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ON OKUYAMA'S THEOREMS ABOUT ALVIS-CURTIS DUALITY

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Abstract. We report on theorems by T. Okuyama about complexes generalizing the Coxeter complex and the action of parabolic subgroups on them, both for finite BN-pairs and finite dimensional Hecke algebras. Several simplifications, using mainly the surjections of [CaRi], allow a more compact treatment than the one in [O].

The purpose of this paper is to report on the unpublished manuscript [O] by T. Okuyama where are proved some conjectures generalizing to homotopy categories the theorems of [CaRi] and [LS] holding in derived categories. We refer to the latter references and [CaEn, §4] for a broader introduction to the subject.

The main theme is the one of complexes related with the Coxeter complex and the action of parabolic subgroups on them, either for finite groups with BN-pairs or for finite dimensional Hecke algebras. Okuyama's contractions prove a quite efficient tool in a number of situations (see the proof of Solomon-Tits theorem in Section 6).

We often stray away from Okuyama's proofs when it allows simplifications. We also emphasize some statements that may be of independent interest (see Section 1 below), and actually re-prove [Ri1, §8], [LS], and [CaRi].

Most proofs are selfcontained apart from basic facts about split BN-pairs and Hecke algebras.

Notations. Let A be a ring, one denotes by $\mathbf{mod}(A)$ the associated category of finitely generated left A-modules. If B is another ring, A,B-bimodules are just objects of $\mathbf{mod}(A\otimes B^\circ)$ where B° denotes the opposite algebra and $\otimes$ denotes (commutative) tensor product over $\mathbb{Z}$. Whenever

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M is an A,B-bimodule and N is a B-module, one denotes by $M\otimes_B N$ the usual tensor product over B, considered as an A-module. When B is a group algebra RG with R a commutative ring and G a finite group, one may use the abbreviation $M\otimes_G N$ for $M\otimes_{RG} N$.

We denote by $\mathbf{C^b}(A)$, $\mathbf{K^b}(A)$, and $\mathbf{D^b}(A)$ the categories of bounded (cochain) complexes of A-modules, its homotopy and derived categories, respectively. We allow to view a given complex of A-modules $(\cdots C^i \xrightarrow{\partial^i} C^{i+1}\cdots)$ as a $\mathbb{Z}$-graded A-module $C:=\bigoplus_i C^i$ endowed with a homogeneous endomorphism $\partial_C=(\partial^i)_i$ of degree 1 satisfying $\partial_C\circ\partial_C=0$.

Let S be a finite set, we denote by 2^S the set of subsets of S. A *coefficient system* on 2^S is a family of A-modules $(M^I)_{I\subseteq S}$ and A-homomorphisms $\varphi^M_{JI}\colon M^I\to M^J$ defined when $I\subseteq J$ ("restriction maps") and satisfying $\varphi^M_{KJ}\circ\varphi^M_{JI}=\varphi^M_{KI}$ whenever $I\subseteq J\subseteq K\subseteq S$.

Choose a total ordering $\leq$ on S. When $I\subseteq S$ and $s\in S$, denote by $n(I,s)$ the number of elements of I that are $<s$. A coefficient system $((M^I)_I,(\varphi^M_{JI})_{I,J})$ gives rise to an object $(\cdots\to M^0\to M^1\to\cdots)$ of $\mathbf{C^b}(A)$, where $M^i=\bigoplus_{I;|I|=i}M^I$ and $\partial^i\colon M^i\to M^{i+1}$ is defined on M^I by $\partial^i{}_{|M^I}=\sum_{s\in S\setminus I}(-1)^{n(I,s)}\varphi^M_{I\cup\{s\},I}$. For a more canonical definition, showing independence with regard to the choice of an ordering, see for instance [CaEn, Exercise 4.1].

§1. Reduced elements in Coxeter groups and Okuyama's contractions

Assume (W,S) is a finite Coxeter group. One denotes by l the length map with respect to S. If $I,J\subseteq S$, denote by W_I the subgroup of W generated by I and by D_{IJ} the set of $w\in W$ such that $l(uwv)=l(u)+l(w)+l(v)$ for any $u\in W_I$, $v\in W_J$.

If V, V' are subsets of W, one denotes $VV'=\{vv'\mid v\in V,\ v'\in V'\}$ and $V^{-1}=\{v^{-1}\mid v\in V\}$.

Let $\mathcal{A}=\bigcup_{I\subseteq S}W_I\backslash W=\{W_Iw\mid I\subseteq S,\ w\in W\}$ be the set of right cosets mod. parabolic subgroups.

If $a\in\mathcal{A}$, denote $S(a):=S\cap aa^{-1}$ (that is $S(a)=I$ if $a=W_Iw$ for some $w\in W$) and $l(a)=\min\{l(x)\mid x\in a\}$, which is the length of the only element of $a\cap D_{S(a),\emptyset}$.

If $J\subseteq S$ and $a\in\mathcal{A}$, denote $a\cup J:=W_{J\cup S(a)}a$. If $s\in S$, we write $a\cup s=a\cup\{s\}$.

The *Coxeter complex* $A(W,S)$ is the complex associated to the coefficient system on 2^S defined by

$$I \longmapsto \mathbb{Z}(W_I\backslash W) = \bigoplus_{a\in\mathcal{A},\, S(a)=I} \mathbb{Z}a$$

with restriction maps φ_{JI} defined by $\varphi_{JI}(a) = a \cup J$ for $I \subseteq J$. Then $A(W,S) = \mathbb{Z}\mathcal{A}$ graded by $a \mapsto |S(a)|$ and the differential is defined by $\partial(a) = \sum_{s\in S\setminus S(a)}(-1)^{n(S(a),s)}a \cup s$ for $a \in \mathcal{A}$ (recall that S is ordered and $n(I,s)$ denotes the number of elements of I that are $< s$).

Fix $I_0 \subseteq S$, $I_0 \neq S$ (the case $I_0 = S$ is trivial for what follows but could create ambiguities).

NOTATION 1. Let $\mathcal{A}(I_0) := \{a \in \mathcal{A} \mid a \cap D_{\emptyset,I_0} \neq \emptyset\}$. Note that the condition defining $\mathcal{A}(I_0)$ is equivalent to the element of minimal length in a being in $D_{\emptyset,I_0}$, hence in $D_{S(a),I_0}$.

Let $\mathcal{A}(I_0)^+ := \{a \in \mathcal{A}(I_0) \mid a \not\subseteq W_{I_0}\}$, that is the classes $W_I w$ for $I \subseteq S$, $w \in D_{I,I_0}$, and $w \neq 1$ if moreover $I \subseteq I_0$.

If $a \in \mathcal{A}(I_0)$, denote $I_0(a) = I_0 \cap a^{-1}a$.

Note the following property

(P2) If $a \in \mathcal{A}(I_0)$, then $I_0(a) = I_0 \cap I^w$ for $I = S(a)$ and the unique $w \in a \cap D_{I,I_0}$ (see for instance [CaEn, 2.6]).

PROPOSITION 3. *$\mathbb{Z}\mathcal{A}(I_0)^+$ is a subcomplex of $A(W,S)$ in $\mathbf{C^b}(\mathbb{Z})$. There exists a linear map $\sigma\colon \mathbb{Z}\mathcal{A}(I_0)^+ \to \mathbb{Z}\mathcal{A}(I_0)^+$ homogeneous of degree -1 such that*

(i) $\sigma\partial + \partial\sigma = \mathrm{Id}$.

(ii) *for any $a \in \mathcal{A}(I_0)^+$, $\sigma(a) \in \bigoplus \mathbb{Z}b$ where the sum is over $b \in \mathcal{A}(I_0)^+$ such that $I_0(b) \supseteq I_0(a)$.*

The Proposition will be used in Sections 2–4 through the following.

THEOREM 4. *Keep (W,S) a finite Coxeter group, and $I_0 \subseteq S$. Let A be a ring and $((M^I)_I, (\varphi^M_{JI})_{I,J})$ be a coefficient system of A-modules on 2^{I_0}.*

Assume that for all $b \in \mathcal{A}(I_0)^+$, we are given a submodule $Z_b \subseteq M^{I_0(b)}$ such that, for all $b,b' \in \mathcal{A}(I_0)^+$ with $b \subseteq b'$ (and therefore $I_0(b) \subseteq I_0(b')$), one has $\varphi^M_{I_0(b'),I_0(b)}(Z_b) \subseteq Z_{b'}$.

Let a coefficient system Z on 2^S be defined by $I \mapsto Z^I := \prod_{b\in\mathcal{A}(I_0)^+;S(b)=I} Z_b$, and $\varphi^Z_{I',I}(x) = \varphi^M_{I_0(b\cup I'),I_0(b)}(x) \in Z_{b\cup I'} \subseteq Z^{I'}$ if $I \subseteq I' \subseteq S$ and $x \in Z_b$ with $S(b) = I$.

Then the complex associated to Z is contractible in $\mathbf{C^b}(A)$.

Proof. Let $\sigma\colon \mathbb{Z}\mathcal{A}(I_0)^+ \to \mathbb{Z}\mathcal{A}(I_0)^+$ be a map as in Proposition 3. This implies the existence of integers $m_{b,b'}$ for $b, b' \in \mathcal{A}(I_0)^+$ with $I_0(b) \subseteq I_0(b')$ and $|S(b')| = |S(b)| - 1$ such that $\sigma(b) = \sum_{b'} m_{b,b'} b'$. Set $m_{b,b'} = 0$ for pairs (b, b') not satisfying the conditions above.

Since the differential on $\mathbb{Z}\mathcal{A}(I_0)^+$ is defined by $\partial(b) = \sum_{s\in S\setminus S(b)} (-1)^{n(S(b),s)}(b\cup s)$, the relation in Proposition 3.(i) reads, for any $b \in \mathcal{A}(I_0)^+$,

$$(\mathrm{E5}_b)\quad b = \sum_{\substack{s\in S\setminus S(b)\\ b'\in\mathcal{A}(I_0)^+}} (-1)^{n(S(b),s)} m_{b\cup s,b'} b' + \sum_{\substack{b'\in\mathcal{A}(I_0)^+\\ s\in S\setminus S(b')}} (-1)^{n(S(b'),s)} m_{b,b'}.(b'\cup s)$$

in $\mathbb{Z}\mathcal{A}(I_0)^+$.

When $b \in \mathcal{A}(I_0)^+$ and $z \in Z_b$, we write z_b to mean the element of $Z^{S(b)}$ in the factor Z_b. Define $\bar\sigma$ on $Z_b \subseteq \prod_{b\in\mathcal{A}(I_0)^+} Z_b$ by $\bar\sigma(z) = \sum_{b'\in\mathcal{A}(I_0)^+} m_{b,b'}\varphi^M_{I_0(b'),I_0(b)}(z)_{b'}$. This is well defined since $m_{b,b'} \neq 0$ implies $I_0(b) \subseteq I_0(b')$.

The differential on Z is defined by

$$\partial^Z(z) = \sum_{s\in S\setminus S(b)} (-1)^{n(S(b),s)} \varphi^M_{I_0(b\cup s),I_0(b)}(z)_{b\cup s}$$

for $z \in Z_b$ (remember that Z_b is at degree $|S(b)|$). Then, keeping $z \in Z_b$,

$$(\bar\sigma\partial^Z + \partial^Z\bar\sigma)(z) = \bar\sigma\Big(\sum_{s\in S\setminus S(b)} (-1)^{n(S(b),s)} \varphi^M_{I_0(b\cup s),I_0(b)}(z)_{b\cup s}\Big) + \partial^Z\Big(\sum_{b'\in\mathcal{A}(I_0)^+} m_{b,b'}\varphi^M_{I_0(b'),I_0(b)}(z)_{b'}\Big)$$

which equals

$$\sum_{\substack{s\in S\setminus S(b)\\ b'\in\mathcal{A}(I_0)^+}} (-1)^{n(S(b),s)} m_{b\cup s,b'}\varphi^M_{I_0(b'),I_0(b)}(z)_{b'} + \sum_{\substack{b'\in\mathcal{A}(I_0)^+\\ s\in S\setminus S(b')}} (-1)^{n(S(b'),s)} m_{b,b'}\varphi^M_{I_0(b'\cup s),I_0(b)}(z)_{b'\cup s}$$

thanks to the composition of restriction maps. By ($\mathrm{E5}_b$) above, this equals $\varphi^M_{I_0(b),I_0(b)}(z)_b = (z)_b$. Then $\bar\sigma\partial + \partial\bar\sigma = \mathrm{Id}$, that is the contractibility of Z in $\mathbf{C^b}(A)$. □

Proof of Proposition 3. That $\mathbb{Z}\mathcal{A}(I_0)$ and $\mathbb{Z}\mathcal{A}(I_0)^+$ are sub-coefficient systems of $\mathbb{Z}\mathcal{A}$ is clear by the definition of restriction maps and the fact that both $\mathcal{A}(I_0)$ and $\mathcal{A}(I_0)^+$ are closed for supsets.

In order to check the particular contractibility that is announced about $\mathbb{Z}\mathcal{A}(I_0)^+$, it seems handy to apply the following symmetry of $D_{\emptyset,I_0}$ and $\mathcal{A}(I_0)$.

Let $\theta\colon W \to W$ defined by $\theta(w) = w_S w w_{I_0}$, where $w_I \in W_I$ denotes the element of maximal length for $I \subseteq S$. This θ clearly preserves $\mathcal{A}$ with $S(\theta(a)) = S(a)^{w_S}$ and $\varphi_{JI}\circ\theta = \theta\circ\varphi_{J^{w_S},I^{w_S}}$ for any $a\in\mathcal{A}$ and $I\subseteq J\subseteq S$. Moreover θ preserves $D_{\emptyset,I_0}$ (exchanging 1 and $w_S w_{I_0}$) and therefore $\mathcal{A}(I_0)$, with $\theta(D_{I,I_0}) = D_{I^{w_S},I_0}$. One has also $I_0(\theta(a)) = I_0(a)^{w_{I_0}}$ for all $a\in\mathcal{A}(I_0)$.

Denote $\mathcal{B} = \theta(\mathcal{A}(I_0)^+) = \{W_I w \mid I\subseteq S,\ w\in D_{I,I_0},\ w\neq w_S w_{I_0}\}$.

Note that if $w\in D_{\emptyset,I_0}\setminus\{w_S w_{I_0}\}$, there is some $s_w\in S\setminus(S\cap wI_0w^{-1})$ such that $l(s_w w) > l(w)$ (take the w_S-conjugate of the first term in any reduced decomposition of $\theta(w)\neq 1$).

If $b\in\mathcal{B}$, $b = W_I w$ with $I\subseteq S$ and $w\in D_{II_0}$, one denotes $s_b := s_w$ and $b\setminus s_b = W_{I\setminus\{s_b\}}w$.

Lemma 6. *Let $\tau\colon \mathbb{Z}\mathcal{B}\to\mathbb{Z}\mathcal{B}$ be the linear map homogeneous of degree -1 defined as follows on $b\in\mathcal{B}$*

$$\tau(b) = (-1)^{n(S(b),s_b)}(b\setminus s_b) \text{ if } s_b\in S(b), \qquad \tau(b) = 0 \text{ otherwise.}$$

One has $\tau^2 = 0$ and
(i) *If $\tau(b)\neq 0$ then $\tau(b) = \pm b'$ where $b'\in\mathcal{B}$, $l(b') = l(b)$ and $I_0(b') = I_0(b)$.*
(ii) *$(\tau\partial+\partial\tau)(b)\in b+\bigoplus_{b'}\mathbb{Z}b'$ the sum being over $b'\in\mathcal{B}$ such that $l(b') < l(b)$ and $I_0(b')\supseteq I_0(b)$.*

Let us show how this implies Proposition 3.

By (i) of Lemma 6, $\tau(\mathcal{B})\subseteq\mathbb{Z}\mathcal{B}$. By (ii), $\tau\partial+\partial\tau = \mathrm{Id}+\rho$ where ρ is nilpotent and of degree 0. Now define

$$\tau' := \tau - \tau\rho + \tau\rho^2 - \tau\rho^3 + \cdots = \tau(\tau\partial+\partial\tau)^{-1}$$

and let us check that it satisfies the conditions (i) and (ii) of Proposition 3.

First $\tau'\partial + \partial\tau' = \mathrm{Id}$ since ∂ clearly commutes with $\tau\partial + \partial\tau$, hence with its inverse.

As for (ii), that is $\tau'(b) \in \bigoplus_{b'} \mathbb{Z}b'$ where the sum is over $b' \in \mathcal{B}$ such that $I_0(b') \supseteq I_0(b)$, this is a consequence of $\tau' = \tau - \tau\rho + \tau\rho^2 - \cdots$ with τ and ρ having the corresponding property by the Lemma.

This implies now Proposition 3 by defining $\sigma = \theta \circ \tau' \circ \theta$ thanks to the elementary properties of θ with regard to the restriction maps and $a \mapsto I_0(a)$. Note however that $\theta \circ \partial \circ \theta$ is not exactly ∂ but the same twisted by w_S-conjugacy due to the property of θ with regard to restriction maps φ_{JI}. A correction consists in adapting the ordering on S: choose first the w_S-conjugate of the ordering implicit in Proposition 3. □

Proof of Lemma 6. (i) Write $b = W_I w$ with $w \in D_{II_0}$, $w \neq w_S w_{I_0}$. One must assume $s_w \in I$. Then $\tau(b) = \pm W_{I\setminus\{s_w\}} w$ with $w \in D_{I\setminus\{s_w\},I_0}$, $w \neq w_S w_{I_0}$ and $I_0(b \setminus s_b) = (I \setminus \{s_w\})^w \cap I_0 = I^w \cap I_0 = I_0(b)$ by (P) and the definition of s_w. It is also clear that $s_{b\setminus s_b} = s_w = s_b$, so $\tau^2(b) = 0$.

(ii) Note that if $b, b' \in \mathcal{A}$ and $b \subseteq b'$ (inclusion of cosets), then $I_0(b) \subseteq I_0(b')$ and $l(b') \leq l(b)$.

From (i) and the definition of ∂ it is then clear that we don't have to worry about sets $I_0(b')$'s. So we concentrate on lengths $l(b')$. Thanks to the above and (i) just proved, on evaluating $(\partial\tau + \tau\partial)(b)$ we must check that only one term, the one producing b, has length remaining equal to $l(b)$ upon applying τ and ∂.

First case: $s_b \in S(b)$. Then $\tau(b) = (-1)^{n(b,s_b)}(b \setminus s_b)$ with $S(b \setminus s_b) = S(b) \setminus \{s_b\}$ and $((b \setminus s_b) \cup s_b) = b$. Then

$$\begin{aligned}(\partial\tau + \tau\partial)(b) = b + &\sum_{s\in S\setminus S(b)} (-1)^{n(b,s_b)+n(b\setminus s_b,s)}((b \setminus s_b) \cup s) \\ &+ \sum_{s\in S\setminus S(b)} (-1)^{n(b,s)}\tau(b \cup s).\end{aligned}$$

In the above sums, one must spot the terms with length $l(b)$ $(= l(w))$. In the first $\sum$, $b \setminus s_b = W_{I\setminus\{s_b\}}w$ and one may have $l(W_{I\setminus\{s_b\}\cup\{s\}}w) = l(w)$ only if $l(sw) > l(w)$, thus producing $((b \setminus s_b) \cup s) = W_{I\setminus\{s_b\}\cup\{s\}}w$ in the above sum. For the second $\sum$, by (i), length $l(b)$ is kept only if $l(b \cup s) = l(b)$. This means again $l(sw) > l(w)$, thus giving a term $\tau(b \cup s) = (-1)^{n(b\cup s,s_b)}W_{I\setminus\{s_b\}\cup\{s\}}w$.

In all, the two $\sum$ contribute terms of length $l(b)$ by the sum over $s \in S \setminus S(b)$ with $l(sw) > l(w)$ of the terms $((-1)^{n(b,s_b)+n(b\setminus s_b,s)} + (-1)^{n(b,s)+n(b\cup s,s_b)})$

$(b \setminus s_b \cup s)$. This is 0 as can be seen by calculating the component on $b \cup s$ of $\partial \circ \partial(b \setminus s_b)$, one finds $(-1)^{n(b,s_b)+n(b,s)} + (-1)^{n(b\setminus s_b,s)+n(b\cup s,s_b)} = 0$.

Second case: $b = W_I w$ with $w \in D_{II_0}$ and $s_b \notin S(b) = I$. Then $\tau(b) = 0$ and $b = \pm\tau(b \cup s_b)$ where $b \cup s_b = W_{I\cup\{s_w\}} w \in \mathcal{B}$ with $w \in D_{I\cup\{s_w\},I_0}$ and $l(b \cup s_b) = l(b) = l(w)$. To get our claim, it suffices to show that $\partial\tau + \tau\partial - \mathrm{Id}$ sends $\tau(b \cup s_b)$ into $\bigoplus_{b';l(b')<l(b)} \mathbb{Z}b'$. We have $(\partial\tau + \tau\partial - \mathrm{Id})\tau(b \cup s_b) = (\tau\partial\tau - \tau)(b \cup s_b) = \tau(\partial\tau + \tau\partial - \mathrm{Id})(b \cup s_b)$ and our claim follows from the first case and Lemma 6.(i). ∎

Remark 7. When $a \in \mathcal{A}(I_0)$, denote by $v_0(a)$ the (unique) element of $a \cap D_{S(a),I_0}$, so that $a = W_{S(a)} v_0(a)$ with $l(a) = l(v_0(a))$.

Denote by $\leq_{\mathbf{r}}$ the right divisibility in W, that is $w' \leq_{\mathbf{r}} w$ if and only if $w = w''w'$ with lengths adding. Any inclusion $b \subseteq b'$ in $\mathcal{A}(I_0)$ clearly implies $v_0(b') \leq_{\mathbf{r}} v_0(b)$.

A quick inspection shows that in the above proof some relation $l(b') < l(b)$ may occur only when in addition $v_0(b') <_{\mathbf{r}} v_0(b)$. So the map $\tau\colon \mathbb{Z}\mathcal{B} \to \mathbb{Z}\mathcal{B}$ of Lemma 6 satisfies $(\tau\partial + \partial\tau)(b) \in b + \bigoplus_{b'} \mathbb{Z}b'$ where the sum is over $b' \in \mathcal{B}$ such that $v_0(b') <_{\mathbf{r}} v_0(b)$ (instead of just $l(b') < l(b)$).

So Proposition 3 holds with a map σ satisfying $\sigma(a) \in \bigoplus_b \mathbb{Z}b$ where the sum is over $b \in \mathcal{A}(I_0)^+$ such that $w_S v_0(b) w_{I_0} \leq_{\mathbf{r}} w_S v_0(a) w_{I_0}$ and $I_0(b) \supseteq I_0(a)$.

§2. A theorem of Curtis type in the homotopy category

Let G be a finite group endowed with a split BN-pair of characteristic the prime p (see [CaEn, 2.20]). We have subgroups N, B, $T \subseteq B \cap N$. The quotient $W := N/T$ is a Coxeter group for the subset $S \subseteq W$. When $I \subseteq S$, the associated parabolic subgroup $P_I = BW_IB$ is a semi-direct product $U_I.L_I$ for U_I the largest normal p-subgroup of P_I and L_I a group with a split BN-pair associated to the subgroups $N \cap L_I$, $B \cap L_I$, T and the Coxeter group W_I.

Let $R = \mathbb{Z}[p^{-1}]$ or any commutative ring where p is invertible.

Notation 8. If $I \subseteq S$, denote $e_I = |U_I|^{-1} \sum_{u\in U_I} u$, an idempotent in the group algebra RG. Define the coefficient system $X(G)$ of RG-bimodules on 2^S by

$$X(G)^I = RGe_I \otimes_{P_I} e_I RG$$

and restriction maps $\varphi_{JI}\colon X(G)^I \to X(G)^J$ defined by $x \otimes_{P_I} y \mapsto x \otimes_{P_J} y$ whenever $x \in RGe_I$ and $y \in e_I RG$. We keep the same notation for the associated complex of RG-bimodules.

Theorem 9. (Okuyama, [O, 3.1]) *Let* $I_0 \subseteq S$. *Then* $X(G)e_{I_0} \cong RGe_{I_0}\otimes_{L_{I_0}} X(L_{I_0})$ *in* $\mathbf{K^b}(RG\otimes(RL_{I_0})^\circ)$ *and* $e_{I_0}X(G) \cong X(L_{I_0})\otimes_{L_{I_0}} e_{I_0}RG$ *in* $\mathbf{K^b}(RL_{I_0}\otimes RG^\circ)$.

The proof consists in giving a description of the kernel of the surjection $X(G)e_{I_0} \to RGe_{I_0}\otimes_{L_{I_0}} X(L_{I_0})$ introduced in [CaRi, 3.5], allowing to apply Theorem 4. One will use repeatedly the following (see [HL, 3.1], [CaEn, Ex. 5, p. 53]).

Proposition 10.

- $e_Ie_J = e_Je_I = e_I$ *when* $I \subseteq J \subseteq S$.
- *If* $I, J \subseteq S$, $w \in D_{IJ}$ *and* $n \in N$ *with* $nT = w$, *then* $e_Ine_J = e_{I\cap {}^wJ}ne_J = e_Ine_{I^w\cap J} = e_{I\cap {}^wJ}ne_{I^w\cap J}$.

Definition. If $I \subseteq S$, $w \in D_{I,I_0}$, let

$$X_{I,w} := RGe_I \otimes_{P_I} e_IRP_IwRP_{I_0}e_{I_0}$$

and

$$Y_{I,w} := RGe_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0}RP_{I_0} = RGe_{I_0}\otimes_{L_{I_0}} X(L_{I_0})^{I^w\cap I_0}$$

both $RG\otimes RP_{I_0}^\circ$-modules.

In the following propositions, keep $I \subseteq S$ and $w \in D_{II_0}$.

Proposition 11. $X_{I,w} \cong Y_{I,w}$ *as* $RG\otimes RP_{I_0}^\circ$*-module by a map sending* $x\otimes_{P_I} \dot wy$ *to* $xe_{I\cap {}^wI_0}\dot we_{I^w\cap I_0}\otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0}y$ *for any* $x \in RGe_I$, $y \in RP_{I_0}e_{I_0}$, $\dot w \in N$ *such that* $\dot wT = w$.

Proposition 12. *If* $J \subseteq S$, $w' \in D_{J,I_0}$ *and* $I^w \cap I_0 \subseteq J^{w'} \cap I_0$, *there is a* $RG\otimes RP_{I_0}^\circ$*-morphism* $Y_{I,w} \to Y_{J,w'}$ *sending* $x\otimes_{P_{I^w\cap I_0}} y$ *to* $x\otimes_{P_{J^{w'}\cap I_0}} y$ *for* $x \in RGe_{I^w\cap I_0}$, $y \in e_{J^{w'}\cap I_0}RP_{I_0}$.

If $I \subseteq J \subseteq S$ *and* $w' \in D_{JI_0}\cap W_Jw$ *(which ensures* $I^w\cap I_0 \subseteq J^{w'}\cap I_0$*) the above morphism corresponds to the restriction map* $X(G)^Ie_{I_0} \to X(G)^Je_{I_0}$ *of* $X(G)e_{I_0}$ *through the isomorphism of Proposition* 11 *and the decomposition* $X(G)^Ie_{I_0} = RGe_I\otimes_{P_I} e_IRGe_{I_0} = \bigoplus_{P_IwP_{I_0}} RGe_I\otimes_{P_I} e_IRP_IwRP_{I_0}e_{I_0}$.

The above will allow to describe the kernel of the surjection $X(G)e_{I_0} \to RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$ defined in [CaRi, §6] (see also [CaEn, 4.10]).

The following will be useful to deduce an isomorphism in the homotopy category.

PROPOSITION 13. *Let $\mathcal{A}$ be an abelian category. Let $0 \to Z \to Y \to Y' \to 0$ be an exact sequence in $\mathbf{C^b}(\mathcal{A})$ which is split in each degree and such that Z is contractible. Then $Y \cong Y'$ in $\mathbf{K^b}(\mathcal{A})$.*

Proof of Proposition 13. More generally, it is well known that in a short exact sequence which is split in each degree, the third term is always homotopy equivalent to the mapping cone of the monomorphism. □

Proof of Theorem 9. Let us consider Y the complex associated to the coefficient system on 2^S defined by $I \mapsto \bigoplus_{w\in D_{II_0}} Y_{I,w}$ and restriction maps $Y_{I,w} \to Y_{J,w'}$ as in Proposition 12 when $I \subseteq J$ and $W_Jw = W_Jw'$, 0 otherwise.

Then $X(G)e_{I_0}$ is isomorphic to Y thanks to the isomorphisms of Proposition 11.

In order to apply Proposition 13, we show that there is surjection of complexes $Y \to RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$ in $\mathbf{C^b}(RG \otimes (RL_{I_0})^\circ)$, split in each degree, and with contractible kernel.

Note that $Y_{I,w}$ only depends on the class $W_Iw \in \mathcal{A}(I_0)$ (see Notation 1). We then use the notation $Y_b = RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})^{I_0(b)}$ for $b \in \mathcal{A}(I_0)$ (see (P2)). Note that Y is a coefficient system on 2^S defined by $I \mapsto \bigoplus_b Y_b$ where the sum is over $b \in \mathcal{A}(I_0)$ such that $S(b) = I$, and the restriction map is defined by sending Y_b to $Y_{b'}$ only when $b' \supseteq b$ and is then $RGe_{I_0} \otimes_{L_{I_0}} \varphi_{I_0(b'),I_0(b)}$ where φ is the restriction map of $X(L_{I_0})$.

Let us define $Y' = \bigoplus_b Y_b$ where the sum is over $b \in \mathcal{A}(I_0)$ such that $b \subseteq W_{I_0}$ and same restriction maps as in Y. Then $Y' \cong RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$ since each $b \subseteq W_{I_0}$ is equal to $W_{S(b)}$ with $I_0(b) = S(b) \subseteq I_0$.

We have a surjective map of coefficient systems of $RG \otimes RL_{I_0}^\circ$-modules on 2^S: $Y \to Y' \cong RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$ sending Y_b to 0 whenever $b \in \mathcal{A}(I_0)$ and $b \not\subseteq W_{I_0}$. The kernel is $\bigoplus_b Y_b$ where the sum is over $b \in \mathcal{A}(I_0)^+$.

One may now apply Theorem 4 with $A = RG \otimes RL_{I_0}^\circ$, $M = RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$ and $Z_b = Y_b$ for $b \in \mathcal{A}(I_0)^+$. This tells us that $\bigoplus_{b\in\mathcal{A}(I_0)^+} Y_b$ is a contractible complex.

We then have the first isomorphism of the theorem. In order to deduce the second one, one uses the (covariant) functor $M \to M^\iota$ from $\mathbf{mod}(RG)$

to $\mathbf{mod}(RG^\circ)$ which consists in keeping the same R-module structure and composing action of group elements $g \in G$ with inversion (see [CaEn, 4.16] and proof). It is exact, commutes with tensor products, extends to complexes, and induces involutory equivalences of $\mathbf{K^b}$ categories. It is easy to see that $X(G)^\iota \cong X(G)$, $(RGe_{I_0})^\iota \cong e_{I_0}RG$ and $X(L_{I_0})^\iota \cong X(L_{I_0})$ by compatible isomorphisms. Then we get the second isomorphism from the first. □

Proof of Propositions 11 *and* 12. Using the isomorphism $RHK \cong RH \otimes_{H\cap K} RK$ as $RH \otimes RK^\circ$-modules (by the evident maps) whenever H, K are subgroups of a finite group G, one has $e_I RP_I wRP_{I_0} e_{I_0} \cong RL_I wRL_{I_0} \cong RL_I w \otimes_{L_{I^w\cap I_0}} RL_{I_0} = RP_I e_I w e_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0} e_{I_0} P_{I_0}$.

Applying Proposition 10, we get $e_I RP_I \dot{w} RP_{I_0} e_{I_0} \cong RP_I e_{I\cap {}^w I_0} w e_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0} RP_{I_0}$ by the map of $RP_I \otimes RP_{I_0}^\circ$-modules sending $e_I x\dot{w} y e_{I_0}$ to $xe_{I\cap {}^w I_0}\dot{w} e_{I^w\cap I_0} \otimes e_{I^w\cap I_0} y$ for $x \in RP_I$, $y \in RP_{I_0}$ and $\dot{w} \in N$ such that $\dot{w}T = w$. Note that our map does not depend on the choice of $\dot{w}$ inside the class $w \in N/T$.

On applying the functor $RGe_I \otimes_{P_I} -$, we get $RGe_I \otimes_{P_I} e_I RP_I wRP_{I_0} e_{I_0} \cong RGe_{I\cap {}^w I_0} w e_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0} RP_{I_0}$ by the map $xe_I \otimes e_I y\dot{w} z e_{I_0} \mapsto xye_{I\cap {}^w I_0} \dot{w} e_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0} z$ for $x \in RG$, $y \in RP_I$, $z \in RP_{I_0}$. On the other hand, $RGe_{I\cap {}^w I_0} \dot{w} e_{I^w\cap I_0} = RGe_{I^w\cap I_0}$ by Dipper-Du-Howlett-Lehrer's theorem of independence (see [HL, 2.4], [CaEn, 3.10]). So the map of Proposition 11 is indeed defined and an isomorphism as announced.

Assume now the hypotheses of Proposition 12. Using the restriction map of $X(L_{I_0})$, it is clear that the map announced in the first statement of Proposition 12 is the map $RGe_{I_0} \otimes_{L_{I_0}} \varphi_{J^{w'}\cap I_0, I^w\cap I_0}$ on $RGe_{I_0} \otimes_{L_{I_0}} X(L_{I_0})$, where φ denotes the restriction maps of $X(L_{I_0})$ as a coefficient system on 2^{I_0}.

To verify the second statement, we assume $W_J w' = W_J w$ with $w' \in D_{JI_0}$ (unique). It suffices to check that the following square is commutative:

$$\begin{array}{ccc} X_{I,w} & \longrightarrow & Y_{I,w} \\ \downarrow & & \downarrow \\ X_{J,w'} & \longrightarrow & Y_{J,w'} \end{array}$$

where horizontal arrows are the isomorphism of Proposition 11, the first vertical arrow is the restriction map $(X(G)e_{I_0})^I \to (X(G)e_{I_0})^J$ (which actually sends the term $X_{I,w}$ in the term $X_{J,w'}$) and the second vertical arrow is the one we have seen above, that is $RGe_{I_0} \otimes \varphi_{J^{w'}\cap I_0, I^w\cap I_0}$.

We check the images on a $RG \otimes RP_{I_0}^\circ$-generator of $X_{I,w}$, namely $e_I \otimes e_I\dot{w}e_{I_0}$. Going right then down, one gets $e_I \otimes_{P_I} e_I\dot{w}e_{I_0} \mapsto e_{I\cap {}^wI_0}\dot{w}e_{I^w\cap I_0} \otimes_{P_{I^w\cap I_0}} e_{I^w\cap I_0} \mapsto e_{I\cap {}^wI_0}\dot{w}e_{I^w\cap I_0} \otimes_{P_{J^{w'}\cap I_0}} e_{I^w\cap I_0}$. Going down then right yields $e_I \otimes_{P_I} e_I\dot{w}e_{I_0} \mapsto e_I \otimes_{P_J} e_I\dot{w}e_{I_0} = e_I \otimes_{P_J} e_I\dot{v}\dot{w}'e_{I_0} = e_I\dot{v} \otimes_{P_J} \dot{w}'e_{I_0} \mapsto e_I\dot{v}e_{J\cap {}^{w'}I_0}\dot{w}'e_{J^{w'}\cap I_0} \otimes_{P_{J^{w'}\cap I_0}} e_{I^w\cap I_0}$ where $w = vw'$ with $v \in W_J$ such that w' is J-reduced and one defines $\dot{v} = \dot{w}(\dot{w}')^{-1} \in L_J$.

But $e_{J\cap {}^{w'}I_0}\dot{w}'e_{J^{w'}\cap I_0} = e_J\dot{w}'e_{I_0} = e_J\dot{w}'e_{J^{w'}\cap I_0}$ (Proposition 10 above), so that $e_I\dot{v}e_{J\cap {}^{w'}I_0}\dot{w}'e_{J^{w'}\cap I_0} = e_I\dot{v}e_J\dot{w}'e_{J^{w'}\cap I_0} = e_Ie_J\dot{w}e_{J^{w'}\cap I_0} = e_I\dot{w}e_{J^{w'}\cap I_0}$ so the last term on the second composition (down then right) is $e_I\dot{w}\otimes_{P_{J^{w'}\cap I_0}} e_{I^w\cap I_0}$. This is what we expect, again by Proposition 10. □

§3. Alvis-Curtis duality in the homotopy category

The main result of this section shows that the derived equivalence of [CaRi] actually holds in the homotopy category ([O, Th. 1]).

Let us recall two general results that will be useful. Let A be a ring and $X, Y \in \mathbf{C^b}(A)$, $X_0 \in \mathbf{C^b}(A^\circ)$. We refer to [CaEn, §A1] for the usual notations $X[i] \in \mathbf{C^b}(A)$ ($i \in \mathbb{Z}$), $\mathrm{Homgr}_A(X, Y)$, $X_0 \otimes_A X \in \mathbf{C^b}(\mathbb{Z})$. If A is an R-algebra for R a commutative ring, one denotes $X^\vee := \mathrm{Homgr}_R(X, R[0])$ as an object of $\mathbf{C^b}(A^\circ)$ in the usual way.

The same proof as [Ri1, 1.1.(a)] gives the following

LEMMA 14. *Let C be in $\mathbf{C^b}(A)$ such that its homology is concentrated in degree* 0 (*i.e.* $H(C) \cong H^0(C)[0]$ *in* $\mathbf{C^b}(A)$) *and, for any term C' of C, both* $\mathrm{Homgr}_A(C', C)$ *and* $\mathrm{Homgr}_A(C, C')$ *have their homology concentrated in degree* 0.

Then $C \cong H^0(C)[0]$ in $\mathbf{K^b}(A)$.

For the strong adjunction below, see [Ro, §2.2.6], [Ri2, 9.2.5]. Note that the isomorphisms can be made explicit (see [Ro, §2.2], [O, §5.1]).

PROPOSITION 15. *Assume A, B and Λ are R-free R-algebras of finite rank. Assume A and B are symmetric* (*see* [CaEn, 1.19], [Ri2, 9.2.1], [Ro, 2.2.3]). *All modules are assumed to be R-free of finite rank.*

Let M be a bounded complex of $A\otimes B^\circ$-modules projective on each side. Let N_1, N_2 be objects of $\mathbf{C^b}(B \otimes \Lambda^\circ)$, $\mathbf{C^b}(A \otimes \Lambda^\circ)$ respectively. Then

$$\mathrm{Homgr}_{A\otimes\Lambda^\circ}(M \otimes_B N_1, N_2) \cong \mathrm{Homgr}_{B\otimes\Lambda^\circ}(N_1, M^\vee \otimes_A N_2).$$

THEOREM 16. (Okuyama, [O, Th. 1]) *Let G be a finite group endowed with a split BN-pair of characteristic p, let $R = \mathbb{Z}[p^{-1}]$. Recall $X(G)$ in $\mathbf{C^b}(RG \otimes RG^\circ)$ (see Notation 8). Then*

$$X(G) \otimes_{RG} X(G)^\vee \cong X(G)^\vee \otimes_{RG} X(G) \cong RG \quad \textit{in } \mathbf{K^b}(RG \otimes RG^\circ)$$

and therefore $X(G) \otimes_{RG} -$ induces a splendid equivalence $\mathbf{K^b}(RG) \to \mathbf{K^b}(RG)$ in the sense of [Ri1].

Proof. We have the claimed isomorphisms in $\mathbf{D^b}(RG \otimes RG^\circ)$ (see [CaRi, 5.1] or [CaEn, 4.18]). Note that in both references, the main argument is the following fact:

(F) *if $I \subseteq S$ and N is a cuspidal RL_I-module, then $M := RGe_I \otimes_{L_I} N \in \mathbf{mod}(RG)$ satisfies $X(G) \otimes_G M \cong M[-|I|]$ in $\mathbf{D^b}(RG)$ (and even in $\mathbf{K^b}(RG)$).*

Here, *cuspidal* means that, considering N as a U_I-trivial RP_I-module, one has $e_J N = 0$ for any $J \subset I$, $J \neq I$. The rest of the proof of [CaRi, 5.1], essentially consists in reducing to R being a field, then use the fact that simple RG-modules are quotients of those M's.

A homotopy equivalence is preserved by any additive functor, so we may apply $- \otimes_{L_I} N \colon \mathbf{mod}(RG \otimes RL_I^\circ) \to \mathbf{mod}(RG)$ to the first isomorphism of Theorem 9, and get $X(G) \otimes_G M \cong RGe_I \otimes_{L_I} X(L_I) \otimes_{L_I} N$ in $\mathbf{K^b}(RG)$. By cuspidality of N, one has clearly $X(L_I) \otimes_{L_I} N = N[-|I|]$. Whence (F).

Let us now prove the isomorphisms of Theorem 16 in $\mathbf{K^b}(RG \otimes RG^\circ)$. We use induction on $|S|$, the case $G = T$ being trivial.

For the first isomorphism $X(G) \otimes_G X(G)^\vee \cong RG$, in view of Lemma 14, we have to check that, for any direct summand C of a term of $X(G) \otimes_G X(G)^\vee$, the $\mathrm{Homgr}_{RG \otimes RG^\circ}$'s between C and $X(G) \otimes_G X(G)^\vee$ have their homology in degree 0 only.

The terms of $X(G) \otimes_G X(G)^\vee$ are of the type $RGe_I \otimes_{P_I} e_I RG \otimes_G RGe_J \otimes_{P_J} e_J RG$ for $I, J \subseteq S$. This rearranges as $RGe_I \otimes_{P_I} e_I RGe_J \otimes_{P_J} e_J RG = \bigoplus_{w \in D_{IJ}} RGe_I \otimes_{P_I} e_I RP_I w RP_J e_J \otimes_{P_J} e_J RG$. Applying Proposition 11, one finds a sum of modules $Y_{I,w} \otimes_{P_J} e_J RG = RGe_{I^w \cap J} \otimes_{P_{I^w \cap J}} e_{I^w \cap J} RG$, each isomorphic to some $X(G)^{I_0} = E_{I_0} \otimes_{L_{I_0}} E_{I_0}^\vee$ where $E_{I_0} = RGe_{I_0}$ for $I_0 \subseteq S$.

So we have to check that, for all $I \subseteq S$, both $\mathrm{Homgr}_{G \times G^\circ}(E_I \otimes_{L_I} E_I^\vee, X(G) \otimes_G X(G)^\vee)$ and $\mathrm{Homgr}_{G \times G^\circ}(X(G) \otimes_G X(G)^\vee, E_I \otimes_{L_I} E_I^\vee)$ have homology in degree 0 only. By Proposition 15, it suffices to check $\mathrm{Endgr}_{L_I \times G^\circ}$

$(E_I^\vee \otimes_G X(G))$. The bi-additivity of the functor $\mathrm{Homgr}_A(-,-)$ along with Theorem 9 give $\mathrm{Endgr}_{L_I\times G^\circ}(E_I^\vee \otimes_G X(G)) \sim \mathrm{Endgr}_{L_I\times G^\circ}(X(L_I)\otimes_{L_I} E_I^\vee)$ where $\sim$ denotes homotopy equivalence in $\mathbf{C^b}(\mathbb{Z})$.

Case $I \neq S$. Our induction hypothesis tells us that $X(L_I)^\vee \otimes_{L_I} X(L_I)$ is homotopically equivalent to $RL_I[0]$ in $\mathbf{C^b}(RL_I \otimes RL_I^\circ)$. Then Proposition 15 gives $\mathrm{Endgr}_{L_I\times G^\circ}(X(L_I)\otimes_{L_I} E_I^\vee) \cong \mathrm{Homgr}_{L_I\times G^\circ}(X(L_I)^\vee \otimes_{L_I} X(L_I)\otimes_{L_I} E_I^\vee, E_I^\vee) \sim \mathrm{Homgr}_{L_I\times G^\circ}(E_I^\vee, E_I^\vee)$ which is in degree zero.

Case $I = S$. One has to check the homology of $\mathrm{Endgr}_{G\times G^\circ}(X(G))$. There is a spectral sequence ${}_IE_i^{pq}$ for the double complex $\mathrm{Hom}_{G\times G^\circ}(X(G), X(G))$, such that $E_0^{pq} = \mathrm{Hom}_{G\times G^\circ}(X(G)^{-p}, X(G)^q)$ and $E_1^{pq} = H^q(\mathrm{Homgr}_{G\times G^\circ}(X(G)^{-p}, X(G)))$ (see for instance [B, §3.4] on spectral sequences of double complexes).

Let $J \subseteq S$. Theorem 9 again gives $\mathrm{Homgr}_{G\times G^\circ}(X(G)^J, X(G)) \cong \mathrm{Homgr}_{G\times L_J^\circ}(E_J, X(G)\otimes_G E_J) \sim \mathrm{Homgr}_{G\times L_J^\circ}(E_J, E_J \otimes_{L_J} X(L_J))$. This has q-th homology $= 0$ for $q > |J|$ since $X(L_J)^q = 0$ for those q. Then $E_1^{pq} = 0$ for $p + q > 0$. Whence $H^i(\mathrm{Endgr}_{G\times G^\circ}(X(G))) = 0$ for $i > 0$. Negative i's are taken care by the spectral sequence ${}_{II}E_i^{pq}$ (satisfying $E_1^{pq} = H^p(\mathrm{Homgr}_{G\times G^\circ}(X(G), X(G)^q)))$.

This completes the proof of $X(G)\otimes_G X(G)^\vee \cong RG$ in $\mathbf{K^b}(RG\otimes RG^\circ)$. As for the isomorphism $X(G)^\vee \otimes_G X(G) \cong RG$, it suffices to do the same with $X(G)^\vee$ instead of $X(G)$. Note that $X(G)^\vee$ has the same terms as $X(G)$. The only non trivial fact that we need is a version of the second isomorphism of Theorem 9 for $X(G)^\vee$. This in turn is a consequence of the first on applying the functor $M \to M^\vee$. □

§4. Hecke algebras

Let (W, S) be a finite Coxeter group. Let R be a commutative ring. Let $(q_s)_{s\in S} \in (R^\times)^S$ be a family of invertible elements of R such that $q_s = q_t$ whenever $s, t \in S$ are W-conjugate.

Recall the definition of the Hecke algebra $\mathcal{H} = \bigoplus_{w\in W} Rh_w$ (see for instance [GP, 4.4.6]) with multiplication obeying the rules

$h_w h_{w'} = h_{ww'}$ when $w, w' \in W$ and lengths add (therefore $h_1 = 1_{\mathcal{H}}$),
$(h_s)^2 = (q_s - 1)h_s + q_s$ when $s \in S$.

Note that $\mathcal{H}$ is symmetric for the linear form giving the coordinate on h_1 in the above basis (see [GP, 8.1.1]). When $I \subseteq S$, $\mathcal{H}_I = \bigoplus_{w\in W_I} Rh_w$ is a subalgebra of $\mathcal{H}$ and is also the Hecke algebra associated to (W_I, I) and same coefficients q_s.

Following [LS], one defines the complex $X(\mathcal{H})$ of $\mathcal{H}$-bimodules, from the following coefficient system on 2^S. For $I \subseteq J \subseteq S$, let $X(\mathcal{H})^I = \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ with restriction maps $\varphi_{JI}\colon X(\mathcal{H})^I \to X(\mathcal{H})^J$ defined by $\varphi_{JI}(h \otimes_{\mathcal{H}_I} h') = h \otimes_{\mathcal{H}_J} h'$.

THEOREM 17. (Okuyama, [O, 4.1]) *Let $I_0 \subseteq S$. The restriction of $X(\mathcal{H})$ to $\mathcal{H} \otimes_R (\mathcal{H}_{I_0})^\circ$ is isomorphic in $\mathbf{K^b}(\mathcal{H} \otimes (\mathcal{H}_{I_0})^\circ)$ with $\mathcal{H} \otimes_{\mathcal{H}_{I_0}} X(\mathcal{H}_{I_0})$.*

Proof. The proof is very similar to the one of Theorem 9. Let us abbreviate $\otimes_{\mathcal{H}_I}$ as $\otimes_I$ in what follows. Since h_w's multiply as elements of W when lengths add, one has $\mathcal{H} = \bigoplus_{w \in D_{II_0}} \mathcal{H}_I h_w \mathcal{H}_{I_0}$ as $\mathcal{H}_I, \mathcal{H}_{I_0}$-bimodule. Then the restriction to $\mathcal{H} \otimes (\mathcal{H}_{I_0})^\circ$ of $X(\mathcal{H})^I$ is $\mathcal{H} \otimes_I \bigoplus_{w \in D_{II_0}} \mathcal{H}_I h_w \mathcal{H}_{I_0} \cong \bigoplus_{w \in D_{II_0}} \mathcal{H}.(1 \otimes_I h_w).\mathcal{H}_{I_0}$.

Corresponding to Propositions 11 and 12, one has the following.

Let $I \subseteq S$, $w \in D_{II_0}$. Denote $X(\mathcal{H})_{I,w} = \mathcal{H} \otimes_I \mathcal{H}_I h_w \mathcal{H}_{I_0}$.

Then $X(\mathcal{H})_{I,w} \cong \mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0}$ by a map sending $1 \otimes_I h_w$ to $h_w \otimes_{I^w \cap I_0} 1$. To see that, note first that if $v \in W_{I^w \cap I_0}$, then ${}^w v \in W_I$ and therefore $h_w h_v = h_{wv} = h_{{}^w v} h_w$. The claimed isomorphism then corresponds to the composition of the following (explicit) isomorphisms $\mathcal{H} \otimes_I \mathcal{H}_I h_w \mathcal{H}_{I_0} = \mathcal{H} \otimes_I \mathcal{H}_I \otimes_{I \cap {}^w I_0} \mathcal{H}_{I \cap {}^w I_0} h_w \mathcal{H}_{I_0} = \mathcal{H} \otimes_I \mathcal{H}_I h_w \otimes_{I^w \cap I_0} \mathcal{H}_{I_0} = \mathcal{H} h_w \otimes_{I^w \cap I_0} \mathcal{H}_{I_0} = \mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0}$.

If $I^w \cap I_0 \subseteq J^{w'} \cap I_0$ with $J \subseteq S$ and $w' \in D_{JI_0}$, we have the restriction map $\mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0} \to \mathcal{H} \otimes_{J^{w'} \cap I_0} \mathcal{H}_{I_0}$.

If moreover $W_J w = W_J w'$, the above corresponds to the restriction map of $X(\mathcal{H})$, $\varphi_{JI}\colon X(\mathcal{H})_{I,w} \to X(\mathcal{H})_{J,w'}$ through the above isomorphism. To check this, one evaluates at $1 \otimes h_w$. Denote $v \in W_J$ such that $w = vw'$ with lengths adding. One composition (isomorphism, then restriction map) gives $1 \otimes_I h_w \mapsto h_w \otimes_{I^w \cap I_0} 1 \mapsto h_w \otimes_{J^{w'} \cap I_0} 1$, the other composition gives $1 \otimes_I h_w \mapsto 1 \otimes_J h_w = 1 \otimes_J h_v h_{w'} = h_v \otimes_J h_{w'} \mapsto h_v h_{w'} \otimes_{J^{w'} \cap I_0} 1$.

One may now replace ${}_{\mathcal{H}}X(\mathcal{H})_{\mathcal{H}_{I_0}}$ by the complex associated to the coefficient system on 2^S associating to I the bi-module $\bigoplus_w \mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0}$, a sum over D_{II_0}, with restriction maps defined by all the maps $\mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0} \to \mathcal{H} \otimes_{J^{w'} \cap I_0} \mathcal{H}_{I_0}$ for $I \subseteq J$, $w' \in D_{JI_0}$ and $W_J w = W_J w'$ (which implies $I^w \cap I_0 \subseteq J^{w'} \cap I_0$).

Then ${}_{\mathcal{H}}X(\mathcal{H})_{\mathcal{H}_{I_0}} \twoheadrightarrow \mathcal{H} \otimes_{I_0} X(\mathcal{H}_{I_0})$ by a map sending $\mathcal{H} \otimes_{I^w \cap I_0} \mathcal{H}_{I_0}$ to 0 if $W_I w \not\subseteq W_{I_0}$, and $\mathcal{H} \otimes_I \mathcal{H}_{I_0} \to \mathcal{H} \otimes_{I_0} \mathcal{H}_{I_0} \otimes_I \mathcal{H}_{I_0}$ the evident (onto) map when $I \subseteq I_0$. The image is the complex associated to the coefficient system $\mathcal{H} \otimes_{I_0} X(\mathcal{H}_{I_0})$ on 2^{I_0}.

Using now the notation of Section 1, the kernel corresponds to the complex associated with the coefficient system on 2^S, $I \mapsto Z^I = \bigoplus_{b\in\mathcal{A}(I_0)^+, S(b)=I} \mathcal{H} \otimes_{I_0(b)} \mathcal{H}_{I_0}$ with restriction maps sending $\mathcal{H} \otimes_{I_0(b)} \mathcal{H}_{I_0}$ into $\mathcal{H} \otimes_{I_0(b')} \mathcal{H}_{I_0}$ by $\mathcal{H} \otimes_{I_0} \varphi_{I_0(b'),I_0(b)}$ only when $S(b') \supseteq S(b)$ and $b' = b \cup S(b')$. The contractibility of Z is immediate by Theorem 4. Then Proposition 13 implies our claim. ∎

Remark 18. The above theorem allows to give a proof of the main result in [LS] and [PS, §3.1] in the following way.

Taking $I_0 = \emptyset$ in Theorem 17, one gets that $X(\mathcal{H})$ has its homology concentrated in degree 0. So this homology is the kernel of $\partial^0 \colon \mathcal{H} \otimes_R \mathcal{H} \to \bigoplus_{s\in S} \mathcal{H} \otimes_{\mathcal{H}_s} \mathcal{H}$, that is the intersection of the kernels of the restriction maps $\varphi_{s,\emptyset} \colon \mathcal{H} \otimes_R \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{H}_s} \mathcal{H}$ defined by $h \otimes h' \mapsto h \otimes_{\mathcal{H}_s} h'$.

An element $x \in \mathcal{H} \otimes_R \mathcal{H}$ writes in a unique way $x = \sum_{w\in W} h_w \otimes x_w$ with x_w's in $\mathcal{H}$. For $s \in S$, using the partition $W = D_{\emptyset,s} \sqcup D_{\emptyset,s}s$ and the law of $\mathcal{H}$, it is clear that $\varphi_{s,\emptyset}(x) = 0$ if and only if $x_{ws} = -(h_s)^{-1} x_w$ for any $w \in D_{\emptyset,s}$. Using a reduced decomposition of each $w \in W$, one then gets that $x \in \ker(\partial^0)$ if and only if $x_w = (-1)^{l(w)}(h_w)^{-1} x_1$ for any $w \in W$.

Define $\xi := \sum_{w\in W} (-1)^{l(w)} h_w \otimes (h_w)^{-1} \in \mathcal{H} \otimes_R \mathcal{H}$. We now have $\ker(\partial^0) = \xi.\mathcal{H} \cong \mathcal{H}$ as right $\mathcal{H}$-module.

Using the partition $W = D_{s,\emptyset} \sqcup sD_{s,\emptyset}$ and the formula for $(h_s)^2$, it is clear that $h_s \xi h_s = -q_s \xi$, for any $s \in S$. So we get

$$H^0(X(\mathcal{H})) = \ker(\partial^0) = \mathcal{H}.\xi = \xi.\mathcal{H} \cong {}_\alpha\mathcal{H}$$

where α is the automorphism of $\mathcal{H}$ sending h_s to $-q_s(h_s)^{-1}$.

The R-dual $X(\mathcal{H})^\vee$ has similar properties, so $H(X(\mathcal{H})) \otimes_{\mathcal{H}} H(X(\mathcal{H})^\vee) \cong {}_{\mathcal{H}}\mathcal{H}_{\mathcal{H}}[0]$. On the other hand, the bi-projectivity of the terms of $X(\mathcal{H})$ implies that the natural map $H(X(\mathcal{H}) \otimes_{\mathcal{H}} X(\mathcal{H})^\vee) \to H(X(\mathcal{H})) \otimes_{\mathcal{H}} H(X(\mathcal{H})^\vee) = {}_{\mathcal{H}}\mathcal{H}_{\mathcal{H}}[0]$ is an isomorphism (see [B, 3.4.4]).

Theorem 19. (Okuyama, [O, Th. 2]) *$X(\mathcal{H}) \otimes_{\mathcal{H}} -$ induces an auto-equivalence of the homotopy category $\mathbf{K}^{\mathbf{b}}(\mathcal{H})$.*

Proof. The proof is similar to the one of Theorem 16 with $\mathcal{H}$ replacing RG, ${}_{\mathcal{H}}\mathcal{H}_{\mathcal{H}_{I_0}}$ replacing RGe_{I_0}, Theorem 17 replacing Theorem 9, and Remark 18 giving the equivalence in the derived category. Also $\mathcal{H}$ is symmetric, which allows to use Proposition 15.

Note that we need to have identical statements for $X(\mathcal{H})$ and $X(\mathcal{H})^\vee$. As in the end of proof of Theorem 16, this is done by applying the contravariant functor $M \mapsto M^\vee$ and using the isomorphism $X(\mathcal{H}) \cong X(\mathcal{H}_{I_0}) \otimes_{\mathcal{H}_{I_0}} \mathcal{H}$ in $\mathbf{K^b}(\mathcal{H}_{I_0} \otimes \mathcal{H}^\circ)$. This in turn is deduced from Theorem 17 by noting that there is a (covariant) functor $M \mapsto M^\iota$ from $\mathbf{mod}(\mathcal{H})$ to $\mathbf{mod}(\mathcal{H}^\circ)$ corresponding to the isomorphism $\mathcal{H} \cong \mathcal{H}^\circ$ (as R-algebras) which is defined by $h_s \mapsto h_s$ (and therefore $h_w \mapsto h_{w^{-1}}$). □

§5. Generalized Steinberg module and the Solomon-Tits theorem

The context (and notations) are now again the ones of Sections 2 and 3, where G is a finite group with a split BN-pair. Define $\mathrm{St}(G)$ as the object of $\mathbf{C^b}(\mathbb{Z}G)$ associated to the coefficient system on 2^S defined by $\mathrm{St}(G)^I = \mathbb{Z}G/P_I$ and $\varphi_{JI}(gP_I) = gP_J$ for $I \subseteq J \subseteq S$ and $g \in G$.

Compare the following with an old lemma on the Steinberg character ([DM, 9.2]).

Theorem 20. (Okuyama, [O, 3.7]) *Let $I \subseteq S$. Then*

$$\mathrm{Res}^G_{P_I} \mathrm{St}(G) \cong \mathrm{Ind}^{P_I}_{L_I} \mathrm{St}(L_I)$$

in $\mathbf{K^b}(\mathbb{Z}P_I)$.

Remark. Note that, for $I = \emptyset$ (and therefore $P_I = B$, $L_I = T$), the theorem implies that $\mathrm{St}(G)$ has homology only in degree 0 (a theorem of Solomon-Tits, see [CuRe, 66.33]). Note also that the proof below simplifies a lot when $I = \emptyset$.

Proof of Theorem 20. We prove the statement with I_0 instead of I. We actually check a right module version of (i) with $\mathrm{St}(G)^I = \mathbb{Z}(P_I \backslash G)$, noting that the left module version follows by the same type of considerations as in the end of proofs of Theorem 9 and Theorem 16.

Note also that the definition of $\mathrm{St}(G)$ can be made using any system of parabolic subgroups containing a given Borel subgroup B^{g_0}. Denote $B^- = B^{w_S}$, $P_I^- = B^- W_I B^- = U_I^- . L_I$ where $U_I^- = U^{w_S} \cap U^{w_S w_I}$.

One may assume that $\mathrm{St}(G)$ (resp. $\mathrm{St}(L_{I_0})$) is associated to the coefficient system on 2^S (resp. 2^{I_0}) defined by $I \mapsto [P_I^-]\mathbb{Z}G$, (resp. $I \mapsto [L_{I_0} \cap P_I^-]\mathbb{Z}L_{I_0}$) where $[F]$ denotes the sum of elements of F in $\mathbb{Z}G$ for any subset $F \subseteq G$.

Let us define a surjective map $\mathrm{Res}^G_{P_{I_0}}\mathrm{St}(G)\to\mathrm{St}(L_{I_0})\otimes_{L_{I_0}}\mathbb{Z}P_{I_0}$ in $\mathbf{C^b}(\mathbb{Z}P_{I_0}{}^\circ)$, split in each degree and with contractible kernel. Then our claim will follow by Proposition 13.

Noting that $P_I^-\backslash G/P_{I_0}=W_I\backslash W/W_{I_0}$ is in bijection with D_{II_0}, one has $\mathrm{Res}^G_{P_{I_0}}(\mathrm{St}(G)^I)=\bigoplus_{w\in D_{II_0}}[P_I^-]w\mathbb{Z}P_{I_0}$ with connecting map $\mathrm{Res}^G_{P_{I_0}}(\mathrm{St}(G)^I\to\mathrm{St}(G)^J)$ sending $[P_I^-]wg$ to $[P_J^-]wg$ for any $g\in P_{I_0}$ and $I\subseteq J\subseteq S$.

Let us define a map $\pi\colon \mathrm{Res}^G_{P_{I_0}}\mathrm{St}(G)\to\mathrm{St}(L_{I_0})\otimes_{L_{I_0}}\mathbb{Z}P_{I_0}$ by sending $[P_I^-]w\mathbb{Z}P_{I_0}$ to 0 except when $I\subseteq I_0$ and $w=1$ in which case we use the isomorphism $\pi_I\colon [P_I^-]\mathbb{Z}P_{I_0}\cong[P_I^-\cap L_{I_0}]\mathbb{Z}P_{I_0}=[P_I^-\cap L_{I_0}]\mathbb{Z}L_{I_0}\otimes_{L_{I_0}}\mathbb{Z}P_{I_0}$ due to the fact that $P_I^-\cap P_{I_0}=P_I^-\cap L_{I_0}$ (apply for instance [CaEn, Exercise 2.4], or see proof of the Lemma below). This is clearly a surjective $\mathbb{Z}P_{I_0}^\circ$-homomorphism on coefficients, split at each I. In order to show that it commutes with connecting maps, we have to check for any $I\subseteq J\subseteq S$, the equality $\pi_J\varphi_{JI}=\varphi_{JI}\pi_I$ where the first φ is the connecting map in $\mathrm{St}(G)$ and the second is the one in $\mathrm{St}(L_{I_0})\otimes_{L_{I_0}}\mathbb{Z}P_{I_0}$ (seen as a coefficient system on 2^S by extending trivially on $2^S\setminus 2^{I_0}$). If $J\not\subseteq I_0$, both sides are 0. If $I\subseteq J\subseteq I_0$, one gets $[P_I^-]g\mapsto[P_J^-]g\mapsto[P_J^-\cap L_{I_0}]g$ and $[P_I^-]g\mapsto[P_I^-\cap L_{I_0}]g\mapsto[P_J^-\cap L_{I_0}]g$ if $g\in P_{I_0}$, $[P_I^-]g\mapsto[P_J^-]g\mapsto 0$ and $[P_I^-]g\mapsto 0\mapsto 0$ if $g\in G\setminus P_{I_0}$.

The kernel Z of our map is a graded sum of $\mathbb{Z}P_{I_0}^\circ$-modules $Z_b=[P_I^-]w\mathbb{Z}P_{I_0}$ for $b=W_Iw\in\mathcal{A}(I_0)^+$, Z_b being at degree $|I|$. To show that Z is contractible, one imitates the proof of Theorem 4. Recall $\sigma\in\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{A}(I_0)^+)$ from Proposition 3, which is expressed by $\sigma(b)=\sum_{b'\in\mathcal{A}(I_0)^+}m_{bb'}b'$ with $m_{bb'}\in\mathbb{Z}$ for $b,b'\in\mathcal{A}(I_0)^+$.

Fix $b=W_Iw$ and $b'=W_{I'}w'$ in $\mathcal{A}(I_0)^+$ with $I,I'\subseteq S$ and $w\in D_{II_0}$, $w'\in D_{I'I_0}$ such that $m_{bb'}\neq 0$. Proposition 3 and Remark 7 tell us that $I_0\cap I^w\subseteq I_0\cap(I')^{w'}$ and $w_Sww_{I_0}\leq_{\mathbf{r}} w_Sw'w_{I_0}$. We use that through the following consequence

Lemma. $(P_I^-)^w\cap P_{I_0}\subseteq(P_{I'}^-)^{w'}\cap P_{I_0}$.

This allows to define $\varphi_{b'b}\colon Z_b=[P_I^-]w\mathbb{Z}P_{I_0}\to Z_{b'}=[P_{I'}^-]w'\mathbb{Z}P_{I_0}$ as the only $(\mathbb{Z}P_{I_0})^\circ$-homomorphism sending $[P_I^-]w$ to $[P_{I'}^-]w'$. This behaves like restriction maps: $\varphi_{b''b'}\circ\varphi_{b'b}=\varphi_{b''b}$ whenever $b,b',b''\in\mathcal{A}(I_0)^+$ with $m_{bb'}\neq 0$ and $m_{b'b''}\neq 0$. Defining now $\bar\sigma\colon Z\to Z$ as $\sum_{b'\in\mathcal{A}(I_0)^+}m_{bb'}\varphi_{b'b}$ on Z_b, the same proof as for Theorem 4 shows that the equations (E5$_b$) satisfied by the $m_{bb'}$'s imply $\bar\sigma\partial+\partial\bar\sigma=\mathrm{Id}$ on Z. ∎

Proof of the Lemma. Replacing I and I' by their w_S-conjugate and w, w' by $w_S w w_{I_0}$, $w_S w' w_{I_0}$ respectively, we have to check that $(P_I)^w \cap P_{I_0} \subseteq (P_{I'})^{w'} \cap P_{I_0}$ as soon as $w \in D_{II_0}$, $w' \in D_{I'I_0}$ satisfy $I^w \cap I_0 \subseteq (I')^{w'} \cap I_0$ and $w' \leq_{\mathbf{r}} w$.

Applying twice [CaEn, 2.27.(i)], one gets $L_{I^w \cap I_0} \subseteq P_I^w \cap P_{I_0} \subseteq P_{I^w \cap I_0} = L_{I^w \cap I_0}.U$ and ${}^wU \cap P_I \subseteq U$. Therefore $P_I^w \cap P_{I_0} = L_{I^w \cap I_0}.(U^w \cap U)$. Now $I^w \cap I_0 \subseteq (I')^{w'} \cap I_0$, while $w' \leq_{\mathbf{r}} w$ implies $U^w \cap U \subseteq U^{w'} \cap U$ (apply [CaEn, 2.3.(iii), 2.23.(i)]). ∎

References

[B] D. Benson, Representations and Cohomology II: Cohomology of Groups and Modules, Cambridge Univ. Press, Cambridge, 1991.

[CaEn] M. Cabanes and M. Enguehard, Representation Theory of Finite Reductive Groups, Cambridge Univ. Press, Cambridge, 2004.

[CaRi] M. Cabanes and J. Rickard, *Alvis-Curtis duality as an equivalence of derived categories*, Modular Representation Theory of Finite Groups (M. J. Collins, B. J. Parshall, L. L. Scott, eds.), de Gruyter, 2001, pp. 157–174.

[CuRe] C. W. Curtis and I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Wiley, 1987.

[DM] F. Digne and J. Michel, Representations of finite groups of Lie type, Cambridge, 1991.

[GP] M. Geck and G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, Oxford, 2000.

[HL] B. Howlett and G. Lehrer, *On Harish-Chandra induction for modules of Levi subgroups*, J. Algebra, **165** (1994), 172–183.

[LS] M. Linckelmann and S. Schroll, *A two-sided q-analogue of the Coxeter complex*, J. Algebra, **289** (2005), no. 1, 128–134.

[O] T. Okuyama, *On conjectures on complexes of some module categories related to Coxeter complexes*, preprint, August 2006, 25 p.

[PS] B. Parshall and L. Scott, *Quantum Weyl reciprocity for cohomology*, Proc. London Math. Soc. (3), **90** (2005), 655–688.

[Ri1] J. Rickard, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math. Soc. (3), **72** (1996), 331–358.

[Ri2] J. Rickard, *Triangulated categories in the modular representation theory of finite groups*, S. König and A. Zimmermann, Derived Equivalences for Group Rings, LNM **1685**, Springer, 1998, pp. 177–198.

[Ro] R. Rouquier, *Block theory via stable and Rickard equivalences*, Modular Representation Theory of Finite Groups (M. J. Collins, B. J. Parshall, L. L. Scott, eds.), Walter de Gruyter, 2001, pp. 101–146.

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