

## ACYCLICITY OF COMPLEXES OF FLAT MODULES

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*Dedicated to Professor Masayoshi Nagata on his eightieth birthday*

**Abstract.** Let  $R$  be a noetherian commutative ring, and

$$\mathbb{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

a complex of flat  $R$ -modules. We prove that if  $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$  is acyclic for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{F}$  is acyclic, and  $H_0(\mathbb{F})$  is  $R$ -flat. It follows that if  $\mathbb{F}$  is a (possibly unbounded) complex of flat  $R$ -modules and  $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$  is exact for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{G} \otimes_R^\bullet \mathbb{F}$  is exact for every  $R$ -complex  $\mathbb{G}$ . If, moreover,  $\mathbb{F}$  is a complex of projective  $R$ -modules, then it is null-homotopic (follows from Neeman's theorem).

### §1. Introduction

Throughout this paper,  $R$  denotes a noetherian commutative ring. The symbol  $\otimes$  without any subscript means  $\otimes_R$ . For  $\mathfrak{p} \in \text{Spec } R$ , let  $-(\mathfrak{p})$  denote the functor  $\kappa(\mathfrak{p}) \otimes -$ , where  $\kappa(\mathfrak{p})$  is the field  $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ . An  $R$ -complex of the form

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

is said to be *acyclic* if  $H_i(\mathbb{F}) = 0$  for every  $i > 0$ .

In this paper, we prove:

**THEOREM 1.** *Let*

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

*be a complex of  $R$ -flat modules. If  $\mathbb{F}(\mathfrak{p})$  is acyclic for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{F}$  is acyclic, and  $H_0(\mathbb{F})$  is  $R$ -flat. In particular,  $M \otimes \mathbb{F}$  is acyclic for every  $R$ -module  $M$ .*

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It has been known that, for an  $R$ -linear map of  $R$ -flat modules  $\varphi : F_1 \rightarrow F_0$ , if  $\varphi(\mathfrak{p})$  is injective for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\varphi$  is injective and  $\text{Coker } \varphi$  is  $R$ -flat (see [1, Lemma 4.2], [2, Lemma I.2.1.4] and Corollary 6). This is the special case of the theorem where  $F_i = 0$  for every  $i \geq 2$ . The new proof of the theorem is simpler than the proofs of the special case in [1] and [2].

By the theorem, it follows immediately that if  $\mathbb{F}$  is an (unbounded) complex of  $R$ -flat modules and  $\mathbb{F}(\mathfrak{p})$  is exact for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{F}$  is  $K$ -flat (to be defined below) and exact. Combining this and Neeman's result, we can also prove that an (unbounded) complex  $\mathbb{P}$  of  $R$ -projective modules is null-homotopic if  $\mathbb{P}(\mathfrak{p})$  is exact for every  $\mathfrak{p} \in \text{Spec } R$ .

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## §2. Main results

We give a proof of Theorem 1.

*Proof of Theorem 1.* It suffices to prove that  $R/I \otimes \mathbb{F}$  is acyclic for every ideal  $I$  of  $R$ . Indeed, if so, then considering the case that  $I = 0$ , we have that  $\mathbb{F}$  is acyclic so that it is a flat resolution of  $H_0(\mathbb{F})$ . Since  $R/I \otimes \mathbb{F}$  is acyclic for every ideal  $I$ , we have that  $\text{Tor}_i^R(R/I, H_0(\mathbb{F})) = 0$  for every  $i > 0$ . Thus  $H_0(\mathbb{F})$  is  $R$ -flat. So  $\text{Tor}_i^R(M, H_0(\mathbb{F})) = 0$  for every  $i > 0$ , and the last assertion of the theorem follows.

Assume the contrary, and let  $I$  be maximal among the ideals  $J$  such that  $R/J \otimes \mathbb{F}$  is not acyclic. Then replacing  $R$  by  $R/I$  and  $\mathbb{F}$  by  $R/I \otimes \mathbb{F}$ , we may assume that  $R/I \otimes \mathbb{F}$  is acyclic for every nonzero ideal  $I$  of  $R$ , but  $\mathbb{F}$  itself is not acyclic.

Assume that  $R$  is not a domain. There exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = R$$

such that for each  $i$ ,  $M_i/M_{i-1} \cong R/\mathfrak{p}_i$  for some  $\mathfrak{p}_i \in \text{Spec } R$ . Since each  $\mathfrak{p}_i$  is a nonzero ideal,  $R/\mathfrak{p}_i \otimes \mathbb{F}$  is acyclic. So  $M_i \otimes \mathbb{F}$  is acyclic for every  $i$ . In particular,  $\mathbb{F} \cong M_r \otimes \mathbb{F}$  is acyclic, and this is a contradiction. So  $R$  must be a domain.

For each  $x \in R \setminus 0$ , there is an exact sequence

$$0 \rightarrow \mathbb{F} \xrightarrow{x} \mathbb{F} \rightarrow R/Rx \otimes \mathbb{F} \rightarrow 0.$$

Since  $R/Rx \otimes \mathbb{F}$  is acyclic, we have that  $x : H_i(\mathbb{F}) \rightarrow H_i(\mathbb{F})$  is an isomorphism for every  $i > 0$ . In particular,  $H_i(\mathbb{F})$  is a  $K$ -vector space, where  $K = \kappa(0)$  is the field of fractions of  $R$ . So

$$H_i(\mathbb{F}) \cong K \otimes H_i(\mathbb{F}) \cong H_i(K \otimes \mathbb{F}) = H_i(\mathbb{F}(0)) = 0 \quad (i > 0),$$

and this is a contradiction. □

Let  $A$  be a ring. A complex  $\mathbb{F}$  of left  $A$ -modules is said to be  $K$ -flat if the tensor product  $\mathbb{G} \otimes_A^\bullet \mathbb{F}$  is exact for every exact complex  $\mathbb{G}$  of right  $A$ -modules, see [4, Definition 5.1].

For a chain complex

$$\mathbb{H} : \cdots \rightarrow H_{i+1} \xrightarrow{d_{i+1}} H_i \xrightarrow{d_i} H_{i-1} \rightarrow \cdots$$

of left or right  $A$ -modules, we denote the complex

$$\cdots \rightarrow H_{i+1} \rightarrow \text{Ker } d_i \rightarrow 0$$

by  $\tau_{\geq i} \mathbb{H}$  or  $\tau^{\leq -i} \mathbb{H}$ . Since  $\mathbb{G} \cong \varinjlim \tau^{\leq n} \mathbb{G}$ ,  $\mathbb{F}$  is  $K$ -flat if and only if  $\mathbb{G} \otimes_A^\bullet \mathbb{F}$  is exact for every exact complex  $\mathbb{G}$  of right  $A$ -modules bounded above (i.e.,  $\mathbb{G}_{-i} = \mathbb{G}^i = 0$  for  $i \gg 0$ ). A complex  $\mathbb{F}$  of flat left  $A$ -modules is  $K$ -flat if it is bounded above, as can be seen easily from the spectral sequence argument. A null-homotopic complex  $\mathbb{F}$  is  $K$ -flat, since  $\mathbb{G} \otimes_A^\bullet \mathbb{F}$  is null-homotopic for every complex  $\mathbb{G}$ .

LEMMA 2. *Let  $A$  be a ring, and*

$$\mathbb{F} : \cdots \rightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \rightarrow \cdots$$

*a complex of flat left  $A$ -modules. Then the following are equivalent.*

- (1)  $M \otimes_A \mathbb{F}$  is exact for every right  $A$ -module  $M$ .
- (2)  $\mathbb{F}$  is exact, and  $\text{Im } d_i$  is flat for every  $i$ .
- (3) For every complex  $\mathbb{G}$  of right  $A$ -modules,  $\mathbb{G} \otimes_A^\bullet \mathbb{F}$  is exact.
- (4)  $\mathbb{F}$  is  $K$ -flat and exact.

*Proof.* (1)  $\Rightarrow$  (2). Obviously,  $\mathbb{F} \cong A \otimes_A \mathbb{F}$  is exact. Thus

$$\mathbb{F}' : \cdots \rightarrow F_{i+1} \rightarrow F_i \rightarrow 0$$

is a flat resolution of  $\text{Im } d_i$ , where  $F_{n+i}$  has the homological degree  $n$  in  $\mathbb{F}'$ . For every  $i \in \mathbb{Z}$ ,

$$\text{Tor}_1^A(M, \text{Im } d_i) \cong H_1(M \otimes_A \mathbb{F}') \cong H_{i+1}(M \otimes_A \mathbb{F}) = 0$$

for every right  $A$ -module  $M$ . Thus  $\text{Im } d_i$  is  $A$ -flat.

(2)  $\Rightarrow$  (1). For every  $i \in \mathbb{Z}$ ,

$$H_{i+1}(M \otimes_A \mathbb{F}) \cong H_1(M \otimes_A \mathbb{F}') \cong \text{Tor}_1^A(M, \text{Im } d_i) = 0,$$

where  $\mathbb{F}'$  is as above.

(1), (2)  $\Rightarrow$  (3). Since  $\mathbb{G} \cong \varinjlim \tau^{\leq n} \mathbb{G}$ , we may assume that  $\mathbb{G}$  is bounded above. Since  $\mathbb{F} \cong \varinjlim \tau^{\leq n} \mathbb{F}$  and  $\tau^{\leq n} \mathbb{F}$  satisfies (2) (and hence (1)), we may assume that  $\mathbb{F}$  is also bounded above. Then by an easy spectral sequence argument,  $\mathbb{G} \otimes_A^\bullet \mathbb{F}$  is exact.

(3)  $\Rightarrow$  (4) is trivial.

(4)  $\Rightarrow$  (1). Let  $\mathbb{P}$  be a projective resolution of  $M$ . Since  $\mathbb{P}$  is a bounded above complex of flat left  $A^{\text{op}}$ -modules and  $\mathbb{F}$  is an exact complex of right  $A^{\text{op}}$ -modules,  $\mathbb{P} \otimes_A^\bullet \mathbb{F}$  is exact. Let  $\mathbb{Q}$  be the mapping cone of  $\mathbb{P} \rightarrow M$ . Then  $\mathbb{Q} \otimes_A^\bullet \mathbb{F}$  is also exact, since  $\mathbb{Q}$  is exact and  $\mathbb{F}$  is  $K$ -flat. By the exact sequence of homology groups

$$H_i(\mathbb{P} \otimes_A^\bullet \mathbb{F}) \rightarrow H_i(M \otimes_A \mathbb{F}) \rightarrow H_i(\mathbb{Q} \otimes_A^\bullet \mathbb{F}),$$

we have that  $M \otimes_A \mathbb{F}$  is also exact. □

In [4, Proposition 5.7], (4)  $\Rightarrow$  (3) above is proved essentially.

**COROLLARY 3.** *Let*

$$\mathbb{F} : \cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots$$

*be a (possibly unbounded) complex of flat  $R$ -modules. If  $\mathbb{F}(\mathfrak{p})$  is exact for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{F}$  is  $K$ -flat and exact.*

*Proof.* By Lemma 2, it suffices to show that for every  $n \in \mathbb{Z}$  and every  $R$ -module  $M$ ,  $H_n(M \otimes \mathbb{F}) = 0$ . But this is trivial by Theorem 1 applied to the complex

$$\cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow 0.$$

□

The following was proved by A. Neeman [3, Corollary 6.10].

**THEOREM 4.** *Let  $A$  be a ring, and  $\mathbb{P}$  a complex of projective left  $A$ -modules. If  $\mathbb{P}$  is  $K$ -flat and exact, then  $\mathbb{P}$  is null-homotopic.*

By Corollary 3 and Theorem 4, we have

**COROLLARY 5.** *Let  $\mathbb{P}$  be a complex of  $R$ -projective modules. If  $\mathbb{P}(\mathfrak{p})$  is exact for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\mathbb{P}$  is null-homotopic.*

The following also follows.

**COROLLARY 6.** ([1, Lemma 4.2], [2, Lemma I.2.1.4]) *Let  $\varphi : F_1 \rightarrow F_0$  be an  $R$ -linear map between  $R$ -flat modules. Then the following are equivalent.*

- 1**  $\varphi$  is injective and  $\text{Coker } \varphi$  is  $R$ -flat.
- 2**  $\varphi$  is pure.
- 3**  $\varphi(\mathfrak{p})$  is injective for every  $\mathfrak{p} \in \text{Spec } R$ .

*Proof.* **1**  $\Rightarrow$  **2**  $\Rightarrow$  **3** is obvious. **3**  $\Rightarrow$  **1** is a special case of Theorem 1.  $\square$

**COROLLARY 7.** ([2, Corollary I.2.1.6]) *Let  $F$  be a flat  $R$ -module. If  $F(\mathfrak{p}) = 0$  for every  $\mathfrak{p} \in \text{Spec } R$ , then  $F = 0$ .*

*Proof.* Consider the zero map  $F \rightarrow 0$ , and apply Corollary 6. We have that this map is injective, and hence  $F = 0$ .  $\square$

If  $M$  is a finitely generated  $R$ -module and  $M(\mathfrak{m}) = 0$  for every maximal ideal  $\mathfrak{m}$  of  $R$ , then  $M = 0$ . This is a consequence of Nakayama's lemma.

**COROLLARY 8.** *Let  $\varphi : F_1 \rightarrow F_0$  be an  $R$ -linear map between  $R$ -flat modules. If  $\varphi(\mathfrak{p})$  is an isomorphism for every  $\mathfrak{p} \in \text{Spec } R$ , then  $\varphi$  is an isomorphism.*

*Proof.* By Corollary 6,  $\varphi$  is injective and  $C := \text{Coker } \varphi$  is  $R$ -flat. Since  $C(\mathfrak{p}) \cong \text{Coker}(\varphi(\mathfrak{p})) = 0$  for every  $\mathfrak{p} \in \text{Spec } R$ , we have that  $C = 0$  by Corollary 7.  $\square$

**COROLLARY 9.** *Let  $M$  be an  $R$ -module. If  $\mathrm{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$  for every  $i > 0$  and every prime ideal  $\mathfrak{p} \in \mathrm{Spec} R$ , then  $M$  is  $R$ -flat. If  $\mathrm{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$  for every  $i \geq 0$  and every prime ideal  $\mathfrak{p} \in \mathrm{Spec} R$ , then  $M = 0$ .*

*Proof.* For the first assertion, Let  $\mathbb{F}$  be a projective resolution of  $M$ , and apply Theorem 1. The second assertion follows from the first assertion and Corollary 7.  $\square$

**COROLLARY 10.** *Let  $M$  be an  $R$ -module. If  $\mathrm{Ext}_R^i(M, \kappa(\mathfrak{p})) = 0$  for every  $i > 0$  (resp.  $i \geq 0$ ) and every prime ideal  $\mathfrak{p} \in \mathrm{Spec} R$ , then  $M$  is  $R$ -flat (resp.  $M = 0$ ).*

*Proof.* This is trivial by Corollary 9 and the fact

$$\mathrm{Ext}_R^i(M, \kappa(\mathfrak{p})) \cong \mathrm{Hom}_{\kappa(\mathfrak{p})}(\mathrm{Tor}_i^R(\kappa(\mathfrak{p}), M), \kappa(\mathfrak{p})).$$

$\square$

### §3. Some examples

**EXAMPLE 11.** There is an acyclic projective complex

$$\mathbb{P} : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

over a noetherian commutative ring  $R$  such that  $H_0(\mathbb{P})$  is  $R$ -flat and  $h_0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H_0(\mathbb{P}(\mathfrak{p}))$  is finite and constant, but  $H_0(\mathbb{P})$  is neither  $R$ -finite nor  $R$ -projective.

*Proof.* Set  $R = \mathbb{Z}$ ,  $M = \sum_{p \text{ prime}} (1/p)\mathbb{Z} \subset \mathbb{Q}$ , and  $\mathbb{P}$  to be a projective resolution of  $M$ . Then  $M$  is  $R$ -torsion free, and is  $R$ -flat. Since  $M_{(p)} = (1/p)\mathbb{Z}_{(p)}$ ,  $h_0(\mathfrak{p}) = 1$  for every  $\mathfrak{p} \in \mathrm{Spec} \mathbb{Z}$ . A finitely generated nonzero  $\mathbb{Z}$ -submodule of  $\mathbb{Q}$  must be rank-one free, but  $M$  is not a cyclic module, and is not rank-one free. This shows that  $M$  is not  $R$ -finite. As  $R$  is a principal ideal domain, every  $R$ -projective module is free. If  $M$  is projective, then it is free of rank  $h_0((0)) = 1$ . But  $M$  is not finitely generated, so  $M$  is not projective.  $\square$

*Remark 12.* Let  $(R, \mathfrak{m})$  be a noetherian local ring,  $F$  a flat  $R$ -module, and  $c$  a non-negative integer. If  $\dim_{\kappa(\mathfrak{p})} F(\mathfrak{p}) = c$  for every  $\mathfrak{p} \in \mathrm{Spec} R$ , then  $F \cong R^c$ , see [2, Corollary III.2.1.10].

*Remark 13.* Let

$$\mathbb{P} : 0 \rightarrow P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \dots$$

be a complex of  $R$ -flat modules such that  $P^0$  is  $R$ -projective. Assume that  $\mathbb{P}(\mathfrak{p})$  is acyclic (i.e.,  $H^i(\mathbb{P}(\mathfrak{p})) = 0$  for every  $i > 0$ ) and  $h_{\mathbb{P}}^0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{P}(\mathfrak{p}))$  is finite for every  $\mathfrak{p} \in \text{Spec } R$ . If  $h_{\mathbb{P}}^0$  is a locally constant function on  $\text{Spec } R$ , then  $H^0(\mathbb{P})$  is  $R$ -finite  $R$ -projective,  $H^i(M \otimes \mathbb{P}) = 0$  ( $i > 0$ ), and the canonical map  $M \otimes H^0(\mathbb{P}) \rightarrow H^0(M \otimes \mathbb{P})$  is an isomorphism for every  $R$ -module  $M$ , see [2, Proposition III.2.1.14]. If, moreover,  $\mathbb{P}$  is a complex of  $R$ -projective modules, then  $\text{Im } d^i$  is  $R$ -projective for every  $i \geq 0$ , as can be seen easily from Theorem 4.

**EXAMPLE 14.** Let  $M$  be an  $R$ -module. Even if  $M(\mathfrak{p}) = 0$  for every  $\mathfrak{p} \in \text{Spec } R$ ,  $M$  may not be zero. Even if  $\text{Tor}_1^R(\kappa(\mathfrak{p}), M) = 0$  for every  $\mathfrak{p} \in \text{Spec } R$ ,  $M$  may not be  $R$ -flat.

Indeed, let  $(R, \mathfrak{m}, k)$  be a  $d$ -dimensional regular local ring, and  $E$  the injective hull of  $k$ . Then

$$\text{Tor}_i^R(\kappa(\mathfrak{p}), E) \cong \begin{cases} k & \text{for } i = d \text{ and } \mathfrak{p} = \mathfrak{m} \\ 0 & \text{otherwise.} \end{cases}$$

$E$  is not  $R$ -flat unless  $d = 0$ .

*Proof.* Since  $\text{supp } E = \{\mathfrak{m}\}$ ,  $\text{Tor}_i^R(\kappa(\mathfrak{p}), E) = 0$  unless  $\mathfrak{p} = \mathfrak{m}$ .

Let  $\mathbf{x} = (x_1, \dots, x_d)$  be a regular system of parameters of  $R$ , and  $\mathbb{K}$  the Koszul complex  $K(\mathbf{x}; R)$ , which is a minimal free resolution of  $k$ . Note that  $\mathbb{K}$  is self-dual. That is,  $\mathbb{K}^* \cong \mathbb{K}[-d]$ , where  $\mathbb{K}^* = \text{Hom}_R^\bullet(\mathbb{K}, R)$ , and  $\mathbb{K}[-d]^n = \mathbb{K}^{n-d}$ . So

$$\begin{aligned} \text{Tor}_i^R(k, E) &\cong H^{-i}(\mathbb{K} \otimes E) \cong H^{-i}(\mathbb{K}^{**} \otimes E) \\ &\cong H^{-i}(\text{Hom}_R^\bullet(\mathbb{K}[-d], E)) \cong H^{-i}(\text{Hom}_R^\bullet(k[-d], E)) \\ &\cong \begin{cases} k & (i = d) \\ 0 & (i \neq d). \end{cases} \end{aligned}$$

□

**EXAMPLE 15.** There is a complex  $\mathbb{P}$  of projective modules over a noetherian commutative ring  $R$  such that for each  $\mathfrak{m} \in \text{Max}(R)$ ,  $\mathbb{P}(\mathfrak{m})$  is exact, but  $\mathbb{P}$  is not exact, where  $\text{Max}(R)$  denotes the set of maximal ideals of  $R$ .

*Proof.* Let  $R$  be a DVR with its field of fractions  $K$ , and  $\mathbb{P}$  a projective resolution of  $K$ . □

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