

**THE FILTERED POINCARÉ LEMMA IN HIGHER
LEVEL
(WITH APPLICATIONS TO ALGEBRAIC GROUPS)**

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Abstract. We show that the Poincaré lemma we proved elsewhere in the context of crystalline cohomology of higher level behaves well with regard to the Hodge filtration. This allows us to prove the Poincaré lemma for transversal crystals of level m . We interpret the de Rham complex in terms of what we call the Berthelot-Lieberman construction and show how the same construction can be used to study the conormal complex and invariant differential forms of higher level for a group scheme. Bringing together both instances of the construction, we show that crystalline extensions of transversal crystals by algebraic groups can be computed by reduction to the filtered de Rham complexes. Our theory does not ignore torsion and, unlike in the classical case ($m = 0$), not all invariant forms are closed. Therefore, close invariant differential forms of level m provide new invariants and we exhibit some examples as applications.

Introduction

In a series of articles, starting with [9] and [10], we have been using the partial divided powers of Berthelot to study the geometry of algebraic varieties of positive characteristic. This gives new insight into the p -adic cohomological theories. Unlike other works on the subject ([7], [13] and [5]), we do not use crystalline cohomology of higher level as a tool to obtain results in rigid cohomology and, in particular, we do not ignore torsion. In fact, torsion is very rich in this theory and provides new invariants that help understand the geometry of algebraic varieties. For example, we will show how the sheaf of closed invariant differential forms of higher level can tell you exactly where the supersingular locus of a family of elliptic curves is.

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Following A. Ogus in [11], we introduced in [9] the notion of transversal crystal of higher level. Although we could slightly improve on some of Ogus' results, progress was hampered by the lack of de Rham techniques for computing exactly crystalline cohomology in higher level. A natural answer to this problem was provided in [10], where we developed ideas of P. Berthelot, introducing the de Rham complex of higher level and proving the exact Poincaré lemma. In the present article, we extend the Poincaré lemma of higher level to transversal crystals by paying close attention to filtrations. This is used to give a precise description of the group of extensions of a transversal crystal by a commutative group scheme. In a forthcoming article, we want to use these results to show that Dieudonné crystals of higher level are transversal.

After reviewing in Section 1 a few results on filtrations, and especially fixing terminology about filtered derived categories, in Section 2 we verify that the formal Poincaré lemma in higher level behaves as expected with respect to filtrations. In doing so, we reinterpret the de Rham and linearized de Rham complexes introduced in [10] as particular cases of what we call a Berthelot-Lieberman complex. This general construction will be used in Section 4 to define the conormal complex of higher level of a group scheme. But first we prove the filtered Poincaré lemma for transversal crystals. This is done in Section 3 and requires careful attention to the behavior of the filtrations all along the process. In Section 4, which is completely independent of the preceding one, we study the relation between the conormal sheaf of higher level and invariant differential forms of higher level. Unlike in the classical case (level 0), not all invariant forms are closed. Actually, the module of closed invariant forms is isomorphic to the first cohomology group of the conormal complex. We present concrete examples, including the Legendre family of elliptic curves, and give the relation with de Rham cohomology of higher level in the case of abelian schemes. Section 5 brings together the two previous sections. More precisely, we show that crystalline extensions of transversal crystals by abelian groups can be computed by reduction to filtered de Rham complexes. As an application, we show that the extension group of the partial divided power ideal by a smooth abelian group is nothing other than a lifting of the module of closed invariant differentials of higher level.

Most results here are inspired by theorems that have been well known for a long time in the case of usual divided powers and classical crystalline

cohomology. In particular, this work owes much to P. Berthelot, L. Breen, L. Illusie, W. Messing and A. Ogus.

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Conventions

Starting at Section 2, we let p be a prime, $m \in \mathbf{N}$ and, unless $m = 0$, all schemes are assumed to be $\mathbf{Z}_{(p)}$ -schemes.

§1. Generalities about filtrations

Concerning filtrations, we use the terminology of [6], 1.1. In particular, we will only consider filtrations of type \mathbf{Z} in additive categories. Also, filtration will always mean decreasing filtration.

DEFINITION 1.1. A filtration Fil^\bullet on an object M is *effective* if $\mathrm{Fil}^0 M = M$.

Unless otherwise specified, we will only consider effective decreasing filtrations. Note that the filtration induced on a subquotient (cf. [6], 1.1.10) by an effective filtration is still effective. One can also check that the image of an effective filtration by a semi-exact (meaning left or right exact) multiadditive functor (cf. [6], 1.1.12) is still effective. This applies in particular to the tensor product filtration.

DEFINITION 1.2. The *trivial (effective) filtration* on an object M is given by

$$\mathrm{Fil}^k M = \begin{cases} M & \text{if } k \leq 0 \\ 0 & \text{if } k > 0. \end{cases}$$

In [6] (Definition 1.3.6), filtered quasi-isomorphisms are only defined for so-called biregular filtrations. This definition does not generalize well. As in [11], Section 4.4, we will need a more restrictive notion.

DEFINITION 1.3. A morphism of filtered complexes $M^\bullet \rightarrow N^\bullet$ is a *true filtered quasi-isomorphism* if, for each $k \in \mathbf{Z}$, the induced morphism $\mathrm{Fil}^k M^\bullet \rightarrow \mathrm{Fil}^k N^\bullet$ is a quasi-isomorphism. A *filtered homotopy* is a homotopy $s : M^\bullet \rightarrow N^\bullet$ such that

$$\forall k \in \mathbf{Z}, \quad \forall n \in \mathbf{Z}, \quad s(\mathrm{Fil}^k M^n) \subset \mathrm{Fil}^k N^{n-1}.$$

Two morphisms of filtered complexes $f, g : M^\bullet \rightarrow N^\bullet$ are *homotopic* if there exists a filtered homotopy $h : M^\bullet \rightarrow N^\bullet$ such that g is homotopic to f with respect to h .

As explained in [11], page 80, the above notions of filtered homotopy and true filtered quasi-isomorphisms are suitable to define the filtered derived category of a Grothendieck category, e.g. a category of modules. Moreover, any left exact additive functor F between such categories gives rise to a filtered derived functor and this construction is completely compatible with the non-filtered situation in the sense that we always have

$$RF(\text{Fil}^k M^\bullet) = \text{Fil}^k RFM^\bullet.$$

In particular, there exists a canonical spectral sequence

$$E_1^{i,j} = R^{i+j}F(Gr^i M^\bullet) \Rightarrow R^n F(M^\bullet)$$

that endows, for each $n \in \mathbf{Z}$, $R^n F(M^\bullet)$ with a canonical filtration.

We now need to recall some definitions and results from [11] and [9] on transversal filtrations.

DEFINITION 1.4. A *ring filtration* on a ring A is an effective filtration by ideals $I^{(k)}$ such that $I^{(k)}I^{(l)} \subset I^{(k+l)}$. A filtered A -module (M, Fil^\bullet) is *transversal* (resp. *almost transversal*) to $I^{(\bullet)}$ if for each $k \in \mathbf{Z}$,

$$I^{(1)}M \cap \text{Fil}^k M = (\text{resp. } \subset) \sum_{i+j=k, i>0} I^{(i)} \text{Fil}^j.$$

If (M, Fil^\bullet) is a filtered A -module, the *saturation* of the filtration with respect to $I^{(\bullet)}$ is the tensor product filtration $\overline{\text{Fil}}^\bullet$ on M under the identification $A \otimes_A M = M$.

When A is filtered by the powers of an ideal I , we simply say transversal (resp. almost transversal, resp. saturation with respect) to I . Note that a module filtration is always transversal to 0 and that, if a filtration is almost transversal, then its saturation is transversal.

DEFINITION 1.5. The *trivial transversal filtration* on an A -module M is the saturation of the trivial filtration.

Note that the trivial transversal filtration is given by

$$\overline{\text{Fil}}^k M = \begin{cases} M & \text{if } k \leq 0 \\ I^{(k)}M & \text{if } k > 0. \end{cases}$$

§2. The formal filtered Poincaré lemma

The aim of this section is to give a filtered version of Theorem 3.3 of [10]. We first recall some properties of m -PD-envelopes and introduce the notion of Berthelot-Lieberman complex.

If $X \hookrightarrow Y$ is an immersion of schemes, then we will denote by $P_{Xm}(Y)$ its divided power envelope of level m . We will write $\mathcal{P}_{Xm}(Y)$ for the structural sheaf and $\mathcal{I}_{Xm}(Y)$ for the m -PD-ideal. We will use a superscript n to denote the same objects modulo $\mathcal{I}_{Xm}^{\{n+1\}}(Y)$. If X is an S -scheme, we will denote by $P_{X/Sm}(r)$ the m -PD envelope of the diagonal embedding of X in X^{r+1} and modify all other notations accordingly.

The notion of m -PD envelope is functorial in the sense that any commutative diagram

$$\begin{array}{ccc} X' & \hookrightarrow & Y' \\ \downarrow g & & \downarrow g' \\ X & \hookrightarrow & Y \end{array}$$

canonically extends to

$$\begin{array}{ccccc} X' & \hookrightarrow & P_{X'm}(Y') & \hookrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & P_{Xm}(Y) & \hookrightarrow & Y \end{array}$$

We recall the following fundamental results of Berthelot:

PROPOSITION 2.1. *Let $f : X \hookrightarrow Y$ be an immersion of schemes with a smooth retraction $h : Y \rightarrow X$. We use h to endow $\mathcal{P}_{Xm}^n(Y)$ and $\mathcal{P}_{Xm}(Y)$ with the structure of an \mathcal{O}_X -algebra. Then,*

1. *For all n , $\mathcal{P}_{Xm}^n(Y)$ is a locally free \mathcal{O}_X -module of finite rank. Actually, $\mathcal{P}_{Xm}(Y)$ itself is locally free when Y is uniformly killed by a power of p .*
2. *Let*

$$\begin{array}{ccc} X' & \hookrightarrow & Y' \\ \downarrow g & & \downarrow g' \\ X & \hookrightarrow & Y \end{array}$$

be a cartesian diagram (both ways). Then, we have for all n ,

$$g^* \mathcal{P}_{X_m}^n(Y) \simeq \mathcal{P}_{X'_m}^n(Y')$$

and even

$$g^* \mathcal{P}_{X_m}(Y) \simeq \mathcal{P}_{X'_m}(Y')$$

if Y is uniformly killed by a power of p .

Proof. This follows from Propositions 1.4.6 and 1.5.3 of [2].

Let $X \hookrightarrow Y(\bullet)$ be an immersion of a scheme into a simplicial scheme. In other words, we are given a family of immersions of $X \hookrightarrow Y(r)$ compatible with the differentials d_i and the degeneracy arrows s_i of $Y(\bullet)$. Taking m -PD envelopes gives rise to a simplicial scheme $\mathcal{P}_{X_m}(Y\bullet)$ from which we derive a complex $(\mathcal{P}_{X_m}(Y\bullet), \mathcal{I}_{X_m}^{\{k\}}(Y\bullet))$ of filtered rings. We then consider as in [8] the normalization of $\mathcal{P}_{X_m}(Y\bullet)$ which is the subcomplex of ideals defined by $N\mathcal{P}_{X_m}(Yr) := \bigcap \ker s_i^*$. Finally, we are interested in the quotient of $N\mathcal{P}_{X_m}(Y\bullet)$ by the differential subalgebra generated by the ideal $\mathcal{I}_{X_m}^{\{p^m+1\}}(Y1)$ of $\mathcal{P}_{X_m}(Y1)$, which we write $\Omega_{X_m}^\bullet(Y)$.

DEFINITION 2.2. The complex $\Omega_{X_m}^\bullet(Y)$ is the *Berthelot-Lieberman complex* of $X \hookrightarrow Y(\bullet)$. The *Hodge filtration* on $\Omega_{X_m}^\bullet(Y)$ is the filtration Fil_H^k induced by the m -PD-filtration $\mathcal{I}_{X_m}^{\{k\}}(Y\bullet)$ of $\mathcal{P}_{X_m}(Y\bullet)$.

DEFINITION 2.3.

1. In a category with products, the *product simplicial object* $X^{\text{prod}}(\bullet)$ of an object X is defined by $X^{\text{prod}}(r) = X^{r+1}$,

$$d_i(x_1, \dots, x_{r+2}) = (x_1, \dots, x_i, x_{i+2}, \dots, x_{r+2}) \quad \text{for } i = 0, 1, \dots, r+1$$

and

$$s_i(x_1, \dots, x_r) = (x_1, \dots, x_i, x_{i+1}, x_{i+1}, x_{i+2}, \dots, x_{r+2})$$

for $i = 0, 1, \dots, r$.

2. If $X(\bullet)$ is a simplicial object in any category, the *shifted simplicial object* is defined by

$$X^+(r) = X(r+1), \quad d_i^+ = d_{i+1}, \quad s_i^+ = s_{i+1}.$$

We can now reformulate some definitions from Section 1 of [10].

Let S be a scheme with p locally nilpotent and X an S -scheme.

DEFINITION 2.4. The Berthelot-Lieberman complex Ω_{X/S_m}^\bullet of the diagonal embedding $X \hookrightarrow (X/S)^{prod}(\bullet)$, is the *de Rham complex of level m* . The Berthelot-Lieberman complex $L_X(\Omega_{X/S_m}^\bullet)$ of the shifted simplicial scheme $(X/S)^{prod+}(\bullet)$ is the *linearized de Rham complex of level m* .

Note that $L_X(\Omega_{X/S_m}^\bullet)$ is what we called Ω_{P/S_m}^\bullet in [10].

For the rest of this section, we assume that X is a smooth scheme over S .

Remarks 2.5.

1. Recall from Section 1 of [10] that, if we have local coordinates t_1, \dots, t_n on X and, as usual, we set $\tau_i := 1 \otimes t_i - t_i \otimes 1$, then $\mathcal{P}_{X/S_m}(r)$ is a free \mathcal{O}_X -module on generators $\tau^{\{J_1\}} \otimes \dots \otimes \tau^{\{J_r\}}$ with $|J_i| \geq 0$ (we use the standard multiindex convention). Using Proposition 1.5.3 of [2], one easily checks that $\mathcal{I}_{X/S_m}^{\{k\}}(r)$ is generated by the $\tau^{\{J_1\}} \otimes \dots \otimes \tau^{\{J_r\}}$ with $|J_1| + \dots + |J_r| \geq k$.
2. Now, if as usual again, dt_i denotes the image of τ_i in Ω_{X/S_m}^1 , we know that Ω_{X/S_m}^1 is a free \mathcal{O}_X -module on generators $(dt)^J$ with $0 < |J| \leq p^m$. As we discussed in Section 1 of [10], for $r > 1$, even if Ω_{X/S_m}^r is generated by the $(dt)^{J_1} \otimes \dots \otimes (dt)^{J_r}$ with $0 < |J_i| \leq p^m$, there are some relations among these generators. Actually, Ω_{X/S_m}^\bullet is the quotient of the tensor algebra $T^\bullet(\Omega_{X/S_m}^1)$ by the relations

$$(*) \quad \sum_{0 < V < J} \left\langle \begin{matrix} J \\ V \end{matrix} \right\rangle (dt)^{J-V} \otimes (dt)^V = 0.$$

By definition, we see that the Hodge filtration $\text{Fil}_H^k \Omega_{X/S_m}^r$ is generated by the

$$(dt)^{J_1} \otimes \dots \otimes (dt)^{J_r} \quad \text{with } 0 < |J_i| \leq p^m \quad \text{and} \quad |J_1| + \dots + |J_r| \geq k.$$

In particular, $\text{Fil}_H^k = 0$ for $k > rp^m$.

3. Similarly, $L_X(\Omega_{X/S_m}^r)$ is generated as $\mathcal{P}_{X/S_m}(1)$ -module by the $(d\tau)^{J_1} \otimes \cdots \otimes (d\tau)^{J_r}$ with $0 < |J_i| \leq p^m$, and $\text{Fil}_H^k L_X(\Omega_{X/S_m}^r)$ is the \mathcal{O}_X -submodule generated by the $\tau^{\{I\}}(d\tau)^{J_1} \otimes \cdots \otimes (d\tau)^{J_r}$ with $|I| + |J_1| + \cdots + |J_r| \geq k$.
4. The Hodge filtration on Ω_{X/S_m}^r is a filtration where, for each $k \geq 0$, both $\text{Fil}^k \Omega_{X/S_m}^r$ and $\text{Gr}^k \Omega_{X/S_m}^r$ are locally free. To see this, one proceeds as in the proof of Proposition 1.4 in [10]. First of all, the question is local. Then, the process consists in writing some of the $(dt)^{J_1} \otimes \cdots \otimes (dt)^{J_r}$ with $|J_1| + \cdots + |J_r| = k$ in term of the others using the relations (*) which is possible because $|J - V| + |V| = |J|$ for all V .
5. Note that when X is smooth, the de Rham complex of level 0 is the usual de Rham complex with its Hodge filtration. The reason is that the generators of Ω_{X/S_0}^r are the $dt_{i_1} \otimes \cdots \otimes dt_{i_r}$ and the relations (*) become simply

$$dt_i \otimes dt_i = 0 \quad \text{and} \quad dt_i \otimes dt_j + dt_j \otimes dt_i = 0.$$

Recall that if \mathcal{F} is any $\mathcal{D}_{X/S}^{(m)}$ -module, we can form the de Rham complex

$$\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S_m}^\bullet.$$

of \mathcal{F} . In particular, we have the m -connection $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S_m}^1$. For further use, recall also that the relative de Rham cohomology of level m of \mathcal{F} is

$$\mathcal{H}_{dRm}^n(X, \mathcal{F}) := \mathbf{R}^n p_{X*} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S_m}^\bullet$$

where $p_X : X \rightarrow S$ is the structural map.

Concerning Griffiths transversality, we refer to Section 2.2 of [9]. Let us just recall that a filtration Fil^\bullet on a $\mathcal{D}_{X/S}^{(m)}$ -module \mathcal{F} is a filtration by \mathcal{O}_X -submodules and that it is said *Griffiths transversal* if we always have

$$\mathcal{D}_{X/S_j}^{(m)} \text{Fil}^k \mathcal{F} \subset \text{Fil}^{k-j} \mathcal{F}.$$

PROPOSITION 2.6. *A filtered $\mathcal{D}_{X/S}^{(m)}$ -module is Griffiths transversal if and only if the m -connection is compatible with the filtrations.*

Proof. This is a local question and, if we have local coordinates, the m -connection on \mathcal{F} is given by

$$s \longmapsto \sum_{0 < |J| \leq p^m} \partial^{[J]} s \otimes (dt)^J.$$

If the filtration is Griffiths transversal and $s \in \text{Fil}^k \mathcal{F}$, we have

$$\partial^{[J]} s \otimes (dt)^J \in \text{Fil}^{k-|J|} \mathcal{F} \otimes \text{Fil}^{|J|} \Omega_{X/S_m}^1.$$

Since both the filtration and the graded pieces are locally free, we have

$$\text{Fil}^{k-|J|} \mathcal{F} \otimes \text{Fil}^{|J|} \Omega_{X/S_m}^1 \subset \text{Fil}^k (\mathcal{F} \otimes \Omega_{X/S_m}^1).$$

It follows that the connection preserves the filtrations.

Conversely, if the m -connection is compatible with the filtrations and $s \in \text{Fil}^k \mathcal{F}$, then, necessarily, for all $i = 1, \dots, n$ and $j \leq m$, we have

$$\partial_i^{[p^j]} s \otimes (dt)^{p^j} \in \text{Fil}^k (\mathcal{F} \otimes \Omega_{X/S_m}^1)$$

so that, necessarily, $\partial_i^{[p^j]} s \in \text{Fil}^{k-|p^j|} \mathcal{F}$ and by Remark 2.2.2 (ii) of [9], this is sufficient for Griffiths transversality.

Since the differential on a de Rham complex can be described by the Leibnitz rule, we have the following:

COROLLARY 2.7. *If \mathcal{F} is a filtered $\mathcal{D}_{X/S}^{(m)}$ -module and if, for each r , $\mathcal{F} \otimes \Omega_{X/S_m}^r$ is endowed with the tensor product filtration, then \mathcal{F} is Griffiths transversal if and only if the de Rham complex of \mathcal{F} is a filtered complex.*

LEMMA 2.8. *For all r , if we endow Ω_{X/S_m}^r and $L_X(\Omega_{X/S_m}^r)$ with their Hodge filtration and $\mathcal{P}_{X/S_m}(1)$ with its m -PD-filtration, we have an isomorphism of filtered modules*

$$L_X(\Omega_{X/S_m}^r) = \mathcal{P}_{X/S_m}(1) \otimes_{\mathcal{O}_X} \Omega_{X/S_m}^r.$$

Moreover, the filtration on $\mathcal{P}_{X/S_m}(1) \otimes_{\mathcal{O}_X} \Omega_{X/S_m}^r$ is the saturation with respect to $\mathcal{I}_{X/S_m}^{\{\bullet\}}(1)$ of the inverse image by $\mathcal{P}_{X/S_m}(1) \rightarrow \mathcal{O}_X$ of the Hodge filtration on Ω_{X/S_m}^r .

Proof. Since we assume that X is smooth over S , then as mentioned in Section 1.5 of [10], $\mathcal{P}_{X/S_m}^+(r)$ is canonically isomorphic as filtered algebra to $\mathcal{P}_{X/S_m}(1) \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S_m}(r)$. The assertion then follows from the functoriality of our construction. The second assertion is local in nature and follows directly from the local description of our filtrations in Remark 2.5.3.

PROPOSITION 2.9. *The Hodge filtration on $L_X(\Omega_{X/S_m}^\bullet)$ is transversal to $\mathcal{I}_{X/S_m}^{\{\bullet\}}(1)$.*

Proof. The morphism of filtered rings $(\mathcal{P}_{X/S_m}(1), \mathcal{I}_{X/S_m}^{\{\bullet\}}(1)) \rightarrow (\mathcal{O}_X, 0)$ obviously satisfies the assumptions of Proposition 1.1.8 of [9]. Since any filtration is transversal to the 0-ideal, we see that the inverse image of the Hodge filtration is almost transversal and it follows that its saturation is transversal. Our assertion now results from Lemma 2.8.

The following is a generalization of Theorem 3.3 of [10].

PROPOSITION 2.10. *If \mathcal{O}_X is endowed with the trivial filtration and $L_X(\Omega_{X/S_m}^\bullet)$ with the Hodge filtration, the canonical map*

$$\mathcal{O}_X \longrightarrow L_X(\Omega_{X/S_m}^\bullet)$$

is a true filtered quasi-isomorphism. More precisely, locally on X , it is a filtered homotopy equivalence.

Proof. The proof works exactly as in Theorem 3.3 of [10] once one notices that the homotopy of 2.1 in [10] is a filtered homotopy.

§3. The filtered Poincaré lemma

We will explain here how the results of [10], Section 4 extend to the case of transversal m -crystals.

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a m -PD-scheme with p locally nilpotent and $p \in \mathfrak{a}$. Let X be an S -scheme to which the m -PD-structure of S extends. We will consider the crystalline site of level m of X/S that was introduced in Section 4.1 of [9] and the corresponding topos $(X/S)_{\text{cris}}^{(m)}$.

Unless otherwise specified, we will assume in this section that X is smooth over S .

NOTATIONS 3.1. If Y is any object in a topos \mathcal{T} , we will denote by $\mathcal{T}|_Y$ the localized category and by $j_Y : \mathcal{T}|_Y \rightarrow \mathcal{T}$ the restriction map.

In particular, we will consider here the localization $(X/S)_{\text{cris}|X}^{(m)}$ at the trivial immersion of X in itself and the restriction morphism

$$j_X : (X/S)_{\text{cris}|X}^{(m)} \longrightarrow (X/S)_{\text{cris}}^{(m)}.$$

By definition, we have $(j_{X*}\mathcal{F})_{(U,T)} = \mathcal{F}_{(U,T \times U)}$ and we see in this case that j_{X*} is exact.

We will also consider the projection

$$u_X : (X/S)_{\text{cris}}^{(m)} \longrightarrow X_{Zar}$$

on the Zariski site, given by $\Gamma(U, u_{X*}\mathcal{F}) = \Gamma((U/S)_{\text{cris}}^{(m)}, \mathcal{F})$. Note that this is not a morphism of ringed topoi because there is no natural map $\mathcal{O}_X \rightarrow u_{X*}\mathcal{O}_{X/S}^{(m)}$. However, the composite

$$u|_X := u_X \circ j_X : (X/S)_{\text{cris}|X}^{(m)} \longrightarrow X_{Zar}$$

is in a natural way a morphism of ringed topoi because it is nothing but the realization morphism: $u|_{X*}\mathcal{F} = \mathcal{F}_X$. We also see that $u|_{X*}$ is an exact functor.

Details can be found in Chapter 5 of [4] for the $m = 0$ case. The generalization is straightforward.

PROPOSITION 3.2. *With the above notations,*

1. *The ideal $\mathcal{K}_{X/S}^{(m)} := j_{X*}j_X^{-1}\mathcal{I}_{X/S}^{(m)}$ of $j_{X*}j_X^{-1}\mathcal{O}_{X/S}^{(m)}$ is an m -PD-ideal.*
2. *There is a canonical exact sequence*

$$0 \longrightarrow \mathcal{K}_{X/S}^{(m)} \longrightarrow j_{X*}j_X^{-1}\mathcal{O}_{X/S}^{(m)} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0.$$

3. *For all $k \in \mathbf{Z}$, we have $\mathcal{K}_{X/S}^{(m)\{k\}} = j_{X*}j_X^{-1}(\mathcal{I}_{X/S}^{(m)\{k\}})$.*

Proof. One easily checks that, if \mathcal{F} is any \mathcal{O}_X -module, the adjunction map

$$j_{X*}j_X^{-1}u_X^{-1}\mathcal{F} \longrightarrow u_X^{-1}\mathcal{F}$$

is an isomorphism. In particular, if we apply the exact functor $j_{X*}j_X^{-1}$ to the exact sequence

$$0 \longrightarrow \mathcal{I}_{X/S}^{(m)} \longrightarrow \mathcal{O}_{X/S}^{(m)} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0,$$

we get the expected exact sequence

$$0 \longrightarrow j_{X*}j_X^{-1}\mathcal{I}_{X/S}^{(m)} \longrightarrow j_{X*}j_X^{-1}\mathcal{O}_{X/S}^{(m)} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0.$$

Thus, we see that if $U \subset X$ is any open subset and $U \hookrightarrow Y$ an m -PD-thickening, we have the following exact sequence

$$0 \longrightarrow (K_{X/S}^{(m)})_Y \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_U \longrightarrow 0$$

where $P := P_{Um}(Y \times_S X)$. Hence, we see that $(K_{X/S}^{(m)})_Y$ is the m -PD-ideal \mathcal{I}_P of P . It follows that $K_{X/S}^{(m)}$ is an m -PD-ideal and that for all $k \in \mathbf{Z}$, $(K_{X/S}^{(m)\{k\}})_Y = \mathcal{I}_P^{\{k\}} = (\mathcal{I}_{X/S}^{(m)\{k\}})_Y$.

DEFINITION 3.3. If \mathcal{F} is a filtered \mathcal{O}_X -module, its *linearization* (of level m) is $L^{(m)}(\mathcal{F}) := j_{X*}u_{|X}^*\mathcal{F}$ endowed with the saturation of the filtration $L^{(m)}(\text{Fil}^k\mathcal{F})$ with respect to the m -PD-ideal $\mathcal{K}_{X/S}^{(m)\{\bullet\}}$ of $L^{(m)}(\mathcal{O}_X) := j_{X*}j_X^{-1}\mathcal{O}_{X/S}^{(m)}$.

LEMMA 3.4. *If \mathcal{F} is a filtered \mathcal{O}_X -module, we have a canonical isomorphism of filtered modules*

$$L^{(m)}(\mathcal{F})_X = \mathcal{P}_{X/Sm}(1) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Proof. Thanks to Proposition 4.3 (1) of [10], only the assertion concerning the filtration has to be checked. Since saturation is just tensor product with the ideal filtration, it is sufficient to note that, as in the proof of Lemma 3.2, $(\mathcal{K}_{X/S}^{(m)})_X = \mathcal{I}_{X/Sm}(1)$.

We recall that $L^{(m)}(\Omega_{X/Sm}^\bullet)$ has a natural structure of complex of crystals whose realization on X is nothing but the linearized de Rham complex $L_X(\Omega_{X/Sm}^\bullet)$. Thanks to the lemma, this is compatible with the filtrations.

We can now state the formal Poincaré lemma in its crystalline form.

THEOREM 3.5. *If E is a filtered $\mathcal{O}_{X/S}^{(m)}$ -module which is saturated with respect to $\mathcal{I}_{X/S}^{(m)\{\bullet\}}$, then the morphism*

$$E \longrightarrow E \otimes_{\mathcal{O}_{X/S}^{(m)}} L^{(m)}(\Omega_{X/S}^{\bullet})$$

is a true filtered quasi-isomorphism. More precisely, locally on $\text{Cris}^{(m)}(X/S)$, it is a filtered homotopy equivalence.

Proof. Since the question is local, we may assume, thanks to Proposition 2.10, that the canonical map $\mathcal{O}_X \rightarrow L_X(\Omega_{X/S}^{\bullet})$ is a filtered homotopy equivalence.

Let's start with the case $E = \mathcal{O}_{X/S}^{(m)}$. We have to check that, for any m -PD-thickening $U \hookrightarrow T$, the morphism

$$\mathcal{O}_T \longrightarrow L^{(m)}(\Omega_{X/S}^{\bullet})_T$$

is locally on T a filtered homotopy equivalence. Thus, we may assume that $U \hookrightarrow T$ has a retraction. Since the pull-back of a filtered homotopy is still a filtered homotopy, we obtain a filtered homotopy equivalence on T from the one on X using this retraction.

In general, tensoring with E gives a morphism

$$E \otimes_{\mathcal{O}_{X/S}^{(m)}} \mathcal{O}_{X/S}^{(m)} \longrightarrow E \otimes_{\mathcal{O}_{X/S}^{(m)}} L^{(m)}(\Omega_{X/S}^{\bullet})$$

which is locally a filtered homotopy equivalence. Since the filtration on E is saturated, the canonical identification

$$E \otimes_{\mathcal{O}_{X/S}^{(m)}} \mathcal{O}_{X/S}^{(m)} = E$$

is compatible with the filtrations. The proof is therefore complete.

The definition of a transversal m -crystal is given in [9], Section 4. Roughly speaking, it is a crystal of transversal modules, but the reader should consider looking at the above reference if he really wants a precise definition as well as a description of its relation with Griffiths transversality.

PROPOSITION 3.6. *Let E be a transversal m -crystal on X/S . If \mathcal{F} is any filtered \mathcal{O}_X -module, there is a canonical isomorphism of filtered modules*

$$E \otimes L^{(m)}(\mathcal{F}) \simeq L^{(m)}(E_X \otimes \mathcal{F}).$$

Proof. We already have an isomorphism of crystals thanks to Proposition 4.3 (3) of [10]. More precisely, if $U \subset X$ is any open subset and $U \hookrightarrow Y$ an m -PD-thickening, we have an isomorphism

$$E_Y \otimes \mathcal{O}_P \otimes \mathcal{F} \simeq E_P \otimes \mathcal{F} \simeq \mathcal{O}_P \otimes E_Y \otimes \mathcal{F},$$

where $P := P_{U^m}(Y \times_S X)$ as in the proof of Proposition 3.2, and we need to show that it is compatible with filtrations. But this follows from the definition in [9], 4.2.2 of a transversal m -crystal.

COROLLARY 3.7. *If E is a transversal m -crystal on X/S , there is a canonical true filtered quasi-isomorphism*

$$E \longrightarrow L^{(m)}(E_X \otimes_{\mathcal{O}_X} \Omega_{X/S}^{\bullet(m)}).$$

More precisely, locally on $\text{Cris}^{(m)}(X/S)$, it is a filtered homotopy equivalence.

In order to obtain the filtered Poincaré lemma at the cohomological level, we need the following:

PROPOSITION 3.8. *If \mathcal{F} is a filtered \mathcal{O}_X -module, we have a canonical isomorphism in the filtered derived category $Ru_{X^*}L^{(m)}(\mathcal{F}) = \mathcal{F}$.*

Proof. We must show that, for all $k \in \mathbf{Z}$, we have $R^i u_{X^*} \text{Fil}^k L^{(m)}(\mathcal{F}) = 0$ for $i > 0$ and $u_{X^*} \text{Fil}^k L^{(m)}(\mathcal{F}) = \text{Fil}^k \mathcal{F}$.

By definition, the filtration of $L^{(m)}(\mathcal{F})$ is the saturation of the filtration $L^{(m)}(\text{Fil}^k \mathcal{F})$ with respect to the m -PD-ideal $\mathcal{K}_{X/S}^{(m)\{\bullet\}}$:

$$\text{Fil}^k L^{(m)}(\mathcal{F}) = \sum_{i+j=k, i \geq 0} \mathcal{K}_{X/S}^{(m)\{i\}} L^{(m)}(\text{Fil}^j \mathcal{F}).$$

Using Proposition 3.2.3, we obtain

$$\text{Fil}^k L^{(m)}(\mathcal{F}) = \sum_{i+j=k, i \geq 0} (j_{X^*} j_X^{-1} \mathcal{I}_{X/S}^{(m)\{i\}})(j_{X^*} u_{|X}^*(\text{Fil}^j \mathcal{F})).$$

Since j_{X^*} is exact, it follows that $\text{Fil}^k L^{(m)}(\mathcal{F}) = j_{X^*} \mathcal{E}_k$ with

$$\mathcal{E}_k := \sum_{i+j=k, i \geq 0} j_X^{-1} \mathcal{I}_{X/S}^{(m)\{i\}} u_{|X}^*(\text{Fil}^j \mathcal{F}).$$

Regarding the higher direct images, it is sufficient to recall that, since j_{X*} and $u|_{X*}$ are exact, we have

$$R^i u_{X*} \mathrm{Fil}^k L^{(m)}(\mathcal{F}) = R^i u_{X*} j_{X*} \mathcal{E}_k = R^i u|_{X*} \mathcal{E}_k = 0$$

for $i > 0$.

Now, we know from Proposition 4.3 (2) of [10] that, for any \mathcal{O}_X -module \mathcal{F} , we have $u_{X*} L^{(m)}(\mathcal{F}) = \mathcal{F}$. Also, u_X^{-1} being fully faithful, $u_{X*} u_X^{-1} \mathcal{F} = \mathcal{F}$. Since u_{X*} is left exact and $\mathcal{K}_{X/S}^{(m)}$ is the kernel of the natural map $L^{(m)}(\mathcal{O}_X) \rightarrow u_X^{-1} \mathcal{O}_X$, we see that $u_{X*} \mathcal{K}_{X/S}^{(m)} = 0$. It follows that u_{X*} ignores saturation with respect to $\mathcal{K}_{X/S}^{(m)}$ and so,

$$u_{X*} \mathrm{Fil}^k L^{(m)}(\mathcal{F}) = u_{X*} L^{(m)}(\mathrm{Fil}^k \mathcal{F}) = \mathrm{Fil}^k \mathcal{F}$$

as asserted.

THEOREM 3.9. *If E is a transversal m -crystal on X/S , there is a canonical isomorphism in the filtered derived category*

$$Ru_{X*} E \simeq E_X \otimes \Omega_{X/S_m}^\bullet.$$

Proof. Using Proposition 3.8, this is an direct consequence of 3.7.

COROLLARY 3.10. *Even if we no longer assume X smooth, but if $i : X \hookrightarrow Y$ is an embedding into a smooth S -scheme and if E is a transversal m -crystal on X , then there is a canonical isomorphism in the filtered derived category*

$$i_* Ru_{X*} E \simeq (i_{\mathrm{cris}*} E)_Y \otimes \Omega_{Y/S_m}^\bullet.$$

Proof. Works exactly as at the end of Section 4 in [10].

COROLLARY 3.11. *In the situation of the previous corollary, there is a canonical filtered isomorphism*

$$R\Gamma((X/S)_{\mathrm{cris}}^{(m)}, E) \simeq R\Gamma(Y, (i_{\mathrm{cris}, m*} E)_Y \otimes \Omega_{Y/S_m}^\bullet).$$

Therefore, for all i , we have

$$H_{\mathrm{cris}, m}^i(X, \mathrm{Fil}^k E) = H^i(\mathrm{Fil}^k((i_{\mathrm{cris}, m*} E)_Y \otimes \Omega_{Y/S_m}^\bullet)).$$

§4. Differentials of higher level on a group scheme

In this section, which is independent of the previous one, we define the conormal complex of level m of a group scheme and study invariant differential forms of higher level.

DEFINITION 4.1. The *simplicial object* $G^{gr}(\bullet)$ associated to a group G in a category with products has components $G^{gr}(r) := G^r$, differentials $d_i : G^{r+1} \rightarrow G^r$ defined by

$$\begin{aligned} d_0(g_1, \dots, g_{r+1}) &= (g_2, \dots, g_{r+1}) \\ &\vdots \\ d_i(g_1, \dots, g_{r+1}) &= (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{r+1}) \\ &\vdots \\ d_{r+1}(g_1, \dots, g_{r+1}) &= (g_1, \dots, g_r) \end{aligned}$$

and degeneracy arrows $s_i : G^{r-1} \rightarrow G^r$ defined by

$$s_i(g_1, \dots, g_{r-1}) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_{r-1}).$$

LEMMA 4.2. *If G is a group in a category with products, the maps*

$$G^{r+1} \longrightarrow G^r : (g_1, \dots, g_{r+1}) \longmapsto (g_1^{-1} g_2, g_2^{-1} g_3, \dots, g_r^{-1} g_{r+1})$$

define a morphism of simplicial objects $\nu(\bullet) : G^{prod}(\bullet) \rightarrow G^{gr}(\bullet)$.

Proof. This is easily checked.

Let G be a group scheme over a scheme S , $p_G : G \rightarrow S$ its structural morphism, $1_G : S \hookrightarrow G$ its unit section and

$$\mu, p_1, p_2 : G \times_S G \longrightarrow G$$

the group law and the projections.

DEFINITION 4.3. The *conormal complex of level m* of G is the Berthelot-Lieberman filtered complex $\omega_{G_m}^\bullet$ associated to the unit embedding of S into $G^{gr}(\bullet)$.

It will be convenient to write $P_{1m}(\bullet)$ for the m -PD-envelope of the unit section in $G^{gr}(\bullet)$ and modify the other notations accordingly.

Remark 4.4. As in the case of the de Rham complex, for which the results are recalled in Remark 2.5, we have a very simple local description of the situation in the case of a smooth group scheme. If s_1, \dots, s_n is a regular sequence of local parameters for the unit section, then $\mathcal{P}_{1m}(r)$ is a free module on generators $s^{\{J_1\}} \otimes \dots \otimes s^{\{J_r\}}$ with $|J_i| \geq 0$ and $\mathcal{I}_{1m}^{\{k\}}(r)$ is generated by the $s^{\{J_1\}} \otimes \dots \otimes s^{\{J_r\}}$ with $|J_1| + \dots + |J_r| \geq k$. Thus, if \bar{s}_i denotes the image of s_i in ω_{Gm}^1 , we see that ω_{Gm}^1 is a free module on generators \bar{s}^J with $0 < |J| \leq p^m$ and that, for bigger r , ω_{Gm}^r is generated by the $\bar{s}^{J_1} \otimes \dots \otimes \bar{s}^{J_r}$ with $0 < |J_i| \leq p^m$, subject to some relations. Of course, the k -th step of the Hodge filtration $\text{Fil}^k \omega_{Gm}^r$ has the same generators subject to the additional condition that $|J_1| + \dots + |J_r| \geq k$. In particular, $\text{Fil}^r \omega_{Gm}^r = \omega_{Gm}^r$.

PROPOSITION 4.5. *The morphism $\nu(\bullet)$ of Lemma 4.2 induces morphisms of filtered complexes*

$$\nu(\bullet) : \mathcal{P}_{1m}(\bullet) \longrightarrow p_{G*} \mathcal{P}_{Gm}(\bullet)$$

and

$$\nu^\bullet : \omega_{Gm}^\bullet \longrightarrow p_{G*} \Omega_{Gm}^\bullet.$$

Proof. This follows from the functoriality of the construction of the Berthelot-Lieberman complex.

Remark 4.6. Of course, we also have for each r morphisms of filtered modules

$$\nu(r) : p_G^* \mathcal{P}_{1m}(r) \longrightarrow \mathcal{P}_{Gm}(r)$$

and

$$\nu^r : p_G^* \omega_{Gm}^r \longrightarrow \Omega_{Gm}^r.$$

There are also morphisms of filtered complexes

$$\mathcal{P}_{1m}(\bullet) = 1_G^* p_G^* \mathcal{P}_{1m}(\bullet) \longrightarrow 1_G^* p_G^* p_{G*} \mathcal{P}_{Gm}(\bullet) \longrightarrow 1_G^* \mathcal{P}_{Gm}(\bullet)$$

and in particular

$$\omega_{Gm}^\bullet \longrightarrow 1_G^* \Omega_{Gm}^\bullet.$$

Without assuming that G is smooth, we cannot really say more.

If $g \in G(S)$, we denote by $T_g : G \rightarrow G$ the left translation map given by $h \mapsto gh$.

DEFINITION 4.7. A section ξ of $\mathcal{P}_{Gm}(r)$, or $\mathcal{P}_{Gm}^k(r)$, is *invariant by translation by $g \in G(S)$* if $T_g^*(\xi) = \xi$. Moreover, ξ is *invariant by translation* if it is invariant by translation by any section of $G_{S'}$ for any extension $S' \rightarrow S$ of the basis.

We indicate with a superscript \bullet^{inv} the subsheaf of translation invariant sections.

PROPOSITION 4.8.

1. If G is smooth, then for all r , $\nu(r)$ is a canonical isomorphism of filtered modules

$$p_G^* \mathcal{P}_{1m}(r) \simeq \mathcal{P}_{Gm}(r)$$

and it follows that

$$p_G^* \omega_{Gm}^r \simeq \Omega_{Gm}^r.$$

2. The morphism $\nu(\bullet)$ gives also isomorphisms of filtered complexes

$$\mathcal{P}_{1m}(\bullet) \simeq 1_G^* \mathcal{P}_{Gm}(\bullet) \simeq \mathcal{P}_{Gm}^{inv}(\bullet)$$

and

$$\omega_{Gm}^\bullet \simeq 1_G^* \Omega_{Gm}^\bullet \simeq \Omega_{Gm}^{\bullet inv}.$$

Proof. Note first that the diagram

$$\begin{array}{ccc} G & \xleftarrow{\quad} & G^{r+1} \\ p_G \downarrow & & \downarrow \nu(r) \\ S & \xleftarrow{\quad} & G^r, \end{array}$$

where the top (resp. bottom) arrows are the first projection (resp. structural map p_G) and the diagonal embedding (resp. unit section 1_G), is cartesian both ways. Since G is smooth, it follows from the second assertion of Proposition 2.1 that, for all r , $\nu(r)$ induces an isomorphism of filtered modules

$$p_G^* \mathcal{P}_{1m}(r) \simeq \mathcal{P}_{Gm}(r).$$

Since the Berthelot-Lieberman complex is obtained as a quotient of the Fil^1 by the differential ideal generated by the Fil^{p^m+1} , we obtain also an isomorphism

$$p_G^* \omega_{Gm}^r \simeq \Omega_{Gm}^r$$

for each r .

It follows from the first part that the morphism of filtered complexes

$$\mathcal{P}_{1m}(\bullet) \longrightarrow 1_G^* \mathcal{P}_{Gm}(\bullet)$$

of Remark 4.6 is an isomorphism. We also have injective maps

$$\mathcal{P}_{1m}(r) \hookrightarrow p_{G*} \mathcal{P}_{Gm}(r).$$

Moreover, since for $g \in G(S)$ we have

$$\nu(r) \circ (T_g \times \cdots \times T_g) = \nu(r) : G^{r+1} \longrightarrow G^r,$$

we see that the image of $\mathcal{P}_{1m}(\bullet)$ in $p_{G*} \mathcal{P}_{Gm}(\bullet)$ is contained in $\mathcal{P}_{Gm}^{inv}(\bullet)$. We want to show that the map

$$\mathcal{P}_{1m}(\bullet) \hookrightarrow \mathcal{P}_{Gm}^{inv}(\bullet)$$

is bijective. Since

$$p_{G*} \mathcal{P}_{Gm}(r) \simeq p_{G*} \mathcal{O}_G \otimes_{\mathcal{O}_S} \mathcal{P}_{1m}(r)$$

and G is flat, we are reduced to showing that the canonical map $\mathcal{O}_S \rightarrow \mathcal{O}_G^{inv}$ is bijective. If we consider translation by the identity

$$G \times G \longrightarrow G \times G, \quad (g, h) \longmapsto (g, gh),$$

we see that, if f is a section of \mathcal{O}_G^{inv} , then $\mu^*(f) = p_1^*(f)$. Pulling back by $(1_G \times \text{Id}_G)^*$, we get $f = p_G^* 1_G^*(f)$ which is what we want.

PROPOSITION 4.9. *If G is a smooth abelian group scheme and if we let*

$$\delta := p_2^* - \mu^* + p_1^* : p_{G*} \mathcal{P}_{Gm}(1) \longrightarrow p_{G^{2*}} \mathcal{P}_{G^{2m}}(1),$$

then the morphism $\nu(1)$ induces an isomorphism of filtered modules

$$p_{G*} \mathcal{I}_{Gm}(1) \cap \ker \delta = \mathcal{I}_{Gm}^{inv}(1) \cap \ker d$$

and therefore also

$$p_{G*} \Omega_{Gm}^1 \cap \ker \delta = \Omega_{Gm}^{1inv} \cap \ker d.$$

Proof. If $g \in G(S)$ and $i_g : G \rightarrow G \times G$, $h \mapsto (g, h)$, we have $p_2 \circ i_g = \text{Id}_G$, $\mu \circ i_g = T_g$ and $p_1 \circ i_g = g \circ p_G$. It follows that

$$i_g^* \circ \delta = (\text{Id} - T_g^*) + p_G^* \circ g^*.$$

If $\xi \in p_{G^*} \mathcal{I}_{G^m}(1)$ then, trivially, $g^*(\xi) = 0$ and it follows that $i_g^*(\delta(\xi)) = \xi - T_g^*(\xi)$. Thus, any $\xi \in p_{G^*} \mathcal{I}_{G^m}(1) \cap \ker \delta$ is g -invariant. This is true for any g and after any base change. It follows that

$$p_{G^*} \mathcal{I}_{G^m}(1) \cap \ker \delta \subset \mathcal{I}_{G^m}^{inv}(1).$$

Since G is a commutative group scheme, the diagrams

$$\begin{array}{ccc} G & \xleftarrow{f} & G^2 \\ \nu_{G(1)} \uparrow & & \uparrow \nu_{G^2(1)} \\ G^2 & \xleftarrow{f \times f} & G^4 \end{array}$$

are all commutative for $f = p_1$, $f = p_2$ and $f = \mu$. Observe that $\mathcal{P}_{1_{G^2}m}(1) = \mathcal{P}_{1m}(2)$. By functoriality, the diagram

$$\begin{array}{ccc} \mathcal{P}_{1m}(1) & \xrightarrow{d} & \mathcal{P}_{1m}(2) \\ \nu_{G(1)} \downarrow & & \downarrow \nu_{G^2(1)} \\ p_{G^*} \mathcal{P}_{G^m}(1) & \xrightarrow{\delta} & p_{G^2} \mathcal{P}_{G^2m}(1) \end{array}$$

is also commutative.

Since we assumed that G is smooth, the vertical arrows induce filtered isomorphisms with the invariant part in the bottom and we obtain

$$\begin{array}{ccc} \mathcal{P}_{1m}(1) & \xrightarrow{d} & \mathcal{P}_{1m}(2) \\ \parallel & & \parallel \\ \mathcal{P}_{G^m}^{inv}(1) & \xrightarrow{\delta} & \mathcal{P}_{G^2m}^{inv}(1) \end{array}$$

from which it follows that $d = \delta$ on $\mathcal{P}_{G^m}^{inv}(1)$ and the result follows.

NOTATIONS 4.10. The sheaf

$$p_{G^*} \Omega_{G^m}^1 \cap \ker \delta = \Omega_{G^m}^{1inv} \cap \ker d \simeq \omega_{G^m}^1 \cap \ker d$$

of closed invariant forms will be generally written $\omega_{G^m,0}$. Notice that, since the first differential of the complex $\omega_{G^m}^\bullet$ is just the zero map, we also have $\omega_{G^m,0} \simeq \mathcal{H}^1(\omega_{G^m}^\bullet)$.

When $m = 0$, in which case we drop m from the notations, the situation is a lot simpler since invariant forms are automatically closed, and in particular, the kernel of δ is exactly the sheaf of invariant differential forms. In general, the group inclusions $i_1, i_2 : G \rightarrow G \times G$ induce morphisms $i_k^* : \omega_{G^{2m}}^1 \rightarrow \omega_{G^m}^1$ and the composite maps

$$\omega_{G^m}^1 \xrightarrow{\delta} \omega_{G^{2m}}^1 \xrightarrow{i_k^*} \omega_{G^m}^1$$

are zero. When $m = 0$, ω_G^1 is nothing but the conormal sheaf to the unit section and we have an isomorphism $(i_1^*, i_2^*) : \omega_{G^2}^1 \simeq \omega_G^1 \oplus \omega_G^1$. It follows that $\delta = 0$ on ω_G^1 . The first example below already shows that this is not necessarily the case when $m > 0$.

EXAMPLES 4.11.

1. If $G = \mathbf{G}_{aS}$ with parameter t , then $\omega_{G^m}^1$ is the free \mathcal{O}_S -module on $\bar{t}, \bar{t}^2, \dots, \bar{t}^m$. The group law is given by $t \mapsto t \otimes 1 + 1 \otimes t$ and it follows that

$$\delta(t^k) = t^k \otimes 1 - (t \otimes 1 + 1 \otimes t)^k + 1 \otimes t^k = - \sum_{i=1}^{k-1} \binom{k}{i} t^i \otimes t^{k-i}.$$

In particular, we have $\delta(t) = 0$ and therefore, we always have $\bar{t} \in \omega_{Gm,0}$.

But we also see that, if $\text{char } S = p$ (where we mean $p\mathcal{O}_S = 0$), then $\bar{t}^{p^j} \in \omega_{Gm,0}$ for all j . Actually, in this case, $\omega_{Gm,0}$ is free on the generators $\bar{t}, \bar{t}^p, \dots, \bar{t}^{p^m}$. In particular, we see that the filtration on $\omega_{Gm,0}$ can have length exactly $p^m - 1$. We also see that when $p \neq 2$, then \bar{t}^2 is not closed.

In the case $S = \text{Spec } \mathbf{Z}/4$ (so that $p = 2$) and $m = 1$, we see that $\delta(\bar{t}^2) = -2\bar{t} \otimes \bar{t} \neq 0$ but that $2\bar{t}^2 \in \omega_{G1,0}$. Thus, $\omega_{Gm,0}$ is not always locally free.

The opposite map in \mathbf{G}_a is given by $t \mapsto -t$ and the difference is therefore given by $t \mapsto -t \otimes 1 + 1 \otimes t$. It follows that the canonical inclusion $\omega_{G^m}^1 \hookrightarrow p_*\Omega_{G^m}^1$ sends \bar{t} to $d\bar{t}$ and therefore, $d\bar{t}$ is a closed invariant differential of level m on \mathbf{G}_{aS} . On the other hand, for $m > 0$, $(d\bar{t})^2$ is an invariant differential form which is not closed in general.

2. If $G = \mathbf{G}_{mS}$ with parameter t , then ω_{Gm}^1 is the free module on $\bar{s}, \bar{s}^2, \dots, \bar{s}^{p^m}$ with $s = t - 1$. Let us consider $\log t^{p^m} := \log(1 + s)^{p^m} \in \widehat{\mathcal{P}_{1m}(1)}$. Using the fact that the group law is given by $t \mapsto t \otimes t$, it is not difficult to check that $\delta(\log t^{p^m}) = 0$. Actually, this is a purely formal calculation that can be done over \mathbf{C} where this is well known. It follows that

$$\sum_{i=1}^{p^m} (-1)^i \frac{p^m}{i} \bar{s}^i \in \omega_{Gm,0}.$$

Note that, unlike the case $m = 0$, $\text{Fil}^2 \omega_{Gm,0} \neq 0$ in general. For example, if $p = 2$, $m = 1$ and $S = \text{Spec } \mathbf{Z}/4$, then we have $2\bar{s} - \bar{s}^2 \in \text{Fil}^1$ and $2\bar{s}^2 \in \text{Fil}^2$.

Since the inverse on \mathbf{G}_m is given by $t \mapsto t^{-1}$, the difference is given by $t \mapsto t^{-1} \otimes t$. Thus we see that $t - 1$ is sent to $t^{-1} \otimes t - 1 \otimes 1 = t^{-1}(1 \otimes t - t \otimes 1)$ and it follows that the canonical inclusion $\omega_{Gm}^1 \hookrightarrow p_* \Omega_{Gm}^1$ sends \bar{s} to dt/t . Finally, we get that

$$\sum_{i=1}^{p^m} (-1)^i \frac{p^m}{i} \left(\frac{dt}{t}\right)^i$$

is a closed invariant differential of level m on \mathbf{G}_{mS} .

3. We assume now $p \neq 2$ and consider the Legendre elliptic curve E given by the equation

$$y^2 = x(x - 1)(x - \lambda)$$

over $\mathbf{A}_S^1 \setminus \{0, 1\}$. Since we are interested in the behavior at O which is the point at infinity, we make the usual change of coordinates $z = -x/y$, $w = -1/y$ and the equation becomes

$$w = z(z - w)(z - \lambda w).$$

Thus, $P_{1m}(1)$ is the m -th divided power envelope of $\mathcal{O}_{\mathbf{A}_S^1 \setminus \{0,1\}}[z]$ with respect to z and ω_{Em}^1 is the free module on $\bar{z}, \dots, \bar{z}^{p^m}$. A quick calculation as explained in Chapter 4 of [12], for example, shows that the group law is given by

$$z \longmapsto 1 \otimes z + z \otimes 1 + (\lambda + 1)(z \otimes z^2 + z^2 \otimes z) + \dots.$$

Assume now that $p = 3$ and $m = 1$. Then, ω_{E1}^1 is the free \mathcal{O}_S -module on $\bar{z}, \bar{z}^2, \bar{z}^3$ and ω_{E1}^2 is generated by the $\bar{z}^i \otimes \bar{z}^j$ for $1 \leq i, j \leq 3$ subject

to the relations

$$\begin{aligned}\bar{z}^3 \otimes \bar{z}^3 &= 0, \\ \bar{z}^2 \otimes \bar{z}^3 + \bar{z}^3 \otimes \bar{z}^2 &= 0, \\ 2\bar{z} \otimes \bar{z}^3 + 3\bar{z}^2 \otimes \bar{z}^2 + 2\bar{z}^3 \otimes \bar{z} &= 0.\end{aligned}$$

Taking into account the symmetry of the series above due to the commutativity of the group law, one sees that higher powers don't play any role and that

$$\begin{aligned}\delta(\bar{z}) &= -(\lambda + 1)(\bar{z} \otimes \bar{z}^2 + \bar{z}^2 \otimes \bar{z}), \\ \delta(\bar{z}^2) &= -2\bar{z} \otimes \bar{z} - (\lambda + 1)(\bar{z}^2 \otimes \bar{z}^2), \\ \delta(\bar{z}^3) &= -3(\bar{z} \otimes \bar{z}^2 + \bar{z}^2 \otimes \bar{z}).\end{aligned}$$

It turns out that $-3\bar{z} + (1 + \lambda)\bar{z}^3$ is in $\omega_{E1,0}$.

Assume moreover that $\text{char } S = 3$. Then we get $(1 + \lambda)\bar{z}^3$ which is in Fil^3 . It follows from the formulas that $\omega_{E1,0}$ is actually free of rank 1 with generator \bar{z}^3 outside $\lambda = -1$. However, at the fiber $\lambda = -1$, both \bar{z} and \bar{z}^3 are in $\omega_{E1,0}$. In other words, the supersingular fiber is characterized by the fact that $\omega_{E1,0}$ is of rank 2 in contrast with the general case where it has rank 1. This is a phenomenon that is specific to higher level because in the classical situation, $\omega_{E0,0} = \omega_E$ is globally free of rank 1. Note however that in the case $m = 1$ and $p = 3$, then $\text{Fil}^m \omega_{Em,0}$ is also globally free of rank 1.

The next question would be to describe the canonical inclusion $\omega_{Em}^1 \hookrightarrow p_*\Omega_{Em}^1$. Since taking opposite on E is given by horizontal symmetry, it sends z to $-z$, and it follows that the difference in E is given by

$$z \longmapsto 1 \otimes z - z \otimes 1 + (\lambda + 1)(-z \otimes z^2 + z^2 \otimes z) + \dots$$

When $m = 0$, one can check that \bar{z} is sent to $dx/2y$ but we have been too lazy to do the computations in other cases.

PROPOSITION 4.12. *If A is an abelian scheme over S , the E_2 -term of the Hodge to de Rham spectral sequence of level m of A/S is*

$$E_2^{ij} = R^j p_{A*} \mathcal{O}_A \otimes_{\mathcal{O}_S} \mathcal{H}^i(\omega_{Am}^\bullet).$$

Proof. Consider the Hodge to de Rham spectral sequence of level m :

$$E_1^{ij} = R^j p_{A*} \Omega_{Am}^i \Rightarrow \mathcal{H}_{dRm}^n(A/S).$$

We know that, for all i , $\Omega_{Am}^i = p_A^* \omega_{Am}^i$. Since the sheaves ω_{Am}^i are locally free, the projection formula tells us that

$$R^j p_{A*} \Omega_{Am}^i = R^j p_{A*} \mathcal{O}_A \otimes_{\mathcal{O}_S} \omega_{Am}^i$$

and we get the spectral sequence

$$E_2^{ij} = R^j p_{A*} \mathcal{O}_A \otimes_{\mathcal{O}_S} \mathcal{H}^i(\omega_{Am}^\bullet) \Rightarrow \mathcal{H}_{dRm}^n(A/S).$$

COROLLARY 4.13. *The sequence*

$$0 \longrightarrow \omega_{Am,0} \longrightarrow \mathcal{H}_{dRm}^1(A/S) \longrightarrow \mathcal{R}^1 p_{A*} \mathcal{O}_A$$

is (left) exact.

Proof. Just note that $\omega_{Am,0} = H^1(\omega_{Am}^\bullet)$.

Remark 4.14. When S is smooth, using some results from [13], we can show that we actually get a short exact sequence. We hope to be able to prove this in general in a forthcoming article.

§5. Crystalline extension groups

In this section, we generalize to higher level the first results of the second chapter of [3]. The point is to describe the “mod p filtration on crystalline Dieudonné modules of level m ” (Proposition 5.14). The situation is way more complicated than in the classical $m = 0$ case because the filtration has more than one step and not all invariant forms are closed.

DEFINITIONS 5.1. A category I is said *very small* if $\text{Ob}(I)$ is countable and for each $\alpha, \beta \in \text{Ob}(I)$, $\text{Hom}(\alpha, \beta)$ is finite. A *decoration* on a very small category I is a pair comprising a *degree map* $d : \text{Ob}(I) \rightarrow \mathbf{Z}$ with finite fibers and a *sign map* $\epsilon : \text{Arr}(I) \rightarrow \{\pm\}$.

NOTATIONS 5.2. If Y is any object in a topos \mathcal{T} , we will denote by $\mathbf{Z}^{(Y)}$ the free abelian group on Y in \mathcal{T} .

Let (I, d, ϵ) be a decorated category and $Y_\bullet := (Y_\alpha, f_\lambda)$ a diagram indexed by I in \mathcal{T} . We set

$$C_n(Y_\bullet) := \bigoplus_{d(\alpha)=n} \mathbf{Z}^{(Y_\alpha)}$$

and let $\delta_n : C_n \rightarrow C_{n-1}$ be given by

$$\sum_{\lambda:\alpha\rightarrow\beta} \epsilon(\lambda) f_\lambda : \mathbf{Z}^{(Y_\alpha)} \longrightarrow \mathbf{Z}^{(Y_\beta)}$$

whenever $\deg(\alpha) = n$ and $\deg(\beta) = n - 1$.

DEFINITION 5.3. A diagram Y_\bullet indexed by a decorated category (I, d, ϵ) is *nice* if $C(Y_\bullet)$ is a complex.

Remark 5.4. P. Deligne has shown that any abelian group G in a topos \mathcal{T} has a canonical left resolution $C_\bullet(G)$ that fits in the above setting. More precisely, there exists a decorated category I and a nice diagram $\Delta(G)_\bullet := (G^{n_\alpha}, f_\lambda)$ indexed by I in the subcategory of \mathcal{T} generated by the powers of G , the projections and the group law such that $C_\bullet(G) = C_\bullet(\Delta(G)_\bullet)$.

To the best of our knowledge, this result of Deligne is unpublished, and we can offer no better reference than what is said in the remark at the end of 2.1.5 of [3]. We have taken the option of using this Deligne construction because we think it is the most elegant and natural way to state Theorem 5.12. For readers who find uncomfortable using unpublished results, we should mention that our main application (Proposition 5.14) can be deduced from a weaker version of the theorem (Remark 5.13.1) that relies only on a truncated resolution as in [3], Chapter 2, Section 1. The point is that Proposition 5.14 involves only the first three terms of the Deligne resolution that we now describe.

The diagram has only one object G in degree 0. In degree 1, there is also a unique object G^2 and the morphisms $G^2 \rightarrow G$ are

$$\begin{array}{rcl} G^2 & \longrightarrow & G & \text{sign} \\ (g_1, g_2) & \longmapsto & g_1 & - \\ (g_1, g_2) & \longmapsto & g_1 + g_2 & + \\ (g_1, g_2) & \longmapsto & g_2 & - \end{array}$$

In degree 2, there are two objects, namely G^2 and G^3 . The morphisms from degree 2 to degree 1 are given by

$$\begin{array}{rcl} G^2 & \longrightarrow & G^2 & \text{sign} \\ (g_1, g_2) & \longrightarrow & (g_1, g_2) & + \\ (g_1, g_2) & \longrightarrow & (g_2, g_1) & - \end{array}$$

and by

$$\begin{array}{rcl} G^3 & \longrightarrow & G^2 & \text{sign} \\ (g_1, g_2, g_3) & \longmapsto & (g_2, g_3) & - \\ (g_1, g_2, g_3) & \longmapsto & (g_1 + g_2, g_3) & + \\ (g_1, g_2, g_3) & \longmapsto & (g_1, g_2 + g_3) & - \\ (g_1, g_2, g_3) & \longmapsto & (g_1, g_2) & + \end{array}$$

Thus we get

$$\mathbf{Z}^{(G^3)} \oplus \mathbf{Z}^{(G^2)} \longrightarrow \mathbf{Z}^{(G^2)} \longrightarrow \mathbf{Z}^{(G)}$$

and the maps are given by

$$[g_1, g_2] \longmapsto -[g_1] + [g_1 + g_2] - [g_2]$$

from degree 1 to degree 0 and

$$\begin{array}{l} [g_1, g_2] \longmapsto [g_1, g_2] - [g_2, g_1] \\ [g_1, g_2, g_3] \longmapsto -[g_2, g_3] + [g_1 + g_2, g_3] - [g_1, g_2 + g_3] + [g_1, g_2] \end{array}$$

from degree 2 to degree 1.

LEMMA 5.5. *Let Y_\bullet be a nice diagram indexed by a decorated category (I, d, ϵ) in some topos \mathcal{T} and E an abelian sheaf in \mathcal{T} . Then, there is a spectral sequence*

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s j_{Y_\alpha} * j_{Y_\alpha}^{-1} E \Rightarrow \mathcal{E}xt^{r+s}(C(Y_\bullet), E).$$

Proof. If Y is any object of \mathcal{T} , we have

$$\mathcal{H}om_{G_r}(\mathbf{Z}^{(Y)}, E) = \mathcal{H}om(Y, E) = j_{Y*} j_Y^{-1} E.$$

It follows that the complex $\mathcal{H}om_{G_r}(C_\bullet(Y_\bullet), E)$ is canonically isomorphic to a complex whose terms are all of the form $\bigoplus_{\deg \alpha=n} j_{Y_\alpha} * j_{Y_\alpha}^{-1} E$. Since pulling back by localization is exact and preserves injective sheaves, we get the spectral sequence by applying this remark to an injective resolution of E .

Let $(S, \mathfrak{a}, \mathfrak{b})$ be an m -PD-scheme with p locally nilpotent and $p \in \mathfrak{a}$.

Let X be an S -scheme to which the m -PD-structure of S extends.

We denote by $\text{Sch}'_{/X}$ the category of X -schemes to which the m -PD-structure of S extends. The Zariski topology on this category is coarser than the canonical topology and we obtain an embedding of $\text{Sch}'_{/X}$ into the corresponding topos $X_{\text{ZAR}'}$. We will now use the big crystalline topos of level m , $(X/S)_{\text{CRIS}}^{(m)}$ that was introduced in [7] (but considering only schemes to which the m -PD structure extends). Composing the embedding $\text{Sch}'_{/X} \hookrightarrow X_{\text{ZAR}'}$ with the canonical map

$$v_{X/S*} : X_{\text{ZAR}'} \longrightarrow (X/S)_{\text{CRIS}}^{(m)}$$

from Section 1.10 of [7] gives a functor

$$\begin{array}{ccc} \text{Sch}'_{/X} & \longrightarrow & (X/S)_{\text{CRIS}}^{(m)} \\ Y & \longmapsto & \underline{Y} \end{array}$$

Remark 5.6. Note that if $Y \in \text{Sch}'_{/X}$ with structural morphism $f_Y : Y \rightarrow X$ then the canonical morphism

$$f_{Y/X} : (Y/S)_{\text{CRIS}}^{(m)} \longrightarrow (X/S)_{\text{CRIS}}^{(m)}$$

factors as an isomorphism $(Y/S)_{\text{CRIS}}^{(m)} \simeq (X/S)_{\text{CRIS}|_{\underline{Y}}}^{(m)}$ followed by the localization map

$$j_{\underline{Y}} : (X/S)_{\text{CRIS}|_{\underline{Y}}}^{(m)} \longrightarrow (X/S)_{\text{CRIS}}^{(m)}.$$

PROPOSITION 5.7. *Let Y_{\bullet} be a nice diagram in $\text{Sch}'_{/X}$ indexed by a decorated category (I, d, ϵ) . If E is an abelian sheaf on $\text{CRIS}^{(m)}(X/S)$, there is a spectral sequence*

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{Y_{\alpha}/X*} f_{Y_{\alpha}/X}^{-1} E \Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_{\bullet}), E).$$

Proof. Taking into account the previous remark, this immediately follows from Lemma 5.5.

Remarks 5.8. Just as in Chapter III, Section 4 of [1], if $Y \in \text{Sch}'_X$, there is a morphism of topoi

$$(Y/S)_{\text{cris}}^{(m)} \longrightarrow (X/S)_{\text{CRIS}}^{(m)}$$

whose inverse image functor $E \mapsto E_Y$ might be called restriction. For any sheaf E on $\text{CRIS}^{(m)}(X/S)$ and any morphism $g : Y' \rightarrow Y$, there is a canonical transition map $g^{-1}E_Y \rightarrow E_{Y'}$ and these data uniquely determine E . An m -crystal on the big site can be defined as an $\mathcal{O}_{X/S}^{(m)}$ -module E such that all E_Y are m -crystals and the transition maps induce isomorphisms $g^*E_Y \simeq E_{Y'}$. In particular, the functor $E \mapsto E_X$ is an equivalence of categories between m -crystals on $\text{CRIS}^{(m)}(X/S)$ and m -crystals on $\text{Cris}^{(m)}(X/S)$. Finally, note that any filtered (resp. transversal) $\mathcal{O}_{X/S}^{(m)}$ -module (E, Fil^\bullet) on $\text{CRIS}^{(m)}(X/S)$ restricts for each Y to a filtered (resp. transversal) $\mathcal{O}_{Y/S}^{(m)}$ -module $(E_Y, \text{Fil}^\bullet E_Y)$ on $\text{Cris}^{(m)}(Y/S)$.

For future reference, note also that, as for level 0, in which case this is proved in 1.1.16.4 of [3], we have for any abelian sheaf on $\text{CRIS}^{(m)}(Y/S)$,

$$(Rf_{Y/X*}E)_{(U,T)} = Rf_{U \times_X Y/T*}E_{U \times_X Y}.$$

DEFINITION 5.9. A *big transversal m -crystal* on X/S is a crystal E on $\text{CRIS}^{(m)}(X/S)$, endowed with a filtration Fil^\bullet such that for each $Y \in \text{Sch}'_X$, the filtered m -crystal $(E_Y, \text{Fil}^\bullet E_Y)$ is a transversal m -crystal.

LEMMA 5.10. *Let $Y \in \text{Sch}'_X$ and $(U \hookrightarrow T) \in \text{CRIS}^{(m)}(X/S)$. Let $i : U \times_X Y \hookrightarrow Z$ be a closed immersion into a smooth Z in Sch'_T . If E is a big transversal m -crystal on X/S , we have a canonical isomorphism*

$$(R^s f_{Y/X*} f_{Y/X}^{-1} \text{Fil}^k E)_{(U,T)} = R^s f_{Z*} \text{Fil}^k [(i_{\text{cris}*} E_{U \times_X Y})_Z \otimes \Omega_{Z/Tm}^\bullet].$$

Proof. First of all, Corollary 3.10 tells us that

$$\text{Fil}^k i_* Ru_{Y*} E_{U \times_X Y} \simeq \text{Fil}^k [(i_{\text{cris}*} E_{U \times_X Y})_Z \otimes \Omega_{Z/Tm}^\bullet].$$

Actually, since we work with filtered derived categories and i_* is exact, we have

$$\text{Fil}^k i_* Ru_{Y*} E_{U \times_X Y} = i_* Ru_{Y*} \text{Fil}^k E_{U \times_X Y}$$

and therefore

$$i_* Ru_{Y*} \text{Fil}^k E_{U \times_X Y} \simeq \text{Fil}^k [(i_{\text{cris}*} E_{U \times_X Y})_Z \otimes \Omega_{Z/Tm}^\bullet].$$

Since i_* is exact, applying Rf_{Z*} gives

$$Rf_{U \times_X Y/T*} \text{Fil}^k E_{U \times_X Y} \simeq Rf_{Z*} \text{Fil}^k [(i_{\text{cris}*} E_{U \times_X Y})_Z \otimes \Omega_{Z/Tm}^\bullet]$$

and, taking cohomology, we obtain

$$R^s f_{U \times_X Y/T*} \mathrm{Fil}^k E_{U \times_X Y} \simeq R^s f_{Z*} \mathrm{Fil}^k [(i_{\mathrm{cris}*} E_{U \times_X Y})_Z \otimes \Omega_{Z/Tm}^\bullet].$$

Finally, Remark 5.8 tells us that

$$(R^s f_{Y/X*} f_{Y/X}^{-1} \mathrm{Fil}^k E)_{(U,T)} = R^s f_{U \times_X Y/T*} \mathrm{Fil}^k E_{U \times_X Y}.$$

PROPOSITION 5.11. *Let Y_\bullet be a nice diagram in Sch'_X and $(U \hookrightarrow T) \in \mathrm{CRIS}^{(m)}(X/S)$. Let Z_\bullet be a nice diagram in Sch'_T with all Z_α smooth and $i_\bullet : U \times_X Y_\bullet \hookrightarrow Z_\bullet$ a compatible family of closed immersions. If E is a big transversal m -crystal on X/S , we have a canonical spectral sequence*

$$\begin{aligned} E_1^{r,s} &= \bigoplus_{d(\alpha)=r} R^s f_{Z_\alpha*} \mathrm{Fil}^k [(i_{\alpha \mathrm{cris}*} E_{U \times_X Y_\alpha})_{Z_\alpha} \otimes \Omega_{Z_\alpha/Tm}^\bullet] \\ &\Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_\bullet), \mathrm{Fil}^k E)_{(U,T)}. \end{aligned}$$

Proof. Since taking value on some object (U, T) is an exact functor, we know from Proposition 5.7 that there is a spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} (R^s f_{Y_\alpha/X*} f_{Y_\alpha/X}^{-1} \mathrm{Fil}^k E)_{(U,T)} \Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_\bullet), \mathrm{Fil}^k E)_{(U,T)}.$$

It is therefore sufficient to prove that for each α , we have

$$(R^s f_{Y_\alpha/X*} f_{Y_\alpha/X}^{-1} \mathrm{Fil}^k E)_{(U,T)} = R^s f_{Z_\alpha*} \mathrm{Fil}^k [(i_{\alpha \mathrm{cris}*} E_{U \times_X Y_\alpha})_{Z_\alpha} \otimes \Omega_{Z_\alpha/Tm}^\bullet]$$

and this follows from the lemma.

Applying this to the nice diagram $\Delta(G)_\bullet$ of Remark 5.4, that gives rise to Deligne resolution, we immediately get

THEOREM 5.12. *Let G be an abelian group scheme in Sch'_X and $(U \hookrightarrow T) \in \mathrm{CRIS}^{(m)}(X/S)$. Let H be a smooth abelian group scheme on T and $i : G_U \hookrightarrow H_U$ an immersion of groups. If E is a big transversal m -crystal on X/S , we have a canonical spectral sequence*

$$\begin{aligned} E_1^{r,s} &= \bigoplus_{d(\alpha)=r} R^s f_{H^{n_\alpha}*} \mathrm{Fil}^k [(i_{\alpha \mathrm{cris}*} E_{G_U^{n_\alpha}})_{H^{n_\alpha}} \otimes \Omega_{H^{n_\alpha}/Tm}^\bullet] \\ &\Rightarrow \mathcal{E}xt^{r+s}(\underline{G}, \mathrm{Fil}^k E)_{(U,T)}. \end{aligned}$$

Remarks 5.13.

1. If we use the truncated complex, we obtain a spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{H^{n\alpha}*} \mathrm{Fil}^k[(i_{\alpha \text{ cris}*} E_{G_U^{n\alpha}})_{H^{n\alpha}} \otimes \Omega_{H^{n\alpha}/T_m}^\bullet] \Rightarrow \mathcal{H}^{r+s}$$

with

$$\mathcal{H}^i = \mathcal{E}xt^i(\underline{G}, \mathrm{Fil}^k E)_{(U,T)} \quad \text{for } i \leq 1.$$

2. If E is any m -crystal on X/S , then the trivial transversal filtration $\mathcal{I}_{X/S}^{\{k\}} E$ turns it into a big transversal m -crystal and both Proposition 5.11 and Theorem 5.12 apply. Also, the theorem is still valid if we replace G with a complex of abelian groups.
3. When H is affine, if we denote by $[K^{\bullet\bullet}]_{\text{sing}}$ the simple complex associated to a bicomplex $K^{\bullet\bullet}$, we have, as in Theorem 2.1.8 of [3], an isomorphism in the derived category

$$\begin{aligned} \mathrm{Fil}^k \left[\bigoplus_{d(\alpha)=\bullet} f_{H^{n\alpha}*} [(i_{\alpha \text{ cris}*} E_{G_U^{n\alpha}})_{H^{n\alpha}} \otimes \Omega_{H^{n\alpha}/T_m}^\bullet]_{\text{sing}} \right] \\ \simeq R\mathcal{H}om(\underline{G}, \mathrm{Fil}^k E)_{(U,T)}. \end{aligned}$$

The proof goes exactly as in [3].

PROPOSITION 5.14. *Let G be a smooth abelian group scheme in Sch'_X and $U \in \mathrm{Sch}'_X$. If $k > 0$, we have*

$$\begin{aligned} \mathcal{H}om(\underline{G}, \mathcal{I}_{X/S}^{\{k\}})_{(U,U)} &= 0 \\ \mathcal{E}xt^1(\underline{G}, \mathcal{I}_{X/S}^{\{k\}})_{(U,U)} &= \mathrm{Fil}^k \omega_{G_U m, 0} \end{aligned}$$

where $\omega_{G_U m, 0}$ denotes, as in 4.10, the sheaf of closed invariant forms of level m on G_U . In particular,

$$\mathcal{E}xt^1(\underline{G}, \mathcal{I}_{X/S})_{(U,U)} = \omega_{G_U m, 0}.$$

Proof. We consider the spectral sequence of Theorem 5.12 (see also Remark 5.13.1)

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{G_U^{n\alpha}*} \mathrm{Fil}^k \Omega_{G_U^{n\alpha}/U_m}^\bullet \Rightarrow \mathcal{E}xt^{p+q}(\underline{G}, \mathcal{I}_{X/S}^{\{k\}})_{(U,U)}.$$

Note first that $E_1^{r,s} = 0$ for $r < 0$ or $s < 0$. Actually, since $k > 0$, we have $\text{Fil}^k \mathcal{O}_{G_U^{n\alpha}} = 0$ and it follows that $E_1^{r,s} = 0$ for $s = 0$ also. Therefore

$$\mathcal{H}om(\underline{G}, \mathcal{I}_{X/S}^{\{k\}})_{(U,U)} = 0$$

and

$$\mathcal{E}xt^1(\underline{G}, \mathcal{I}_{X/S}^{\{k\}})_{(U,U)} = E_2^{0,1} = \ker d_1^{0,1} : E_1^{0,1} \longrightarrow E_1^{1,1}.$$

Recall that if K^\bullet is any complex with $K^i = 0$ for $i < s$ and f a left exact functor, then

$$R^s f_* K^\bullet = R^s f_* \ker d^s[-s] = f_* \ker d^s = \ker f_* d^s.$$

Thus, here, we have for all α ,

$$\begin{aligned} R^1 f_{G_U^{n\alpha}*} \text{Fil}^k \Omega_{G_U^{n\alpha}/U_m}^\bullet \\ = \ker [d : f_{G_U^{n\alpha}*} \text{Fil}^k \Omega_{G_U^{n\alpha}/U_m}^1 \rightarrow f_{G_U^{n\alpha}*} \text{Fil}^k \Omega_{G_U^{n\alpha}/U_m}^2]. \end{aligned}$$

Since filtration commutes with kernel and direct image, we have

$$R^1 f_{G_U^{n\alpha}*} \text{Fil}^k \Omega_{G_U^{n\alpha}/U_m}^\bullet = \text{Fil}^k \ker [d : f_{G_U^{n\alpha}*} \Omega_{G_U^{n\alpha}/U_m}^1 \rightarrow f_{G_U^{n\alpha}*} \Omega_{G_U^{n\alpha}/U_m}^2].$$

It follows from the description of the first terms of the Deligne resolution that

$$\begin{aligned} E_1^{0,1} &= \text{Fil}^k \ker [d : f_{G_U*} \Omega_{G_U/U_m}^1 \rightarrow f_{G_U*} \Omega_{G_U/U_m}^2], \\ E_1^{1,1} &= \text{Fil}^k \ker [d : f_{G_U^2*} \Omega_{G_U^2/U_m}^1 \rightarrow f_{G_U^2*} \Omega_{G_U^2/U_m}^2] \end{aligned}$$

and that

$$d_1^{0,1} = -p_1^* + \mu^* - p_2^* = -\delta$$

with the notations of Proposition 4.9. Hence, we obtain

$$\begin{aligned} E_2^{0,1} &= \text{Fil}^k \ker [d : f_{G_U*} \Omega_{G_U/U_m}^1 \rightarrow f_{G_U*} \Omega_{G_U/U_m}^2] \\ &\quad \cap \ker [\delta : f_{G_U*} \Omega_{G_U/U_m}^1 \rightarrow f_{G_U^2*} \Omega_{G_U^2/U_m}^1] \end{aligned}$$

which is exactly $\text{Fil}^k \omega_{G_U m, 0}$.

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