SOME NUMERICAL CRITERIA FOR THE NASH PROBLEM ON ARCS FOR SURFACES

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Abstract. Let (X, O) be a germ of a normal surface singularity, $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of $\tilde{X}$. In an old preprint Nash proves that the set of arcs on a surface singularity is a scheme $\mathcal{H}$, and defines a map $\mathcal{N}$ from the set of irreducible components of $\mathcal{H}$ to the set of exceptional components of the minimal resolution of singularities of (X, O). He proved that this map is injective and ask if it is surjective. In this paper we consider the canonical decomposition $\mathcal{H} = \bigcup_{i=1}^{n} \bar{\mathcal{N}}_i$:

- For any couple (E_i, E_j) of distinct exceptional components, we define Numerical Nash condition $(NN_{(i,j)})$. We have that $(NN_{(i,j)})$ implies $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$. In this paper we prove that $(NN_{(i,j)})$ is always true for at least the half of couples (i, j).

- The condition $(NN_{(i,j)})$ is true for all couples (i, j) with $i \neq j$, characterizes a certain class of negative definite matrices, that we call Nash matrices. If A is a Nash matrix then the Nash map $\mathcal{N}$ is bijective. In particular our results depend only on A and not on the topological type of the exceptional set.

- We recover and improve considerably almost all results known on this topic and our proofs are new and elementary.

- We give infinitely many other classes of singularities where Nash Conjecture is true.

The proofs are based on my old work [8] and in Plenat [10].

§1. Introduction

Let (X, O) be a germ of a normal surface singularity. In an old preprint, published recently by Duke [9], Nash proved that the set of arcs on a surface singularity is a scheme $\mathcal{H}$, and defined a map $\mathcal{N}$ from the set of irreducible components of $\mathcal{H}$ to the set of exceptional components of the minimal res-

olution of singularities of (X, O). He proved that this map is injective and ask if it is surjective.

Among the principal contributions to this subject we can cite Monique Lejeune-Jalabert [6], [7], Ana Reguera [7], [13], [14], S. Ishii and J. Kollar [5], G. Gonzalez-Sprinberg and Monique Lejeune-Jalabert [4], Camille Plenat [10] and C. Plenat and P. Popescu-Pampu [12]. The study of arcs spaces was further developed by Kontsevich, Denef and Loeser [1] in the theory of motivic integration.

Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities, and $E_1, \ldots,$ E_n be the components of the exceptional divisor, Ana Reguera [13] associates to every E_i the family of arcs $\mathcal{N}_i$ such that the proper transform cuts properly E_i, the spaces $\bar{\mathcal{N}}_i$ are irreducible and give a decomposition of the space of arcs $\mathcal{H} = \bigcup \bar{\mathcal{N}}_i$. In order to give an affirmative answer to the Nash problem it is sufficient to prove that for any $i \neq j$ then $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$. In [13] Theorem 1.10, a valuative sufficient condition is given to separate two Nash components, that has been further developed by Camille Plenat [10], Proposition 2.2:

PROPOSITION. ([10]) *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and $E_1, \ldots, E_n$ be the components of the exceptional divisor, if there exist some $f \in \mathcal{O}_{X,O}$ such that $\mathrm{ord}_{E_i}(f) < \mathrm{ord}_{E_j}(f)$ then $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$.*

The following Theorem follows from my work [8] Theorem 1.1 and Lemma 2.2. Remark that in [12] C. Plenat and P. Popescu-Pampu have recently rediscover a similar condition.

THEOREM. ([8]) *Let (X, O) be a germ of normal surface singularity, $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and $E_1, \ldots, E_n$ be the components of the exceptional divisor. Let $K_{\tilde{X}}$ the canonical divisor on $\tilde{X}$. Let E be an exceptional effective divisor and $Q = \pi_* \mathcal{O}_{\tilde{X}}(-E)$,*

1. *If $-E \cdot E_i \geq 2K \cdot E_i$ for all $i = 1, \ldots, n$ then $Q\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$.*
2. *For any general linear combination f of a set of generators of Q we have $div(f \circ \pi) = \tilde{H} + E$, where $\tilde{H}$ is the proper transform of the cycle defined by f.*

In Section 2 we will recall some properties on minimal resolutions of singularities, in Section 3 we will introduce the definition of Nash numerical conditions, this is the central point of this work, in Section 4 we will prove

that Nash numerical conditions depend only on the intersection matrix of the exceptional set and we will establish a connection between the numerical Nash condition and the Gauss's method to squaring a quadratic form. A Nash matrix will be a matrix satisfying the Nash numerical conditions. In Section 5, we characterize some Nash matrices, in Section 6 we consider like star shaped graphs.

In this paper singularities will be considered only through the dual graph of the minimal resolution of singularities and the intersection matrix. The existence of a formal singularity with a given resolution graph (with negative definite intersection matrix) is guaranteed by Grauert [3]. Let us remark that provide equations corresponding to the dual graph of one singularity is in general an open problem.

§2. On the minimal resolution of singularities

Remark 1.

• *Since $\pi : \tilde{X} \to X$ is the minimal resolution of singularities of X we have $-K_{\tilde{X}} \cdot E_i \leq 0$ for any $i = 1, \ldots, n$. This fact will be very important in our proofs.* More precisely, for any irreducible component E_i of the exceptional divisor, we consider the adjunction formula for (eventually singular) curves

$$p(E_i) = \frac{E_i \cdot (E_i + K_{\tilde{X}})}{2} + 1$$

where $p(E_i)$ is the genus of E_i. Recall that $p(E_i) \geq 0$ and $p(E_i) = 0$ if and only if E_i is a curve of genus zero and self intersection equal to -1, which is impossible by Castelnuovo theorem since we are assuming that $\pi : \tilde{X} \to X$ is the minimal resolution of singularities of X. As a consequence $K_{\tilde{X}} \cdot E_i = 2(p(E_i) - 1) - E_i^2 \geq 0$ for any $i = 1, \ldots, n$.

• Since the graph of the resolution is connected we have that for any $1 \leq i < k \leq n$ the intersection number $E_i \cdot E_k \geq 0$ and for each index k there are at least one index i such that $E_i \cdot E_k > 0$.

It follows from the previous item that if $E = \sum_{k=1}^{n} n_k E_k$, $n_k \in \mathbb{N}$ is an exceptional divisor such that $E \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k \leq 0$, for all $k = 1, \ldots, n$, then E has full support, i.e. $n_k > 0$ for all $k = 1, \ldots, n$.

• If $E = \sum_{k=1}^{n} n_k E_k$ with $n_k \in \mathbb{N}^*$ for $k = 1, \ldots, n$, is an exceptional divisor such that $E \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k$, then for any $\alpha \in \mathbb{N}^*$ we have $(\alpha E) \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k$.

DEFINITION 1. Let (X, O) be a germ of normal surface singularity, $\pi : \tilde{X} \to X$ be the minimal resolution of singularities, $E_1, \ldots, E_n$ be the components of the exceptional divisor and $A = (a_{i,j})$ with $a_{i,j} = E_i \cdot E_j$, be the $n \times n$ symmetrical intersection matrix of the exceptional set of $\tilde{X}$. The dual graph Γ of the intersection matrix A is defined as follows:

- The vertices of the graph Γ are $E_1, \ldots, E_n$.
- For $i \neq j$ there is an edge between E_i, and E_j if and only if $a_{i,j} \neq 0$.

Remark 2. The graph Γ is connected and conversely by a theorem due to Grauert [3], given a $n \times n$ symmetrical negative definite matrix $A = (a_{i,j})$ with integer entries, such that $a_{i,i} < 0$ and $a_{i,j} \geq 0$ for any $i, j = 1, \ldots, n$, $i \neq j$, with a connected graph there exist a singularity with A as intersection matrix.

§3. Numerical Nash condition

DEFINITION 2. Let (X, O) be a germ of normal surface singularity, $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and $E_1, \ldots, E_n$ be the components of the exceptional divisor. Let $K_{\tilde{X}}$ the canonical divisor on $\tilde{X}$. We say that (X, O) satisfies numerical Nash condition for (i, j) if the following condition is fulfilled

$$(NN_{(i,j)}) \qquad \exists E = \sum_{k=1}^{n} n_k E_k, n_k \in \mathbb{N}^*$$

$$\text{with } n_i < n_j \text{ and } -E \cdot E_k \geq 2K_{\tilde{X}} \cdot E_k, \forall k = 1, \ldots, n.$$

We also say that (X, O) satisfies numerical Nash condition, (NN), if $(NN_{(i,j)})$ is true for all couples (i, j), with $i \neq j$.

As an immediate consequence of Proposition of Plenat [10] and my Theorem [8] we have:

COROLLARY 1. *With the above notations, if (X, O) satisfy numerical Nash condition for (i, j) then $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$. In particular if (NN) is true then the Nash problem on arcs has a positive answer.*

PROPOSITION 1. *With the notations as above. Let Γ be the dual graph of the intersection matrix of the exceptional set. If (NN) is true for Γ, then*

- (NN) *is true for any subgraph of* Γ.
- (NN) *is true by decreasing the self intersection numbers.*

Proof.

- Let us consider a subgraph G of Γ and let I be its support. Since (NN) is true for Γ, for any $i, j \in I$, $i \neq j$, there exist $E = \sum_{k=1}^{n} n_k E_k$, $n_k \in \mathbb{N}^*$ with $n_i < n_j$ such that $E \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k$, $\forall k = 1, \ldots, n$. It then follows that for any $k \in I$,

$$\left(\sum_{l \in I} n_l E_l \right) \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k - \sum_{l \notin I} n_l E_l \cdot E_k \leq -2K_{X'} \cdot E_k,$$

  where $K_{X'}$ is the canonical divisor of the minimal resolution singularity X', having G as dual graph of the exceptional set. Remark that $K_{X'} \cdot E_k = K_{\tilde{X}} \cdot E_k$.

- In order to prove the second assertion it will be enough to consider one index $k \in \{1, \ldots, n\}$ and the intersection matrix $A' = (a'_{i,j})$ defined by $a'_{i,j} = a_{i,j}$ if $(i,j) \neq (k,k)$ and $a'_{k,k} = a_{k,k} - 1$. Let remark that the matrix A' corresponds to a minimal resolution of some isolated singularity, $\pi' : \tilde{X}' \to X'$, call $E'_1, \ldots, E'_n$ the irreducible components of the exceptional set in $\tilde{X}'$ and we can assume that $p(E'_i) = p(E_i)$. Let $E = \sum_{l=1}^{n} n_l E_l$, $n_l \in \mathbb{N}^*$ with $n_i < n_j$ such that $E \cdot E_l \leq -2K_{\tilde{X}} \cdot E_l$, $\forall l = 1, \ldots, n$ and set $E' = \sum_{l=1}^{n} n_l E'_l$. By the Remark 1 we can assume that $n_l \geq 2$ for any $l = 1, \ldots, n$. It follows that

$$\begin{aligned} K_{\tilde{X}'} \cdot E'_i &= K_{\tilde{X}} \cdot E_i \quad \text{for } i \neq k \\ K_{\tilde{X}'} \cdot E'_k &= K_{\tilde{X}} \cdot E_k + 1 \\ E' \cdot E'_i &= E \cdot E_i \leq -2K_{\tilde{X}} \cdot E_i = -2K_{\tilde{X}} \cdot E_i \quad \text{for } i \neq k \\ E' \cdot E'_k &= E \cdot E_k - n_k \leq -2K_{\tilde{X}} \cdot E_k - n_k \\ &= -2K_{\tilde{X}'} \cdot E'_k - n_k + 2 \leq -2K_{\tilde{X}'} \cdot E'_k. \end{aligned}$$

  This completes the proof of the second assertion.

§4. Nash matrices, Gauss sequences

Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X, consider an exceptional effective divisor $E = x_1 E_1 + \cdots + x_n E_n$, then

$$E \cdot E_k = x_1 E_1 \cdot E_k + \cdots + x_n E_n \cdot E_k = x_1 a_{k,1} + \cdots + x_n a_{k,n}.$$

Set $^tX = (x_1, \ldots, x_n)$, $^tC = (-2K_{\tilde{X}} \cdot E_1, \ldots, -2K_{\tilde{X}} \cdot E_n)$ and $c_i = -2K_{\tilde{X}} \cdot E_i$, then

1. Corollary 1 can be translated into linear algebra:
   If the inequality: $AX \le C$ has a solution $(x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $x_i < x_j$, then $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$

2. The condition $(NN_{(i,j)})$ is equivalent to the condition:
   the inequality: $AX \le C$ has solutions $(x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $x_i < x_j$.

   Remark that since $\tilde{X}$ is the minimal resolution we have $-K_{\tilde{X}} \cdot E_i \le 0$ for any i. In what follows we allow the intersection matrix A to have rational terms, remark that after multiplication by a convenient integer it will correspond to a singularity.

   LEMMA 1. *Let (X, O) be a germ of a normal surface singularity, $\pi : \tilde{X} \to X$ be the minimal resolution of singularities. Assume that π has only two exceptional components E_1, E_2. Let $A = \begin{pmatrix} -a & c \\ c & -b \end{pmatrix}$ the intersection matrix of E_1, E_2. Then*

1. $c < a$ *if and only if $(NN_{(1,2)})$ is true*
2. $c < b$ *if and only if $(NN_{(2,1)})$ is true*
3. $c < \min\{a, b\}$ *if and only if (NN) is true.*

In particular since the quadratic form associated to the matrix A is negative definite, we have $c^2 < ab$, which implies that either $\bar{\mathcal{N}}_1 \not\subset \bar{\mathcal{N}}_2$ or $\bar{\mathcal{N}}_2 \not\subset \bar{\mathcal{N}}_1$.

   Proof. We are looking for solutions $(x, y) \in \mathbb{N}^*$ of the system:

$$(*) \qquad \begin{aligned} -ax + cy \le c_1 \le 0 \\ cx - by \le c_2 \le 0 \end{aligned}$$

let D_1 the line of equation $-ax + cy = c_1$ and D_2 the line with equation $cx - by = c_2$, since A is negative definite we have $c^2 < ab$, which implies $c/b < a/c$, so the relative positions of the lines D_1, D_2, and the set of solutions of the system $(*)$ are represented in figures below. Since these are the unique possible cases we are done.

(i) $c < a$ and $b \leq c$

(ii) $c < b$ but $a \leq c$

(iii) $c < a$ and $c < b$.
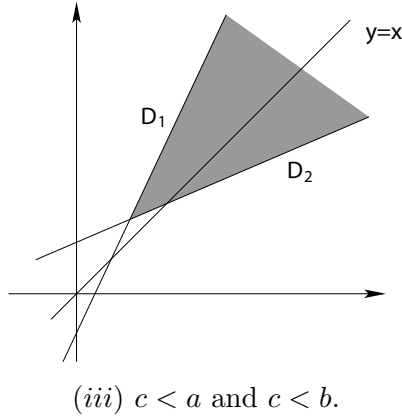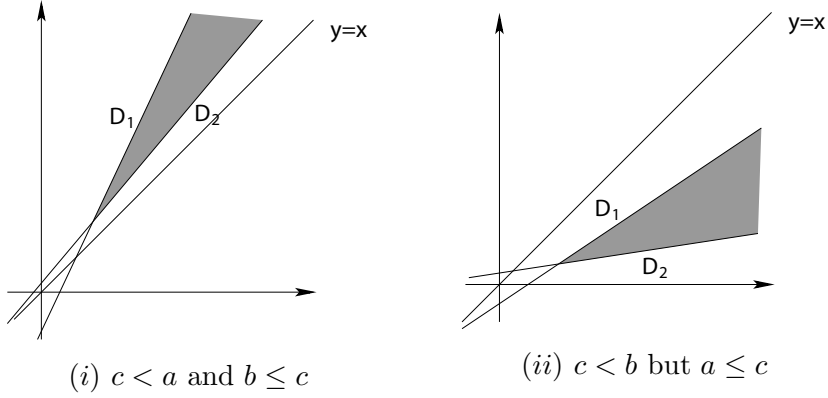
COROLLARY 2. *Let* $\pi : \tilde{X} \to X$ *be the minimal resolution of singularities and* $E_1, \ldots, E_n$ *be the components of the exceptional divisor, then for any* $i \neq j$ *either* $\bar{\mathcal{N}}_i \not\subset \bar{\mathcal{N}}_j$ *or* $\bar{\mathcal{N}}_j \not\subset \bar{\mathcal{N}}_i$. *In any case if* $i \neq j$ *then* $\bar{\mathcal{N}}_i \neq \bar{\mathcal{N}}_j$. *In particular after considering numerical Nash conditions, in order to check if Nash is true, we will be reduced to check at most the half of non inclusion conditions.*

We prove the Corollary by induction on n. For $n = 2$ it was proved in Lemma 1.

Assume $n \geq 3$, by changing the order in the set $E_1, \ldots, E_n$, we can suppose that $i = 1$ and $j = 2$, now pick k a positive integer such that $ka_{n,n} < c_n$ and put $-a_{n,n}x_n = \sum_{1 \leq i \leq n-1} a_{n,i}x_i - ka_{n,n}$ in our system $AX \leq C$, then

we have the inequality: $A'X \leq C'$ where $a'_{i,j} = a_{n,i}a_{n,j} - a_{i,j}a_{n,n}$ for all i, j and $c'_j = (-c_j + ka_{n,j})a_{n,n}$. By induction hypothesis there exist a vector $S = (s_1, \ldots, s_{n-1}) \in \mathbb{N}^n$ solution of the in-equation

$$A'X \leq C'$$

with $s_1 \neq s_2$. Let $s_n = a_{n,1}s_1 + \cdots + a_{n,n-1}s_{n-1} + k$, then a simple computation shows that the vector $T = (-a_{n,n}s_1, \ldots, -a_{n,n}s_{n-1}, s_n)$ is a solution of $AX \leq C$ for k large enough.

Remark that by construction the vector T has strict positive components.

Now we consider the sequences appearing in the proof of the last Corollary.

DEFINITION 3. Let: $a_{i,j}^{(n)} = a_{i,j}$ and for any $2 \leq l \leq n-1$ set $a_{i,j}^{(l)} = a_{i,j}^{(l+1)} - \frac{a_{l+1,i}^{(l+1)}a_{l+1,j}^{(l+1)}}{a_{l+1,l+1}^{(l+1)}}$, $1 \leq i,j \leq l$. Also for any $2 \leq l \leq n$ let $C(A)_i^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l)}$. We will also use the notation $C(A)_i = C(A)_i^{(n)}$.

LEMMA 2. *The matrices $A^{(l)} = (a_{i,j}^{(l)})$ appear naturally when we use the Gauss method to decompose the quadratic form associated to A into a sum of squares. In particular the matrix $A^{(l)}$ are negative definite. For this reason we will call the terms $a_{i,j}^{(l)}$ the Gauss sequence associated to A.*

Proof. The quadratic form associated to the matrix A is:

$$Q = \sum_{i=1}^{n} a_{i,i}x_i^2 + 2 \sum_{1 \leq i < j \leq n} a_{i,j}x_ix_j$$

we follow Gauss method to squaring a quadratic form:

$$Q = \sum_{i=1}^{n-1} a_{i,i}x_i^2 + 2 \sum_{1 \leq i < j \leq n-1} a_{i,j}x_ix_j + a_{n,n}x_n^2 + 2\sum_{i=1}^{n-1} a_{i,n}x_ix_n$$

but

$$a_{n,n}x_n^2 + 2\sum_{i=1}^{n-1} a_{i,n}x_ix_n$$

$$= a_{n,n}\left(x_n + \sum_{i=1}^{n-1} \frac{a_{i,n}}{a_{n,n}}x_i\right)^2 - \sum_{i=1}^{n-1} \frac{a_{i,n}^2}{a_{n,n}}x_i^2 - 2 \sum_{1 \leq i < j \leq n-1} \frac{a_{i,n}a_{j,n}}{a_{n,n}}x_ix_j$$

Hence

$$Q = a_{n,n}\Big(x_n + \sum_{i=1}^{n-1} \frac{a_{i,n}}{a_{n,n}} x_i\Big)^2 + \sum_{i=1}^{n-1}\Big(a_{i,i} - \frac{a_{i,n}^2}{a_{n,n}}\Big)x_i^2$$
$$+ 2 \sum_{1 \le i < j \le n-1}\Big(a_{i,j} - \frac{a_{i,n}a_{j,n}}{a_{n,n}}\Big)x_i x_j$$

and A is negative definite if and only if $a_{n,n} < 0$ and $A^{(n-1)}$ is negative definite.

>   *Remark* 3. 1. By multiplying by a convenient natural number the matrix A has integer coefficients and correspond to some singularities. Our definition does not depend on the topological type of the components of the exceptional divisor.
>
> 2. In what follows we will say that for $l \ge 3$ the operation $A^{(l)} \mapsto A^{(l-1)}$ is an *algebraic contraction* of the exceptional component E_l in the graph Γ_l corresponding to $A^{(l)}$, it is an algebraic operation and this algebraic contraction has no geometry meaning.

We have immediately from Lemma 1 and Corollary 2 that

PROPOSITION 2. *Let* $\pi : \tilde{X} \to X$ *be the minimal resolution of singularities and let* $A = (a_{i,j})$ *be the* $n \times n$ *symmetrical intersection matrix of the exceptional set of* X. *Then*

> 1. $a_{1,2}^{(2)} < -a_{1,1}^{(2)}$ *if and only if* $(NN_{(1,2)})$ *is true*
> 2. $a_{1,2}^{(2)} < -a_{2,2}^{(2)}$ *if and only if* $(NN_{(2,1)})$ *is true*
> 3. $a_{1,2}^{(2)} < \min\{-a_{1,1}^{(2)}, -a_{2,2}^{(2)}\}$ *if and only if both* $(NN_{(1,2)})$, $(NN_{(2,1)})$ *are true.*

THEOREM 1. *Let* $\pi : \tilde{X} \to X$ *be the minimal resolution of singularities, let* $A = (a_{i,j})$ *be the* $n \times n$ *symmetrical intersection matrix of the exceptional set of* X *and let* $C(A)_i^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l)}$. *For* $l \ge 1$, *we consider the property:*

$$(*_{l+1}) \qquad\qquad C(A)_i^{(l+1)} < 0, \quad for\ i = 1,\dots,l+1.$$

If $(*_{l+1})$ *is true for some* $l \ge 2$ *then* $(*_l)$ *is true.*

Let $\sigma \in S_n$ any permutation of $E_1, \ldots, E_n$, we denote by A^σ the corresponding intersection matrix obtained from A by permuting lines and columns. Then (NN) is true if and only if there exist a natural integer $l \geq 1$ such that

$$(*_{l+1}) \qquad C(A^\sigma)_i^{(l+1)} < 0, \quad for \ i = 1, \ldots, l+1, \ \forall \sigma \in S_n.$$

In particular we recover the following result from [12]: if $C(A)_i^{(n)} < 0$, for $i = 1, \ldots, n$ then the Nash map $\mathcal{N}$ is bijective.

Note that condition $(*_l)$ has a meaning only if $l \geq 2$.

Proof. Assume that $C(A)_i^{(l+1)} < 0$, for $i = 1, \ldots, l+1$, let $i \leq l$, by definition

$$C(A)_i^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l)} = \sum_{j=1}^{l} \left(a_{i,j}^{(l+1)} - \frac{a_{l+1,i}^{(l+1)} a_{l+1,j}^{(l+1)}}{a_{l+1,l+1}^{(l+1)}} \right)$$

$$C(A)_i^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l+1)} - \frac{a_{l+1,i}^{(l+1)}}{a_{l+1,l+1}^{(l+1)}} \sum_{j=1}^{l} a_{l+1,j}^{(l+1)},$$

$$C(A)_i^{(l)} = C(A)_i^{(l+1)} - \frac{a_{l+1,i}^{(l+1)}}{a_{l+1,l+1}^{(l+1)}} C(A)_{l+1}^{(l+1)} < 0.$$

The second assertion follows from Proposition 3. Remark that it is not necessary to consider all permutation of $E_1, \ldots, E_n$.

DEFINITION 4. Let $A = (a_{i,j})$ be the $n \times n$ symmetrical negative definite matrix with rational coefficients with $a_{i,i} < 0$, $a_{i,j} \geq 0$ for all i, j, $i \neq j$. We say that A is a Nash matrix if for any permutation σ of the set $\{1, \ldots, n\}$ $C(A^\sigma)_1^{(2)} < 0$, $C(A^\sigma)_2^{(2)} < 0$.

§5. Trees, cycles, generalized cycles

We look now for some necessary or sufficient conditions in order to have the condition (NN) true. For the moment we need to recall some notation on graphs.

DEFINITION 5. Let $A = (a_{i,j})$ be the $n \times n$ symmetrical negative definite matrix with rational coefficients with $a_{i,i} < 0$, $a_{i,j} \geq 0$ for all i, j,

$i \neq j$. Let Γ the dual graph associated to A. We say that E_i is a leaf of Γ if $a_{i,j} \neq 0$ for exactly one index $j \neq i$ i.e. E_i is connected to only one other vertex of Γ. A cycle of Γ is a subgraph $\mathcal{C}$ where every vertex is connected to exactly two others vertices of $\mathcal{C}$. A tree is a subgraph with no cycles. Finally a complete subgraph is a subset of Γ, where every two points are connected.

LEMMA 3. *Assume that for any point E_j of Γ, we have $C(A)_j \leq 0$.*

1. *For any $l \leq n$ and $j \leq l$ we have $C(A)_j^{(l)} \leq 0$*
2. *If $C(A)_i^{(l+1)} < 0$ then $C(A)_i^{(l)} < 0$*
3. *Let us consider a path $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$ in Γ, and $C(A)_{i_k} < 0$. After the algebraic contraction of $E_{i_k}, E_{i_{k-1}}, \ldots, E_{i_3}$ we will have $C(A)_{i_2}^{(2)} < 0$.*

Proof. The first two assertions follow immediately from the following formula, which is true for any $l \geq 2$, and $1 \leq i \leq l$:

$$C(A)_i^{(l)} = C(A)_i^{(l+1)} - \frac{a_{l+1,i}^{(l+1)}}{a_{l+1,l+1}^{(l+1)}} C(A)_{l+1}^{(l+1)} < 0.$$

We prove the third assertion by induction on k the length of the path, if $k = 2$, there is nothing to do. Now take any $k \geq 3$, then using the above formula we have that $C(A)_{i_{k-1}}^{(n-1)} < 0$, by the induction hypothesis we get $C(A)_{i_2}^{(n-k+1)} < 0$, so by the assertion 1 we are done.

THEOREM 2. *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X.*

1. *If (NN) is true then $C(A)_i < 0$ for any leaf E_i of Γ.*
2. *Assume that Γ is a tree and $C(A)_i \leq 0$ for any vertex E_i of Γ. Then $C(A)_i < 0$ for any leaf E_i of Γ if and only if (NN) is true.*
3. *In particular if the above conditions are satisfied the Nash map $\mathcal{N}$ is bijective.*

Proof.

1. Suppose that (NN) is true. Let E_i be a leaf of Γ, we can assume that $i = 1$ and E_2 is the unique vertex connected to E_1, by the algebraic contraction of all other vertices of Γ, we will have $a_{1,1}^{(2)} = a_{1,1}$, $a_{1,2}^{(2)} = a_{1,2}$. By Proposition 2 (or Theorem 1) we must have $C(A)_1 = C(A)_1^{(2)} < 0$. This concludes the proof of the first statement.

2. The necessary condition was proved before. The proof of the other implication is by induction on n. If $n = 2$ the hypothesis implies that (NN) is true by Lemma 1. So assume the case $n-1$ is solved and we prove the case n. Take any $i \neq j$. We have two cases

   1) Both E_i, E_j are leaves of Γ, then $C(A)_i^{(n)} < 0$, $C(A)_j^{(n)} < 0$, by the algebraic contraction of all vertices in Γ except E_i, E_j and applying Lemma 3, we get that $C(A)_i^{(2)} < 0$, $C(A)_j^{(2)} < 0$, and we are done.

   2) At most one of E_i, E_j is a leaf, then there exist a leaf E_k, different from E_i, E_j, so after changing the order of the exceptional components we can assume that $i = 1$, $j = 2$, $k = n$, let E_l be unique component connected to E_n. By the algebraic contraction of E_n, we get the matrix $A^{(n-1)} = (a_{i,j}^{(n-1)})$, with $a_{i,j}^{(n-1)} = a_{i,j}$ for any $(i,j) \neq (l,l)$ and $a_{l,l}^{(n-1)} = a_{l,l} - \frac{a_{l,n}^2}{a_{n,n}} < 0$. It follows that $C(A)_i^{(n-1)} = C(A)_i \leq 0$ for $i \neq l$ and $C(A)_l^{(n-1)} = C(A)_l - \left(\frac{a_{l,n}^2}{a_{n,n}} + a_{l,n}\right) < C(A)_l \leq 0$. Also the graph corresponding to the matrix $A^{(n-1)}$ is a tree, so by induction hypothesis $(NN_{(1,2)})$ and $(NN_{(2,1)})$ are true, and we are done.

Remark 4. Inside the class of rational singularities, rational minimal singularities are exactly those for which the graph satisfies the hypothesis of the above theorem. Note that Nash problem's on arcs for (rational) minimal singularities has a positive solution by the work of Ana Reguera [13], also C. Plenat [11] and Fernandez-Sanchez [2] gave different proofs. Our Theorem applies without any restriction on the topological type of the exceptional components and so extends to non rational singularities the mentioned results.

DEFINITION 6. We say that a subgraph G of Γ is a generalized cycle if any two vertices of G are connected by a cycle. Remark that a cycle or a complete graph are generalized cycles.

A generalized cycle is a leaf of Γ if at most one vertex of G is connected to one vertex of $\Gamma \setminus G$.

THEOREM 3. *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X.*

1. *Assume that the graph Γ of the exceptional set is a cycle, with $n \geq 3$, and $C(A)_i \leq 0$ for all i. Then (NN) is true if and only if $C(A)_i < 0$ for at least two exceptional components.*

2. *Suppose that Γ is a generalized cycle, $n \geq 3$ and $C(A)_i \leq 0$, for any vertex E_i. Then (NN) is true if and only if $C(A)_i < 0$, for at least two vertices.*

3. *In particular if these conditions are fulfilled the Nash map $\mathcal{N}$ is bijective.*
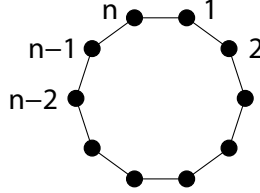
Proof.

1. Assume first that $C(A)_i < 0$ for at least two exceptional components. We prove that (NN) is true by induction on n.

   If $n = 3$, we make the algebraic contraction of the exceptional fiber E_3 and we get the matrix:

   $$A^{(2)} = \begin{pmatrix} a_{1,1} - \frac{a_{1,3}^2}{a_{3,3}} & a_{1,2} - \frac{a_{1,3}a_{2,3}}{a_{3,3}} \\ a_{1,2} - \frac{a_{1,3}a_{2,3}}{a_{3,3}} & a_{2,2} - \frac{a_{2,3}^2}{a_{3,3}} \end{pmatrix}$$

   It follows that $C(A)_1^{(2)} = C(A)_1 - \left(\frac{a_{1,3}}{a_{3,3}}(C(A)_3)\right) < 0$ and $C(A)_2^{(2)} = C(A)_2 - \left(\frac{a_{2,3}}{a_{3,3}}(C(A)_3)\right) < 0$ since by hypothesis two over the three numbers $C(A)_1$, $C(A)_2$, $C(A)_3$ are strictly negative. So the case $n = 3$ is over.

   Consider now the case $n \geq 4$. By the algebraic contraction of E_n, we get the matrix $A^{(n-1)} = (a_{i,j}^{(n-1)})$, with $a_{i,j}^{(n-1)} = a_{i,j}$ if $i, j \in$

$\{2, 3, \ldots, n-2\}$ and $a_{1,1}^{(n-1)} = a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}}$, $a_{1,n-1}^{(n-1)} = -\frac{a_{1,n}a_{n-1,n}}{a_{n,n}}$ and $a_{n-1,n-1}^{(n-1)} = a_{n-1,n-1} - \frac{a_{n-1,n}^2}{a_{n,n}}$.

It follows that $C(A)_i^{(n-1)} = C(A)_i \leq 0$ for $i \in \{2, 3, \ldots, n-2\}$, $C(A)_{n-1}^{(n-1)} = C(A)_{n-1} - \left(\frac{a_{n-1,n}}{a_{n,n}}(C(A)_n)\right)$ and $C(A)_1^{(n-1)} = C(A)_1 - \left(\frac{a_{1,n}}{a_{n,n}}(C(A)_n)\right)$. We have to consider three cases:

i) $C(A)_1 = C(A)_{n-1} = C(A)_n = 0$ then there are two indexes $i, j \in \{2, 3, \ldots, n-2\}$ such that $C(A)_i^{(n-1)} < 0$, $C(A)_j^{(n-1)} < 0$.

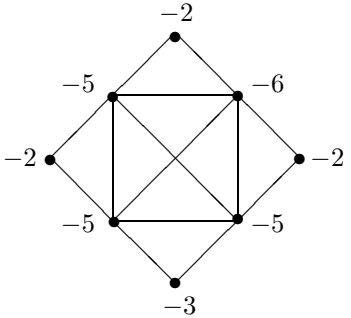ii) $C(A)_n < 0$, then $C(A)_{n-1}^{(n-1)}$, $C(A)_1^{(n-1)}$ are strictly negative.

iii) at least one of $C(A)_1 = 0$, $C(A)_{n-1}$ and $C(A)_n$ is zero, then either $C(A)_{n-1}^{(n-1)} < 0$ or $C(A)_1^{(n-1)} < 0$.

So the induction hypothesis is verified by $A^{(n-1)}$ and we are done.

Conversely, if (NN) is true and $C(A)_i < 0$ for at most one index i, take any index $j \neq i$, by the algebraic contraction of all other components E_k, $k \neq i, j$ we will have $C(A)_i^{(2)} = C(A)_i = 0$, $C(A)_j^{(2)} = C(A)_j < 0$, this is a contradiction by Proposition 3.

2. If Γ is a generalized cycle, then any two vertices are connected by a cycle and the statement follows from the first item.

EXAMPLE 1.   The following matrix and graph correspond to a generalized cycle, for which Nash's problem has an affirmative answer.

$$A = \begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -6 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -5 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -5 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -3 \end{pmatrix}$$

§6. Like star graphs

We can improve the above result in the some special situations:

THEOREM 4.   *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X. Assume that X has a polygon singularity, i.e. the graph of the exceptional set is a star with root E_n and all other vertices are leaves.*
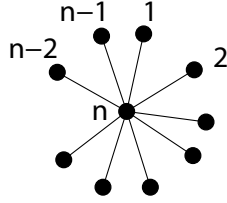
Then (NN) *is true if and only if we have the following conditions*:

- $\forall i = 1, \ldots, n-1 \quad a_{i,i} + a_{i,n} < 0$
- $\forall i, j = 1, \ldots, n-1 \quad a_{i,i} a_{j,j} \Delta_n + a_{j,n}(a_{i,i} a_{j,n} - a_{j,j} a_{i,n}) < 0$
- $\forall i = 1, \ldots, n-1 \quad \frac{a_{i,n}}{a_{i,i}}(a_{i,i} + a_{i,n}) + \Delta_n < 0$

where $\Delta_n = a_{n,n} - \sum_{i=1}^{n-1} \frac{a_{i,n}^2}{a_{i,i}}$. *We note that* A *is negative definite if and only if* $\Delta_n < 0$.

Proof. It is enough to compare any two leaves and any leaf with the root. So we consider the following order: $E_1, E_2, E_n, E_3, \ldots, E_{n-1}$. After applying the construction above we are reduced to the matrix

$$A^{(3)} = \begin{pmatrix} a_{1,1} & 0 & a_{1,n} \\ 0 & a_{2,2} & a_{2,n} \\ a_{1,n} & a_{2,n} & a_{n,n}^{(3)} \end{pmatrix}$$

where $a_{n,n}^{(3)} = a_{n,n} - \sum_{i=3}^{n-1} \frac{a_{i,n}^2}{a_{i,i}}$. We are reduced to consider two cases:

first case: Comparison of E_1, E_2 Again by the construction above we are reduced to the matrix:

$$A^{(2)} = \begin{pmatrix} a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}^{(3)}} & -\frac{a_{1,n} a_{2,n}}{a_{n,n}^{(3)}} \\ -\frac{a_{1,n} a_{2,n}}{a_{n,n}^{(3)}} & a_{2,2} - \frac{a_{2,n}^2}{a_{n,n}^{(3)}} \end{pmatrix}$$

So $(NN)_{(1,2)}$ and $(NN)_{(2,1)}$ are true if and only if:

$$a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}^{(3)}} - \frac{a_{1,n} a_{2,n}}{a_{n,n}^{(3)}} < 0$$

$$a_{2,2} - \frac{a_{2,n}^2}{a_{n,n}^{(3)}} - \frac{a_{1,n} a_{2,n}}{a_{n,n}^{(3)}} < 0$$

by simple computations these are equivalent to:

$$a_{i,i}a_{j,j}\Delta_n + a_{j,n}(a_{i,i}a_{j,n} - a_{j,j}a_{i,n}) < 0$$

for $\{i,j\} = \{1,2\}$.

Let us consider now the second case: Comparison of E_1, E_n

By the construction above we are reduced to the matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,n} \\ a_{1,n} & a_{n,n}^{(3)} - \frac{a_{2,n}^2}{a_{2,2}} \end{pmatrix}$$

So $(NN)_{(1,n)}$ and $(NN)_{(n,1)}$ are true if and only if: $a_{1,1} + a_{1,n} < 0$ and $a_{1,n} + a_{n,n}^{(3)} - \frac{a_{2,n}^2}{a_{2,2}} < 0$.

After simple computations these are equivalent to: $a_{1,1} + a_{1,n} < 0$ and $\frac{a_{1,n}}{a_{1,1}}(a_{1,1} + a_{1,n}) + \Delta_n < 0$ Since the choice of the leaves were arbitrary, we are done.

The next corollary follows immediately from the theorem.

COROLLARY 3.   *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X. Assume that X has a polygon singularity, the graph of the exceptional set is a star shaped with root E_n, and $a_{i,n} = 1$, $a_{i,i} = -2$, for $i = 1, \ldots, n-1$. Then the matrix A is negative definite if and only if $-a_{n,n} > \frac{n-1}{2}$ and (NN) is true if and only if $-a_{n,n} > \frac{n}{2}$. So if n is odd (NN) is always true, but if n is even it remains open the case $-a_{n,n} = \frac{n}{2}$. By the above theorem only the cases $(NN_{(n,i)})$ for $i = 1, \ldots, n-1$ are not true.*

EXAMPLE 2.   Our theorem cannot be applied to the following graph of a sandwich singularity where $a \geq 3$. In fact it follows from the theorem that only $(NN)_{(1,3)}$ is not true. Note that the Nash problem's on arcs for (rational) sandwich singularities has a positive solution by the work of Ana Reguera [14].

$$A = \begin{pmatrix} -a & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
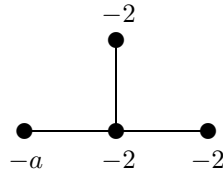
THEOREM 5.   *Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of X. Assume that the singularity is like a star, i.e. the graph of the exceptional set is a star with root E_0 having $s \geq 3$ branches.*
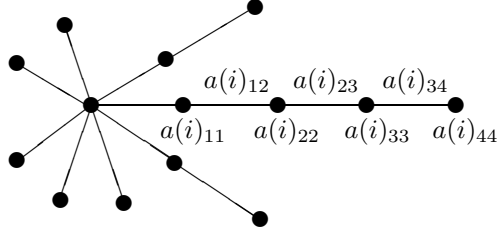
To any branch of the star we associate some continuous fractions expansions ($i = 1, \ldots, s$; $j \geq 1$):

$$q_i := a(i)_{1,1} - \cfrac{a(i)_{1,2}^2}{a(i)_{2,2} - \cfrac{a(i)_{2,3}^2}{a(i)_{3,3} - \cfrac{a(i)_{3,4}^2}{\ldots}}};$$

$$q_{i,j} := a(i)_{j,j} - \cfrac{a(i)_{j,j+1}^2}{a(i)_{j+1,j+1} - \cfrac{a(i)_{j+1,j+2}^2}{a(i)_{j+2,j+2} - \cfrac{a(i)_{j+2,j+3}^2}{\ldots}}}$$

Then (NN) is true if we have the following conditions:

- *for any leaf E_i we have $C(A)_i < 0$*
- *for any vertex E_i which is not the root $C(A)_i \leq 0$*
- *$\forall i, j = 1, \ldots, s$, $i \neq j$, $a_{0,0} + a_{0,i} + a_{0,j} - \sum_{k=1,\, k \neq i,j}^{s} \frac{a_{0,k}^2}{q_k} \leq 0$*

Let $\Delta_n = a_{0,0} - \sum_{i=1}^{s} \frac{a_{i,0}^2}{q_i}$. We note that A is negative definite if and only if $\Delta_n < 0$, and $q_{i,j} < 0$ $\forall i = 1, \ldots, s$; $j \geq 1$.

Proof.   Our first step consists to make the algebraic contraction of a whole branch G_k of the star Γ. We reorder the irreducible components of the exceptional set by letting E_n to be the leaf of the branch G_k, E_{n-1} be the unique vertex connected to E_n, E_{n-2} be the unique vertex connected

to E_{n-1} but distinct from E_n, and so on until we arrive to the root named always by E_0, the order in the other branches are arbitrary. We also denote $a_{0,k} := a(k)_{0,1}$. By the algebraic contraction of E_n we will get again a like star graph and a new matrix $A^{(n-1)} = (a_{i,j}^{(n-1)})$ given by

$$a_{i,j}^{(n-1)} = a_{i,j}^{(n)} - \frac{a_{n,i}^{(n)} a_{n,j}^{(n)}}{a_{n,n}^{(n)}},$$

regarding that our graph is a star we get:

$$a_{i,j}^{(n-1)} = a_{i,j} \ \ \text{if} \ \ (i,j) \neq (n-1, n-1) \ \ \text{and}$$

$$a_{n-1,n-1}^{(n-1)} = a_{n-1,n-1} - \frac{a_{n,n-1}^2}{a_{n,n}}.$$

Proceeding in this way we can make the algebraic contraction of all the vertices of G_k, then we will get again a like star graph and a new matrix $A' = (a'_{i,j})$ given by

$$a'_{i,j} = a_{i,j} \ \ \text{if} \ \ (i,j) \neq (0,0) \ \ \text{and}$$

$$a'_{0,0} = a_{0,0} - \frac{a_{0,k}^2}{q_k}.$$

Now we are ready to prove the claim: we need to compare any two elements in the graph Γ, these elements are in at most two branches G_α, G_β of the star, we make the algebraic contraction of $s - 2$ branches (indexed by a set I) of the star distinct from G_α, G_β and we get a new graph of type A_n and a new matrix $A'' = (a''_{i,j})$ given by

$$a''_{i,j} = a_{i,j} \ \ \text{if} \ \ (i,j) \neq (0,0) \ \ \text{and}$$

$$a''_{0,0} = a_{0,0} - \sum_{k \in I} \frac{a_{0,k}^2}{q_k},$$

the hypothesis of the theorem imply that this special tree satisfies the hypothesis of Theorem 2 and we are done.

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