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CANONICAL BASES OF BORCHERDS-CARTAN TYPE

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Abstract. We study the canonical basis for the negative part $\mathbf{U}^-$ of the quantum generalized Kac-Moody algebra associated to a symmetric Borcherds-Cartan matrix. The algebras $\mathbf{U}^-$ associated to two different matrices satisfying certain conditions may coincide (6.3). We show that the canonical bases coincide provided that the algebras $\mathbf{U}^-$ coincide (Theorem 6.3.5). We also answer partially a question by Lusztig in [L3] (Theorem 7.1.1).

§1. Introduction

Let $\mathbf{U}^-(C)$ be the negative part of the quantized enveloping algebra of the Kac-Moody Lie algebra associated to a generalized Cartan matrix C. In [L1], [L2], [L4], Lusztig defined and studied the canonical basis $\mathbf{B}(C)$ for $\mathbf{U}^-(C)$. The canonical basis $\mathbf{B}(C)$ has many remarkable properties such as positivity and integrality ([L4]). The appearance of the canonical basis is one of the major achievements in Lie theory. (Kashiwara developed his *crystal base* theory in [K]. It was shown by Grojnowski and Lusztig ([GL]) that the global crystal base by Kashiwara coincides with the canonical basis by Lusztig.)

There are two quite different approaches, algebraic and geometric, to define the canonical basis $\mathbf{B}(C)$ of $\mathbf{U}^-(C)$. For the geometric approach, the algebra $\mathbf{U}^-(C)$ is realized as an algebra $\mathcal{K}_Q$, which is contained in the direct sum of the Grothendieck groups of the bounded derived categories of complexes of sheaves on representation varieties of certain quiver Q without loops ([L2]). The irreducible perverse sheaves appearing in $\mathcal{K}_Q$ give rise to the canonical basis $\mathbf{B}(C)$ of $\mathbf{U}^-(C)$. The "simplest" irreducible perverse sheaves in $\mathcal{K}_Q$, which generate $\mathcal{K}_Q$, correspond to the Chevalley generators of $\mathbf{U}^-(C)$.

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When the matrix C is replaced by a Borcherds-Cartan matrix, the algebra $\mathbf{U}^-(C)$ becomes the negative part of the quantized enveloping algebra of a generalized Kac-Moody algebra (quantum generalized Kac-Moody algebra for short). In this more general setting, the algebraic approach is carried out in [JKK], and the global crystal base $\mathbb{B}(C)$ for $\mathbf{U}^-(C)$ is defined. The geometric approach is taken recently in [KS1], and the canonical basis $\mathbf{B}(C)$ for $\mathbf{U}^-(C)$ is defined. The two bases $\mathbb{B}(C)$ and $\mathbf{B}(C)$ coincide under certain condition on C. It is conjectured that this condition can be removed.

One notices that in this more general setting, the Borcherds-Cartan matrix C is not determined by the algebra $\mathbf{U}^-(C)$ uniquely. Under certain conditions on two given Borcherds-Cartan matrices C and D, the algebras $\mathbf{U}^-(C)$ and $\mathbf{U}^-(D)$ coincide (6.3). It is not clear if $\mathbf{B}(C) = \mathbf{B}(D)$ or $\mathbb{B}(C) = \mathbb{B}(D)$ when $\mathbf{U}^-(C) = \mathbf{U}^-(D)$.

The first main result in this paper says that the canonical bases $\mathbf{B}(C)$ and $\mathbf{B}(D)$ coincide provided that the algebras $\mathbf{U}^-(C)$ and $\mathbf{U}^-(D)$ coincide for two symmetric Borcherds-Cartan matrices C and D. The second main result in this paper answers partially a question in [L3]. This question asked if the algebra $\mathcal{K}_Q$ mentioned above is still generated by the "simplest" irreducible perverse sheaves in $\mathcal{K}_Q$ when the quiver Q has loops. In this more general setting, the algebra $\mathcal{K}_Q$ is bigger than $\mathbf{U}^-(C)$. We show that $\mathcal{K}_Q$ is generated by the simplest possible elements in $\mathcal{K}_Q$ when each imaginary vertex has at least two loops (Theorem 7.1.1). In general, the subalgebra generated by these simplest possible elements in $\mathcal{K}_Q$ has a canonical-type basis, and a characterization of these subalgebras is also given (Theorem 7.1.1, Proposition 7.1.2).

To prove the first result, we investigate in detail a factorization of Lusztig's resolution of the representation varieties of a quiver with possible loops. Such a factorization was first appeared in [KS1]. This factorization resolves the imaginary parts of the representation varieties first and then resolves the real part. In the case when the flags used in the resolution on the imaginary part are full flags, we are able to construct three based algebras from certain classes of simple perverse sheaves on the representation varieties and the intermediate varieties in the factorization. It turns out that the three algebras are isomorphic and the bases of the three algebras are compatible with each other. One of the algebras is the one constructed in [KS1], which is shown to be the geometric realization of the quantum generalized Kac-Moody algebras of the Borcherds-Cartan matrix of the quiver. The other two algebras are new. One of the two new algebras is independent of the number of the loops on each imaginary vertex. This leads us to a proof of the first result. (See Section 6.3.)

In order to prove the second result, we show that the subalgebra generated by the generators $L_{i,n}$ has a basis, which we call the canonical basis for this subalgebra, consisting of certain semisimple perverse sheaves. In the case when the number of loops is at least two for each imaginary vertex, semisimplicity can be strengthen to simplicity. In this case the subalgebra coincides with Lusztig's algebra. Lusztig's question is then answered affirmatively. To show that the semisimple perverse sheaves, as basis elements for the subalgebra, are generated by the generators $L_{i,n}$, we build up a bridge between the full-flag resolutions and the partial-flag resolutions. There is no obvious relation between the full-flag and partial-flag resolutions. This was done by reducing to a certain resolution of the representation variety associated to the subquiver of the original quiver consisting only non-loop arrows. Then we investigate the compatibility of several induction functors defined on various varieties along the bridge. This leads us to the proof of the second result. The proof of the second result is the difficult part of this paper.

The paper is organized as follows. In Section 2, we define the quantum generalized Kac-Moody algebra, and its negative part. In Section 3, we study the factorization of the resolutions of the representation varieties of a quiver with loops. Then we relate this factorization with the resolution of the representation varieties of the subquiver consisting of all non-loop arrows. Section 4 is devoted to the construction of classes of (semi)simple perverse on various varieties in Section 3. In Section 5, we study Lusztig's induction functor, and its several variations. We also show the compatibility of the various induction functors defined. In Section 6, we constructed various algebras from the classes of (semi)simple perverse sheaves defined in Section 5. The first result is then proved. In Section 7, we prove the second result, and give a characterization for Lusztig's algebra.

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§2. Quantum generalized Kac-Moody algebras

2.1. Symmetric Borcherds-Cartan matrices

Let I be a countable set. A *symmetric Borcherds-Cartan* matrix is a matrix $C = (c_{ij})_{i,j\in I}$ satisfying

$$c_{ii} \in \{2, 0, -2, -4, \dots\} \text{ and } c_{ij} = c_{ji} \in \mathbb{Z}_{\leq 0} \quad \text{for } i \neq j \in I.$$

We set $I^+ = \{i \in I \mid c_{ii} = 2\}$ and $I^- = I - I^+$.

2.2. Quantum generalized Kac-Moody algebras

Let $\mathbb{Q}(v)$ be the field of rational functions. For any $m \leq n$, we set

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \text{ and } [n]^! = [n][n-1]\cdots[1].$$

The *quantum generalized Kac-Moody algebra* $\mathbf{U} = \mathbf{U}(C)$ attached to the symmetric Borcherds-Cartan matrix $C = (c_{ij})_{i,j\in I}$ is an associative $\mathbb{Q}(v)$-algebra with generators

$$E_i,\ F_i,\ K_i \text{ and } K_i^{-1} \quad (i \in I)$$

and satisfying the following relations

$$\begin{aligned}
&K_iK_i^{-1} = 1, \quad K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i,\\
&K_iE_j = v^{c_{ij}}E_jK_i, \quad K_iF_j = v^{-c_{ij}}F_jK_i,\\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v - v^{-1}},\\
&\sum_{p=0}^{1-c_{ij}} (-1)^p E_i^{(p)}E_jE_i^{(1-c_{ij}-p)} = 0 \quad \text{for any } i \in I^+,\ j \in I,\ i \neq j,\\
&\sum_{p=0}^{1-c_{ij}} (-1)^p F_i^{(p)}F_jF_i^{(1-c_{ij}-p)} = 0 \quad \text{for any } i \in I^+,\ j \in I,\ i \neq j,\\
&E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \quad \text{if } c_{ij} = 0.
\end{aligned}$$

Here we use the notations $E_i^{(n)} = E_i^n/[n]^!$ and $F_i^{(n)} = F_i^n/[n]^!$ for $i \in I^+$ and $n \in \mathbb{N}$.

Let $\mathbf{U}^-$ be the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by F_i for $i \in I$. Set $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$, the ring of Laurent polynomials. Let ${}_{\mathbb{A}}\mathbf{U}^-$ denote the $\mathbb{A}$-subalgebra of $\mathbf{U}^-$ generated by the elements $F_i^{(n)}$ for $i \in I^+$, and F_i for $i \in I^-$.

§3. Varieties associated to quivers

3.1. Quivers

A *quiver* is a quadruple $Q = (I, \Omega, s, t : \Omega \to I)$ where I is the vertex set, Ω is the arrow set, and $s(\omega)$ and $t(\omega)$ are the starting and terminating vertices for $\omega \in \Omega$, respectively. In this paper, we assume that the quiver is *locally finite*: the vertex set I is countable and the set $\{\omega \in \Omega \mid s(\omega), t(\omega) \in J\}$ is finite for any finite subset $J \subseteq I$.

A vertex $i \in I$ is called *imaginary* if $s(\omega) = t(\omega) = i$ for some $\omega \in \Omega$. Let I^- be the set of all imaginary vertices in I. We set $I^+ = I \backslash I^-$. Vertices in I^+ will be called *real* vertices.

Let $\Omega(i)$ be the subset of Ω consisting of all ω such that $s(\omega) = t(\omega) = i$. For any $i \in I$, let

$$l_i = \#\Omega(i).$$

Note that when $i \in I^+$, $l_i = 0$. we set

$$\Omega^- = \bigcup_{i \in I^-} \Omega(i) \quad \text{and} \quad \Omega^+ = \Omega \backslash \Omega^-.$$

(In other words, Ω^- consists of all loop arrows, while Ω^+ consists of all loop free arrows.)

The (nonsymmetric) *Euler form* of Q $\langle\ ,\ \rangle : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{\omega \in \Omega} \alpha_{s(\omega)} \beta_{t(\omega)} \quad \text{for } \alpha, \beta \in \mathbb{Z}[I].$$

The *symmetric Euler form* $(\ ,\ ) : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}[I]$ is defined by

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \quad \text{for all } \alpha, \beta \in \mathbb{Z}[I].$$

Let

$$c_{ij} = (i, j) \quad \text{for } i, j \in I.$$

Then the matrix $C(Q) = (c_{ij})_{i,j \in I}$ is a symmetric Borcherds-Cartan matrix (see 2.1). We call $C = C(Q)$ the Borcherds-Cartan matrix associated to Q.

3.2. Representation varieties associated to quivers

We fix an algebraically closed field k. Given any I-graded k-vector space $V = \bigoplus_{i \in I} V_i$, let

$$G_V = \prod_{i \in I} \mathrm{GL}(V_i)$$

where $\mathrm{GL}(V_i)$ is the general linear group of V_i. Let

$$E_V = \bigoplus_{\omega \in \Omega} \mathrm{Hom}_k(V_{s(\omega)}, V_{t(\omega)}).$$

G_V acts on E_V by conjugation:

$$(g.x)_\omega = g_{t(\omega)} x_\omega g_{s(\omega)}^{-1}$$

for any $g \in G_V$ and $x \in E_V$. Given any subset $\Delta \subseteq \Omega$, let

$$E_{V,\Delta} = \bigoplus_{\omega \in \Delta} \mathrm{Hom}(V_{s(\omega)}, V_{t(\omega)}).$$

In particular, $E_{V,\Omega^+} = \bigoplus_{\omega \in \Omega^+} \mathrm{Hom}(V_{s(\omega)}, V_{t(\omega)})$.

3.3. Flag varieties

Given any $\nu \in \mathbb{N}[I]$, let $\mathcal{X}_\nu$ be the set of all pairs $(\mathbf{i}, \mathbf{a})$ where

$$\mathbf{i} = (i_1, \dots, i_n) \ (i_l \in I) \quad \text{and} \quad \mathbf{a} = (a_1, \dots, a_n) \ (a_l \in \mathbb{N})$$

such that $a_1 i_1 + \cdots + a_n i_n = \nu$. Let V be an I-graded vector space of dimension ν, a flag

$$\mathbf{F}^\bullet = (V = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^n = 0)$$

is called *of type* $(\mathbf{i}, \mathbf{a})$ $(\in \mathcal{X}_\nu)$ if $\dim V^{l-1}/V^l = a_l i_l$ for $l = 1, \dots, n$. We set

$$\mathcal{F}_{\mathbf{i},\mathbf{a}} = \{\text{all flags } \mathbf{F}^\bullet \text{ of type } (\mathbf{i}, \mathbf{a})\}.$$

Note that $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ is a smooth, irreducible projective variety. (Indeed, it is a product of the partial flag varieties.)

G_V acts on $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ transitively, i.e.,

$$g.F^\bullet = (V = V^0 \supseteq g(V^1) \supseteq g(V^2) \supseteq \cdots \supseteq V^n = 0)$$

for any $g \in G_V$ and $\mathbf{F}^\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a}}$.

3.4. (Partial) Resolutions of representation varieties

For any $x \in E_V$ and $\mathbf{F}^\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a}}$, $\mathbf{F}^\bullet$ is *x-stable* if $x_\omega(V_{s(\omega)}^{l-1}) \subseteq V_{t(\omega)}^l$ for any $\omega \in \Omega$ and $l = 1, \dots, n$. Let

$$\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} = \{(x, \mathbf{F}^\bullet) \in E_V \times \mathcal{F}_{\mathbf{i},\mathbf{a}} \mid \mathbf{F}^\bullet \text{ is } x\text{-stable}\}.$$

G_V acts on $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ by $g.(x, \mathbf{F}^\bullet) = (g.x, g.\mathbf{F}^\bullet)$ for any $g \in G_V$ and $(x, \mathbf{F}^\bullet) \in \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$. We have the following diagram

$$\begin{array}{ccc} \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} & \xrightarrow{\pi_{\mathbf{i},\mathbf{a}}} & E_V \\ {\scriptstyle p_{\mathbf{i},\mathbf{a}}}\big\downarrow & & \\ \mathcal{F}_{\mathbf{i},\mathbf{a}} & & \end{array}$$

where $\pi_{\mathbf{i},\mathbf{a}}$ and $p_{\mathbf{i},\mathbf{a}}$ are the first and second projections, respectively. From the above diagram, one deduces the following well-known facts

LEMMA 3.4.1. ([L2])

(1) $\pi_{\mathbf{i},\mathbf{a}}$ *is a* G_V*-equivariant, projective morphism.*
(2) $p_{\mathbf{i},\mathbf{a}}$ *is a vector bundle.*
(3) $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ *is a smooth, irreducible variety.*

3.5. A factorization of $\pi_{\mathbf{i},\mathbf{a}}$

For any $\nu \in \mathbb{N}[I]$, we set $\nu^- = \sum_{i \in I^-} \nu_i i$, the component of ν supported on I^-. (Recall that I^- is the set of all imaginary vertices.) For any pair $(\mathbf{i}, \mathbf{a}) \in \mathcal{X}_\nu$, let $(\mathbf{i}^-, \mathbf{a}^-) \in \mathcal{X}_{\nu^-}$ be the pair obtained from $(\mathbf{i}, \mathbf{a})$ by deleting the real vertices in $\mathbf{i}$ and the corresponding entries in $\mathbf{a}$. Let

$$\mathcal{E}_{\mathbf{i},\mathbf{a}} = \{(x, \mathbf{D}^\bullet) \in E_V \times \mathcal{F}_{(\mathbf{i}^-,\mathbf{a}^-)} \mid x_\omega(D^{l-1}_{s(\omega)}) \subseteq D^l_{t(\omega)}, \forall\, \omega \in \Omega^- \text{ and } 1 \le l \le n\}.$$

Suppose that V is an I-graded vector space of dimension ν, we set $V^- = \bigoplus_{i \in I^-} V_i$, the imaginary component of V. For any flag $\mathbf{F}^\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a}}$, we set $D^l = V^l \cap V^-$. Then the flag

$$\mathbf{F}^\bullet \cap V^- = (V^- = D^0 \supseteq D^1 \supseteq D^2 \supseteq \cdots \supseteq 0)$$

can be regarded as a flag in $\mathcal{F}_{(\mathbf{i}^-,\mathbf{a}^-)}$. We have the following diagram

$$\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \xrightarrow{\phi_{\mathbf{i},\mathbf{a}}} \mathcal{E}_{\mathbf{i},\mathbf{a}} \xrightarrow{\psi_{\mathbf{i},\mathbf{a}}} E_V,$$

where $\phi_{\mathbf{i},\mathbf{a}} : (x, \mathbf{F}^\bullet) \mapsto (x, \mathbf{F}^\bullet \cap V^-)$ and $\psi_{\mathbf{i},\mathbf{a}} : (x, \mathbf{D}^\bullet) \mapsto x$ are natural projections. By definition, we have

$$\pi_{\mathbf{i},\mathbf{a}} = \psi_{\mathbf{i},\mathbf{a}} \phi_{\mathbf{i},\mathbf{a}}.$$

Note that $\phi_{\mathbf{i},\mathbf{a}}$ is proper and G_V-equivariant, and $\psi_{\mathbf{i},\mathbf{a}}$ is semismall and G_V-equivariant. Further, when $l_i \ge 2$ for all $i \in I^-$, $\psi_{\mathbf{i},\mathbf{a}}$ is a small resolution. For a proof of these facts, see [L3], [KS1] and [KS2].

3.6. Relation with varieties associated to quivers without loops

Similar to $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ and $\mathcal{E}_{\mathbf{i},\mathbf{a}}$, we define

$$\widetilde{\mathcal{F}}^+_{\mathbf{i},\mathbf{a}} = \{(x, \mathbf{F}^\bullet) \in E_{V,\Omega^+} \times \mathcal{F}_{\mathbf{i},\mathbf{a}} \mid \mathbf{F}^\bullet \text{ is } x\text{-stable}\}, \text{ and}$$
$$\mathcal{E}^+_{\mathbf{i},\mathbf{a}} = E_{V,\Omega^+} \times \mathcal{F}_{\mathbf{i}^-,\mathbf{a}^-}.$$

Combining with the diagram in Section 3.5, we have the diagram

$$\begin{tikzcd}
\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \arrow[r, "\phi_{\mathbf{i},\mathbf{a}}"] \arrow[d, "\gamma_{\mathbf{i},\mathbf{a}}"'] & \mathcal{E}_{\mathbf{i},\mathbf{a}} \arrow[r, "\psi_{\mathbf{i},\mathbf{a}}"] \arrow[d, "\delta_{\mathbf{i},\mathbf{a}}"'] & E_V \arrow[d] \\
\widetilde{\mathcal{F}}^+_{\mathbf{i},\mathbf{a}} \arrow[r, "\phi^+_{\mathbf{i},\mathbf{a}}"] & \mathcal{E}^+_{\mathbf{i},\mathbf{a}} \arrow[r, "\psi^+_{\mathbf{i},\mathbf{a}}"] & E_{V,\Omega^+},
\end{tikzcd}$$

where the first row is the diagram in Section 3.5, the second row is defined in a similar manner as the first row, and the vertical maps are the self-explained projections. Observe that

(1) $\gamma_{\mathbf{i},\mathbf{a}}$ and $\delta_{\mathbf{i},\mathbf{a}}$ are vector bundles.
(2) The left square is cartesian.

3.7. A bridge

For any pair $(\mathbf{i}, \mathbf{a}) = ((i_1, \dots, i_n), (a_1, \dots, a_n)) \in \mathcal{X}_\nu$ with the imaginary vertex $i_{j_1}, i_{j_2}, \dots, i_{j_s}$ $(j_1 < \cdots < j_s)$ in $\mathbf{i}$, we set

$$\mathbf{j} = (i_1, \dots, i_{j_1-1}, \overbrace{i_{j_1}, \dots, i_{j_1}}^{a_{j_1} \text{ copies}}, i_{j_1+1}, \dots, i_{j_2-1}, \dots, \overbrace{i_{j_s}, \dots, i_{j_s}}^{a_{j_s} \text{ copies}}, i_{j_s+1}, \dots, i_n).$$

If we set $\mathbf{b}$ to be the sequence of the same length as $\mathbf{j}$ and with all entries corresponding to the imaginary vertex in $\mathbf{b}$ are 1 and all entries corresponding to the real vertex in $\mathbf{b}$ equal to a_l, then the pair $(\mathbf{j}, \mathbf{b}) \in \mathcal{X}_\nu$. By abusing the notation, We write $\mathbf{j}$ for $(\mathbf{j}, \mathbf{b})$. Consider the following diagram

$$\begin{tikzcd}
\widetilde{\mathcal{F}}^+_{\mathbf{j}} \arrow[r, "\phi^+_{\mathbf{j}}"] \arrow[d, "\alpha_{\mathbf{i},\mathbf{a}}"'] & \mathcal{E}^+_{\mathbf{j}} \arrow[d, "\beta_{\mathbf{i},\mathbf{a}}"'] \\
\widetilde{\mathcal{F}}^+_{\mathbf{i},\mathbf{a}} \arrow[r, "\phi^+_{\mathbf{i},\mathbf{a}}"] & \mathcal{E}^+_{\mathbf{i},\mathbf{a}}
\end{tikzcd}$$

where the rows are defined in Section 3.5 and the vertical maps are obvious projections. Observe that

(1) $\alpha_{\mathbf{i},\mathbf{a}}$ and $\beta_{\mathbf{i},\mathbf{a}}$ are smooth with connected fibres. The square is cartesian.

§4. Perverse sheaves on varieties associated to quivers

4.1. Notations

We fix some notations, most of them are taken from [L5].

Fix a prime l that is invertible in k. Given any algebraic variety X over k, denote by $\mathcal{D}(X)$ the bounded derived category of complexes of l-adic sheaves on X ([BBD]). Let $\mathcal{M}(X)$ be the full subcategory of $\mathcal{D}(X)$ consisting of all perverse sheaves on X ([BBD]).

Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of the field of l-adic numbers. By abuse of notation, denote by $\bar{\mathbb{Q}}_l = (\bar{\mathbb{Q}}_l)_X$ the complex concentrated on degree zero, corresponding to the constant l-adic sheaf over X. For any complex $K \in \mathcal{D}(X)$ and $n \in \mathbb{Z}$, let $K[n]$ be the complex such that $K[n]^i = K^{n+i}$ and the differential is multiplied by a factor $(-1)^n$. Denote by $\mathcal{M}(X)[n]$ the full subcategory of $\mathcal{D}(X)$ whose objects are of the form $K[n]$ with $K \in \mathcal{M}(X)$. For any $K \in \mathcal{D}(X)$ and $L \in \mathcal{D}(Y)$, denote by $K \boxtimes L$ the external tensor product of K and L in $\mathcal{D}(X \times Y)$.

Let $f : X \to Y$ be a morphism of varieties, denote by $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ and $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ the inverse image functor and the direct image functor with compact support, respectively.

Let G be a connected algebraic group. Assume that G acts on X algebraically. Denote by $\mathcal{D}_G(X)$ the full subcategory of $\mathcal{D}(X)$ consisting of all G-equivariant complexes over X. Similarly, denote by $\mathcal{M}_G(X)$ the full subcategory of $\mathcal{M}(X)$ consisting of all G-equivariant perverse sheaves ([L5]). If G acts on X algebraically and f is a principal G-bundle, then f^* induces a functor (still denote by f^*) of equivalence between $\mathcal{M}(Y)[\dim G]$ and $\mathcal{M}_G(X)$. Its inverse functor is denoted by $f_\flat : \mathcal{M}_G(X) \to \mathcal{M}(Y)[\dim G]$ ([L5]).

4.2. A class of simple perverse sheaves on E_V

The complex $\bar{\mathbb{Q}}_l[\dim \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}]$ on $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ is a simple G_V-equivariant perverse sheaf due to Lemma 3.4.1 (3). By Lemma 3.4.1 (1) and the Decomposition theorem in [BBD], the complex

$$L_{\mathbf{i},\mathbf{a}} = (\pi_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l[\dim \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}])$$

is a G_V-equivariant semisimple complex on E_V.

Let $\mathcal{P}_V$ be the set of isomorphism classes of all simple perverse sheaves P appearing in $L_{\mathbf{i},\mathbf{a}}$ as a summand with a possible shift for all $(\mathbf{i},\mathbf{a}) \in \mathcal{X}_\nu$.

Let $\mathcal{Q}_V$ be the full subcategory of $\mathcal{D}(E_V)$ whose objects are finite direct sums of shifts of simple perverse sheaves coming from $\mathcal{P}_V$. Note that all complexes in $\mathcal{Q}_V$ are semisimple and G_V-equivariant.

Let $\mathcal{Q}_T \boxtimes \mathcal{Q}_W$ be the full subcategory of $\mathcal{D}(E_T \times E_W)$ whose objects are of the form $P' \boxtimes P''$ for any $P' \in \mathcal{Q}_T$ and $P'' \in \mathcal{Q}_W$.

4.3. A class of *semisimple* perverse sheaves on E_V

Similar to the complex $L_{\mathbf{i},\mathbf{a}}$ in Section 4.2, we see that the complex

$$L^-_{\mathbf{i},\mathbf{a}} = (\phi_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l[\dim \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}])$$

is a G_V-equivariant semisimple complex on $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ (see Section 3.5 for notations).

Let $\widehat{\mathcal{E}}_{\mathbf{i},\mathbf{a}}$ be the set of isomorphism classes of all simple perverse sheaves on $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ appearing as direct summands in the complex $L^-_{\mathbf{i},\mathbf{a}}$ with possible shifts.

Lemma 4.3.1. ([KS1], [KS2])

(1) $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$ *is a semisimple perverse sheaf on* E_V *for any* $R \in \widehat{\mathcal{E}}_{\mathbf{i},\mathbf{a}}$.

(2) $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$ *is simple if* $l_i \geq 2$ *for all* $i \in I^-$.

It is proved in [L3] when Q has one vertex and multiple loops. When the entries in $\mathbf{a}$ are 1, the Lemma is proved in [KS1, Proposition 4.1] and [KS2, Section 1]. For arbitrary $\mathbf{a}$, one can repeat the proof in [KS1] and [KS2] essentially word by word.

Let $\mathcal{R}_V$ be the set of isomorphism classes of the *semisimple* perverse sheaves of the form: $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$ where $R \in \widehat{\mathcal{E}}_{\mathbf{i},\mathbf{a}}$ and $(\mathbf{i},\mathbf{a}) \in \mathcal{X}_\nu$.

Let $\mathcal{S}_V$ be the full subcategory of $\mathcal{D}(E_V)$ whose objects are finite direct sums of shifts of the semisimple perverse sheaves coming from $\mathcal{R}_V$.

Similar to Section 4.2, let $\mathcal{S}_T \boxtimes \mathcal{S}_W$ be the full subcategory of $\mathcal{D}(E_T \times E_W)$ whose objects are of the form $S' \boxtimes S''$ for any $S' \in \mathcal{S}_T$ and $S'' \in \mathcal{S}_W$. Observe that

$$L_{\mathbf{i},\mathbf{a}} = (\psi_{\mathbf{i},\mathbf{a}})_!(L^-_{\mathbf{i},\mathbf{a}}),$$

one sees that $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$ is a direct sum of simple perverse sheaves in $\mathcal{P}_V$. Thus $\mathcal{S}_V$ (resp. $\mathcal{S}_T \boxtimes \mathcal{S}_W$) is a full subcategory of $\mathcal{Q}_V$ (resp. $\mathcal{Q}_T \boxtimes \mathcal{Q}_W$).

4.4. Certain classes of simple perverse sheaves on $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ and $\mathcal{E}^+_{\mathbf{i},\mathbf{a}}$

Let $\mathcal{Y}_\nu$ be the subset of $\mathcal{X}_\nu$ (3.3) consisting of all pairs $(\mathbf{i},\mathbf{a})$ such that the (imaginary) entries in $\mathbf{a}$ are 1. For simplification, we write $\mathbf{i}$ for $(\mathbf{i},\mathbf{a})$.

For any $\mathbf{i} \in \mathcal{Y}_\nu$, the variety $\mathcal{E}_\mathbf{i}$ can be identified with the variety

$$\begin{aligned}\mathcal{E}_\nu = \{(x, \Pi_{i\in I^-} D_i^\bullet) \in E_V \times \Pi_{i\in I^-} \mathcal{F}_{(i,\dots,i)} \mid \\ x_\omega(D_i^{l-1}) \subseteq D_i^l,\ \forall\, \omega \in \Omega(i),\ 1 \le l \le \nu_i\}\end{aligned}$$

where $\mathcal{F}_{(i,\dots,i)}$ is the full flag variety of V_i, $\dim V_i = \nu_i$ and $\Omega(i)$ is defined in Section 3.1. (Note that the varieties $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ in Section 3.5 depend on the pair $(\mathbf{i},\mathbf{a}) \in \mathcal{X}_\nu$ in general.)

Now the diagram in Section 3.6 reads (for $\mathbf{i} \in \mathcal{Y}_\nu$)

$$\begin{array}{ccccc} \widetilde{\mathcal{F}}_\mathbf{i} & \xrightarrow{\phi_\mathbf{i}} & \mathcal{E}_\nu & \xrightarrow{\psi_\nu} & E_V \\ {\scriptstyle\gamma_\mathbf{i}}\big\downarrow & & {\scriptstyle\delta_\nu}\big\downarrow & & {\scriptstyle\epsilon}\big\downarrow \\ \widetilde{\mathcal{F}}^+_\mathbf{i} & \xrightarrow{\phi^+_\mathbf{i}} & \mathcal{E}^+_\nu & \xrightarrow{\psi^+_\nu} & E_{V,\Omega^+}, \end{array}$$

where $\mathcal{E}^+_\nu = E_{V,\Omega^+} \times \prod_{i\in I^-} \mathcal{F}_{(i,\dots,i)}$, ψ_ν and ψ^+_ν are first projections.

Similar to $\widehat{\mathcal{E}}_{\mathbf{i},\mathbf{a}}$, $\mathcal{R}_V$ and $\mathcal{S}_V$ defined in Section 4.3, we define the following objects.

$\widehat{\mathscr{E}}_\nu$ is the set of isomorphism classes of all simple perverse sheaves on $\mathcal{E}_\nu$ appearing as direct summands in the complex $L^-_\mathbf{i}$ with possible shifts for all $\mathbf{i} \in \mathcal{Y}_\nu$.

$\mathscr{R}_V$ is the set of isomorphism classes of the semisimple perverse sheaves of the form: $(\psi_\nu)_!(R)$ for all $R \in \widehat{\mathscr{E}}_\nu$.

$\mathscr{S}_V$ be the full subcategory of $\mathcal{D}(E_V)$ whose objects are finite direct sums of shifts of the semisimple perverse sheaves coming from $\mathscr{R}_V$.

Finally, we define the following objects.

Let $\widehat{\mathscr{F}_\nu}$ be the full subcategory of $\mathcal{D}(\mathcal{E}_\nu)$ whose objects are finite direct sums of the simple perverse sheaves coming from $\widehat{\mathscr{E}}_\nu$.

Let $\widehat{\mathscr{E}}^+_\nu$ be the set of isomorphism classes of all simple perverse sheaves on $\mathcal{E}^+_\nu$ appearing as direct summands in the complex $(\phi^+_\mathbf{i})_!(\bar{\mathbb{Q}}_l)$ with possible shifts for all $\mathbf{i} \in \mathcal{Y}_V$.

Let $\widehat{\mathscr{F}^+_\nu}$ be the full subcategory of $\mathcal{D}(\mathcal{E}^+_\nu)$ whose objects are finite direct sums of shifts of the semisimple perverse sheaves coming from $\widehat{\mathscr{E}}^+_\nu$.

These objects will be used in Section 6.3 to construct various algebras.

§5. Induction functors

5.1. Induction functors on E_V

Let $W \subseteq V$ be an I-graded subspace. Let $T = V/W$ and $p : V \to T$ the natural projection. We sometimes write p_T to avoid confusion.

Given any $x \in E_V$, we write $x(W) \subseteq W$ if $x_\omega(W_{s(\omega)}) \subseteq W_{t(\omega)}$, for all $\omega \in \Omega$.

If $x(W) \subseteq W$, it induces two elements x_W and x_T in E_W and E_T respectively as follows. $(x_W)_\omega$ is the restriction of x_ω to W for all $\omega \in \Omega$. $(x_T)_\omega$ is defined such that $p_{t(\omega)}\, x_\omega = (x_T)_\omega\, p_{s(\omega)}$ for all $\omega \in \Omega$.

We consider the following diagram

$$E_T \times E_W \xleftarrow{q_1} E' \xrightarrow{q_2} E'' \xrightarrow{q_3} E_V$$

where $E'' = \{(x, V') \mid x(V') \subseteq V', \dim V' = \dim W\}$, and E' is the variety consisting of all quadruples (x, V', r', r'') such that $(x, V') \in E''$, $r' : V/V' \to T$ and $r'' : V' \to W$ are I-graded linear isomorphisms.

The maps are defined as follows. $q_3 : (x, V') \mapsto x$, $q_2 : (x, V', r', r'') \mapsto (x, V')$ and $q_1 : (x, V', r', r'') \mapsto (x', x'')$ with $x'_\omega = r'_{t(\omega)}\, (x_{V/V'})_\omega\, (r')^{-1}_{s(\omega)}$ and $x''_\omega = r''_{t(\omega)}\, (x_{V'})_\omega\, (r'')^{-1}_{s(\omega)}$ for all $\omega \in \Omega$.

It is well-known that q_3 is proper, q_2 is principal $G_T \times G_W$ bundle of fiber dimension $d_2 = \sum_{i\in I}(|T_i|^2 + |W_i|^2)$ and q_1 is smooth with connected fibers of fiber dimension $d_1 = \sum_{i\in I}(|T_i|^2 + |W_i|^2) + \sum_{\omega\in\Omega} |T_{s(\omega)}|\, |W_{t(\omega)}| + \sum_{i\in I} |T_i|\, |W_i|$. Here we use the notation

$$|V| = \dim V.$$

From the diagram in this section and the above properties, we have a functor

$$(q_3)_!\, (q_2)_\flat\, q_1^* : \mathcal{Q}_T \boxtimes \mathcal{Q}_W \longrightarrow \mathcal{D}(E_V).$$

Given any $K_1 \in \mathcal{Q}_T$ and $K_2 \in \mathcal{Q}_W$ (see 4.2), we set

$$K_1 \circ K_2 = (q_3)_!\, (q_2)_\flat\, q_1^*(K_1 \boxtimes K_2)\, [d_1 - d_2].$$

Lemma 5.1.1.

(1) $K_1 \circ K_2 \in \mathcal{Q}_V$.

(2) $K_1 \circ K_2 \in \mathcal{S}_V$ *if* $K_1 \in \mathcal{S}_T$ *and* $K_2 \in \mathcal{S}_W$ (*see* 4.3).

(3) $L_{\mathbf{i}',\mathbf{a}'} \circ L_{\mathbf{i}'',\mathbf{a}''} = L_{\mathbf{i}'\mathbf{i}'',\mathbf{a}'\mathbf{a}''}$, *where* $(\mathbf{i}', \mathbf{a}') \in \mathcal{X}_{|T|}$, $(\mathbf{i}'', \mathbf{a}'') \in \mathcal{X}_{|W|}$, *and* $(\mathbf{i}'\mathbf{i}'', \mathbf{a}'\mathbf{a}'') \in \mathcal{X}_{|V|}$ *is the concatenation of the two sequences.*

Lemma 5.1.1 (1) and (3) are proved in [L2]. Lemma 5.1.1 (2) is proved in [KS1], but see Lemma 5.2.1.

5.2. Compatibility of induction functors on E_V, $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ and $\mathcal{E}^+_{\mathbf{i},\mathbf{a}}$

In this section, we assume that $(\mathbf{i},\mathbf{a})$ is the concatenation of the pairs $(\mathbf{i}',\mathbf{a}')$ and $(\mathbf{i}'',\mathbf{a}'')$. Consider the following diagram

$$\text{(A)}\qquad \begin{array}{ccccccc} \mathcal{E}_{\mathbf{i}',\mathbf{a}'}\times\mathcal{E}_{\mathbf{i}'',\mathbf{a}''} & \xleftarrow{r_1} & \mathcal{E}' & \xrightarrow{r_2} & \mathcal{E}'' & \xrightarrow{r_3} & \mathcal{E}_{\mathbf{i},\mathbf{a}} \\ \downarrow{\scriptstyle \psi_{\mathbf{i}',\mathbf{a}'}\times\psi_{\mathbf{i}'',\mathbf{a}''}} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi''} & & \downarrow{\scriptstyle \psi_{\mathbf{i},\mathbf{a}}} \\ E_T\times E_W & \xleftarrow{q_1} & E' & \xrightarrow{q_2} & E'' & \xrightarrow{q_3} & E_V \end{array}$$

where the bottom row is the diagram defined in Section 5.1, $\mathcal{E}_{\mathbf{i},\mathbf{a}}$ is defined in Section 3.5.

$$\mathcal{E}''=\{(x,\mathbf{D}^\bullet;V')\mid(x,\mathbf{D}^\bullet)\in\mathcal{E}_{\mathbf{i},\mathbf{a}},\ x(V')\subseteq V',\ V'\in\mathbf{D}^\bullet,\ \dim V'=\dim W\},$$

where V' is an I-graded subspace of V, and $V'\in\mathbf{D}^\bullet$ means that $D^l\supseteq V'\cap V^-\supseteq D^{l+1}$ for some l and

$$\mathcal{E}'=\{(x,\mathbf{D}^\bullet;V';r',r'')\mid(x,\mathbf{D}^\bullet;V')\in\mathcal{E}'',\ r':V/V'\xrightarrow{\sim}T,\ r'':V'\xrightarrow{\sim}W\}.$$

The vertical morphisms are the obvious projections and

$$r_3:(x,\mathbf{D}^\bullet;V')\longmapsto(x,\mathbf{D}^\bullet),\quad r_2:(x,\mathbf{D}^\bullet;V';r',r'')\longmapsto(x,\mathbf{D}^\bullet;V'),\ \text{and}$$
$$r_1:(x,\mathbf{D}^\bullet;V';r',r'')\longmapsto((x',\mathbf{D}'^\bullet),(x'',\mathbf{D}''^\bullet)),$$

where (x',x'') is defined in Section 5.1, $\mathbf{D}''^\bullet=r''(\mathbf{D}^\bullet\cap V')$, and $\mathbf{D}'^\bullet=r'(p_T(\mathbf{D}^\bullet))$. (Note that the imaginary parts of the pairs $(\mathbf{i}',\mathbf{a}')$ and $(\mathbf{i}'',\mathbf{a}'')$ are uniquely determined by the types of the flags $p_T(\mathbf{D}^\bullet)$ and $\mathbf{D}^\bullet\cap V'$, respectively, although $(\mathbf{i}',\mathbf{a}')$ and $(\mathbf{i}'',\mathbf{a}'')$ are not.)

Lemma 5.2.1. ([KS1]) *For any $R_1\in\widehat{\mathcal{E}}_{\mathbf{i}',\mathbf{a}'}$ and $R_2\in\widehat{\mathcal{E}}_{\mathbf{i}'',\mathbf{a}''}$ (see Section 4.3), we have*

$$(\psi_{\mathbf{i}',\mathbf{a}'})_!(R_1)\circ(\psi_{\mathbf{i}'',\mathbf{a}''})_!(R_2)=(\psi_{\mathbf{i},\mathbf{a}})_!(r_3)_!(r_2)_\flat r_1^*(R_1\boxtimes R_2)[d_1-d_2].$$

Proof. Observe that all squares in the diagram above are commutative. Moreover, the left and the middle squares are cartesian. By base change for proper morphism, we have

$$\begin{aligned}&(\psi_{\mathbf{i}',\mathbf{a}'})_!(R_1)\circ(\psi_{\mathbf{i}'',\mathbf{a}''})_!(R_2)\\ &\quad=(q_3)_!(q_2)_\flat(q_1)^*(\psi_{\mathbf{i}',\mathbf{a}'}\times\psi_{\mathbf{i}'',\mathbf{a}''})_!(R_1\boxtimes R_2)[d_1-d_2]\\ &\quad=(q_3)_!(q_2)_\flat(\psi')_!(r_1)^*(R_1\boxtimes R_2)[d_1-d_2]\\ &\quad=(q_3)_!(\psi'')_!(r_2)_\flat(r_1)^*(R_1\boxtimes R_2)[d_1-d_2]\\ &\quad=(\psi_{\mathbf{i},\mathbf{a}})_!(r_3)_!(r_2)_\flat(r_1)^*(R_1\boxtimes R_2)[d_1-d_2].\end{aligned}$$

Lemma follows. ∎

We set $R_1 \circ R_2 = (r_3)_!(r_2)_\flat r_1^*(R_1 \boxtimes R_2)[d_1 - d_2]$. Then the identity in Lemma above reads

$$(\psi_{\mathbf{i}',\mathbf{a}'})_!(R_1) \circ (\psi_{\mathbf{i}'',\mathbf{a}''})_!(R_2) = (\psi_{\mathbf{i},\mathbf{a}})_!(R_1 \circ R_2). \tag{1}$$

Similarly, consider the diagram

$$\text{(B)}\qquad \begin{array}{ccccccc} \mathcal{E}_{\mathbf{i}',\mathbf{a}'} \times \mathcal{E}_{\mathbf{i}'',\mathbf{a}''} & \xleftarrow{r_1} & \mathcal{E}' & \xrightarrow{r_2} & \mathcal{E}'' & \xrightarrow{r_3} & \mathcal{E}_{\mathbf{i},\mathbf{a}} \\ {\scriptstyle \delta_{\mathbf{i}',\mathbf{a}'} \times \delta_{\mathbf{i}'',\mathbf{a}''}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{i},\mathbf{a}}} \\ \mathcal{E}^+_{\mathbf{i}',\mathbf{a}'} \times \mathcal{E}^+_{\mathbf{i}'',\mathbf{a}''} & \xleftarrow{r_1^+} & \mathcal{E}'^+ & \xrightarrow{r_2^+} & \mathcal{E}''^+ & \xrightarrow{r_3^+} & \mathcal{E}^+_{\mathbf{i},\mathbf{a}}, \end{array}$$

where the top row is the top row in Diagram A, $\mathcal{E}^+_{\mathbf{i},\mathbf{a}}$ is defined in Section 3.6, the objects $\mathcal{E}'^+$ and $\mathcal{E}''^+$ are defined in the same way as the varieties $\mathcal{E}'$ and $\mathcal{E}''$ with $x \in E_V$ replaced by $x \in E_{V,\Omega^+}$, and the vertical maps are projections.

Observe that the squares in Diagram B are commutative and the right and middle squares are cartesian. By using a similar argument in the Proof of Lemma 5.2.1, one has

$$\delta^*_{\mathbf{i}',\mathbf{a}'}(S_1^+) \circ \delta^*_{\mathbf{i}'',\mathbf{a}''}(S_2^+) = \delta^*_{\mathbf{i},\mathbf{a}}(r_3^+)_!(r_2^+)_\flat(r_1^+)^*(S_1^+ \boxtimes S_2^+)[d_1 - d_2],$$

for any semisimple complex $S_1^+ \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}',\mathbf{a}'})$ and $S_2^+ \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}'',\mathbf{a}''})$, which are, up to shifts, direct summands of $(\phi^+_{\mathbf{i}',\mathbf{a}'})_!(\bar{\mathbb{Q}}_l)$ and $(\phi^+_{\mathbf{i}'',\mathbf{a}''})_!(\bar{\mathbb{Q}}_l)$ respectively. For simplicity, we set $S_1^+ \circ S_2^+ = (r_3^+)_!(r_2^+)_\flat(r_1^+)^*(S_1^+ \boxtimes S_2^+)[d_1^+ - d_2]$, where d_1^+ is the fibre dimension of r_1^+. (Note that the fibre dimension of r_1 and r_1^+ are not the same in general.) Then the above identity reads

$$\delta^*_{\mathbf{i}',\mathbf{a}'}(S_1^+) \circ \delta^*_{\mathbf{i}'',\mathbf{a}''}(S_2^+) = \delta^*_{\mathbf{i},\mathbf{a}}(S_1^+ \circ S_2^+)[d_1^+ - d_1]. \tag{2}$$

Let $d_{\mathbf{i},\mathbf{a}}$ be the fibre dimension of $\delta_{\mathbf{i},\mathbf{a}}$. By direct computation,

$$d_{\mathbf{i},\mathbf{a}} = d_1 - d_1^+ + d_{\mathbf{i}',\mathbf{a}'} + d_{\mathbf{i}'',\mathbf{a}''}.$$

From this and equation (2), one has

LEMMA 5.2.2. *For any semisimple complex $S_1^+ \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}',\mathbf{a}'})$ and $S_2^+ \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}'',\mathbf{a}''})$ which are shifts of direct summands of $(\phi_{\mathbf{i}',\mathbf{a}'})_!(\bar{\mathbb{Q}}_l)$ and $(\phi_{\mathbf{i}'',\mathbf{a}''})_!(\bar{\mathbb{Q}}_l)$ respectively,*

$$\delta^*_{\mathbf{i}',\mathbf{a}'}[d_{\mathbf{i}',\mathbf{a}'}](S_1^+) \circ \delta^*_{\mathbf{i}'',\mathbf{a}''}[d_{\mathbf{i}'',\mathbf{a}''}](S_2^+) = \delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](S_1^+ \circ S_2^+). \tag{3}$$

Finally, consider the following commutative diagram

$$
\text{(C)}\qquad
\begin{array}{ccccccc}
\mathcal{E}^+_{\mathbf{j}'} \times \mathcal{E}^+_{\mathbf{j}''} & \xleftarrow{r^+_{\mathbf{j},1}} & \mathcal{E}'^+_{\mathbf{j}} & \xrightarrow{r^+_{\mathbf{j},2}} & \mathcal{E}''^+_{\mathbf{j}} & \xrightarrow{r^+_{\mathbf{j},3}} & \mathcal{E}^+_{\mathbf{j}} \\
{\scriptstyle \beta_{\mathbf{i}',\mathbf{a}'}\times\beta_{\mathbf{i}'',\mathbf{a}''}}\big\downarrow & & \big\downarrow & & \big\downarrow & & {\scriptstyle \beta_{\mathbf{i},\mathbf{a}}}\big\downarrow \\
\mathcal{E}^+_{\mathbf{i}',\mathbf{a}'} \times \mathcal{E}^+_{\mathbf{i}'',\mathbf{a}''} & \xleftarrow{r^+_1} & \mathcal{E}'^+ & \xrightarrow{r^+_2} & \mathcal{E}''^+ & \xrightarrow{r^+_3} & \mathcal{E}^+_{\mathbf{i},\mathbf{a}},
\end{array}
$$

where the sequence $\mathbf{j}$ is defined in Section 3.7 with respect to the sequence $(\mathbf{i}, \mathbf{a})$, the bottom row is defined in Diagram (B), and the top row is defined the same as the bottom row with the pair $(\mathbf{i}, \mathbf{a})$ replaced by $\mathbf{j}$. The vertical maps are obvious projections. To avoid confusion, we put a subscript $\mathbf{j}$ on the morphisms and the varieties in the middle in the top row. Note that again all squares are commutative, and the left and middle squares are cartesian.

LEMMA 5.2.3. *For any semisimple complex $R^+_1 \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}',\mathbf{a}'})$ and $R^+_2 \in \mathcal{D}(\mathcal{E}^+_{\mathbf{i}'',\mathbf{a}''})$ which are shifts of direct summands of $(\phi_{\mathbf{i}',\mathbf{a}'})_!(\bar{\mathbb{Q}}_l)$ and $(\phi_{\mathbf{i}'',\mathbf{a}''})_!(\bar{\mathbb{Q}}_l)$ respectively,*

$$
(\beta_{\mathbf{i},\mathbf{a}})_!(R^+_1 \circ R^+_2) = (\beta_{\mathbf{i}',\mathbf{a}'})_!(R^+_1) \circ (\beta_{\mathbf{i}'',\mathbf{a}''})_!(R^+_2). \tag{4}
$$

The proof goes exactly the same as the proof of Lemma 5.2.1.

§6. Based algebras

6.1. Lusztig's algebras

Let $\mathcal{K}_V = \mathcal{K}(\mathcal{Q}_V)$ be the Grothendieck group of the category $\mathcal{Q}_V$, i.e., it is the abelian group with one generator $\langle L\rangle$ for each isomorphism class of objects in $\mathcal{Q}_V$ with relations: $\langle L\rangle + \langle L'\rangle = \langle L''\rangle$ if $L'' \cong L \oplus L'$.

Recall that $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$. $\mathcal{K}_V$ has an $\mathbb{A}$-module structure defined by $v^n\langle L\rangle = \langle L[n]\rangle$ for any generator $\langle L\rangle \in \mathcal{Q}_V$ and $n \in \mathbb{Z}$. Observe that $\mathcal{K}_V$ is a free $\mathbb{A}$-module with basis $\langle L\rangle$ where $\langle L\rangle$ runs over $\mathcal{P}_V$. Moreover, $\mathcal{K}_V \cong \mathcal{K}_{V'}$, for any V and V' such that $|V| = |V'|$.

For each $\nu \in \mathbb{N}[I]$, we fix an I-graded vector space V of dimension ν. Let

$$
\mathcal{K}_\nu = \mathcal{K}_V, \quad \mathcal{K} = \textstyle\bigoplus_{\nu\in\mathbb{N}[I]} \mathcal{K}_\nu, \quad \mathcal{K}_Q = \mathbb{Q}(v) \otimes_{\mathbb{A}} \mathcal{K},
$$
$$
\mathcal{P}_\nu = \mathcal{P}_V \quad \text{and} \quad \mathcal{P} = \textstyle\bigsqcup_{\nu\in\mathbb{N}[I]} \mathcal{P}_\nu.
$$

For any $\alpha, \beta \in \mathbb{N}[I]$, the functor $\circ$ defined in Section 5.1 induces an $\mathbb{A}$-linear map

$$\circ : \mathcal{K}_\alpha \otimes_{\mathbb{A}} \mathcal{K}_\beta \longrightarrow \mathcal{K}_{\alpha+\beta}.$$

By adding up these linear maps, we have an $\mathbb{A}$-linear map

$$\circ : \mathcal{K} \otimes_{\mathbb{A}} \mathcal{K} \longrightarrow \mathcal{K}.$$

Similarly, the operation $\circ$ induces a $\mathbb{Q}(v)$-linear map $\circ : \mathcal{K}_Q \otimes_{\mathbb{Q}(v)} \mathcal{K}_Q \to \mathcal{K}_Q$.

PROPOSITION 6.1.1.

(1) *$(\mathcal{K}, \circ)$ is an associative algebra over $\mathbb{A}$, while $(\mathcal{K}_Q, \circ)$ is a $\mathbb{Q}(v)$-algebra.*
(2) *$\mathcal{P}$ is an $\mathbb{A}$-basis of $(\mathcal{K}, \circ)$ and a $\mathbb{Q}(v)$-basis of $(\mathcal{K}_Q, \circ)$.*

See [L2] or [L4] for a proof of associativity.

Following Lusztig, we call $\mathcal{P}$ the *canonical basis* of the algebras $(\mathcal{K}, \circ)$ and $(\mathcal{K}_Q, \circ)$. From now on, we simply write $\mathcal{K}$ (resp. $\mathcal{K}_Q$) for the algebra $(\mathcal{K}, \circ)$ (resp. $(\mathcal{K}_Q, \circ)$).

6.2. The algebras $\mathcal{M}$ and $\mathcal{M}_Q$

Similar to the algebras $\mathcal{K}$ and $\mathcal{K}_Q$, we define algebras $\mathcal{M}$ and $\mathcal{M}_Q$ as follows.

Let $\mathcal{M}_V$ be the $\mathbb{A}$-submodule of $\mathcal{K}_V$ generated by elements in $\mathcal{R}_V$ (see 4.3). For each $\nu \in \mathbb{N}[I]$, we fix an I-graded vector space V of dimension ν. Let

$$\mathcal{M}_\nu = \mathcal{M}_V, \quad \mathcal{M} = \bigoplus\nolimits_{\nu \in \mathbb{N}[I]} \mathcal{M}_\nu, \quad \mathcal{M}_Q = \mathbb{Q}(v) \otimes_{\mathbb{A}} \mathcal{M},$$
$$\mathcal{R}_\nu = \mathcal{R}_V \quad \text{and} \quad \mathcal{R} = \bigsqcup\nolimits_{\nu \in \mathbb{N}[I]} \mathcal{R}_\nu.$$

By Lemma 5.1.1 (2), the functor $\circ$ defined in Section 5.1 induces an $\mathbb{A}$-linear map

$$\circ : \mathcal{M}_\alpha \otimes_{\mathbb{A}} \mathcal{M}_\beta \longrightarrow \mathcal{M}_{\alpha+\beta}, \quad \text{for } \alpha, \beta \in \mathbb{N}[I].$$

By adding up these linear maps, we have linear maps

$$\circ : \mathcal{M} \otimes_{\mathbb{A}} \mathcal{M} \longrightarrow \mathcal{M} \quad \text{and} \quad \circ : \mathcal{M}_Q \otimes_{\mathbb{Q}(v)} \mathcal{M}_Q \longrightarrow \mathcal{M}_Q,$$

where the first map is $\mathbb{A}$-linear while the second map is $\mathbb{Q}(v)$-linear.

PROPOSITION 6.2.1.

(1) *The pair* $(\mathcal{M}, \circ)$ *is an* $\mathbb{A}$*-subalgebra of* $\mathcal{K}$*, and* $(\mathcal{M}_Q, \circ)$ *is a* $\mathbb{Q}(v)$*-subalgebra of* $\mathcal{K}_Q$*.*

(2) $\mathcal{R}$ *is an* $\mathbb{A}$*-basis of* $(\mathcal{M}, \circ)$*, and a* $\mathbb{Q}(v)$*-basis of* $(\mathcal{M}_Q, \circ)$*.*

(3) $(\mathcal{M}, \circ) = (\mathcal{K}, \circ)$ *and* $\mathcal{R} = \mathcal{P}$ *if* $l_i \geq 2$ *for all* $i \in I^-$ *(see* 3.1*).*

Proposition 6.2.1 (1) is by definition. The proof of Proposition 6.2.1 (2) follows essentially word by word from the proof of Proposition 4.3 in [KS1]. We leave it to the reader. Proposition 6.2.1 (3) is due to Lemma 4.3.1 (2).

We call $\mathcal{R}$ the *canonical basis* of $\mathcal{M}$ and $\mathcal{M}_Q$.

6.3. By replacing the pair $(\mathcal{R}_V, \mathcal{S}_V)$ with the pair $(\mathscr{R}_V, \mathscr{S}_V)$ (4.4) and following the construction in Section 6.2, we obtain similar based algebras, denoted by $\mathscr{M} = (\mathscr{M}, \circ)$ and $\mathscr{M}_Q = (\mathscr{M}_Q, \circ)$, respectively, with basis $\mathscr{R} = \bigsqcup_{\nu \in \mathbb{N}[I]} \mathscr{R}_\nu$.

By replacing the pair $(\mathcal{P}_V, \mathcal{Q}_V)$ with the pair $(\widehat{\mathscr{E}}_\nu, \widehat{\mathscr{F}_\nu})$ (4.4), the diagram in Section 5.1 with the first row in Diagram (A) in Section 5.2, and following the construction in Section 6.1, we obtain similar based algebras, denoted by $\mathscr{K} = (\mathscr{K}, \circ)$ and $\mathscr{K}_Q = (\mathscr{K}_Q, \circ)$, respectively, with basis $\widehat{\mathscr{E}} = \bigsqcup_{\nu \in \mathbb{N}[I]} \widehat{\mathcal{E}}_\nu$.

Finally, the data $(\widehat{\mathscr{E}}_\nu, \widehat{\mathscr{F}_\nu})$ (4.4) and the second row in Diagram (B) give rise to a based algebra, denoted by $\mathscr{K}^+ = (\mathscr{K}^+, \circ)$ and $\mathscr{K}_Q^+ = (\mathscr{K}_Q^+, \circ)$, respectively, with basis $\widehat{\mathscr{E}}^+ = \bigsqcup_{\nu \in \mathbb{N}[I]} \widehat{\mathcal{E}}_\nu^+$.

The algebras $\mathscr{M}$, $\mathscr{K}$ and $\mathscr{K}^+$ are related as follows. The bijective map

$$\psi : \widehat{\mathscr{E}}_\nu \longrightarrow \mathscr{R}_V \quad \widehat{R} \longmapsto (\psi_\nu)_!(\widehat{R}) \quad \forall \widehat{R} \in \widehat{\mathscr{E}}_\nu$$

extends to an isomorphism of $\mathbb{A}$-modules $\psi : \mathscr{K} \xrightarrow{\sim} \mathscr{M}$. The bijective map

$$\delta_\nu^*[d_\nu] : \widehat{\mathscr{E}}_\nu^+ \longrightarrow \widehat{\mathscr{E}}_\nu \quad \widehat{R}^+ \longmapsto \delta_\nu^*[d_\nu](\widehat{R}^+) \quad \forall \widehat{R}^+ \in \widehat{\mathscr{E}}_\nu^+$$

(d_ν is the fibre dimension of δ_ν) defines an isomorphism of $\mathbb{A}$-modules $\delta : \mathscr{K}^+ \xrightarrow{\sim} \mathscr{K}$.

Now equation (1) in Section 5.2 implies that ψ is an $\mathbb{A}$-algebra homomorphism, while Lemma 5.2.2 implies that δ is an $\mathbb{A}$-algebra homomorphism. In summary,

Proposition 6.3.1. *The maps* $\mathscr{K}^+ \xrightarrow{\delta} \mathscr{K} \xrightarrow{\psi} \mathscr{M}$ *are isomorphisms of* $\mathbb{A}$*-algebras. Moreover,* $\delta(\widehat{\mathscr{E}}^+) = \widehat{\mathscr{E}}$ *and* $\psi(\widehat{\mathscr{E}}) = \mathscr{R}$*.*

Remark 6.3.2. The pair $(\mathscr{M}, \mathscr{R})$ first appeared in [KS1].

Let $C = C(Q)$ be the Borcherds-Cartan matrix associated to the quiver Q (3.1). Let $\mathbf{U}^-$ be the negative part of the quantum generalized Kac-Moody algebra associated to C (2.2). Kang and Schiffmann showed that

THEOREM 6.3.3. ([KS1])

(1) *The assignment $L_{i,1} \mapsto F_i$ defines an $\mathbb{A}$-algebra isomorphism $\mathscr{M} \xrightarrow{\phi} {}_{\mathbb{A}}\mathbf{U}^-$, and a $\mathbb{Q}(v)$-algebra isomorphism $\mathscr{M}_Q \to \mathbf{U}^-$. Here $L_{i,1}$ is the unique element in $\mathscr{R}_i$ for any $i \in I$.*

(2) *$\mathbf{B} = \phi(\mathscr{R})$ is the canonical basis of ${}_{\mathbb{A}}\mathbf{U}^-$ and $\mathbf{U}^-$. Moreover, $\mathbf{B}$ coincides with the global crystal base of $\mathbf{U}^-$ in* [JKK] *if $l_i \geq 2$ $(i \in I^-)$.*

(3) *For any $R \in \mathscr{R}_\nu$ with $n = n_i(R) > 0$ (i is a sink), $R = R_1 \circ R_2 + R_3$ where $R_1 \in \mathscr{R}_{ni}$, $R_2 \in \mathscr{R}_{\nu - ni}$ and $R_3 \in \mathscr{M}_\nu$ with $n_i(R_3) > n$. (see* 7.2 *for notations.)*

In view of Proposition 6.3.1 and Theorem 6.3.3, we have

COROLLARY 6.3.4.

(1) *The algebras $\mathscr{K}_Q^+$ and $\mathscr{K}_Q$ are geometric realizations of $\mathbf{U}^-$ via the maps δ, ψ and ϕ.*

(2) *$\widehat{\mathscr{E}}^+$ and $\widehat{\mathscr{E}}$ are realizations of the canonical basis $\mathbf{B}$ of $\mathbf{U}^-$.*

(3) *For any $S^+ \in \widehat{\mathscr{E}}^+_\nu$ such that $n = n_i(S^+) > 0$ (i a sink), $S^+ = S_1^+ \circ S_2^+ + S_3^+$ for some $S_1^+ \in \widehat{\mathscr{E}}^+_{mi}$ $(n \geq m > 0)$, $S_2^+ \in \widehat{\mathscr{E}}^+_{\nu - mi}$, and $S_3^+ \in \mathscr{K}_\nu^+$ with $n_i(S_3^+) > n$.*

Assume that $C = (c_{ij})_{i,j \in I}$ and $D = (d_{ij})_{i,j \in I}$ are two Borcherds-Cartan matrices. We call $C \approx D$ if

$$c_{ij} = d_{ij} \quad \forall i \neq j \in I$$

and

$$c_{ii} = 2 \quad \text{if and only if} \quad d_{ii} = 2.$$

In other words, $C \approx D$ if C and D coincide at all entries except the imaginary diagonal entries. Although $\mathbf{U}(C)$ is not isomorphic to $\mathbf{U}(D)$, we have (see 2.2)

$$\mathbf{U}^-(C) = \mathbf{U}^-(D) \quad \text{if} \quad C \approx D.$$

Let $\mathbf{B}(C)$ be the canonical basis of $\mathbf{U}^-(C)$ and $\mathbf{B}(D)$ be the canonical basis of $\mathbf{U}^-(D)$. Although $\mathbf{U}^-(C) = \mathbf{U}^-(D)$, the constructions of $\mathbf{B}(C)$ and $\mathbf{B}(D)$ are different. But in view of Corollary 6.3.4 (2), we have

THEOREM 6.3.5. $\mathbf{B}(C) = \mathbf{B}(D)$, *provided that* $\mathbf{U}^-(C) = \mathbf{U}^-(D)$.

This is because they all equal the image of $\widehat{\mathscr{E}}^+$ under the composition $\phi\psi\delta$ of morphisms. Note that $\phi\psi\delta$ is independent of the number of loop arrows on the imaginary vertices, if the quivers for C and D are chosen such that the subquivers containing all non loop arrows are the same.

Remark 6.3.6. As the referee pointed out, the natural scalar products of the various algebras $\mathscr{K}$, $\mathscr{K}^+$ and $\mathscr{M}$ are different, despite the fact that they are isomorphic. This difference explains the fact that only positive halves of the various quantum generalized Kac-Moody algebras are isomorphic to each other.

6.4. Independence of orientations

Let $Q' = (I, \Omega, s', t')$ be a quiver with the sets I and Ω the same as $Q = (I, \Omega, s, t)$ and $\{s'(\omega), t'(\omega)\} = \{s(\omega), t(\omega)\}$ for all $\omega \in \Omega$. Associated to Q' similar algebras defined in Section 6.1, 6.2 and 6.3.

PROPOSITION 6.4.1. *The algebras defined in Section* 6.1, 6.2 *and* 6.3 *are independent of changes of orientations, via Fourier-Deligne transform. Moreover, the Fourier-Deligne transform preserves the various bases under the changes of orientations.*

The independence of the algebras $\mathcal{K}$ under changes of orientations have been investigated by Lusztig in [L2]. Notice the fact that Fourier-Deligne transform commutes with base change ([La]). One can adapt the proof given in [L2] to the rest of the algebras. See [KS2, Section 2] for a concrete proof.

§7. A characterization of the algebra $\mathcal{M}_Q$

7.1. When the pair $(\mathbf{i}, \mathbf{a}) = (i, n)$ for $i \in I$ and $n \in \mathbb{N}$, the isomorphism class of the complex $L_{i,n}$ (see 4.2) is in $\mathcal{R}_{ni}$. (Note that the $L_{i,n}$'s are simple perverse sheaves supported on $\{0\} \subseteq E_V$ ($|V| = ni$).)

THEOREM 7.1.1. *The algebra* $\mathcal{M}_Q$ (*see* 6.2) *is generated by the complex* $L_{i,n}$ *for* $i \in I$ *and* $n \in \mathbb{N}$. *In particular,* $\mathcal{K}_Q = \mathcal{M}_Q$ *is generated by the* $L_{i,n}$*'s if* $l_i \geq 2$ *for all* $i \in I^-$ (*see* 3.1).

The proof will be given in Section 7.2.

Let $C = (c_{ij})_{i,j\in I}$ be the Borcherds-Cartan matrix associated to Q (see 3.1). We set

$$\mathbf{I} = I^+ \sqcup \{ni \mid n \in \mathbb{N},\, i \in I^-\},$$

and define a new Borcherds-Cartan matrix $\widetilde{C} = (\tilde{c}_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in\mathbf{I}}$ by

$$\tilde{c}_{\mathbf{ij}} = (\mathbf{i}, \mathbf{j})$$

where $(-,-)$ is the bilinear form associated to Q (see 3.1).

Proposition 7.1.2. *There exists elements $\xi_{\mathbf{i}} \in \mathcal{M}_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$ ($\xi_{\mathbf{i}} = L_i$ if $i \in I$) such that the assignment $\xi_{\mathbf{i}} \mapsto F_{\mathbf{i}}$ defines a $\mathbb{Q}(v)$-algebra isomorphism*

$$\mathcal{M}_Q \cong \mathbf{U}^-(\widetilde{C}),$$

where $\mathbf{U}^-(\widetilde{C})$ is the negative part of the quantum generalized Kac-Moody algebras associated to the matrix $\widetilde{C}$ (see 2.2).

Proof. Recall that $\mathcal{M}_Q = \mathbb{Q}(v) \otimes_{\mathbb{A}} \mathcal{M}$. $\mathcal{M}_Q$ then has a natural $\mathbb{N}[I]$-grading inherited from $\mathcal{M}$, i.e., $\mathcal{M}_Q = \bigoplus_{\nu\in\mathbb{N}[I]} \mathcal{M}_Q(\nu)$, where $\mathcal{M}_Q(\nu) = \mathbb{Q}(v) \otimes_{\mathbb{A}} \mathcal{M}_\nu$.

Define a $\mathbb{Q}(v)$-algebra structure on $\mathcal{M}_Q \otimes \mathcal{M}_Q$ by

$$(x \otimes y)(x' \otimes y') = v^{(\deg(y),\deg(x'))} xx' \otimes yy',$$

where x, y, x' and y' are homogeneous elements, $\deg(y)$ and $\deg(x')$ are the degree of y and x', respectively.

Following Lusztig ([L4]), we can associate to $\mathcal{M}_Q$ an algebra homomorphism

$$r : \mathcal{M}_Q \longrightarrow \mathcal{M}_Q \otimes \mathcal{M}_Q$$

and a symmetric nondegenerate linear form

$$(-,-) : \mathcal{M}_Q \otimes \mathcal{M}_Q \longrightarrow \mathbb{Q}(v),$$

satisfying

(1) $(1, 1) = 1$, and $(\mathcal{M}_Q(\nu), \mathcal{M}_Q(\nu')) = 0$ if $\nu \neq \nu'$.
(2) $(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in \mathcal{M}_Q$.

Given any $i \in I^-$, we set

$$\mathcal{H}_{mi} = \left\{ x \in \mathcal{M}_Q \;\middle|\; \left(x, \sum_{m',m''} \mathcal{M}_Q(m'i)\mathcal{M}_Q(m''i) \right) = 0 \right\},$$

where the sum runs over all pairs (m', m'') such that $m' + m'' = m$ and $0 < m', m'' < m$. Note that the elements of the following form span $\mathcal{M}_Q(mi)$

$$L_{i,m_1} \circ L_{i,m_2} \circ \cdots \circ L_{i,m_l}, \quad m_1 + \cdots + m_l = m.$$

Note also that all these elements are in $\sum_{m',m''} \mathcal{M}_Q(m'i)\mathcal{M}_Q(m''i)$, except $L_{m,i}$. From these observations and the nondegeneracy of the linear form on $\mathcal{M}_Q$, we have $\dim_{\mathbb{Q}(v)} \mathcal{H}_{mi} = 1$. For a nonzero element x in $\mathcal{H}_{mi}$, x must be of the form $x = a_m F_{mi} + x'$ where $a_m \in \mathbb{Q}(v)$ and $x' \in \sum_{m',m''} \mathcal{M}_Q(m'i)\mathcal{M}_Q(m''i)$. We set $\xi_{mi} = a_m^{-1} x$, for $m \in \mathbb{N}$ and $i \in I^-$. Note that $\xi_i = L_{i,1}$ for all $i \in I^-$. Let $\xi_i = L_{i,1}$ if $i \in I^+$. So the set $\{\xi_{\mathbf{i}} \mid \mathbf{i} \in \mathbf{I}\}$ generates $\mathcal{M}_Q$.

Now Proposition 7.1.2 follows from the argument in [SV]. ∎

7.2. Proof of Theorem 7.1.1

We need some preparations. Let $Path(i)$ be the set of all paths $p = \omega_1 \cdots \omega_n$ for $\omega_1, \ldots, \omega_n \in \Omega(i)$ and $n \in \mathbb{Z}_{>0}$. Given any $x \in E_V$, we set

$$V_{i,x} = \sum_{\omega \in \Omega^+ : t(\omega) = i} x_\omega(V_{s(\omega)}), \quad \text{and} \quad V_i(x) = V_{i,x} + \sum_{p \in Path(i)} x_p(V_{i,x}).$$

(See Section 3.1 for notations.) A vertex i in Q is called a *sink* if $t(\omega) \neq i$ for all $\omega \in \Omega^+$. Given any $x \in E_V$, we set

$$n_i(x) = \operatorname{codim}_{V_i} V_i(x) \quad \text{if } i \text{ is a sink.}$$

More generally, for any semisimple complex $P \in \mathcal{D}(E_V)$, we set

$$n_i(P) = \min\{n_i(x) \mid x \in \operatorname{supp}(P)\},$$

where $\operatorname{supp}(P)$ is the support of P.

For any semisimple perverse sheaf $(\psi_{\mathbf{i},\mathbf{a}})_!(R) \in \mathcal{R}_V$ with $\mathbf{i} = (i_1, \ldots, i_m)$, we may assume that $i = i_1$ is a sink via change of orientations in view of Proposition 6.4.1. Under this assumption, one see that

$$N = n_i((\psi_{\mathbf{i},\mathbf{a}})_!(R)) > 0.$$

On $\mathbb{N}[I]$, we define a partial order $<$ by $\tau < \nu$ if $\tau_j \leq \nu_j$ for all $j \in I$ and $\tau_{j_0} < \nu_{j_0}$ for some $j_0 \in I$.

Now we begin to prove Theorem 7.1.1. Let $\mathfrak{M}$ be the subalgebra of $\mathcal{M}$ generated by $L_{(i,n)}$ for $i \in I$ and $n \in \mathbb{N}$. Assume that all semisimple perverse sheaves in $\mathcal{R}_\tau$ are in $\mathfrak{M}$ for all $\tau < \nu$ and all semisimple perverse sheaves $P' \in \mathcal{R}_\nu$ such that $n_i(P') > N > 0$ are in $\mathfrak{M}$. To show Theorem 7.1.1, it suffices to show that any semisimple perverse sheaf $P = (\psi_{\mathbf{i},\mathbf{a}})_!(R) \in \mathcal{R}_\nu$ with $n_i(P) = N$ is in $\mathfrak{M}$.

Let us consider the diagram in Section 3.6. By property (3.6.2),

$$(\phi_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l) = \delta^*_{\mathbf{i},\mathbf{a}}(\phi^+_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l).$$

Since $\delta_{\mathbf{i},\mathbf{a}}$ is a vector bundle, $\delta^*_{\mathbf{i},\mathbf{a}}$ is fully faithful. So there is a unique simple perverse sheaf R^+ as a summand of $(\phi^+_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l)$ such that $\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](R^+) = R$. Observe that $n_i(R^+) > 0$ since $N > 0$.

Consider the diagram in Section 3.7. By property (3.7.1), we have

$$(\phi^+_{\mathbf{j}})_!(\bar{\mathbb{Q}}_l) = \beta^*_{\mathbf{i},\mathbf{a}}(\phi^+_{\mathbf{i},\mathbf{a}})_!(\bar{\mathbb{Q}}_l).$$

From this and the fact that $\beta_{\mathbf{i},\mathbf{a}}$ is smooth with connected fibres, we see that

$$S^+ = \beta^*_{\mathbf{i},\mathbf{a}}(R^+)$$

is a simple perverse sheaf (up to a shift) on $\mathcal{E}_{\mathbf{j}}$ and a direct summand of $(\phi^+_{\mathbf{j}})_!(\bar{\mathbb{Q}}_l)$. Moreover $n_i(S^+) = n_i(R^+) > 0$. By Corollary 6.3.4 (3),

$$\beta^*_{\mathbf{i},\mathbf{a}}(R^+) = S^+_1 \circ S^+_2 + S^+_3 \tag{5}$$

where $S^+_1 \in \widehat{\mathscr{E}}^+_{mi}$ $(n \geq m > 0)$ and $S^+_2 \in \widehat{\mathscr{E}}^+_{\nu-mi}$, and S^+_3 is a semisimple complex on $\mathcal{E}^+_\nu$ such that $n_i(S^+_3) > n_i(S^+)$. Thus by Lemma 5.2.3,

$$(\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+) = (\beta_{\mathbf{i}',\mathbf{a}'})_!(S^+_1) \circ (\beta_{\mathbf{i}'',\mathbf{a}''})_!(S^+_2) + (\beta_{\mathbf{i},\mathbf{a}})_!(S^+_3), \tag{6}$$

where $(\beta_{\mathbf{i}',\mathbf{a}'})_!(S^+_1) \in \mathcal{M}_{mi}$, $(\beta_{\mathbf{i}'',\mathbf{a}''})_!(S^+_2) \in \mathcal{M}_{\nu-mi}$ and $(\beta_{\mathbf{i},\mathbf{a}})_!(S^+_3) \in \mathcal{M}_\nu$ with $n_i((\beta_{\mathbf{i},\mathbf{a}})_!(S^+_3)) > n_i(R^+)$.

By applying the functor $\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}]$ to equation (6) and Lemma 5.2.2,

$$\begin{aligned} &\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+) \\ &\quad = \delta^*_{\mathbf{i}',\mathbf{a}'}[d_{\mathbf{i}',\mathbf{a}'}](\beta_{\mathbf{i}',\mathbf{a}'})_!(S^+_1) \circ \delta^*_{\mathbf{i}'',\mathbf{a}''}[d_{\mathbf{i}'',\mathbf{a}''}](\beta_{\mathbf{i}'',\mathbf{a}''})_!(S^+_2) \\ &\qquad + \delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!(S^+_3). \end{aligned} \tag{7}$$

By applying the functor $\psi_{\mathbf{i},\mathbf{a}}$ to equation (7) and in view of Lemma 5.2.3,

$$(8)\quad \begin{aligned}&(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+)\\ &\quad= (\psi_{\mathbf{i}',\mathbf{a}'})_!\delta^*_{\mathbf{i}',\mathbf{a}'}[d_{\mathbf{i}',\mathbf{a}'}](\beta_{\mathbf{i}',\mathbf{a}'})_!(S_1^+)\circ(\psi_{\mathbf{i}'',\mathbf{a}''})_!\delta^*_{\mathbf{i}'',\mathbf{a}''}[d_{\mathbf{i}'',\mathbf{a}''}](\beta_{\mathbf{i}'',\mathbf{a}''})_!(S_2^+)\\ &\qquad+(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!(S_3^+).\end{aligned}$$

By assumption, the right hand side of equation (8) is in $\mathfrak{M}$. So is the left hand side.

Since $\beta_{\mathbf{i},\mathbf{a}}$ is smooth with connected fibres, by Corollary 4.2.6.2 in [BBD],

$$R^+ = {}^pH^{-e_{\mathbf{i},\mathbf{a}}}(\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+),$$

where ${}^pH^{-e_{\mathbf{i},\mathbf{a}}}$ is the perverse cohomology functor at degree $-e_{\mathbf{i},\mathbf{a}}$ and $e_{\mathbf{i},\mathbf{a}}$ is the fibre dimension of $\beta_{\mathbf{i},\mathbf{a}}$. But $(\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+)$ is semisimple, so

$$\begin{aligned}(\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+) &= \bigoplus_{-e_{\mathbf{i},\mathbf{a}}\le z\le e_{\mathbf{i},\mathbf{a}}} {}^pH^z(\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+)[-z]\\ &= R^+[e_{\mathbf{i},\mathbf{a}}]\oplus\bigoplus_{-e_{\mathbf{i},\mathbf{a}}\le z< e_{\mathbf{i},\mathbf{a}}} R'[z].\end{aligned}$$

Thus $(\psi_{\mathbf{i},\mathbf{a}})_!(R)[e_{\mathbf{i},\mathbf{a}}]$ is a direct summand of $(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+)$, as the leading term with respect to the degree of shift. Moreover, for any other direct summand P' in $(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R^+)$ is of the form

$$P'[z] = (\psi_{\mathbf{i},\mathbf{a}})_!(R')[z],\quad -e_{\mathbf{i},\mathbf{a}}\le z< e_{\mathbf{i},\mathbf{a}}$$

and $n_i(P')\ge N = n_i(R^+)$.

For those $P' = (\psi_{\mathbf{i},\mathbf{a}})_!(R')$'s such that $n_i(P') = N$, we can apply the same argument again to obtain equations similar to equation (8):

$$(9)\quad \begin{aligned}&(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!\beta^*_{\mathbf{i},\mathbf{a}}(R'^+)\\ &\quad= (\psi_{\mathbf{i}',\mathbf{a}'})_!\delta^*_{\mathbf{i}',\mathbf{a}'}[d_{\mathbf{i}',\mathbf{a}'}](\beta_{\mathbf{i}',\mathbf{a}'})_!(S_1'^+)\circ(\psi_{\mathbf{i}'',\mathbf{a}''})_!\delta^*_{\mathbf{i}'',\mathbf{a}''}[d_{\mathbf{i}'',\mathbf{a}''}](\beta_{\mathbf{i}'',\mathbf{a}''})_!(S_2'^+)\\ &\qquad+(\psi_{\mathbf{i},\mathbf{a}})_!\delta^*_{\mathbf{i},\mathbf{a}}[d_{\mathbf{i},\mathbf{a}}](\beta_{\mathbf{i},\mathbf{a}})_!(S_3'^+).\end{aligned}$$

Again, by induction hypothesis, the right hand side of (9) is in $\mathfrak{M}$, so is the left hand side.

Thus we have a system of finitely many linear equations indexed by P' such that $n_i(P') = N$ in $\mathfrak{M}$. Notice the left hand side of each equation has a leading term of the form $P'[e_{\mathbf{i},\mathbf{a}}]$. So the terms in the left hand sides of the equations are linearly independent. Therefore we can solve for $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$ from this system in $\mathfrak{M}$. In other words, $(\psi_{\mathbf{i},\mathbf{a}})_!(R)\in\mathfrak{M}$. This finishes the proof.

§8. Comments

1. It may be of interest to generalize the main results in this paper to the symmetrizable case.

2. In view of Theorem 6.3.5 and Theorem 6.3.3 (2), the global crystal bases $\mathcal{B}(C)$ and $\mathcal{B}(D)$ in [JKK] coincide if $C \approx D$, $c_{ii} \neq 0$ and $d_{ii} \neq 0$ for all $i \in I^-$. One may conjecture that the conditions $c_{ii} \neq 0$ and $d_{ii} \neq 0$ can be removed.

Conjecture. *$\mathcal{B}(C) = \mathcal{B}(D)$ if $\mathbf{U}^-(C) = \mathbf{U}^-(D)$.*

The conjecture in [KS1] is a consequence of the above conjecture.

3. Lusztig asked in [L3] if $\mathcal{K}_Q = \mathcal{M}_Q$. This is the case if the quiver Q does not have any loop ([L2]). When Q is the Jordan quiver (the quiver with one vertex and one loop arrow), $\mathcal{K}_Q = \mathcal{M}_Q$. This is because every simple perverse sheaves in $\mathcal{K}_Q$ is the leading term of a certain monomial. More generally, when $l_i \geq 2$ for $i \in I^-$, $\mathcal{K}_Q = \mathcal{M}_Q$ by Theorem 7.1.1. For Q such that $l_i = 1$ for some $i \in I^-$, computational evidents show that the simple perverse sheaves in $\mathcal{K}_Q$ are leading terms of some semisimple perverse sheaves $(\psi_{\mathbf{i},\mathbf{a}})_!(R)$. If one can show that this holds, then the question asked by Lusztig in [L3] gets a positive answer.

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