

§5. Main theorem

THEOREM 1. *For any numbers* x_1, x_2, x_3 *satisfying* $0 < x_3 \leq x_2 \leq x_1 \leq 1$, *we have*

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D\Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1-x_1^2, 1-x_2^2, 1-x_3^2\Big)^2,$$

where $\mu(1,x_1,x_2,x_3)$ is the common limit of the quadruple sequence (1) *with initial* $(1,x_1,x_2,x_3)$ *and F_D is Lauricella's hypergeometric function.*

We put

$$F(z_1,z_2,z_3)=F_D\Big(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},1;z_1,z_2,z_3\Big).$$

PROPOSITION 1. *The function F satisfies*

$$\frac{1+x_1+x_2+x_3}{4}F(1-x_1^2,1-x_2^2,1-x_3^2)^2=F(1-y_1^2,1-y_2^2,1-y_3^2)^2$$
$$=F\left(\left(\frac{1-x_1-x_2+x_3}{1+x_1+x_2+x_3}\right)^2,\left(\frac{1-x_1+x_2-x_3}{1+x_1+x_2+x_3}\right)^2,\left(\frac{1+x_1-x_2-x_3}{1+x_1+x_2+x_3}\right)^2\right)^2,$$

where $(y_1,y_2,y_3)=\varphi(x_1,x_2,x_3)$ is defined in Lemma 2.

Proof. Put

$$(\xi_1,\xi_2,\xi_3)=\left(\frac{1-x_1-x_2+x_3}{1+x_1+x_2+x_3},\frac{1-x_1+x_2-x_3}{1+x_1+x_2+x_3},\frac{1+x_1-x_2-x_3}{1+x_1+x_2+x_3}\right).$$

Then we have

$$(x_1,x_2,x_3)=\left(\frac{1-\xi_1-\xi_2+\xi_3}{1+\xi_1+\xi_2+\xi_3},\frac{1-\xi_1+\xi_2-\xi_3}{1+\xi_1+\xi_2+\xi_3},\frac{1+\xi_1-\xi_2-\xi_3}{1+\xi_1+\xi_2+\xi_3}\right),$$
$$\frac{1+x_1+x_2+x_3}{4}=\frac{1}{1+\xi_1+\xi_2+\xi_3},$$
$$(1-x_1^2,1-x_2^2,1-x_3^2)$$
$$=\left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right).$$

Thus the equality in Proposition 1 is equivalent to

$$\sqrt{1+\xi_1+\xi_2+\xi_3}F(\xi_1^2,\xi_2^2,\xi_3^2)$$
$$=F\left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right)$$

for $0\le\xi_1\le\xi_2\le\xi_3<1$. We show that the Pfaffian systems obtained by the functions in the both sides of the above equality coincide.

Let $\Omega(x)$ be the connection 1-form in Fact 4 for $\alpha=\beta_1=\beta_2=\beta_3=1/4$ and $\gamma=1$. Fact 2 implies that the vector valued function

$$g(x)={}^t\!\left(F,\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2},\frac{\partial F}{\partial x_3}\right)$$

satisfies the Pfaffian system $dg = \Omega_1(x)g$, where

$$\Omega_1(x) = P\Omega(x)P^{-1} + dPP^{-1},$$

$$P = \mathrm{diag}\Big(1, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\Big) = \begin{pmatrix} 1 & & & \\ & \frac{1}{x_1} & & \\ & & \frac{1}{x_2} & \\ & & & \frac{1}{x_3} \end{pmatrix}.$$

The vector valued function

$$h(\xi) = {}^t\Big(h_0, \frac{\partial h_0}{\partial \xi_1}, \frac{\partial h_0}{\partial \xi_2}, \frac{\partial h_0}{\partial \xi_3}\Big)$$

for

$$h_0(\xi_1, \xi_2, \xi_3) = \sqrt{1+\xi_1+\xi_2+\xi_3}F(\xi_1^2, \xi_2^2, \xi_3^2)$$

satisfies $h(0,0,0) = {}^t(1, 1/2, 1/2, 1/2)$ and the Pfaffian system $dh = \Omega_2(\xi)h$, where

$$\Omega_2(\xi) = Q[J_1\Omega_1(\xi)J_1^{-1} + dJ_1J_1^{-1}]Q^{-1} + dQQ^{-1}, \quad J_1 = \mathrm{diag}(1, 2\xi_1, 2\xi_2, 2\xi_3),$$

$$Q = \begin{pmatrix} \zeta & & & \\ \frac{1}{2\zeta} & \zeta & & \\ \frac{1}{2\zeta} & & \zeta & \\ \frac{1}{2\zeta} & & & \zeta \end{pmatrix}, \quad \zeta = \sqrt{1+\xi_1+\xi_2+\xi_3},$$

and $\Omega_1(\xi)$ is the pull-back of $\Omega_1(x)$ under the map

$$(\xi_1, \xi_2, \xi_3) \longmapsto (x_1, x_2, x_3) = (\xi_1^2, \xi_2^2, \xi_3^2).$$

On the other hand, the vector valued function

$$h(x) = {}^t\Big(h_0, \frac{\partial h_0}{\partial \xi_1}, \frac{\partial h_0}{\partial \xi_2}, \frac{\partial h_0}{\partial \xi_3}\Big)$$

for

$$h_0(\xi_1, \xi_2, \xi_3) = F\Big(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\Big)$$

satisfies $h(0,0,0) = {}^t(1, 1/2, 1/2, 1/2)$ and the Pfaffian system $dh = \Omega_3(\xi)h$, where

$$\Omega_3(\xi) = J_2\Omega_1'(\xi)J_2^{-1} + dJ_2J_2^{-1}, \quad J_2 = \begin{pmatrix} 1 & \\ & {}^tJ \end{pmatrix},$$

$\Omega_1'(\xi)$ is the pull-back of $\Omega_1(x)$ under the map

$$\varphi' : (\xi_1, \xi_2, \xi_3) \mapsto \left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right),$$

and J is the Jacobi matrix of the map φ'. By a straight forward calculation, we can show that $\Omega_2(\xi) = \Omega_3(\xi)$. Thus we have the required equality around $\xi = (0, 0, 0)$. ∎

Proof of Theorem 1. Consider the quadruple sequence (1) with initial $(a_0, b_0, c_0, d_0) = (1, x_1, x_2, x_3)$. Lemma 2 and Proposition 1 imply that

$$\begin{aligned} &\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 \\ &= \mu(1, y_1, y_2, y_3)F(1 - y_1^2, 1 - y_2^2, 1 - y_3^2)^2. \end{aligned}$$

Thus we have

$$\begin{aligned} &\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 \\ &= \mu\Big(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n}\Big)F\Big(1 - \Big(\frac{b_n}{a_n}\Big)^2, 1 - \Big(\frac{c_n}{a_n}\Big)^2, 1 - \Big(\frac{d_n}{a_n}\Big)^2\Big)^2 \end{aligned}$$

for any $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{c_n}{a_n} = \lim_{n\to\infty} \frac{d_n}{a_n} = 1,$$

and $\mu(1, 1, 1, 1) = F(0, 0, 0) = 1$, we have

$$\begin{aligned} &\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 \\ &= \lim_{n\to\infty} \mu\Big(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n}\Big)F\Big(1 - \Big(\frac{b_n}{a_n}\Big)^2, 1 - \Big(\frac{c_n}{a_n}\Big)^2, 1 - \Big(\frac{d_n}{a_n}\Big)^2\Big)^2 \\ &= \mu(1, 1, 1, 1)F(0, 0, 0)^2 = 1, \end{aligned}$$

which is the desired equality. ∎

COROLLARY 1. *For $1 > x_1 \geq x_2 \geq x_3 \geq 0$, we have*

$$F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2) = \prod_{n=0}^{\infty} \sqrt{\frac{a_n}{a_{n+1}}},$$

where we set the initial of the quadruple sequence (1) *as* $(a_0, b_0, c_0, d_0) = (1, x_1, x_2, x_3)$.

Proof. By Proposition 1, we have

$$
\begin{aligned}
&F(1-x_1^2, 1-x_2^2, 1-x_3^2)\\
&= \frac{2}{\sqrt{1+x_1+x_2+x_3}} F(1-y_1^2, 1-y_2^2, 1-y_3^2)\\
&= \sqrt{\frac{4a_0}{a_0+b_0+c_0+d_0}} F\left(1-\frac{b_1^2}{a_1^2}, 1-\frac{c_1^2}{a_1^2}, 1-\frac{d_1^2}{a_1^2}\right)\\
&= \sqrt{\frac{a_0}{a_1}}\sqrt{\frac{a_1}{a_2}} F\left(1-\frac{b_2^2}{a_2^2}, 1-\frac{c_2^2}{a_2^2}, 1-\frac{d_2^2}{a_2^2}\right)\\
&= \left(\prod_{i=0}^{n-1}\sqrt{\frac{a_i}{a_{i+1}}}\right) F\left(1-\frac{b_n^2}{a_n^2}, 1-\frac{c_n^2}{a_n^2}, 1-\frac{d_n^2}{a_n^2}\right),
\end{aligned}
$$

which implies this corollary. □

§6. A specialization

For the case $b = c = d$, the quadruple sequence reduces to

$$
a_{n+1} = \frac{a_n + 3b_n}{4}, \quad b_{n+1} = c_{n+1} = d_{n+1} = \sqrt{b_n \frac{a_n + b_n}{2}},
$$

which is studied in [BB]. It is shown that the reciprocal of the common limit of the double sequences is $F\left(\frac{1}{4}, \frac{3}{4}, 1; 1-x^2\right)^2$, where $x = b/a$ and $F(\alpha, \beta, \gamma; z)$ is the Gauss hypergeometric function. By our main theorem, we have

$$
\begin{aligned}
&F_D\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1-x^2, 1-x^2, 1-x^2\right)^2\\
&= \frac{1}{\mu(1, x, x, x)} = \frac{1}{M_4(1, x)} = F\left(\frac{1}{4}, \frac{3}{4}, 1; 1-x^2\right)^2.
\end{aligned}
$$

Note that the above reduction of F_D to F can be easily obtained by the integral representation of F_D.

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