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TRUNCATED EULER SYSTEMS OVER IMAGINARY QUADRATIC FIELDS

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Abstract. Let K be an imaginary quadratic field and let F be an abelian extension of K. It is known that the order of the class group Cl_F of F is equal to the order of the quotient U_F/El_F of the group of global units U_F by the group of elliptic units El_F of F. We introduce a filtration on U_F/El_F made from the so-called truncated Euler systems and conjecture that the associated graded module is isomorphic, as a Galois module, to the class group. We provide evidence for the conjecture using Iwasawa theory.

§1. Introduction

Let F be a number field and $\mathcal{O}_F$ the ring of integers of F. The ideal class group Cl_F of F is related with various subgroups of the global units $U_F = \mathcal{O}_F^{\times}$ of F. Among the most fundamental subgroups are the circular units, the elliptic units, and the modular units of F. When F is an abelian field, Sinnott formulated the class number formulas of F after Kummer, Hasse, and Iwasawa (cf. [20] and [21]). For the case of the elliptic units, let $F/\mathbb{Q}$ be an abelian extension containing a quadratic imaginary field K. The argument of the Euler system of Rubin provides us a way to reformulate these units as higher special units coming from the so-called truncated Euler systems of F (cf. [9], [13], [14], [15] and [22]). In this paper, p always denotes an odd prime. In [19], we introduced a filtration to U_F made from the truncated Euler systems having the circular units as the last term and conjectured that the associated graded module is isomorphic, as a Galois module, to the class group of F when F is a real abelian field. We will

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extend the conjecture above to an arbitrary abelian extension of an imaginary quadratic field with elliptic units. Namely, we introduce a filtration to U_F made from the truncated Euler systems having the elliptic units as the last term and conjecture that the associated graded module is isomorphic, as a Galois module, to the class group of F when F is an abelian extension of an imaginary quadratic field. For any subgroup A of U_F, the profinite p-completion $\varprojlim A/A^{p^n}$ of A can be identified with $A \otimes \mathbb{Z}_p$. Since the Leopoldt's conjecture is true for our abelian fields, this will be also identified with the topological closure of A inside the group of local units of $F \otimes_K K_{\mathfrak{p}} = \prod_{\mathfrak{B}|\mathfrak{p}} F_{\mathfrak{B}}$, which are congruent one modulo the primes above $\mathfrak{p}$ via the natural diagonal embedding $F \hookrightarrow \prod_{\mathfrak{B}|\mathfrak{p}} F_{\mathfrak{B}}$.

In Section 2, we will formulate a conjecture on the structure of the ideal class group by using higher special units. This is an analogue of the real abelian extension (cf. [19]). Let $S^r_{F/K}$ denote the higher special units of depth $r \geq 1$ and $S^0_{F/K} = U_F$. For a finite abelian group A we denote by $A^{(p)}$ the p-Sylow subgroup of A. For each natural number n, let $\mathrm{gr}_n(S_{F/K})$ denote the quotient $S^{n-1}_{F/K}/S^n_{F/K}$ of the two consecutive higher special units. We define $\mathrm{gr}(S^{(p)}_{F/K})$ to be the direct sum

$$\mathrm{gr}(S^{(p)}_{F/K}) = \bigoplus_{n\geq 1} \mathrm{gr}_n(S_{F/K})^{(p)}.$$

For a Galois extension L/K, write $G(L/K)$ for its Galois group. Now we suppose that p does not divide the extension degree $[F:K]$. On the structure of the ideal class group of F, we give the following conjecture.

Conjecture. *If $p \nmid [F:K]$, then $\mathrm{Cl}^{(p)}_F \cong \mathrm{gr}(S^{(p)}_{F/K})$ as $\mathbb{Z}_p[G(F/K)]$-modules.*

Let Ξ be the set of all irreducible $\mathbb{Z}_p$-representations of $G(F/K)$. For each $\chi \in \Xi$, let e^χ denote the χ-idempotent

$$e^\chi = 1/[F:K] \sum_{\sigma \in G(F/K)} \mathrm{Tr}(\chi(\sigma))\sigma^{-1}$$

where Tr is the trace map from $\mathbb{Z}_p[\mathrm{image}(\chi)]$ to $\mathbb{Z}_p$. For each $\mathbb{Z}[G(F/K)]$-module $\mathcal{M}$ of finite type and $\chi \in \Xi$, we let $\mathcal{M}^\chi$ denote the χ-component $e^\chi(\mathcal{M} \otimes \mathbb{Z}_p)$ of $\mathcal{M} \otimes \mathbb{Z}_p$. Let $\mathrm{gr}_n(S^\chi_{F/K}) = S^{n-1\,\chi}_{F/K}/S^{n\,\chi}_{F/K}$ and $\mathrm{gr}(S^\chi_{F/K}) = \bigoplus_{n\geq 1} \mathrm{gr}_n(S^\chi_{F/K})$. The conjecture above can be formulated in terms of χ.

CONJECTURE$^{\chi}$. *If $p \nmid [F:K]$, then for each $\chi \in \Xi$, $\mathrm{Cl}_F^{\chi} \cong \mathrm{gr}(S_{F/K}^{\chi})$ as $\mathbb{Z}_p[G(F/K)]^{\chi}$-modules.*

We denote by p^{s_i} the exponent of $\mathrm{gr}_i(S_{F/K}^{\chi})$. Let $\dim(\chi) = (\mathbb{Q}_p(\mathrm{image}(\chi)) : \mathbb{Q}_p)$ denote the dimension of χ. Notice that $\mathrm{gr}_i(S_{F/K}^{\chi})$ is isomorphic to $(\mathbb{Z}/p^{s_i}\mathbb{Z})^{\dim(\chi)}$. As evidence for the conjecture, we give the following theorem.

THEOREM 1.1. *Let F be an abelian extension of an imaginary quadratic field K and p be an odd prime such that $p \nmid [F:K]$. Fix $\chi \in \Xi$. Let $\mathrm{Cl}_F^{\chi} = \bigoplus_{i=1}^{k}(\mathbb{Z}/p^{r_i}\mathbb{Z})^{\dim(\chi)}$ with $0 \neq r_k \leq \cdots \leq r_1$. Then we have $\sum_{i=1}^{a} r_i \leq \sum_{i=1}^{a} s_i$ for $1 \leq a \leq k$ and $\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i$.*

We now compare two conjectures using the class field theory. Suppose now that F is an abelian extension of $\mathbb{Q}$ containing an imaginary quadratic field K. Let F^+ denote the maximal real subfield of F. Then we have the following conjecture as was introduced in [19].

CONJECTURE^{+}. *If $p \nmid [F:K]$, then $\mathrm{Cl}_{F^+}^{(p)} \cong \mathrm{gr}(S_{F^+/\mathbb{Q}}^{(p)})$ as $\mathbb{Z}_p[G(F^+/\mathbb{Q})]$-modules.*

Let H_F and H_{F^+} denote respectively the Hilbert class fields of F and F^+. Let $N_+ = N_{F^+/F}$ denote the norm map from F to F^+. Since H_{F^+} and F are linearly disjoint over F^+, we have the following surjection

$$G(H_F/F) \xrightarrow{\mathrm{res}_{H_{F^+}}} G(H_{F^+}/F^+) \longrightarrow 0$$

where $\mathrm{res}_{H_{F^+}}$ denotes the restriction map from $G(H_F/F)$ to $G(H_{F^+}/F^+)$. Since the Artin symbol satisfies $\mathrm{res}_{H_{F^+}}(\mathfrak{p}, H_F/F) = (N_+\mathfrak{p}, H_{F^+}/F^+)$, we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(N_+) & \longrightarrow & \mathrm{Cl}_F & \xrightarrow{N_+} & \mathrm{Cl}_{F^+} & \longrightarrow & 0 \\
 & & \big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\
0 & \longrightarrow & G(H_F/H_{F^+}F) & \longrightarrow & G(H_F/F) & \xrightarrow{\mathrm{res}_{H_{F^+}}} & G(H_{F^+}/F^+) & \longrightarrow & 0
\end{array}$$

Since all the fields are abelian over $\mathbb{Q}$, the norm N_+ is a Galois equivariant map. Note also that $F = F^+K$ and F^+ and K are linearly disjoint over $\mathbb{Q}$.

Hence, by applying the norm N_+ to CONJECTURE, we have if $p \nmid [F : K]$, then as $\mathbb{Z}_p[G(F/K)] \cong \mathbb{Z}_p[G(F^+/\mathbb{Q})]$-modules

$$\mathrm{Cl}_{F^+}^{(p)} \cong N_+(\mathrm{gr}(S_{F/K}^{(p)})) = \mathrm{gr}(N_+(S_{F/K}^{(p)})).$$

Combining this with CONJECTURE$^+$ above, presumably, we have the following isomorphism

$$\mathrm{gr}_n\, S_{F^+/\mathbb{Q}}^{(p)} \cong N_+\, \mathrm{gr}_n(S_{F/K})^{(p)} \tag{1}$$

for all $n \geq 0$. Thus, CONJECTURE implies *essentially* CONJECTURE$^+$,

$$(1) + \text{CONJECTURE} \Longrightarrow \text{CONJECTURE}^+.$$

Over the cyclotomic $\mathbb{Z}_p$-extension $F_\infty = \bigcup_{n=0}^{\infty} F_n$ of F, we let El_n denote the elliptic units of F_n, and El_∞ denote the inverse limits of the profinite p-completion of El_n with respect to the norm maps. We will use similar notations for various Galois modules. Let Λ denote the Iwasawa algebra as defined in page 5 of the next section. Let $\mathrm{char}(\mathcal{M})$ denote the characteristic ideal of the finitely generated torsion Iwasawa Λ-module $\mathcal{M}$. For a finite abelian group A, we define the p-rank $\mathrm{rk}_p(A)$ to be,

$$\mathrm{rk}_p(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} A \otimes \mathbb{Z}/p\mathbb{Z}.$$

Let Cl_n denote the ideal class group of F_n, and let $w = w_F$ denote the maximum of $\mathrm{rk}_p(\mathrm{Cl}_n)$ as n varies, which is a well-defined invariant of F/K from a theorem of Ferrero-Washington.

THEOREM 1.2. *Suppose $p \nmid [F : K]$. Then for all $i \geq w$, the main conjecture implies*

$$\mathrm{char}(S_{F/K,\infty}^{i}/El_\infty) = 1.$$

Moreover, if for all sufficiently large $m > n \gg 0$, N_{F_m/F_n} induces an epimorphism over $\{S_{F_m/K}^{i} \otimes \mathbb{Z}_p\}_{m \gg 0}$ then

$$(S_{F_n/K}^{i}/El_n) \otimes \mathbb{Z}_p = 1$$

for all $n \geq 0$ and $i \geq w$.

In the following theorems, we allow F to be any abelian field, real or imaginary. Let $\mathcal{C}_n$ denote the group of cyclotomic units of F_n in the sense of Sinnott (cf. [20], [21]). Finally, let $\mathcal{C}_\infty$ denote the inverse limit of $\mathcal{C}_n$ with respect to the norm maps.

THEOREM 1.3. *Let F be an abelian extension of $\mathbb{Q}$ such that $p \nmid [F : \mathbb{Q}]$ and $i \geq w$. Then*

$$\mathrm{char}(S^i_{F/\mathbb{Q},\infty}/\mathcal{C}_\infty) = 1.$$

Let l ($\neq p$) be a fixed prime which is prime to $[F : \mathbb{Q}]$ and let $F_{n,l^\infty} = \bigcup_s F_{n,l^s}$ denote the cyclotomic $\mathbb{Z}_l$-extension of the field F_n.

COROLLARY 1.4. *If for all sufficiently large $m > n \gg 0$, N_{F_m/F_n} induces an epimorphism over $\{S^i_{F_m/\mathbb{Q}} \otimes \mathbb{Z}_p\}_{m \gg 0}$ then $(S^i_{F_n/\mathbb{Q}}/\mathcal{C}_n) \otimes \mathbb{Z}_p = 1$ for all $n \geq 0$ and $i \geq w$. Moreover, if for all sufficiently large $s > t \gg 0$, $N_{F_{n,l^s}/F_{n,l^t}}$ induces an epimorphism over $\{S^i_{F_{n,l^s}} \otimes \mathbb{Z}_l\}_{s \gg 0}$ then $\mathcal{C}_n \otimes \mathbb{Z}_l$ is equal to $S^i_{F_n/\mathbb{Q}} \otimes \mathbb{Z}_l$.*

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§2. Truncated Euler systems over imaginary quadratic fields

We briefly introduce truncated Euler systems of fixed depth r. In this and the next sections, let F be an abelian extension of K containing the Hilbert class field of K and M the maximum of the squares of the cardinalities of the χ-ideal class groups Cl^χ_F over all $\chi \in \Xi$. From now on, we fix a prime p which is prime to $[F : K]$. Let $\mathrm{I}^r_{F/K,M}$ be the set of square-free integral fractional ideals $\mathfrak{a}$ of K, such that each prime $\mathfrak{l}$ dividing $\mathfrak{a}$ has an absolute degree of one and splits completely in F, $N_{K/\mathbb{Q}}(\mathfrak{l}) \equiv 1 \pmod{M}$, and the number of primes dividing $\mathfrak{a}$ is less than or equal to r. Moreover, if F is an abelian extension of the rational field $\mathbb{Q}$, then we delete the finite set of primes dividing the conductor of F. For each prime ideal $\mathfrak{l}$ of K, let $K(\mathfrak{l})$ denote the ray class field of K modulo $\mathfrak{l}$. Then we have the following lemma.

LEMMA 2.1 (= Lemma 1.1 of [14]). *Suppose $\mathfrak{l} \in \mathrm{I}^r_{F/K,M}$. There is a unique extension $F(\mathfrak{l})$ of F of degree M in $FK(\mathfrak{l})$. Furthermore, $F(\mathfrak{l})/F$ is cyclic, totally ramified at all primes above $\mathfrak{l}$, and unramified at all primes not dividing $\mathfrak{l}$.*

Let $F(\mathfrak{a})$ be the composite of $F(\mathfrak{l})$ over all prime divisors $\mathfrak{l}$ of $\mathfrak{a}$. For an integral ideal $\mathfrak{b}$ of K, let $\mathrm{I}^r_{F/K,M}(\mathfrak{b})$ be the set of all integral ideals $\mathfrak{a}$ of $\mathrm{I}^r_{F/K,M}$ such that $\mathfrak{a}$ is prime to $\mathfrak{b}$. Let $\mathcal{E}^r_{F/K,M}(\mathfrak{b})$ be the set of maps ψ from $\mathrm{I}^r_{F/K,M}(\mathfrak{b})$

to a fixed algebraic closure F^{alg}, such that for each $\mathfrak{m}, \mathfrak{n} \in \mathrm{I}^r_{F/K,M}(\mathfrak{b})$ with $\mathfrak{n}|\mathfrak{m}$, $\psi(\mathfrak{m}) \in F(\mathfrak{m})$,

$$N_{F(\mathfrak{m})/F(\mathfrak{n})}\psi(\mathfrak{m}) = \psi(\mathfrak{n})^{\prod_{\mathfrak{p}|\mathfrak{m},\mathfrak{p}\nmid\mathfrak{n}}(\mathrm{Frob}_{\mathfrak{p}}-1)}$$

and $\psi(\mathfrak{n}\mathfrak{l})$ is congruent to $\psi(\mathfrak{n})^{(N_{K/\mathbb{Q}}(\mathfrak{l})-1)/M}$ modulo primes over $\mathfrak{l}$, whenever $\mathfrak{n}$ is prime to $\mathfrak{l}$. These conditions will be called product and congruence conditions, respectively. For the fixed power M of p, we define truncated Euler systems $\mathcal{E}^r_{F/K,M}$ of depth r to be the disjoint union $\coprod \mathcal{E}^r_{F/K,M}(\mathfrak{b})$ of $\mathcal{E}^r_{F/K}(\mathfrak{b})$ over all ideals $\mathfrak{b}$ of $\mathcal{O}_K$. Often, we will denote $\mathcal{E}^r_{F/K,M}$ by $\mathcal{E}^r_{F/K}$ by omitting the subscript M. We define the higher special units $S^r_{F/K}$ of depth r to be

$$S^r_{F/K} = \langle \psi(\mathcal{O}_K) \mid \psi \in \mathcal{E}^r_{F/K,M_K} \rangle \cap U_K.$$

As in the introduction, let $\mathrm{gr}_n(S_{F/K})$ denote the quotient $S^{n-1}_{F/K}/S^n_{F/K}$ of the consecutive higher special units and $\mathrm{gr}(S^{(p)}_{F/K})$ the direct sum

$$\mathrm{gr}(S^{(p)}_{F/K}) = \bigoplus_{n\geq 1} \mathrm{gr}_n(S_{F/K})^{(p)}.$$

Notice that for all n, $\mathrm{gr}_n(S_{F/K})^{(p)}$ is finite and its order is bounded by the p-part of class number of F. Since p is prime to $[F:K]$ and elliptic units are contained in the higher special units of all depths (cf. Proposition 3.1), this follows from the class number formula of elliptic units (cf. Theorem 1.3 of [14]). We are ready to give our conjecture.

Conjecture. *If $p \nmid [F:K]$, then $\mathrm{Cl}^{(p)}_F \cong \mathrm{gr}(S^{(p)}_{F/K})$ as $\mathbb{Z}_p[G(F/K)]$-modules.*

This conjecture is an analogue of the conjecture in [19] for an imaginary quadratic field. If $\overline{U}_F$ and $\overline{S}^i_{F/K}$ denote respectively the natural images of U_F and $S^i_{F/K}$ in $\varprojlim F^\times/(F^\times)^{p^n}$, then the free part $(\overline{U}^\chi_F)_{\mathrm{fr}}$ of $\overline{U}^\chi_F$ is a free $\mathbb{Z}_p[G(F/K)]^\chi$-module of rank one. Since $\mathbb{Z}_p[G(F/K)]^\chi$ is a free $\mathbb{Z}_p$-module of rank $\dim(\chi)$, we have the following isomorphisms,

$$(U_F/S^i_{F/K})^\chi = (\overline{U}^\chi_F)_{\mathrm{fr}}/(\overline{S^i}^\chi_{F/K})_{\mathrm{fr}} = (\mathbb{Z}/p^{e_i}\mathbb{Z})[G(F/K)]^\chi = (\mathbb{Z}/p^{e_i}\mathbb{Z})^{\dim(\chi)}$$

for some e_i. As defined in the introduction, p^{s_i} denotes the exponent of the quotient of the higher special units $(S^{i-1}_{F/K}/S^i_{F/K})^\chi$, and $\beta_i = s_i \dim(\chi)$. We need the following proposition.

PROPOSITION 2.2. *If A_i is any subgroup of $\mathrm{Cl}_F(\chi)$ generated by i-elements, then $\#(A_i) \mid \#(U_F^{\chi}/S_{F/K}^{i\,\chi}) = p^{\sum_{j=1}^{i}\beta_j}$.*

Proof. It follows from the same argument of Proposition 2.2 of [19]. ∎

THEOREM 2.3. *Let F be an abelian extension of an imaginary quadratic field K and p be an odd prime such that $p \nmid [F : K]$. Fix $\chi \in \Xi$. Let $\mathrm{Cl}_F^{\chi} = \bigoplus_{i=1}^{k}(\mathbb{Z}/p^{r_i}\mathbb{Z})^{\dim(\chi)}$ with $0 \neq r_k \leq \cdots \leq r_1$. Then we have $\sum_{i=1}^{a} r_i \leq \sum_{i=1}^{a} s_i$ for $1 \leq a \leq k$ and $\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i$.*

Proof. It follows similarly from Proposition 2.2 above, Theorem 3.2 of [14] and Theorem 2.5 of [19]. ∎

Let $F_\infty = \bigcup_{n=0}^{\infty} F_n \supset \cdots \supset F_1 \supset F_0$ be the cyclotomic $\mathbb{Z}_p$-extension of $F_0 = F$ with $[F_n : F] = p^n$. Note that F_∞ is an abelian extension of $\mathbb{Q}$. The group $G(F_\infty/K)$ has a direct decomposition $G(F_\infty/K) = G(F_\infty/F) \times G(F/K)$ into the p-part $G(F_\infty/F)$ and the prime to p-part $G(F_0/K)$. Let $\Gamma = G(F_\infty/F)$. Let $R = \varprojlim \mathbb{Z}_p[G(F_n/K)]$ be the completed group ring of $\mathbb{Z}_p[G(F_\infty/K)]$. We have $R = \Lambda[G(F/K)]$, where

$$\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\Gamma/\Gamma_n]$$

and $\Gamma_n = \Gamma^{p^n}$ is the unique subgroup of Γ of index p^n. A pseudo-isomorphism of Λ-modules is a map with finite kernel and cokernel. It follows from the structure theorem of finitely generated torsion Λ-modules that every finitely generated torsion Λ-module Y is pseudo-isomorphic to

$$\prod \Lambda/f_i\Lambda.$$

The characteristic ideal $\mathrm{char}(Y)$ of Y is the ideal $(\prod f_i)\Lambda$ which is a well-defined invariant of Y. In order to define the inverse limit $S_{F/K,\infty}^{i}$ of the higher special units $S_{F_n/K}^{i} \otimes \mathbb{Z}_p$, we need to define a certain map from the Euler systems $\mathcal{E}_{L/K}^{r}$ to the Euler systems $\mathcal{E}_{F/K}^{r}$, where L/F is a totally ramified extension at a prime over p and $L/\mathbb{Q}$ is an abelian extension.

Remark. Notice that the primes of K which splits completely in F/K need not splits completely in L/K. However, the Chebotarev density theorem guarantees an existence of infinitely many primes $\mathfrak{l}$ in each element of the ideal class group of F such that $\mathfrak{l}$ splits completely in L/K and the method of Theorem 3.1 of [13] or Theorem 3.1 of [14] works well. In the argument to follow, in order to define Euler systems in the image of $\Psi^{L/K}$, we will use the new set $\tilde{\mathrm{I}}^r_{F/K,M}$ of integral ideals which are divisible by these primes, i.e., the subset of $\mathrm{I}^r_{F/K,M}$ integral ideals whose prime divisors are splits completely in L/K. We will use the same notations for these Euler systems defined over $\tilde{\mathrm{I}}^r_{F/K,M}$ as those defined over $\mathrm{I}^r_{F/K,M}$.

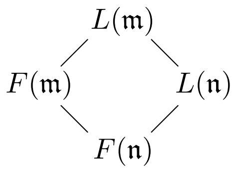
Based on the remark above, we define the map $\Psi^{L/F} : \mathcal{E}^r_{L/K} \to \mathcal{E}^r_{F/K}$ to be

$$\Psi^{L/F}(\alpha)(\mathfrak{n}) = N_{L(\mathfrak{n})/F(\mathfrak{n})}\alpha(\mathfrak{n})$$

for each $\alpha \in \mathcal{E}^r_{L/K}$. We write α_{nor} for $\Psi^{L/F}(\alpha)$. In this setting, we have the following proposition.

Proposition 2.4. *Let L/F be a totally ramified extension at a prime over p and $L/\mathbb{Q}$ is an abelian extension. Then the map $\Psi^{L/F}$ defined above is well defined.*

Proof. For each pair of ideals $\mathfrak{n} \mid \mathfrak{m}$ which are prime to p, we have the following field diagram which is linearly disjoint.



By applying the norm map $N_{L(\mathfrak{m})/L(\mathfrak{n})}$ to $\alpha_{\mathrm{nor}}(\mathfrak{m}) = N_{L(\mathfrak{m})/F(\mathfrak{m})}\alpha(\mathfrak{m})$, we have

$$\begin{aligned} N_{F(\mathfrak{m})/F(\mathfrak{n})}\alpha_{\mathrm{nor}}(\mathfrak{m}) &= N_{L(\mathfrak{n})/F(\mathfrak{n})}N_{L(\mathfrak{m})/L(\mathfrak{n})}\alpha(\mathfrak{m}) \\ &= N_{L(\mathfrak{n})/F(\mathfrak{n})}\alpha(\mathfrak{n})^{\prod_{\mathfrak{l}\mid\mathfrak{m},\,\mathfrak{l}\nmid\mathfrak{n}}(\mathrm{Frob}_{\mathfrak{l}}-1)} \\ &= \alpha_{\mathrm{nor}}(\mathfrak{n})^{\prod_{\mathfrak{l}\mid\mathfrak{m},\,\mathfrak{l}\nmid\mathfrak{n}}(\mathrm{Frob}_{\mathfrak{l}}-1)}. \end{aligned}$$

Hence, we have

$$N_{F(\mathfrak{m})/F(\mathfrak{n})}\alpha_{\mathrm{nor}}(\mathfrak{m}) = \alpha_{\mathrm{nor}}(\mathfrak{n})^{\prod_{\mathfrak{l}\mid\mathfrak{m},\,\mathfrak{l}\nmid\mathfrak{n}}(\mathrm{Frob}_{\mathfrak{l}}-1)}.$$

The congruence conditions can be obtained by a similar method.

$$\begin{aligned} \alpha_{\mathrm{nor}}(\mathfrak{n}\mathfrak{l}) = N_{L(\mathfrak{n}\mathfrak{l})/F(\mathfrak{n}\mathfrak{l})}\alpha(\mathfrak{n}\mathfrak{l}) &\equiv N_{L(\mathfrak{n}\mathfrak{l})/F(\mathfrak{n}\mathfrak{l})}\alpha(\mathfrak{n})^{(N_{K/\mathbb{Q}}(\mathfrak{l})-1)/M} \\ &= N_{L(\mathfrak{n})/F(\mathfrak{n})}\alpha(\mathfrak{n})^{(N_{K/\mathbb{Q}}(\mathfrak{l})-1)/M} \\ &= \alpha_{\mathrm{nor}}(\mathfrak{n})^{(N_{K/\mathbb{Q}}(\mathfrak{l})-1)/M}. \end{aligned}$$

modulo primes over $\mathfrak{l}$. This completes the proof. ∎

Notice that $\alpha_{\mathrm{nor}}(\mathcal{O}_K) = N_{L/K}\alpha(\mathcal{O}_K)$. Hence, Proposition 2.4 induces a natural norm map between the higher special units $S^r_{L/K}$ and $S^r_{F/K}$.

Corollary 2.5. *Let L/F be a totally ramified extension at a prime over p and let $L/\mathbb{Q}$ be an abelian extension. Then the norm map $N_{L/K}$ from $S^r_{L/K}$ to $S^r_{F/K}$ is well defined.*

Applying Corollary 2.5 to various subfields F_n, we denote by

$$S^i_{F/K,\infty} = \varprojlim S^i_{F_n/K} \otimes \mathbb{Z}_p$$

the inverse limits of the higher special units $S^i_{F_n/K} \otimes \mathbb{Z}_p$ of F_n of depth i with respect to the norm maps. In Section 3 we will consider the case when the base field K is an imaginary quadratic field, and in Section 4 the case when the base field K is the rational field.

§3. The higher special units over an imaginary quadratic field

We briefly recall the definition of elliptic units after [3], [5], and [14]. Fix an embedding of the algebraic closure K^{alg} of K into the complex field $\mathbb{C}$ and let $L \subset \mathbb{C}$ be the period of some elliptic curve defined over the Hilbert class field H_K of K with complex multiplication by $\mathcal{O}_K$. For an integral ideal $\mathfrak{g}$ of K prime to 6, a meromorphic function Θ_0 is defined as follows.

$$\Theta_0(z;\mathfrak{g}) = \left(\frac{\Delta(L)^{N(\mathfrak{g})}}{\Delta(\mathfrak{g}^{-1}L)}\right)^{1/12} \prod_u (\wp(z;L) - \wp(u;L))^{-1}$$

where Δ is the Ramanujan Δ-function, $\wp(z;L)$ is the Weierstrass $\wp$-function for the lattice L, and the product is taken over representatives of the nonzero classes u in $(\mathfrak{g}^{-1}L/L)/\pm 1$. For this function $\Theta_0(z;\mathfrak{g})$, an Euler system $\alpha_{\tau,\mathfrak{g}} \in \mathcal{E}^r_{F/K,M}(\mathfrak{f}\mathfrak{g})$ (cf. Section 1 of [14]) is defined as follows.

$$\alpha_{\tau,\mathfrak{g}}(\mathfrak{a}) = N_{FK(\mathfrak{f}\mathfrak{a})/F(\mathfrak{a})}\Theta_0\left(\tau + \sum_{\mathfrak{l}|\mathfrak{a}} x_{\mathfrak{l}};\mathfrak{g}\right)$$

where $x_{\mathfrak{l}}$ is an element of $\mathbb{C}/L$ of order exactly $\mathfrak{l}$ for each $\mathfrak{l} \in \mathrm{I}^r_{F/K,M}$, $\mathfrak{f}$ is an integral ideal of K such that the natural map $\mathcal{O}_K^\times \to \mathcal{O}_K/\mathfrak{f}$ is injective, $\tau \in \mathbb{C}/L$ is an element of order exactly $\mathfrak{f}$, and $\mathfrak{g}$ is an integral ideal of K prime to $6\mathfrak{f}$. Let E_F be the group generated over $\mathbb{Z}[G(F/K)]$ by $\alpha_{\tau,\mathfrak{g}}(\mathcal{O}_K)^{\sigma-1}$

where $\tau, \mathfrak{g}$ is as above and $\sigma \in G(F/K)$. We denote the group of all roots of unity in F by $\mu(F)$. The group of elliptic units El_F is defined as follows.

$$El_F = \mu(F)E_F.$$

In the following proposition, we need to find an element in the truncated Euler systems $\mathcal{E}^r_{K/F,M}$ whose value at $\mathcal{O}_K$ is a given elliptic unit and hence the elliptic units are contained in the higher special units of an arbitrary depth.

PROPOSITION 3.1. *If $u \in El_F$, then for every M there is an element $\alpha \in \mathcal{E}^r_{F/K,M}$ such that $\alpha(\mathcal{O}_K) = u$.*

Proof. This follows immediately from Proposition 1.2 of [14] since the Euler systems are contained in the truncated Euler systems. ∎

We let U_n, El_n, $\mathcal{C}_n$ denote respectively the global units U_{F_n}, the elliptic units U_{F_n} and the circular units $\mathcal{C}_{F_n}$ of F_n. Let Cl_n denote the ideal class group Cl_{F_n} of F_n. We denote by

$$U_\infty = \varprojlim U_n \otimes \mathbb{Z}_p \quad \text{and} \quad El_\infty = \varprojlim El_n \otimes \mathbb{Z}_p$$

the inverse limits of $U_n \otimes \mathbb{Z}_p$ and $El_n \otimes \mathbb{Z}_p$ with respect to the norm maps respectively. Finally, let $\mathrm{Cl}_\infty = \varprojlim \mathrm{Cl}_n \otimes \mathbb{Z}_p$ be the inverse limit of the p-part of the ideal class groups of F_n. The set $\{El_n \otimes \mathbb{Z}_p\}_{n \in \mathbb{N}}$ is said to have the Galois descent property if $El_m^{G(F_m/F_n)}/El_n \otimes \mathbb{Z}_p = 1$ for all $m \geq n$. Let r_n be the exact power of p dividing $\#(\mathrm{Cl}_n)$. In this case, there exist well known invariants, Iwasawa invariants, λ, μ, and ν such that

$$r_n = \lambda n + \mu p^n + \nu$$

for all sufficiently large values of n. On the Iwasawa μ-invariant, we need the following theorem of Ferrero-Washington, whose proof is similar to that of Theorem 4.4 (cf. [4]).

THEOREM 3.2. (Ferrero-Washington) *Let L be an abelian extension of $\mathbb{Q}$, let p be any prime, and let L_∞/L be the cyclotomic $\mathbb{Z}_p$-extension of L. Then the Iwasawa μ-invariant is zero.*

From Theorem 3.2 above, the p-rank $\mathrm{rk}_p \mathrm{Cl}_n$, which is the number of direct summands of p-power order when Cl_n is decomposed into cyclic groups of prime power order, is bounded independently of n. As in the introduction, let $w = w_F$ be the maximum of $\mathrm{rk}_p \mathrm{Cl}_n$ as n varies. Notice that the main conjectures of Iwasawa theory for imaginary quadratic fields are proved by Rubin using methods of Euler systems and Iwasawa theory when p splits in K and under some conditions when p does not split. (cf. Theorem 4.1 of [14]). For the second case, we refer the reader to a preprint of Johnson-Leung and Kings (cf. [7]). We have the following theorem.

THEOREM 3.3. *Suppose* $p \nmid [F : K]$. *Then for all* $i \geq w$, *the main conjecture implies*

$$\mathrm{char}(S^i_{F/K,\infty}/El_\infty) = 1.$$

Moreover, if for all sufficiently large $m > n \gg 0$, N_{F_m/F_n} *induces an epimorphism over* $\{S^i_{F_m/K} \otimes \mathbb{Z}_p\}_{m \gg 0}$ *then*

$$(S^i_{F_n/K}/El_n) \otimes \mathbb{Z}_p = 1$$

for all $n \geq 0$ *and* $i \geq w$.

Proof. Notice that since p divides $[F_n : K]$, we can not apply the argument of the Euler system to conclude $El_n \otimes \mathbb{Z}_p = S^i_{F_n/K} \otimes \mathbb{Z}_p$. The main conjecture of an imaginary quadratic field indicates

$$\mathrm{char}(\mathrm{Cl}_\infty) = \mathrm{char}(U_\infty/El_\infty).$$

The argument of the Euler system of Rubin (cf. [14]) yields, for all $i \geq w$,

$$\mathrm{char}(\mathrm{Cl}_\infty) = \mathrm{char}(U_\infty/S^i_{F/K,\infty}).$$

From the two equations above and the multiplicative property of the characteristic ideals in a short exact sequence, we derive

$$\mathrm{char}(S^i_{F/K,\infty}/El_\infty) = 1.$$

If for all sufficiently large $m > n \gg 0$, N_{F_m/F_n} induces an epimorphism over $\{S^i_{F_m/K} \otimes \mathbb{Z}_p\}_{m \gg 0}$, then $S^i_{F_n/K}/El_n \otimes \mathbb{Z}_p = 1$, for all $n \geq 0$ and $i \geq w$ since $S^i_{F/K,\infty}/El_\infty$ has no finite Λ-submodules. ∎

§4. The higher special units over the rational field

In this section, we suppose that the ground field K is the rational field. Using the same construction of Proposition 2.4, we can define the inverse limit $S^i_{F/\mathbb{Q},\infty} = \varprojlim S^i_{F_n/\mathbb{Q}} \otimes \mathbb{Z}_p$ of the higher special units $S^i_{F_n/\mathbb{Q}} \otimes \mathbb{Z}_p$ coming from the truncated Euler systems $\mathcal{E}^i_{F_n/\mathbb{Q}}$. The following theorem is a natural generalization of Theorem 2.3 of [17] to arbitrary abelian extensions. In *ibid*, we covered only for the case when F is the cyclotomic field. The proof follows in the same way as that for Theorem 3.3, and Theorem 2.3 of *ibid*. We will leave the proof to the reader.

THEOREM 4.1. *Let F be an abelian extension of $\mathbb{Q}$ such that $p \nmid [F : \mathbb{Q}]$ and $i \geq w$. Then*

$$\mathrm{char}(S^i_{F/\mathbb{Q},\infty}/\mathcal{C}_\infty) = 1.$$

In general, the Galois descent property for the circular units over the cyclotomic $\mathbb{Z}_p$-extension fails. However, the following result due to Belliard will be enough for our purpose.

LEMMA 4.2. (Belliard) *Let F be a real abelian field and $p \nmid [F : \mathbb{Q}]$. Then $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ satisfies the Galois descent property over the cyclotomic $\mathbb{Z}_p$-extension $F_\infty = \bigcup F_n$.*

Proof. See [1] and [18]. □

LEMMA 4.3. *Under the same conditions of Theorem 4.1, $\{\mathcal{C}_n \otimes \mathbb{Z}_p\}_{n\in\mathbb{N}}$ satisfies the Galois descent property, i.e., $(\mathcal{C}_m \otimes \mathbb{Z}_p)^{G(F_m/F_n)} = \mathcal{C}_n \otimes \mathbb{Z}_p$ for all $m \geq n$.*

Proof. When F is real, the Galois descent property holds from Lemma 4.2. If F is imaginary, then F_n is a quadratic extension of its maximal real subfield F_n^+. Let N_+ denote the norm map from F_n to F_n^+. Fix $m \geq n \geq 0$. From the exact sequence, $0 \to \mu(F_m) \to \mathcal{C}_m \xrightarrow{N_+} N_+(\mathcal{C}_m) \to 0$, we obtain the following diagram of a long exact sequence,

$$\begin{array}{ccccccc}
0 \longrightarrow \mu_p(F_n) \longrightarrow & \mathcal{C}_n \otimes \mathbb{Z}_p & \xrightarrow{N_+} & \mathcal{C}_n^+ \otimes \mathbb{Z}_p \longrightarrow & 0 \\
\downarrow & \downarrow & & \downarrow & \\
0 \longrightarrow \mu_p(F_n) \longrightarrow & (\mathcal{C}_m \otimes \mathbb{Z}_p)^{G(F_m/F_n)} & \longrightarrow & \mathcal{C}_n^+ \otimes \mathbb{Z}_p \longrightarrow & \widehat{H}^1(G(F_m/F_n), \mu_p(F_m))
\end{array}$$

where $\mu_p(F_n) = \mu(F_n) \otimes \mathbb{Z}_p$. Since $G(F_m/F_n)$ is cyclic, we have

$$\#(\widehat{H}^1(G(F_m/F_n), \mu_p(F_m)) = \#(\widehat{H}^0(G(F_m/F_n), \mu_p(F_m)) = 1.$$

Hence we have, for all $m \geq n$,

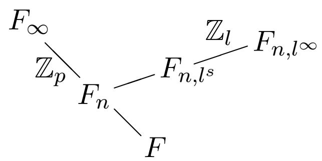
$$(\mathcal{C}_m \otimes \mathbb{Z}_p)^{G(F_m/F_n)} = \mathcal{C}_n \otimes \mathbb{Z}_p$$

which is what we wanted to show. ∎

In the following corollary, we fix a prime l $(\neq p)$ which is prime to $[F : \mathbb{Q}]$ and let $F_{n,l^\infty} = \bigcup_s F_{n,l^s}$ denote the cyclotomic $\mathbb{Z}_l$-extension of the field F_n.

Corollary 4.4. *If for all sufficiently large $m > n \gg 0$, N_{F_m/F_n} induces an epimorphism over $\{S^i_{F_m/\mathbb{Q}} \otimes \mathbb{Z}_p\}_{m\gg 0}$ then $(S^i_{F_n/\mathbb{Q}}/\mathcal{C}_n) \otimes \mathbb{Z}_p = 1$ for all $n \geq 0$ and $i \geq w$. Moreover, if for all sufficiently large $s > t \gg 0$, $N_{F_{n,l^s}/F_{n,l^t}}$ induces an epimorphism over $\{S^i_{F_{n,l^s}} \otimes \mathbb{Z}_l\}_{s\gg 0}$ then $\mathcal{C}_n \otimes \mathbb{Z}_l$ is equal to $S^i_{F_n/\mathbb{Q}} \otimes \mathbb{Z}_l$.*

Proof. From Theorem 4.1 and Lemma 4.3, the p-primary parts of the indices $(S^i_{F_n/\mathbb{Q}} : \mathcal{C}_n)$ are trivial since $S^i_{F/\mathbb{Q},\infty}/\mathcal{C}_\infty$ has no nontrivial finite Λ-submodules. We claim that the l-primary parts of the indices are also trivial whenever $l \nmid [F : \mathbb{Q}]$. This follows from the following observation of the cyclotomic $\mathbb{Z}_l$-extension $F_{n,l^\infty} = \bigcup_s F_{n,l^s}$ of the field F_n.



For the higher special units $S^i_{F_{n,l^s}/\mathbb{Q}}$ and the circular units $\mathcal{C}_{n,l^s}$ of F_{n,l^s}, we obtain the same result that

$$(S^i_{F_{n,l^s}/\mathbb{Q}}/\mathcal{C}_{n,l^s}) \otimes \mathbb{Z}_l = 1$$

for all such $s \gg 0$, as in Theorem 3.3, since l does not divide the degree of the extension of the ground field $F_{n,l^0} = F_n$ over $\mathbb{Q}$. From Lemma 4.3, we have the following exact sequence,

$$0 \longrightarrow (S^i_{F_n/\mathbb{Q}}/\mathcal{C}_n) \otimes \mathbb{Z}_l \longrightarrow (S^i_{F_{n,l^s}/\mathbb{Q}}/\mathcal{C}_{n,l^s}) \otimes \mathbb{Z}_l = 1.$$

Hence, if $l \nmid [F : \mathbb{Q}]$, $\mathcal{C}_n \otimes \mathbb{Z}_l$ is equal to $S^i_{F_n/\mathbb{Q}} \otimes \mathbb{Z}_l$. ∎

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