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VECTOR SEMI-FREDHOLM TOEPLITZ OPERATORS AND MEAN WINDING NUMBERS

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Abstract. For a continuous nonvanishing complex-valued function g on the real line, several notions of a mean winding number are introduced. We give necessary conditions for a Toeplitz operator with matrix-valued symbol G to be semi-Fredholm in terms of mean winding numbers of $\det G$. The matrix function G is assumed to be continuous on the real line, and no other apriori assumptions on it are made.

§1. Introduction and main result

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane in the complex plane $\mathbb{C}$. Let $1 \le p < \infty$. We recall that the classical Hardy space $H^p(\mathbb{C}_+)$ consists of analytic functions f in $\mathbb{C}_+$ such that

$$\|f\| \overset{\text{def}}{=} \left(\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dx \right)^{1/p}$$

is finite. It is a Banach space for any p as above. The space $H^\infty(\mathbb{C}_+)$ is defined as the Banach space of bounded analytic functions in $\mathbb{C}_+$. We refer to the book [18] for an account of the theory of H^p spaces of the upper half-plane and of the unit disc. Functions in $H^p(\mathbb{C}_+)$ have non-tangential boundary limit values on $\mathbb{R}$, which permits us to identify $H^p(\mathbb{C}_+)$ with a closed subspace of $L^p(\mathbb{R})$. We put $H^p = H^p(\mathbb{C}_+)$, $1 \le p \le \infty$.

For any function space Ψ, we denote by $\Psi_{\mathbb{R}}$ the set of its real elements and by Ψ_r, $\Psi_{r\times r}$, respectively, the spaces of $r \times 1$ vector-valued functions and of $r \times r$ matrix-valued functions with entries in Ψ. If $\mathcal{A}$ is a scalar or matrix functional algebra, we denote by $\mathcal{GA}$ the set of all its invertible elements.

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Let the natural number r be fixed and let $G \in L^\infty_{r\times r}(\mathbb{R})$. The vector Toeplitz operator T_G with the symbol G acts on the vector Hardy space H^2_r by the formula

$$T_G x = P_+\big(G \cdot x\big), \quad x \in H^2_r, \tag{1.1}$$

here P_+ is the orthogonal projection of $L^2_r(\mathbb{R})$ onto its closed subspace H^2_r.

A (bounded linear) operator K on a Banach space B is called normally solvable [21], [28] if its image is closed. K is called a Φ_+-operator (a Φ_--operator) if it is normally solvable and $\dim \operatorname{Ker} K < \infty$ ($\dim \operatorname{Coker} K = \dim B/\operatorname{Range} K < \infty$, respectively). We denote by $\Phi_\pm(B)$ these classes of operators on B. Operators in $\Phi_+(B) \cup \Phi_-(B)$ are called semi-Fredholm. Operators in $\Phi(B) = \Phi_-(B) \cap \Phi_+(B)$ are called Fredholm.

The index of a semi-Fredholm operator is defined by

$$\operatorname{Ind} K = \dim \operatorname{Ker} K - \dim \operatorname{Coker} K;$$

its values are integers or $\pm\infty$. A semi-Fredholm operator is Fredholm if and only if its index is finite.

Fredholm and semi-Fredholm operators have several important properties. For instance, the product of two $\Phi_\pm$ operators is again a $\Phi_\pm$ operator, and the formula $\operatorname{Ind}(K_1K_2) = \operatorname{Ind}(K_1) + \operatorname{Ind}(K_2)$ holds for K_1, K_2 both in $\Phi_+(B)$ or in $\Phi_-(B)$. We refer to [21], [28] for detailed expositions of the theory of these classes and for applications.

We put $C^b = C^b(\mathbb{R})$ to be the Banach space of all continuous uniformly bounded functions on $\mathbb{R}$ with the supremum norm. Our paper is devoted to finding necessary conditions for semi-Fredholmness and Fredholmness of T_G for the case when G is an $r\times r$ matrix function, whose entries are in C^b. Such questions appear naturally in connection with the Riemann-Hilbert problem on the real line. This problem appears in many different situations, such as various problems in mechanics of continuous media and hydrodynamics [3], [8], [9], [29], [40], inverse scattering method for integrable equations [1], linear control theory of systems with delays [16], convolution equations and systems on finite intervals (see [13], [25], and others). The case of infinite index often appears in these applications.

First we quote the following well-known result.

Theorem A. (see [13, Thm. 16.3(b)]) *The condition $|\det G| \geq \varepsilon > 0$ is necessary for T_G to be semi-Fredholm.*

We will always assume this condition to be fulfilled.

For a function $G \in C^b_{r\times r}$ which has limits at $\pm\infty$, T_G is semi-Fredholm iff it is Fredholm, and a complete criterion for it is known (see [15] or [13]). In a particular case, when $G(-\infty) = G(+\infty)$, T_G is Fredholm if and only if $|\det G| \geq \varepsilon > 0$ on $\mathbb{R}$, and

$$\operatorname{Ind} T_G = -\operatorname{wind}\det G, \tag{1.2}$$

where wind stands for the winding number (around the origin). So our main concern is about symbols that have no limits at $-\infty$ or at $+\infty$.

Let BMO $=$ $\mathrm{BMO}_{\mathbb{R}}$ be the space of real-valued functions on $\mathbb{R}$ of bounded mean oscillation. We recall that BMO consists of those locally integrable functions f on $\mathbb{R}$ that satisfy

$$\|f\|_{\mathrm{BMO}_{\mathbb{R}}} \stackrel{\text{def}}{=} \sup_J \frac{1}{|J|}\int_J |f - f_J| \leq C, \tag{1.3}$$

where the supremum is taken over all finite subintervals J of the real line and $f_J = \frac{1}{|J|}\int_J f$ is the mean of f on the interval J. We refer to [20] for an exposition of the theory of these spaces.

Let $C_+(\mathbb{R})$ be the class of real continuous (nonstrictly) increasing functions on $\mathbb{R}$, and put

$$\begin{aligned}\mathrm{BMO}^+_{\mathbb{R}} &= \big\{u + v : u \in \mathrm{BMO}_{\mathbb{R}},\ v \in C_+(\mathbb{R})\big\},\\ \mathrm{BMO}^-_{\mathbb{R}} &= \big\{u - v : u \in \mathrm{BMO}_{\mathbb{R}},\ v \in C_+(\mathbb{R})\big\}.\end{aligned}$$

The main result of the paper is as follows.

Theorem 1. *Suppose that* $G \in C^b_{r\times r}$.
(1) *If* $T_G \in \Phi_\pm(H^2_r)$, *then* $\arg\det G \in \mathrm{BMO}^\pm_{\mathbb{R}}$.
(2) *If* $T_G \in \Phi(H^2_r)$, *then* $\arg\det G \in \mathrm{BMO}_{\mathbb{R}}$.

In Section 3, we introduce a system of mean winding numbers of $\det G$ and formulate and prove Theorems 2 and 3 (they will follow from Theorem 1 and can be considered as its applications). In Section 4, we discuss some unresolved questions, related with our results.

Our principal motivation comes from the control theory. In a problem about the complete controllability of delay equations it turned out to be necessary to estimate the number

$$\inf\big\{\tau \in \mathbb{R} : T_{e^{-i\tau x}G(x)} \text{ is onto}\big\} \stackrel{\text{def}}{=} \beta(G) \tag{1.4}$$

in terms of some computable characteristics of a matrix function $G \in \mathcal{G}C^b_{r\times r}$. The number $\beta(G)$ has a meaning of the least time of complete controllability. Theorems 2 and 3 permitted us to give a good estimate of this number. The results on complete controllability were obtained jointly by the author and Sjoerd Lunel and will be published elsewhere.

A great part of the recent book [19] by Dybin and Grudsky treats scalar and matrix functions that are continuous on the real line. This book summarized (and generalized) earlier work by these authors. Several novel tools are used, such as the notion of a u-periodic function, where u is an inner function on $\mathbb{C}_+$, continuous on the real line. Other tools are a construction of an inner function whose argument models an arbitrarily given increasing continuous function and the notion of a generalized factorization with infinite index. These hard analysis tools permitted the authors to give a sufficient condition for semi-Fredholmness (see [19, Theorem 5.10]). By applying this result, Dybin and Grudsky get complete answers in cases of whirls at $\pm\infty$ with different asymptotic, such as power, logarithmic or exponential.

Earlier work on whirled symbols include the works by Govorov [22], Ostrovsky [33], Monakhov, Semenko (see the book [29]) and others. The approach of these authors was based on the theory of analytic functions of completely regular growth. In various works, the behavior of the property of Fredholmness under an orientation preserving homeomorphism of $\mathbb{R}$ have been studied, see [7], [12], [19], [10] and others.

Various mean winding numbers were introduced in the work by Sarason [37] for symbols in QC and by Power [35] for slowly oscillating symbols. For symbols of these classes, these mean winding numbers allow one to formulate nice complete criteria for a Toeplitz operator to be Fredholm or semi-Fredholm. We remark that a wider C^*-algebra of slowly oscillating functions was considered in a recent paper by Sarason [39], where the maximal ideal space of this algebra was studied.

Necessary and sufficient conditions for a Toeplitz operator to be Fredholm and semi-Fredholm are also known if G belongs to various algebras of symbols. For instance, classes PC of piecewise continuous symbols, $\mathrm{QC} = L^\infty \cap \mathrm{VMO}$ of quasicontinuous symbols, and $\mathrm{PQC} = \mathrm{alg}(\mathrm{PC}, \mathrm{QC})$ have been studied both in scalar and matrix case.

Another well-studied cases are that of almost periodic and semi-almost periodic symbols. For matrix symbols of these types, a great breakthrough has been done recently by Böttcher, Karlovich and Spitkovsky, see [13]. Among other things, generalizations of the index formula (1.2) are known for these cases (see [15], [31], [32]). We refer to [14] for an alternative approach. In [2], [5], [12], other classes of symbols are studied. In [6], a Fredholm criterion and an index formula are given for vector Toeplitz operators, whose (matrix) symbols belong to the Banach algebra, generated by semi-almost periodic matrix functions and slowly oscillating matrix functions. See [26] for a connection with the factorization and the Riemann-Hilbert problem.

For symbols in $C^b_{r\times r}$ with no other assumptions, our knowledge is much less complete. We refer to Subsections 2.26 and 4.73 in [15] and to [11] for several relevant results. The criterion for surjectivity of a Toeplitz operator with a nontrivial kernel, given in [23], can also be reformulated as a criterion for a Toeplitz operator to belong to $\Phi_+(H^2)\setminus\Phi(H^2)$. Some additional comments will be given at the end of the article.

Books [13], [15], [21], [26], [28], [31], [32] contain systematic expositions of the spectral theory of Toeplitz operators, with different emphasis.

It is worth to note that recently, Toeplitz operators with symbols like ours have been appeared in papers by Baranov, Havin, Makarov, Mashreghi, Poltoratsky and others in relation with the Beurling-Malliavin theorem, bases in de Branges spaces and related topics (see [4], [24], [27], and references therein). It seems that the ideas and methods of these papers can be applied to achieve a better understanding of semi-Fredholm Toeplitz operators with continuous symbols at least in the case of scalar symbols.

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§2. Proof of Theorem 1

First we need some facts and definitions.

Let $0<\alpha<1$. We put

$$\mathrm{Lip}^{\alpha,\mathrm{loc}}=\{f\in C^b(\mathbb{R}) : f|J\in \mathrm{Lip}^{\alpha}(J)\ \ \forall J\};$$

here J runs over all compact intervals in $\mathbb{R}$ and $\mathrm{Lip}^{\alpha}(J)$ is the Hölder-Lipschitz class on J with the exponent α. Next, we will need the classes

$$C_a(\mathbb{C}_+)=\{f\in C(\operatorname{clos}\mathbb{C}_+) : f|\mathbb{C}_+\in H^\infty\},$$
$$A^{\alpha,\mathrm{loc}}=\{f\in C_a(\mathbb{C}_+) : f|\mathbb{R}\in \mathrm{Lip}^{\alpha,\mathrm{loc}}\}.$$

A function f in $\mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$ or in $A^{\alpha,\mathrm{loc}}_{r\times r}$ is invertible if and only if $|\det f|>\varepsilon>0$ on $\mathbb{R}$ (or on $\operatorname{clos}\mathbb{C}_+$, respectively). Recall that a function g in H^∞ is called *inner* if its modulus is equal to one a.e. on $\mathbb{R}$. An analytic function g on $\mathbb{C}_+$ is called *outer* if it has a form $g(z) = \exp(u(z) + iv(z))$,

$$(u + iv)(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \frac{t}{1 + t^2} \right] \log k(t)\, dt + is,$$

where $k > 0$ a.e. on $\mathbb{R}$, $\log k \in L^1(\mathbb{R})$, and s is a real constant. We assume u and v to be real-valued. These functions are harmonic in $\mathbb{C}_+$. They have boundary limit values a.e. on $\mathbb{R}$, which satisfy $u|\mathbb{R} = \log k$ a.e. and $v|\mathbb{R} = \mathcal{H}(u|\mathbb{R})$, where $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$.

Each function g in $\mathcal{G}H^\infty$ is outer; in this case $\log k \in L^\infty(\mathbb{R})$. We refer to [18], [20] for all these (classical) facts.

Let g be any function in $\mathcal{G}H^\infty$. Then $\arg g(z) = s + v(z)$ is well defined on $\mathbb{C}_+$ (up to an additive constant $2\pi n$). We also see that the function $\arg g(z)$ has boundary limit values a.e. on $\mathbb{R}$, which will be denoted as $\arg g(x)$, $x \in \mathbb{R}$. It follows from standard facts about BMO [20] that $\arg g|\mathbb{R} \in \mathrm{BMO}_{\mathbb{R}}$.

Definition. We define the class H^∞_* as the set of functions $f \in H^\infty$ that have the form

$$f = g \cdot h,$$

where $g \in \mathcal{G}H^\infty$ and h is inner in $\mathbb{C}_+$ and has a continuous extension to $\mathbb{R}$.

It is well-known that the set of points of discontunuity of an inner function in $\mathbb{C}_+$ contains any limit point of its zeros and also any point in the support of the singular inner measure, that defines the singular inner factor of the function, see, for instance, [31, Chapter 3]. Therefore a function h is inner of the above type if and only if it has the form

$$h(z) = Ce^{iaz} \prod_j \frac{|z_j^2 + 1|}{z_j^2 + 1} \, \frac{z - z_j}{z - \bar{z}_j}, \quad z \in \mathbb{C}_+, \tag{2.1}$$

where $|C| = 1$, $a > 0$, and $z_j \in \mathbb{C}_+$, $|z_j| \to \infty$. Take any positive continuous function $y = \psi(x)$ on $\mathbb{R}$ such that the subgraph

$$\Gamma_\psi = \{(x + iy) : 0 < y < \psi(x)\} \subset \mathbb{C}_+$$

does not contain the zeros z_j of h. Then $\arg h(z)$ is well defined and continuous on $\Gamma_\psi \cup \mathbb{R}$.

DEFINITION. Let $f \in H^\infty_*$, and let g, h, Γ_ψ be as above. We define the argument $\arg f$ on $\Gamma_\psi \cup \mathbb{R}$ by

$$\arg f = \arg g + \arg h.$$

So for $f \in H^\infty_*$, the argument $\arg f$ is well defined on Γ_ψ (up to adding $2\pi n$, $n \in \mathbb{Z}$). It is continuous on Γ_ψ and its values on $\mathbb{R}$ exist almost everywhere in the sense of nontangential limits.

PROPOSITION 1. *For any $f \in H^\infty_*$, $\arg f \in \mathrm{BMO}^+_{\mathbb{R}}$.*

Proof. For any $f = g \cdot h \in H^\infty_*$ as above, $\arg g \in \mathrm{BMO}_{\mathbb{R}}$ and $\arg h$ is a continuous increasing function. ∎

LEMMA 1. *Let $f \in H^\infty$. Then $f \in H^\infty_*$ if and only if there is a positive function $\psi \in C(\mathbb{R})$ and some $\varepsilon > 0$ such that $|f| > \varepsilon$ on the subgraph Γ_ψ.*

Proof. If $f \in H^\infty_*$, then it is clear that f satisfies the above property. Conversely, suppose $|f| > \varepsilon > 0$ on Γ_ψ, for a certain positive function $\psi \in C(\mathbb{R})$. Let $f = h \cdot g$ be the inner - outer factorization of f, then $g \in \mathcal{G}H^\infty$. It follows that the inner function $h = f/g$ satisfies an inequality $|h| > \varepsilon_1 > 0$ on Γ_ψ, and consequently, it has a form (2.1), see [31, Chapter 3]. ∎

In many works on Toeplitz operators, the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ instead of the upper half-plane $\mathbb{C}_+$ is considered. If $G \in L^\infty_{r\times r}(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle, then the same formula (1.1) defines a Toeplitz operator $\widehat{T}_G$ on $H^2_r(\mathbb{D})$ (in this setting, P_+ stands for the orthogonal projection of $L^2_r(\mathbb{T})$ onto the vector Hardy space $H^2_r(\mathbb{D})$). Let

$$\varphi(z) = \frac{z - i}{z + i}$$

be the conformal mapping of $\mathbb{C}_+$ onto the unit disc $\mathbb{D}$. The formula

$$T_G = W \widehat{T}_{G\circ\varphi} W^{-1},$$

where $W : H^2_r(\mathbb{D}) \to H^2_r$ is the unitary isomorphism, given by

$$(Wf)(z) = \pi^{-1/2}\,(z+i)^{-1}\big(f \circ \varphi\big)(z)$$

shows that each vector Toeplitz operator on $\mathbb{C}_+$ is unitarily equivalent to a vector Toeplitz operator on $\mathbb{D}$, and vice versa, so there is no difference in the study of Toeplitz operators in these two settings. The symbols on $\mathbb{T}$ that correspond to symbols in $C^b_{r\times r}$ by means of this construction have the only discontinuity at the point 1.

By a result by Pousson [34] and Rabindranathan [36], each function G in $\mathcal{G}L^\infty_{r\times r}(\mathbb{R})$ can be factored as $G = UG_e$, where $G_e \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$ and U is unitary-valued on $\mathbb{R}$ (see also [15, Thm. 6.13]). Then $T_G = T_U T_{G_e}$, and T_{G_e} is invertible, so that Fredholmness or semi-Fredholmness of T_G is equivalent to the corresponding property of T_U. For unitary symbols, the following results hold.

THEOREM B. *Let* $U \in \mathcal{G}L^\infty_{r\times r}(\mathbb{R})$ *be unitary-valued. Then*
(i) T_U *is left-invertible if and only if* $\operatorname{dist}(U, H^\infty_{r\times r}(\mathbb{C}_+)) < 1$.
(ii) T_U *is invertible if and only if* $\operatorname{dist}(U, \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)) < 1$.

For a proof, see [17, Chap. VIII, Lemma 5.1], [15, Corollary 4.36]. We refer to the work by Nakazi [30] for a study of possible spectra of (scalar) Toeplitz operators with unimodular symbols.

We will also make use of the following properties.

PROPOSITION 2. (1) *Each selfadjoint matrix function* $K \in L^\infty_{r\times r}(\mathbb{R})$ *such that* $K(x) \geq \varepsilon I > 0$ *on* $\mathbb{R}$ *has a factorization* $K(x) = G_e^*(x)G_e(x)$ *on* $\mathbb{R}$, *where* $G_e \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$. *This factorization is unique up to multiplying* G_e *on the left by a constant unitary matrix.*

(2) *If the matrix* K *(as above) satisfies additionally* $K\circ\varphi \in \operatorname{Lip}^\alpha_{r\times r}(\mathbb{T})$, *then* $G_e\circ\varphi \in \mathcal{G}A^\alpha_{r\times r}(\operatorname{clos}\mathbb{D})$; *here*

$$A^\alpha(\operatorname{clos}\mathbb{D}) = \{f \in C(\operatorname{clos}\mathbb{D}) : f|\mathbb{D} \in H^\infty(\mathbb{D}),\ f|\mathbb{T} \in \operatorname{Lip}^\alpha(\mathbb{T})\}.$$

For the property (1), see [26, Theorems 7.7 and 7.9]. The proof of (2) is contained in [17, Chap. III, Corollary 2.1].

LEMMA 2. *Let* $G \in L^\infty_{r\times r}(\mathbb{R})$. *Then* $T_G \in \Phi_+(H^2_r)$ *if and only if* $T_{\varphi^n G}$ *is left invertible for some integer* $n \geq 0$.

Proof. It is more transparent to work with $H^2_r(\mathbb{D})$ instead of H^2_r. Suppose $G = G(z) \in L^\infty_{r\times r}(\mathbb{T})$ and $\widehat{T}_G \in \Phi_+$; we have to check that there is some integer $n \geq 0$ such that $\widehat{T}_{z^nG(z)}$ is left invertible. By the assumption, the kernel $\operatorname{Ker}\widehat{T}_G$ is finite dimensional; let $x_1, \dots, x_m \in H^2_r(\mathbb{D})$ be its basis. Put

$$L_n = \left\{(c_1, \dots, c_m) \in \mathbb{C}^m : G \cdot \sum_j c_j x_j \in z^{-n} H^2_{-,r}\right\}, \quad n \geq 0,$$

where $H^2_{-,r} = L^2_r(\mathbb{T}) \ominus H^2_r(\mathbb{D})$. Then $\mathbb{C}^m = L_0 \supset L_1 \supset \cdots \supset L_n \supset \cdots$. Since $\bigcap_0^\infty L_k = \{0\}$, one has $L_n = \{0\}$ for some $n \geq 0$. If $x \in \operatorname{Ker}\widehat{T}_{z^n G(z)}$, then $x = \sum_{j=1}^m c_j x_j$ for some coefficients c_j and $z^n G x \in H^2_{-,r}$, which implies that $c_1 = \cdots = c_m = 0$. Hence $\operatorname{Ker}\widehat{T}_{z^n G(z)} = \{0\}$. Since $\widehat{T}_{z^n G} = \widehat{T}_G \widehat{T}_{z^n}$ is a Φ_+-operator with trivial kernel, it follows that it is left invertible.

Conversely, suppose that $\widehat{T}_{z^n G}$ is left invertible for some $n \geq 0$. Denote by I_n the unit matrix of size n. Then $T_{z^n G} \in \Phi_+(H^2_r)$, and therefore $\widehat{T}_G = \widehat{T}_{z^{-n} I_n} \widehat{T}_{z^n G}$ is also in $\Phi_+(H^2_r)$. ∎

Lemma 3. 1) *Suppose that u_1, u_2 are real increasing functions on $\mathbb{R}$ and $u := u_1 + u_2 \in \mathrm{BMO}_\mathbb{R}$. Then $u_1, u_2 \in \mathrm{BMO}_\mathbb{R}$.*

2) $\mathrm{BMO}^-_\mathbb{R} \cap \mathrm{BMO}^+_\mathbb{R} = \mathrm{BMO}_\mathbb{R}$.

Proof. 1) It is known [20] that, given a real-valued function f on $\mathbb{R}$, if the last inequality in (1.3) holds for some constant C, any finite interval J in $\mathbb{R}$ and arbitrary real numbers f_J, then f belongs to $\mathrm{BMO}_\mathbb{R}$.

For any finite interval $J \subset \mathbb{R}$, one can find a point $c = c_J \in J$ such that $u(x) \leq u_J$ for $x < c_J$ and $u(x) \geq u_J$ for $x > c_J$. There exist numbers α_{1J}, α_{2J} (depending on J) such that $u_k(c_J - 0) \leq \alpha_{kJ} \leq u_k(c_J + 0)$ for $k = 1, 2$ and $\alpha_{1J} + \alpha_{2J} = u_J$. Then for any subinterval $J \subset \mathbb{R}$,

$$\int_J |u_1(x) - \alpha_{1J}|\, dx + \int_J |u_2(x) - \alpha_{2J}|\, dx = \int_J |u(x) - u_J|\, dx \leq C|J|,$$

where $C = \|u\|_{\mathrm{BMO}_\mathbb{R}}$. It follows that $u_1, u_2 \in \mathrm{BMO}_\mathbb{R}$.

2) If $h = w_1 - v_1 = w_2 + v_2 \in \mathrm{BMO}^-_\mathbb{R} \cap \mathrm{BMO}^+_\mathbb{R}$, where $w_1, w_2 \in \mathrm{BMO}_\mathbb{R}$ and $v_1, v_2 \in C_+(\mathbb{R})$, then by part 1), $v_1, v_2 \in \mathrm{BMO}_\mathbb{R}$ because $v_1 + v_2 \in \mathrm{BMO}_\mathbb{R}$. Hence $\mathrm{BMO}^-_\mathbb{R} \cap \mathrm{BMO}^+_\mathbb{R} \subset \mathrm{BMO}_\mathbb{R}$. The inclusion relation $\mathrm{BMO}_\mathbb{R} \subset \mathrm{BMO}^-_\mathbb{R} \cap \mathrm{BMO}^+_\mathbb{R}$ is trivial. ∎

The next lemma is not new; in fact, Spitkovsky gives in [41, Theorem 2] a more general result. We will give a proof for completeness.

LEMMA 4. *Suppose that J is a finite open interval on the real line, $F, G \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$, and $F^*F = G^*G$ a.e. on J. Then there exists a neighbourhood $\mathcal{W}$ of J in $\mathbb{C}$ and a bounded analytic $r \times r$ matrix function V on $\mathcal{W}$ such that $F = VG$ on $\mathcal{W}$ (and a.e. on J) and V is unitary-valued on J.*

Proof. Put $V = FG^{-1}$, then $F = VG$ on $\mathbb{C}_+$ and a.e. on $\mathbb{R}$ and V is unitary on J. We apply the symmetry principle to V. Since $V \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$, it is easy to prove that $\widetilde{V}(z) = V^{*-1}(\bar{z})$ is an analytic continuation of V onto the lower half-plane through the arc J. ∎

LEMMA 5. *Every matrix function $G \in \mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$ such that $\inf_{\mathbb{R}}|\det G| > 0$ has a factorization $G = UG_e$, where G_e, $G_e^{-1} \in A^{\alpha,\mathrm{loc}}_{r\times r}$ and $U \in \mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$ is unitary-valued.*

Proof. Put $K(x) = G^*(x)G(x)$, then $K(x) \geq \varepsilon_1 I > 0$ on $\mathbb{R}$. By the above property (1), K can be factorized as $K(x) = G_e^*(x)G_e(x)$, where $G_e \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$. Hence $G = UG_e$, where $U \in L^\infty_{r\times r}$.

Consider a sequence of matrix functions K_n such that $K_n(x) = K(x)$ on $[-n, n]$, $K_n(x) \geq \varepsilon_1 I > 0$ on $\mathbb{R}$ and $K_n \circ \varphi$ are in the Lipschitz class $\mathrm{Lip}^\alpha_{r\times r}(\mathbb{T})$. By Proposition 2, we arrive at functions $G_{ne} \in \mathcal{G}H^\infty_{r\times r}(\mathbb{C}_+)$ such that $G_{ne} \circ \varphi \in A^\alpha_{r\times r}(\mathbb{D})$ and $K_n = G^*_{ne}G_{ne}$ on $\mathbb{R}$. By Lemma 4, $G_e = V_n G_{ne}$ on $(-n, n)$, where V_n are unitary on $(-n, n)$ and analytic in neighbourhoods of these intervals. It follows that $G_e, G_e^{-1} \in A^{\alpha,\mathrm{loc}}_{r\times r}$. Therefore $U \in \mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$. ∎

LEMMA 6. *Suppose that $H \in C^b_{r\times r}(\mathbb{R})$ and $\Psi \in H^\infty(\mathbb{C}_+)$. Then for any finite interval L on the real line we have*

$$\limsup_{y\to 0+} \|\Psi(\,\cdot\, + iy) - H(\,\cdot\,)\|_{L^\infty_{r\times r}(L)} \leq \|\Psi - H\|_\infty;$$

here $\|\Psi - H\|_\infty = \|\Psi - H\|_{L^\infty_{r\times r}(\mathbb{R})}$.

Proof. Denote by $H(z)$, $z \in \mathbb{C}_+$, the harmonic extension of H by means of the Poisson formula. Then for any $y > 0$,

$$\|\Psi(\,\cdot\, + iy) - H(\,\cdot\, + iy)\|_{L^\infty_{r\times r}(L)} \leq \|\Phi - H\|_\infty.$$

Since $H(x + iy) \to H(x)$ as $y \to 0+$ uniformly on compact subsets of the real line, the result follows. ∎

Proof of Theorem 1. We prove part (1). Suppose $G \in C^b_{r\times r}$ and $T_G \in \Phi_+(H^2_r)$. We have to prove that $\arg\det G \in \mathrm{BMO}^+_{\mathbb{R}}$. By Lemma 2, there is some $k > 0$ such that T_{G_1} is left invertible, where $G_1 = \varphi^k G$. Since $\arg\det G = \arg\det G_1 - kr\arg\varphi$ and $\arg\varphi \in L^\infty(\mathbb{R}) \subset \mathrm{BMO}_{\mathbb{R}}$, we have only to prove that $\arg\det G_1$ is in $\mathrm{BMO}^+_{\mathbb{R}}$. Let $\|T_{G_1}x\| \geq \varepsilon\|x\|$, $x \in H^2_r$, where $\varepsilon > 0$, then for any G_2 with $\|G_1 - G_2\|_\infty < \varepsilon$, T_{G_2} is also left invertible. Take $G_2 = G_1 + R$ such that $G_2 \in \mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$ and $R \in C^b_{r\times r}$ has a small norm $\|R\|_\infty$: $\|R\|_\infty < \varepsilon' < \varepsilon$, where ε' has to be chosen. Since

$$\arg\det G_2 = \arg\det G_1 + \arg\det(I + G_1^{-1}R),$$

it follows that $\arg\det G_2 - \arg\det G_1 \in L^\infty(\mathbb{R})$ if we assume that $\varepsilon' \cdot \|G_1^{-1}\|_\infty < 1$. So it suffices to consider G_2 instead of G.

By Lemma 5, we have a factorization $G_2 = UG_{2e}$, where $U \in \mathrm{Lip}^{\alpha,\mathrm{loc}}_{r\times r}$ is unitary-valued and $G_{2e} \in \mathcal{G}A^{\alpha,\mathrm{loc}}_{r\times r}$. Then

$$T_{G_2} = T_U\, T_{G_{2e}}.$$

Since $T^{-1}_{G_{2e}} = T_{G^{-1}_{2e}}$, we conclude that T_U is left invertible. We apply Theorem B and arrive at a function $F \in H^\infty_{r\times r}(\mathbb{C}_+)$ with $\|U - F\|_\infty < 1 - \varepsilon_0 < 1$. Put $F_y(x) = F(x + iy)$, $y > 0$, $L = L_\rho = [-\rho, \rho]$, where $\rho > 0$. By Lemma 6,

$$\|I - U(x)^{-1}F_y(x)\|_{L^\infty_{r\times r}(L_\rho)} = \big\|U(x) - F_y(x)\big\|_{L^\infty_{r\times r}(L_\rho)} < 1 - \varepsilon_0 \tag{2.2}$$

for $x \in L_\rho$, $y \in (0, \delta)$, where $\delta = \delta(\rho) > 0$. It follows, in particular, that there is a graph $y = \psi(x)$ of a positive function $\psi \in C^b(\mathbb{R})$ such that

$$\|I - U(x)^{-1}F(x + iy)\| < 1 - \varepsilon_0 \quad \text{for } x + iy \in \Gamma_\psi.$$

It follows that $\arg\det F$ is well defined on Γ_ψ. By Lemma 1, $\det F$ belongs to $H^\infty_*(\mathbb{C}_+)$.

One can define a continuous branch of $\arg\det\big(U(x)^{-1}F(x + iy)\big)$ for $x + iy \in \Gamma_\psi$ so that $\big|\arg\det\big(U(x)^{-1}F(x + iy)\big)\big| < r\pi/2$. Therefore there is a continuous branch of $\arg\det F(x + iy)$, $x + iy \in \Gamma_\psi$ such that its limit values satisfy

$$\big|\arg\det F(x) - \arg\det U(x)\big| \leq \frac{r\pi}{2} \quad \text{a.e. on } \mathbb{R}.$$

By Proposition 1, $\arg\det F \in \mathrm{BMO}^+_{\mathbb{R}}$. Hence $\arg\det U \in \mathrm{BMO}^+_{\mathbb{R}}$. Since $G_{2e} \in \mathcal{G}A^{\alpha,\mathrm{loc}}_{r\times r}(\mathbb{C}_+)$, it follows that $\det G_{2e} \in \mathcal{G}H^\infty(\mathbb{C}_+)$, so that $\arg\det G_{2e} \in \mathrm{BMO}_{\mathbb{R}}$. Finally, we deduce from the formula

$$\arg\det G_2 = \arg\det U + \arg\det G_{2e}$$

that $\arg\det G_2 \in \mathrm{BMO}^+_{\mathbb{R}}$.

The case when $T_G \in \Phi_-(H^2_r)$ is obtained by considering G^* instead of G. The assertion (2) follows from (1) and Lemma 3. ∎

I. M. Spitkovsky communicated to the author an outline of an alternative proof of Theorem 1, which is based on some properties of the transplantation of the algebra $H^\infty(\mathbb{D}) + C(\mathbb{T})$ to the real line.

In connection with Theorems 1 and B, we mention for completeness the following well-known result.

Theorem C. *Let $U \in \mathcal{G}L^\infty_{r\times r}(\mathbb{R})$ be unitary-valued. Then*
(i) $T_U \in \Phi_+$ *if and only if* $\operatorname{dist}(U, C_{r\times r} + H^\infty_{r\times r}(\mathbb{C}_+)) < 1$.
(ii) $\widehat{T}_U \in \Phi$ *if and only if* $\operatorname{dist}\big(U, \mathcal{G}(C_{r\times r} + H^\infty_{r\times r}(\mathbb{C}_+))\big) < 1$.

See [15, Corollary 4.37]. We refer to [15, Remark 4.38] for the connection with Fredholmness.

§3. Mean winding numbers

Let $H^1_{\mathbb{R}}$ be *the real Hardy space*,

$$H^1_{\mathbb{R}} = \big\{\operatorname{Re} f : f \in H^1\big\} = \big\{u \in L^1_{\mathbb{R}}(\mathbb{R}) : \mathcal{H}u \in L^1_{\mathbb{R}}(\mathbb{R})\big\}.$$

We put $\|u\|_{H^1_{\mathbb{R}}} = \|u\|_{L^1} + \|\mathcal{H}u\|_{L^1}$.

Definition. Consider the cone

$$\Pi = \big\{\eta \in H^1_{\mathbb{R}} : \eta \text{ has a compact support on } \mathbb{R},\ \textstyle\int_{-\infty}^x \eta \le 0 \ \ \forall x \in \mathbb{R}\big\}.$$

Theorem 2. *Let G be an $r \times r$ matrix function in $C^b_{r\times r}$.*
(1) *If $T_G \in \Phi_-(H^2_r)$, there is a constant $C > 0$ such that for any η in Π,*

$$\int_{\mathbb{R}} \eta(x)\big(\arg\det G\big)(x)\,dx \le C\|\eta\|_{H^1_{\mathbb{R}}}.$$

(2) *If $T_G \in \Phi_+(H^2_r)$, there is a constant $C > 0$ such that for any η in Π,*

$$\int_{\mathbb{R}} \eta(x)\big(\arg\det G\big)(x)\,dx \ge -C\|\eta\|_{H^1_{\mathbb{R}}}.$$

It is well known that $\int_{\mathbb{R}} \eta = 0$ for any function η in $H^1_{\mathbb{R}}$, see [20, Chapter III]. Hence the above integrals do not depend on the additive constant in $\arg\det G$.

As a consequence, we obtain that if $T_G \in \Phi(H^2_r)$, then

$$\left| \int_{\mathbb{R}} \eta(x) \big(\arg\det G\big)(x)\, dx \right| \le C \|\eta\|_{H^1_{\mathbb{R}}}, \quad \eta \in \Pi.$$

In the scalar case, this inequality follows from the Widom-Devinatz theorem (Theorem B), together with the Fefferman duality theorem, and takes place for all $\eta \in H^1_{\mathbb{R}}$ (the integral is to be understood in the sense of the duality $H^1_{\mathbb{R}} - \mathrm{BMO}_{\mathbb{R}}$).

Definition. Let $\eta \in \Pi$, $\eta \not\equiv 0$ be fixed, and let $G \in C^b_{r\times r}$. Define the upper and the lower *mean winding numbers* of $\det G$ (associated with η) by

$$\overline{w}_\eta(G) = \varlimsup_{T\to+\infty} \sup_{y\in\mathbb{R}} \frac{1}{T} \int_{\mathbb{R}} \eta\Big(\frac{x-y}{T}\Big) \cdot \arg\det G(x)\, dx,$$
$$\underline{w}_\eta(G) = \varliminf_{T\to+\infty} \inf_{y\in\mathbb{R}} \frac{1}{T} \int_{\mathbb{R}} \eta\Big(\frac{x-y}{T}\Big) \cdot \arg\det G(x)\, dx.$$

Theorem 3. (1) *If $T_G \in \Phi_+(H^2_r)$, then $\underline{w}_\eta(G) \neq -\infty$;*
(2) *If $T_G \in \Phi_-(H^2_r)$, then $\overline{w}_\eta(G) \neq +\infty$.*

One can also define simpler characteristics

$$\widetilde{w}_\eta(G) = \varlimsup_{T\to+\infty} \frac{1}{T} \int_{\mathbb{R}} \eta\Big(\frac{x}{T}\Big) \cdot \arg\det G(x)\, dx$$

and the number $\underset{\sim}{w}_\eta(G)$, defined as the corresponding lower limit. One has $\underline{w}_\eta(G) \le \underset{\sim}{w}_\eta(G) \le \widetilde{w}_\eta(G) \le \overline{w}_\eta(G)$, so that Theorem 3 implies the same assertions for $\underset{\sim}{w}_\eta(G)$, $\widetilde{w}_\eta(G)$.

Consider a scalar $G \in C^b(\mathbb{R})$, $|G| > \varepsilon > 0$ on $\mathbb{R}$. If $\arg G$ has finite limits at $\pm\infty$, then $\widetilde{w}_\eta(G) = \underset{\sim}{w}_\eta(G) = K \cdot \arg G\big|_{-\infty}^{+\infty}$, where $K = \int_0^{+\infty} \eta(x)\, dx$. One also has $\overline{w}_\eta(G) = L \cdot \big(\arg G\big|_{-\infty}^{+\infty}\big)_+$, $\underline{w}_\eta(G) = L \cdot \big(\arg G\big|_{-\infty}^{+\infty}\big)_-$, where $L = \sup_{y\in\mathbb{R}} \int_y^{+\infty} \eta$, $y_+ = \max(y,0)$, $y_- = \min(y,0)$. So in this case all these winding numbers have a simple sense. For these symbols, each of the conditions $T_G \in \Phi_-(H^2)$, $T_G \in \Phi_+(H^2)$, $T_G \in \Phi(H^2)$ is equivalent to the requirement $\arg G\big|_{-\infty}^{+\infty} \neq \pm\pi, \pm 3\pi, \pm 5\pi$, etc. (see, for instance, [15] or [21, Ch. 9]).

Corollary 1. *Let $\alpha > 0$, and define generalized winding numbers*

$$\overline{w}_{\eta,\alpha}(G) = \overline{\lim_{T\to+\infty}} \sup_{y\in\mathbb{R}} \frac{1}{T^{1+\alpha}} \int_{\mathbb{R}} \eta\Big(\frac{x-y}{T}\Big) \cdot (\arg\det G)(x)\,dx,$$

$$\underline{w}_{\eta,\alpha}(G) = \underline{\lim_{T\to+\infty}} \inf_{y\in\mathbb{R}} \frac{1}{T^{1+\alpha}} \int_{\mathbb{R}} \eta\Big(\frac{x-y}{T}\Big) \cdot (\arg\det G)(x)\,dx.$$

(1) *If $T_G \in \Phi_+(H^2_r)$, then $\underline{w}_{\eta,\alpha}(G) \geq 0$;*
(2) *If $T_G \in \Phi_-(H^2_r)$, then $\overline{w}_{\eta,\alpha}(G) \leq 0$.*

This follows immediately from Theorem 3. ∎

In particular, the function $\eta_\alpha = \frac{1+\alpha}{2}\big(\chi_{[0,1]} - \chi_{[-1,0]}\big)$ is in Π. The corresponding upper winding number is given by

$$\overline{w}_\alpha(G) = \overline{\lim_{T\to+\infty}} \frac{1+\alpha}{2T^{1+\alpha}} \sup_{y\in\mathbb{R}} \Big[\int_y^{T+y} - \int_{y-T}^{y}\Big] \arg\det G(x)\,dx. \tag{3.1}$$

Let us define similarly the lower winding number $\underline{w}_\alpha(G)$, by taking $\inf_{y\in\mathbb{R}}$ and the corresponding lower limit. Corollary 1 holds, in particular, for these characteristics of G. If $r = 1$, $G(x) = \exp\big(i\gamma(\operatorname{sign} x)\cdot |x|^\alpha\big)$, and $0 < \alpha \leq 1$, then $\overline{w}_\alpha(G) = \underline{w}_\alpha(G) = \gamma$.

In fact, we could take instead of $T^{1+\alpha}$ any function $\rho(T)$ such that $\rho(T) > 0$, $T/\rho(T) \to 0$ as $T \to +\infty$ in the above definitions of generalized winding numbers.

Corollary 2 of Theorem 3. *Suppose G is in $\mathcal{G}C_{a,r\times r}(\mathbb{C}_+)$ or in $\mathcal{G}C_{a,r\times r}(\mathbb{C}_-)$, where $\mathbb{C}_- = \{z \in BC : \operatorname{Im} z < 0\}$. Then for any $\alpha > 0$, $\overline{w}_\alpha(G) = \underline{w}_\alpha(G) = 0$.*

Indeed, in both cases $T_G^{-1} = T_{G^{-1}}$, hence $T_G \in \Phi(H^2_r)$, and we can apply Corollary 1. ∎

Corollary 3 of Theorem 3. *Let $G \in \mathcal{G}C^b_{r\times r}(\mathbb{R})$, and define $\overline{w}_1(G)$ by (3.1) and $\beta(G)$ by (1.4). Then $\beta(G) \geq \overline{w}_1(G)/r$.*

Indeed, if $T_{e^{-i\tau x}G}$ is onto, then it is a Φ_--operator, which implies that

$$\overline{w}_1(e^{-i\tau x}G) = \overline{w}_1(G) - r\tau \leq 0. \qquad ∎$$

We remark that if G is a semi-almost periodic $r \times r$ matrix function such that $G, G^{-1} \in C^b_{r\times r}(\mathbb{R})$, then $\det G$ is a scalar semi-almost periodic function, and $\det G$ has almost periodic representatives $(\det G)_{\pm\infty}$ at $+\infty$ and $-\infty$, respectively (see [13, Theorem 1.21]). These representatives, by the Bohr mean motion theorem have the form

$$(\det G)_{\pm\infty}(x) = e^{i\varkappa_\pm x} e^{g_\pm(x)},$$

where $\varkappa_\pm$ are mean motions of $\det G(x)$ at $\pm\infty$ and functions $g_\pm$ are almost periodic (see, for instance, [13, Thm. 2.25]). In this case,

$$\underline{w}_1(G) = \min(\varkappa_-, \varkappa_+), \qquad \overline{w}_1(G) = \max(\varkappa_-, \varkappa_+),$$
$$\underset{\sim}{w}_1(G) = \widetilde{w}_1(G) = \frac{\varkappa_- + \varkappa_+}{2}.$$

If $r = 1$, complete criteria of Fredholmness, as well as the calculation of the Fredholm index are known since the work by Sarason [38]. It follows, in particular, that in this case $T_{e^{-i\tau x}G(x)}$ is not right-invertible if $\tau < \max(\varkappa_-, \varkappa_+)$ and is right-invertible if $\tau > \max(\varkappa_-, \varkappa_+)$. Hence $\beta(G) = \max(\varkappa_-, \varkappa_+)$. So Corollary 3 of Theorem 3 gives an exact estimate for the case of scalar semi-almost periodic functions.

The study of the almost periodic and semi-almost periodic matrix cases depends on the existence of some special factorizations of G. If these factorizations exist, then complete criteria for Fredholmness and formulas for the index are available, see [13, Ch. 10 and §19.6].

Proof of Theorem 2. By Theorem 1, it only has to be proved that if $f \in \mathrm{BMO}^+_{\mathbb{R}}$, then

$$\int_{\mathbb{R}} f(x)\eta(x)\,dx \ge -C\|\eta\|_{H^1_{\mathbb{R}}} \quad \text{for all } \eta \in \Pi.$$

This inequality follows from the Fefferman duality $H^1_{\mathbb{R}} - \mathrm{BMO}_{\mathbb{R}}$ (see [20]) in the case when $f \in \mathrm{BMO}_{\mathbb{R}}$. Now let f be nondecreasing, and take any function $\eta \in \Pi$. Suppose that $\operatorname{supp}\eta \subset I$, where I is a finite interval. Approximate f in $L^\infty(I)$ by a sequence of nondecreasing step functions $\{f_n\}$ of the form

$$f_n = C_n + \sum_k \alpha_{nk}\, \chi_{(-\infty, a_{nk}]},$$

where $C_n, a_{nk} \in \mathbb{R}$ and α_{nk} are negative. Then $\int_{\mathbb{R}} \eta f_n \ge 0$ for all n, hence $\int_{\mathbb{R}} \eta f \ge 0$.

We obtain the result by combining these two cases. ∎

Proof of Theorem 3. Let $\eta_{T,y}(x) = \eta\big(\frac{x-y}{T}\big)$. Since $\mathcal{H}(\eta_{T,y}) = (\mathcal{H}\eta)_{T,y}$, it follows that $\|\eta_{T,y}\|_{H^1_{\mathbb{R}}} = T\|\eta\|_{H^1_{\mathbb{R}}}$. So the assertions follow directly from Theorem 2. ∎

§4. Some related questions

PROBLEM 1. Give a real variable characterization of classes $\mathrm{BMO}^{\pm}_{\mathbb{R}}$.

The next two questions are certainly known for specialists for a long time, however, complete answers are not known.

PROBLEM 2. 1) Let $r = 1$, and let $G \in C(\mathbb{R})$, $\arg G \in C_+(\mathbb{R})$, $\lim_{x\to\pm\infty} \arg G(x) = \pm\infty$. What additional conditions guarantee that $T_G \in \Phi_+(H^2_r)$?

2) What can be said in this respect for the matrix case $r > 1$?

Sufficient conditions for $r = 1$ are given in [11] and in [19, Theorem 5.10]. As it follows from the construction of Lemma 4.9 in [11], there are symbols G of the above type such that T_G is not semi-Fredholm. See also [22, Theorem 28.2 and Section 32] for related counter-examples.

The book [19] also contains results about the matrix valued case. At least for the scalar case, it seems that more complete answers can be found.

PROBLEM 3. Suppose that $T_G \in \Phi_+(H^2_r)$. Can one give some estimates of $\operatorname{Ind} T_G$ in terms of some explicit real variable characteristics of $\arg\det G$?

PROBLEM 4. Suppose that $\eta_1, \eta_2 \in \Pi$. When can one assert that $\overline{w}_{\eta_1}(G) \neq +\infty$ implies that $\overline{w}_{\eta_2}(G) \neq +\infty$ for all $G \in C^b_{r\times r}(\mathbb{R})$ with $|\det G| > \varepsilon > 0$ on $\mathbb{R}$? Is there a "universal" function $\eta_0 \in \Pi$ such that for any G as above, $\overline{w}_{\eta_0}(G) \neq +\infty$ implies that $\overline{w}_{\eta}(G) \neq +\infty$ for all $\eta \in \Pi$?

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