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EVERY CURVE OF GENUS NOT GREATER THAN EIGHT LIES ON A $K3$ SURFACE

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Abstract. Let C be a smooth irreducible complete curve of genus $g \geq 2$ over an algebraically closed field of characteristic 0. An ample $K3$ extension of C is a $K3$ surface with at worst rational double points which contains C in the smooth locus as an ample divisor.

In this paper, we prove that all smooth curve of genera $2 \leq g \leq 8$ have ample $K3$ extensions. We use Bertini type lemmas and double coverings to construct ample $K3$ extensions.

§1. Introduction

Let C be a smooth irreducible complete curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0. An *ample* $K3$ *extension* of C is a $K3$ surface S with at worst rational double points which contains C in the smooth locus as an ample divisor. If C is contained in a smooth $K3$ surface, then we obtain an ample $K3$ extension by contracting all (-2)-curves disjoint from C.

The purpose of this paper is to show

Main Theorem. *All smooth curves of genera* $2 \leq g \leq 8$ *have ample* $K3$ *extensions. Moreover, they have smooth ample extensions except the following cases*;

- $g = 6, 7, 8$ *and* $K_C = 2D$ *where* D *is a* g^2_{g-1}, *or*
- $g = 8$ *and* $K_C = A + 2B$ *where* A *is a* g^1_4 *and* B *is a* g^1_5.

In these exceptional cases, the canonical model $C \subset \mathbb{P}^{g-1}$ is contained in a weighted projective variety. Rational double points come from the singularities of the weighted projective variety (Lemma 2.6).

Since the dimension of the moduli space of curves of genus g is $3g - 3$ and the dimension of the moduli space of pairs (S, C) of a $K3$ surface S

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and a curve $C \subset S$ of genus g is $19 + g$, general smooth curves have no ample $K3$ extensions for $g \geq 12$. For $g = 10$, by [M4], general curves have no ample $K3$ extensions. For $g = 11, 9$, by [MM] and [M4], general curves have ample $K3$ extensions, but special cases are still unknown.

In [ELMS], D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer studied curves of Clifford dimension r, genus $4r - 2$, degree $4r - 3$, and Clifford index $2r - 3$. They made an example of such a curve of Clifford dimension $r = 6$ which does not lie on any $K3$ surfaces. In [W], J. Wahl studied Gaussian map on a curve C, which is the map $\phi : \bigwedge^2 H^0(\omega_C) \to H^0(\omega_C{}^3)$, essentially defined by $f\,dz \wedge g\,dz \mapsto (fg' - f'g)\,dz^3$. And he showed that if ϕ is surjective then C does not lie on any $K3$ surface. An easiest example of a curve with surjective Gaussian map is a complete intersection of two quintic in $\mathbb{P}^3$.

In Section 2, we prepare some lemmas to construct ample $K3$ extensions, namely, double covering and Bertini type lemmas. In Section 3, we study hyperelliptic curves, trigonal curves, and bielliptic curves, and construct $K3$ extensions which preserve the hyperelliptic pencils, trigonal pencils, and 2:1-morphisms onto the elliptic curves respectively by these lemmas. In Section 4, we construct $K3$ extensions of remaining curves.

Notation and conventions. For a smooth variety X, we denote by K_X the canonical divisor class of X and by $\omega_X := \mathcal{O}_X(K_X)$ the canonical line bundle. A g^r_d on a curve is a line bundle $\mathcal{L}$ of degree d such that $h^0(\mathcal{L}) \geq r + 1$.

§2. How to make a $K3$ extension

2.1. $K3$ extension as a double cover

Let X be a scheme and $\mathcal{L}$ a line bundle over X. A global section $s \in H^0(X, \mathcal{L}^{-2})$ yields an algebra structure on $\mathcal{O}_X \oplus \mathcal{L}$. Then $\pi : Y = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L}) \to X$ is a double covering branched along $B = (s)_0$.

Lemma 2.1. *Let X be a smooth regular surface (i.e., smooth complete surface with $H^1(X, \mathcal{O}_X) = 0$). Let B be a smooth member of $|-2K_X|$. Then the double cover $\pi : Y \to X$ branched over B, obtained as above, is a smooth $K3$ surface.*

Proof. The double covering Y is obviously smooth, and has the irregularity

$$h^1(Y, \mathcal{O}_Y) = h^1(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X)) = h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X(K_X)) = 0$$

by our assumption. Since the canonical divisor class K_Y of Y is linearly equivalent to $\pi^*K_X + R$ where R is the ramification divisor class, and R is linearly equivalent to $\pi^*\mathcal{O}_X(-K_X)$ in this situation, we conclude that K_Y is linearly equivalent to zero. □

2.2. Bertini type lemmas for smooth extension

Let S be a surface in $\mathbb{P}^g$ and C a hyperplane section of S. Then we have a commutative diagram;

$$\begin{array}{cccl} & S & \subset & \mathbb{P}^g \\ & \cup & & \cup \quad \text{hyperplane section} \\ S\cap \mathbb{P}^{g-1} = & C & \subset & \mathbb{P}^{g-1}. \end{array}$$

Lemma 2.2. ([R, 3.3]) *Assume that $S \subset \mathbb{P}^g$ is a surface with at worst rational double points. Then the following conditions are equivalent*;

(i) *S is a $K3$ surface embedded by a very ample complete linear system.*
(ii) *Every smooth hyperplane section is a canonical curve of genus g.*
(iii) *One smooth hyperplane section is a canonical curve of genus g.*

According to this lemma, we only need to show that the extension S is smooth or S has at worst rational double points as its singularities for our main theorem. We shall often use Bertini's theorem which guarantees us the existence of smooth extensions; if Λ is a base point free linear system on a smooth variety X, then every general member of Λ is smooth ([GH, p. 137]). The same holds true under the weaker assumption that there exists a member which is smooth at p for every base point p of Λ.

Lemma 2.3. (Bertini type lemma for complete linear sections) *Let Λ be a linear system of dimension n on X. Assume that the base locus B of the system Λ is smooth of codimension $n+1$, i.e., B is a complete intersection of basis divisors of Λ, then general members of Λ are smooth.*

Proof. General members D of a linear system Λ are smooth away from the base loci. Since B is smooth complete intersection of D and n divisors of Λ, D is also smooth around B. □

Lemma 2.4. (Bertini type lemma for two divisors) *Let W be a smooth divisor and $\mathcal{L}$ a line bundle on X. Let $D \subset W$ be a smooth member of the linear system $|\mathcal{L}|_W|$. Assume that $H^1(X, \mathcal{L}(-W)) = 0$ and the linear system $|\mathcal{L}(-W)|$ is base point free. Then D has a smooth extension, i.e., there is a smooth divisor $\widetilde{D} \in |\mathcal{L}|$ on X which satisfies $\widetilde{D} \cap W = D$.*

Proof. Since $H^1(X, \mathcal{L}(-W)) = 0$, the restriction map

$$H^0(X, \mathcal{L}) \longrightarrow H^0(W, \mathcal{L}|_W)$$

is surjective, and therefore there is a divisor $\overline{D} \in |\mathcal{L}|$ such that $\overline{D} \cap W = D$.

Consider the linear subsystem

$$\Lambda = \langle \overline{D}, |\mathcal{L}(-W)| + W \rangle \subset |\mathcal{L}|$$

generated by $\overline{D}$ and the members of $|\mathcal{L}(-W)| + W$. Since $|\mathcal{L}(-D)|$ is base point free, the base locus of Λ is $\overline{D} \cap W = D$. By Bertini's theorem, there is a divisor $\widetilde{D} \in \Lambda$ which is smooth away from $D = \widetilde{D} \cap W$. Since $D = \widetilde{D} \cap W$ is smooth complete intersection, $\widetilde{D}$ is smooth around D, hence smooth everywhere. ∎

LEMMA 2.5. (Bertini type lemma for more divisors) *Let $D_1, \ldots, D_s$, and W be divisors on X. Assume that $C := W \cap D_1 \cap \cdots \cap D_s$ is a smooth complete intersection, and $D_i \cap Bs|D_i - W| = \emptyset$ for $i = 1, \ldots, s$. Then there exist divisors $\widetilde{D}_1, \ldots, \widetilde{D}_s$ such that $\widetilde{D}_i \sim D_i$ for $i = 1, \ldots, s$, $S := \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_s$ is smooth, and $S \cap W = C$.*

Proof. We prove the case $s = 2$. Induction goes for $s \geq 2$.

First, consider the linear system

$$\Lambda_1 = \langle D_1, |D_1 - W| + W \rangle \subset |D_1|$$

on X. Since $D_1 \cap Bs|D_1 - W| = \emptyset$, we have $Bs(\Lambda_1) = D_1 \cap W$. Let $\widetilde{D}_1$ be a general member of Λ_1, then $\widetilde{D}_1$ is smooth away from $D_1 \cap W = \widetilde{D}_1 \cap W$.

Next, consider the linear system

$$\Lambda_2 = (\langle D_2, |D_2 - W| + W \rangle)|_{\widetilde{D}_1} \subset |(D_2|_{\widetilde{D}_1})|$$

on $\widetilde{D}_1$. Since $D_2 \cap Bs|D_2 - W| = \emptyset$, we have $Bs(\Lambda_2) = \widetilde{D}_1 \cap D_2 \cap W = C$ which is a smooth complete intersection. Therefore a general member $D'_2 \in \Lambda_2$ satisfies $D'_2 \cap W = \widetilde{D_1} \cap D_2 \cap W = C$ and is smooth away from $\mathrm{Sing}(\widetilde{D}_1) \cup Bs(\Lambda_2) \subset (W \cap \widetilde{D}_1) \cup C$. Since D'_2 meets W only at C, D'_2 is smooth away from C.

It is clear, from the definition of Λ_2, that there exist an extension $\widetilde{D}_2 \in |D_2|$ of D'_2, i.e., $\widetilde{D}_2 \cap \widetilde{D}_1 = D'_2$. Since $S = \widetilde{D}_1 \cap \widetilde{D}_2 = D'_2$ is smooth away from $C = W \cap \widetilde{D}_1 \cap \widetilde{D}_2$, S is smooth everywhere. ∎

A weighted projective variety $X \subset \mathbb{P}(a_1 : a_2 : \cdots : a_n)$ is said to be *quasi-smooth* if its affine cone $\mathrm{Cone}(X) \subset \mathbb{A}(a_1 : a_2 : \cdots : a_n) = \mathbb{A}^n$ is smooth outside the vertex $0 \in \mathbb{A}^n$. If a weighted projective variety X is quasi-smooth, then X has at worst cyclic quotient singularities.

LEMMA 2.6. (Bertini type lemma for weighted projective varieties) *Let X be a quasi-smooth weighted projective variety. Assume that C is a smooth complete intersection of divisors in X, and satisfies the same assumptions as in Lemma* 2.3*,* 2.4*, or* 2.5*.*

Then there is an extension S of C which has at worst cyclic quotient singularities. Moreover, if C is smooth curve and X is Gorenstein, then the extension S has at worst rational double points.

Proof. Since C is smooth, its affine cone $\mathrm{Cone}(C)$ is smooth outside the vertex. By Bertini type lemmas, we can construct an extension $\mathrm{Cone}(S)$ of $\mathrm{Cone}(C)$, which is smooth outside the vertex. Therefore S has at worst cyclic quotient singularities.

If C is a curve and X is Gorenstein, then the extension S is a surface with at worst Gorenstein cyclic quotient singularities. Therefore these singularities are rational double points. ∎

§3. Curves with very special linear systems

The main tool in this section is the rational normal scrolls $\mathbb{F} = \mathbb{F}(a_1, \ldots, a_n)$. We denote by H (instead of M in [R]) the pull back of the hyperplane section divisor class by the natural projective morphism $\mathbb{F} \to \mathbb{P}^N$ ($N = \sum(a_i + 1) - 1$), and by L the fiber (class) of the projection $\mathbb{F} \to \mathbb{P}^1$. As in [R], we denote by F_i the i-th coordinate divisor $\{x_i = 0\}$, which is a divisor of class $H - a_i L$.

3.1. Hyperelliptic cases

Let C be a smooth hyperelliptic curve of genus g. Then the canonical divisor K_C defines a two-to-one map $\Phi_{|K_C|}$ from C onto a rational normal curve $\overline{C}$ of degree $g-1$ in $\mathbb{P}^{g-1}$. The morphism $\Phi_{|K_C|} : C \to \overline{C}$ ($\subset \mathbb{P}^{g-1}$) is branched over $2g+2$ points $P_1, \ldots, P_{2g+2}$. Since C is smooth, these points are distinct.

We consider a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{F} & \hookrightarrow & \mathbb{P}^g \\ & & \cup & & \cup \\ C & \xrightarrow{2:1} & \overline{C} & \hookrightarrow & \mathbb{P}^{g-1}, \end{array}$$

where $\mathbb{F}$ is the two-dimensional rational normal scroll of degree $g-1$ and $\overline{C}$ is embedded into $\mathbb{F}$ as a hyperplane section. The canonical divisor of $\mathbb{F}$ is $K_{\mathbb{F}} = -2H + (g-3)L$. We take

$$\begin{cases} \mathbb{F}\big(\frac{g-1}{2}, \frac{g-1}{2}\big) & \text{if } g \text{ is odd,} \\ \mathbb{F}\big(\frac{g}{2}, \frac{g}{2}-1\big) & \text{if } g \text{ is even.} \end{cases}$$

as $\mathbb{F}$.

Proposition 3.1. *If $2 \le g \le 9$, there is a smooth curve $B \in |-2K_{\mathbb{F}}|$ which passes through $P_1, \dots, P_{2g+2}$.*

Proof. Since $-2K_{\mathbb{F}} \sim 4H - 2(g-3)L$ and $\overline{C} \sim H$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{F}}(3H - 2(g-3)L) \longrightarrow \mathcal{O}_{\mathbb{F}}(-2_{\mathbb{F}}) \longrightarrow \mathcal{O}_{\overline{C}}(-2K_{\mathbb{F}}) \longrightarrow 0.$$

Since the degree of $\mathcal{O}_{\overline{C}}(-2K_{\mathbb{F}})$ is

$$\begin{aligned}(4H - 2(g-3)L)H &= 4H^2 - 2(g-3)HL \\ &= 4(g-1) - 2(g-3) = 2g+2\end{aligned}$$

on $\overline{C} \cong \mathbb{P}^1$, we have $\mathcal{O}_{\overline{C}}(-2K_{\mathbb{F}}) \cong \mathcal{O}_{\mathbb{P}^1}(2g+2)$ and $P_1 + \cdots + P_{2g+2}$ is a smooth member of the system $|\mathcal{O}_{\overline{C}}(-2K_{\mathbb{F}})|$.

If g is odd, we have

$$\begin{aligned}&H^1(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(3H - 2(g-3)L)) \\ &\quad = H^1(\mathbb{P}^1, (\mathrm{Sym}^3(\mathcal{O}_{\mathbb{P}^1}(\tfrac{g-1}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(\tfrac{g-1}{2})))(-2(g-3))) \\ &\quad = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\tfrac{9-g}{2})^{\oplus 4}),\end{aligned}$$

and this vanishes for $g \le 11$. Moreover, since

$$3H - 2(g-3)L = 3(H - \tfrac{g-1}{2}L) + (\tfrac{9-g}{2})L,$$

the linear system $|3H - 2(g-3)L|$ is base point free for $g \le 9$. Therefore there is a smooth extension $B \in |-2K_{\mathbb{F}}|$ of $P_1 + \cdots + P_{2g+2} \in |-2K_{\mathbb{F}}|_{\overline{C}}|$ by Lemma 2.4.

If g is even, since

$$\begin{aligned}&\pi_* \mathcal{O}_{\mathbb{F}}(3H - 2(g-3)L) \cong \\ &\quad \mathcal{O}_{\mathbb{P}^1}(6 - \tfrac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(5 - \tfrac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(4 - \tfrac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(3 - \tfrac{g}{2}),\end{aligned}$$

$H^1(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(3H - 2(g-3)L))$ vanishes for $g \leq 8$, and the linear system $|3H - 2(g-3)L|$ is base point free for $g \leq 6$. Therefore, by Lemma 2.4, there is a smooth extension $B \in |-2K_{\mathbb{F}}|$ for $g = 2, 4, 6$.

If $g = 8$, the system $|3H - 2(g-3)L| = |3H - 10L|$ has $F_1 \sim H - 4L$ as its base component, and the system $|3H - 10L - F_1| = |2(H - 3L)|$ is base point free. We may assume that P does not intersect F_1, since there is an action of $PGL(1)$ on $\overline{C} \cong \mathbb{P}^1$. Let $B \subset \mathbb{F}$ be an extension of the 18 branch points $P = P_1 + \cdots + P_{18} \subset \overline{C}$ such that $F_1 \not\subset B$. We now consider the linear system

$$\begin{aligned}\Lambda &= \langle B, |3H - 10L| + \overline{C} \rangle \\ &= \langle B, |2(H - 3L)| + F_1 + \overline{C} \rangle.\end{aligned}$$

By Lemma 2.4, we can choose B so general that B is smooth outside $B \cap F_1$. Since $F_1 \cong \mathbb{P}^1$ is smooth, general members of Λ are smooth at $B \cap F_1$. Hence general members of Λ are smooth everywhere. ∎

3.2. Trigonal cases

Let C be a smooth non-hyperelliptic trigonal curve of genus $g \geq 5$. Then C is contained in a 2-dimensional rational normal scroll $\mathbb{F} = \mathbb{F}(a_1, a_2)$ of degree $a_1 + a_2 = g - 2$, and C is a divisor linearly equivalent to $3H - (g-4)L$. By [S], we have a bound

$$\frac{2g-2}{3} \geq a_1 \geq a_2 \geq \frac{g-4}{3}.$$

If $g = 5$, C is contained in $\mathbb{F} = \mathbb{F}(2,1)$ and C is a divisor of class $3H - L$. There is a commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\mathbb{F}} := \mathbb{F}(1,1,1) & \overset{\widetilde{\varphi}}{\hookrightarrow} & \mathbb{P}^5 \\ & & \cup & & \cup \\ C & \subset & \mathbb{F} := \mathbb{F}(2,1) & \overset{\varphi}{\hookrightarrow} & \mathbb{P}^4, \end{array}$$

and $\mathbb{F}$ is a divisor linearly equivalent to the hyperplane section $\widetilde{H}$ on $\widetilde{\mathbb{F}}$. Since $2\widetilde{H} - \widetilde{L} = 2(\widetilde{H} - \widetilde{L}) + \widetilde{L}$, the system $|2\widetilde{H} - \widetilde{L}|$ is base point free. We have

$$\begin{aligned}H^1(\widetilde{\mathbb{F}}, \mathcal{O}_{\widetilde{\mathbb{F}}}(2\widetilde{H} - \widetilde{L})) &= H^1(\mathbb{P}^1, \mathrm{Sym}^2(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})(-1)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 6}) = 0,\end{aligned}$$

and therefore by Lemma 2.4, there is a smooth surface S of class $3\widetilde{H}-\widetilde{L}$ in $\widetilde{\mathbb{F}}$. Thus C has a smooth $K3$ extension.

For a smooth trigonal curve of genus g, what we have to do is;

(1) classify the type (a_1,a_2) of $\mathbb{F}$ and find a type (b_1,b_2,b_3) of $\widetilde{\mathbb{F}}$ suitable for extension,
(2) check the vanishing of $H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(4-g))$, and
(3) check the freeness of the system $|2\widetilde{H}-(g-4)\widetilde{L}|$.

where $\widetilde{\mathcal{E}}=\mathcal{O}_{\mathbb{P}^1}(b_1)\oplus\mathcal{O}_{\mathbb{P}^1}(b_2)\oplus\mathcal{O}_{\mathbb{P}^1}(b_3)$.

The table below is the answer to (1). The condition (2) holds for $5\leq g\leq 9$, and (3) holds for $g=5,6,8$.

Table 1: trigonal curves

genus	$\mathbb{F}$	$\widetilde{\mathbb{F}}$	base locus	vanishing of H^1
5	(2, 1)	(1, 1, 1)	$\emptyset$	$H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-1))=0$
6	(3, 1) (2, 2)	(2, 1, 1)	$\emptyset$	$H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-2))=0$
7	(4, 1) (3, 2)	(2, 2, 1)	$F_1\cap F_2$	$H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-3))=0$
8	(4, 2) (3, 3)	(2, 2, 2)	$\emptyset$	$H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-4))=0$
9	(5, 2) (4, 3)	(3, 2, 2) (3, 3, 1)	F_1 $F_1\cap F_2$	$H^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-5))=0$
10	(6, 2) (5, 3) (4, 4)	(4, 2, 2) (3, 3, 2) (4, 3, 1)	F_1 $F_1\cap F_2$ $F_1\cap F_2$	$h^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-6))=1$ $h^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-6))=0$ $h^1(\mathbb{P}^1,(\mathrm{Sym}^2\widetilde{\mathcal{E}})(-6))=1$

For $g=7$, since $H^1(\widetilde{\mathbb{F}},\mathcal{O}_{\widetilde{\mathbb{F}}}(2\widetilde{H}-3\widetilde{L}))=0$, there is an extension $S'\in|3\widetilde{H}-3\widetilde{L}|$ of C. The linear pencil

$$\Lambda=\langle S',|2\widetilde{H}-3\widetilde{L}|+\mathbb{F}\rangle$$

has the base locus $Bs\Lambda=(S'\cap\mathbb{F})\cup(S'\cap Bs|2\widetilde{H}-3\widetilde{L}|)=C\cup(S'\cap F_1\cap F_2)$.

We can choose the linear embedding $\mathbb{F}\subset\mathbb{F}(2,2,1)$ so that C does not contain $F_1\cap F_2\cap\mathbb{F}$. Therefore S' does not contain $F_1\cap F_2\cong\mathbb{P}^1$. Since S'

and $F_1 \cap F_2$ have the intersection number

$$\begin{aligned}(S')(F_1)(F_2) &= (3\widetilde{H} - 3\widetilde{L})(\widetilde{H} - 2\widetilde{L})^2 \\ &= 3\widetilde{H}^3 - 15\widetilde{H}^2\widetilde{L} \\ &= 3 \cdot 5 - 15 \cdot 1 = 0,\end{aligned}$$

we conclude that $S' \cap F_1 \cap F_2$ is empty. Hence a general member S of Λ is smooth by Lemma 2.4. Thus C has a smooth $K3$ extension S.

3.3. Bielliptic cases

Let $C \subset \mathbb{P}^{g-1}$ be a smooth bielliptic canonical curve of genus g. By definition, there is a two-to-one morphism $f : C \to E$ from C onto an elliptic curve E. For any point p in E, set $f^*(p) = q_1 + q_2$, and define the line l_p in $\mathbb{P}^{g-1}$ as follows;

$$l_p = \begin{cases} \text{the line passing through } q_1 \text{ and } q_2 & \text{if } q_1 \neq q_2, \\ \text{the tangent line to } C \text{ at } q_1 & \text{if } q_1 = q_2. \end{cases}$$

Let p, p' be points in E and set $f^*(p) = q_1 + q_2$ and $f^*(p') = {q'}_1 + {q'}_2$. Then

$$h^0(C, \mathcal{O}_C(q_1 + q_2 + {q'}_1 + {q'}_2)) = h^0(E, \mathcal{O}_E(p + p')) = 2,$$

and therefore q_1, q_2, ${q'}_1$, and ${q'}_2$ are all lie in a 2-plane by the geometric version of Riemann-Roch theorem ([ACGH]). Since C is non-degenerate, this implies that all the lines l_p's pass through a common point $p \in \mathbb{P}^{g-1} \smallsetminus C$. The projection from p gives a two-to-one map $\pi_p : C \to E_{g-1}$ from C onto an elliptic curve $E_{g-1} \subset \mathbb{P}^{g-2}$ of degree

$$\deg E_{g-1} = \frac{1}{2} \deg C = g - 1.$$

Every elliptic curves $E := E_{g-1}$ of degree $g-1$ in $\mathbb{P}^{g-2}$, where $5 \le g-1 \le 8$, is smoothly extended to del Pezzo surfaces $S := S_{g-1}$ of degree $g - 1$ in $\mathbb{P}^{g-1}$. The extension S is the blowing-up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $9 - (g - 1)$ points, and the elliptic curve E is the strict transform of a nonsingular cubic curve which passes through all the center of the blowing-up.

Let $B = B_1 + \cdots + B_{2g-2}$ be the branch locus of $\pi_p : C \to E$, and $R = R_1 + \cdots + R_{2g-2}$ be the ramification locus. Then $K_C \sim {\pi_p}^*(K_E) + R \sim R$ since E is elliptic. We distinguish the ambient spaces $\mathbb{P}^{g-1}$ of C and S, and denote them by $\mathbb{P}_1^{g-1}$ and $\mathbb{P}_2^{g-1}$ respectively. Let H_i $(i = 1, 2)$ be the hyperplane divisor classes of $\mathbb{P}_i^{g-1}$. Then $H_1|_C = K_C \sim R$ and hence

$$2H_2|_E \sim \pi_{p_*} H_1|_C \sim \pi_{p_*} R \sim B.$$

On the other hand, we have $H_2|_E \sim -K_S|_E$, thus we conclude that

$$B \sim (-2K_S)|_E.$$

PROPOSITION 3.2. *There is a smooth curve* $X \in |-2K_S|$ *on* S *which passes through* $B_1, \ldots, B_{2g-2}$.

Proof. Let $h \in \mathrm{Pic}(S)$ be the pull-back of a line of $\mathbb{P}^2$ and $e = e_1 + \cdots + e_{10-g}$ be the sum of all the exceptional divisors. Since $K_S \sim -3h + e$ and $E \sim 3h - e \sim -K_S$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(-K_S) \longrightarrow \mathcal{O}_S(-2K_S) \longrightarrow \mathcal{O}_E(-2K_S) \longrightarrow 0.$$

Since $-K_S \sim H_2|_S$, the system $|-K_S| = |\mathcal{O}_S(H_2)| = |\mathcal{O}_S(1)|$ is base point free and $H^1(\mathcal{O}_S(-K_S)) = H^1(\mathcal{O}_S(1))$ vanishes. Therefore, by Lemma 2.5, $B \in |(-2K_S)|_E|$ extends to a smooth curve $X \in |-2K_S|$. ∎

§4. Curves without very special linear systems

4.1. Genus ≤ 5

Every curve of genus 2 is hyperelliptic, so we have done before. Every non-hyperelliptic curve of genus 3 is a plane quartic, every non-hyperelliptic curve of genus 4 is a complete intersection of hypersurfaces of degree three and four in $\mathbb{P}^3$, and every non-hyperelliptic non-trigonal curve of genus 5 is a complete intersection three quadric hypersurfaces. Hence they are $K3$ by Lemma 2.5.

4.2. Genus 6

Let C be a smooth non-hyperelliptic, non-trigonal, non-bielliptic canonical curve of genus 6. There are two cases remaining;

1. C is not plane quintic, and
2. C is smooth plane quintic.

Case 1. In this case, by [M2], there is a commutative diagram

$$\begin{array}{ccccc} & & G = \mathrm{Grass}(5,2) & \subset & \mathbb{P}^9 \\ & & \cup & & \cup \\ C & \subset & S_5 = G \cap \mathbb{P}^5 & \subset & \mathbb{P}^5, \end{array}$$

where S_5 is a quintic del Pezzo surface and C is a hyperquadric section of S_5.

Let H_1, H_2, H_3, H_4 be the hyperplanes and Q the hyperquadric in $\mathbb{P}^9$ such that $C = G \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap Q$. Then the systems $|(H_i - H_1)|_G| = |\mathcal{O}_G|$ and $|(Q - H_1)|_G| = |\mathcal{O}_G(1)|$ are base point free and $H^1(\mathcal{O}_G) = H^1(\mathcal{O}_G(1)) = 0$. Therefore there are extensions $\widetilde{H}_2$, $\widetilde{H}_3$, $\widetilde{H}_4$ and $\widetilde{Q}$ such that $S := G \cap \widetilde{H}_2 \cap \widetilde{H}_3 \cap \widetilde{H}_4 \cap \widetilde{Q}$ is a smooth surface. Thus C has a smooth $K3$ extension.

Case 2. If C has a g^2_5, then there is an isomorphism from C onto a smooth plane quintic $C_5 = \{f(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$, and the canonical model is the image of C_5 under the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Let $L = \{l(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$ be a line which meets C_5 transversally at 5 distinct points. Let $S \to \mathbb{P}^2$ be the blowing-up at $L \cap C_5$, and $\overline{L}$ and $\overline{C_5}$ be the strict transform of L and C_5 respectively. Then $\overline{L} + \overline{C_5}$ is a smooth member of $|-2K_S|$, and therefore the double covering $X \to S$ is the smooth $K3$ surface which contains a curve isomorphic to C.

Remark. The pull back of L is a (-2)-curve on the smooth $K3$ surface X. Collapsing this and we get a singular ample $K3$ extension $\widetilde{X} = \{l(x)y^2 + f_5(x) = 0\}$ in the weighted projective space $\mathbb{P}(1:1:1:2)$.

4.3. Genus 7

Let C be a smooth non-hyperelliptic non-trigonal non-bielliptic curve of genus 7. There are three cases remaining;

1. C has a g^1_4 but no g^2_6,
2. C has a g^2_6 but is not bielliptic.
3. C is non-tetragonal (i.e., C has no g^1_4's)

For Case 3, our main theorem is immediate from the Bertini type lemma 2.3 and the Mukai linear section theorem.

Theorem 4.1. ([M3]) *A curve C of genus 7 is a transversal linear section of the 10-dimensional orthogonal Grassmannian $X \subset \mathbb{P}^{15}$ if and only if C is not tetragonal.*

Case 1. Let α be a g^1_4 and $\beta := \omega_C \alpha^{-1}$ its Serre adjoint. Then β is a g^3_8 by the Riemann-Roch theorem. Since C has no g^2_6 the morphism $\Phi_{|\beta|} : C \to \mathbb{P}^3 = \mathbb{P}^* H^0(\beta)$ is an embedding and the multiplication map

$$\mu : H^0(\alpha) \otimes H^0(\beta) \longrightarrow H^0(\omega_C)$$

is surjective by [M3]. Hence we have a linear embedding

$$\mu^* : \mathbb{P}^6 = \mathbb{P}^*(H^0(\omega_C)) \longrightarrow \mathbb{P}^*(H^0(\alpha) \otimes H^0(\beta))$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^3 & \xrightarrow{\text{Segre}} & \mathbb{P}^7 \\ {\scriptstyle \Phi_{|\alpha|}\times\Phi_{|\beta|}}\uparrow & & \uparrow {\scriptstyle \mu^*} \\ C & \xrightarrow{\text{canonical}} & \mathbb{P}^6. \end{array}$$

By [M3], C is a complete intersection of divisors of bidegrees $(1,1)$, $(1,2)$ and $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. Let $W = (\mathbb{P}^1 \times \mathbb{P}^3) \cap \mu^*(\mathbb{P}^6)$ be the divisor of bidegree $(1,1)$ and D_1, D_2 the divisors of degree $(1,2)$ such that $C = W \cap D_1 \cap D_2$. Since $|D_i - W| = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(0,1)|$ is base point free for $i = 1,2$ and $H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(0,1)) = 0$, by Lemma 2.5, we have extensions $\widetilde{D}_1$ and $\widetilde{D}_2$ of D_1 and D_2 respectively such that $S = \widetilde{D}_1 \cap \widetilde{D}_2$ is a smooth surface. Thus C has a $K3$ extension.

Case 2. Let α be a g^2_6 and $\beta = \omega_C\alpha^{-1}$ its Serre adjoint. Then β is also a g^2_6 by the Riemann-Roch theorem.

If α is not isomorphic to β, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\text{Segre}} & \mathbb{P}^8 \\ {\scriptstyle \Phi_{|\alpha|}\times\Phi_{|\beta|}}\uparrow & & \uparrow {\scriptstyle \mu^*} \\ C & \xrightarrow{\text{canonical}} & \mathbb{P}^6. \end{array}$$

By [M3], all morphisms in the diagram are embeddings, and C is a complete intersection of divisors of bidegrees $(1,1)$, $(1,1)$ and $(2,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$. Let H_1 and H_2 be divisors of bidegree $(1,1)$ and D a divisor of bidegree $(2,2)$ such that $C = H_1\cap H_2\cap D$. Then the systems $|H_2-H_1| = |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}|$ and $|D-H_1| = |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)|$ are base point free and $H^1(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}) = H^1(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 0$. Therefore, by Lemma 2.5, we have extensions $\widetilde{H}_2$ and $\widetilde{D}$ such that $S := \widetilde{H}_2 \cap \widetilde{D}$ is a smooth $K3$ extension of C.

If α is isomorphic to β, then by [M3] the canonical embedding $C \hookrightarrow \mathbb{P}^6$ factors through the weighted projective space $\mathbb{P}(1:1:1:2)$, and C is a complete intersection of two divisors D_3 and D_4 in $\mathbb{P}(1:1:1:2)$ of degree 3 and 4 respectively. By Lemma 2.6, we can extend these divisors to $\widetilde{D}_3$ and $\widetilde{D}_4$ in $\mathbb{P}(1:1:1:2:2)$ of degree 3 and 4 such that $S = \widetilde{D}_3 \cap \widetilde{D}_4$ has at worst cyclic quotient singularities. These singularities are Gorenstein since $\mathbb{P}(1:1:1:2:2)$ is so. Thus S has only rational double points as its singularities and S is an ample $K3$ extension of C.

4.4. Genus 8

Let C be a non-hyperelliptic, non-trigonal, non-bielliptic smooth curve of genus 8. We have one of the following;

1. C has a g^1_4 but has no g^2_6,
2. C has a g^2_6 but is not bielliptic,

3-1. C has a g^2_7 α such that $\alpha^2 \not\cong \omega_C$, but C has no g^1_4,

3-2. C has a g^2_7 α such that $\alpha^2 \cong \omega_C$, but C has no g^1_4, or

4. C has no g^2_7.

For Case 4, it is immediate from Bertini type lemma 2.3 and the Mukai linear section theorem.

Theorem 4.2. ([M2]) *A curve C of genus 8 is a transversal linear section of the 8-dimensional Grassmannian variety $Gr(2,6) \subset \mathbb{P}^{14}$ if and only if it has no g^2_7*

Case 1. In this case we have

Theorem 4.3. ([M1], [MI]) *The canonical curve C is the complete intersection of four divisors in $\mathbb{P}^1 \times \mathbb{P}^4$ of bidegrees $(1,1)$, $(1,1)$, $(1,2)$ and $(0,2)$.*

Let X be the unique irreducible divisor of bidegree $(0,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ which contains C. Let D'_1, D'_2, and E' be the divisors on X of bidegrees $(1,1)$, $(1,1)$ and $(1,2)$ respectively, such that $C = D'_1 \cap D'_2 \cap E'$ in X. Since $|E' - D'_2|$ and $|D'_1 - D'_2|$ are base point free linear systems and since $H^1(\mathcal{O}_X(D'_1 - D'_2)) = 0$, there are divisors D'_0 and E'_0 of bidegrees $(1,1)$ and $(1,2)$ such that $S = D'_0 \cap E'_0$ is smooth away from the singular locus $\mathrm{Sing}(X)$ of X.

If X is $\mathbb{P}^1 \times \mathbb{P}(1:1:2:2)$, then $\dim \mathrm{Sing}(X) = 2$ and we can choose D'_0 and E'_0 so general that $S = D'_0 \cap E'_0$ has at worst ordinally double points as its singularities.

If X is $\mathbb{P}^1 \times \mathrm{Cone}(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3)$ or $\mathbb{P}^1 \times$ (smooth quadric), then $\dim \mathrm{Sing}(X) \leq 1$ and therefore a general intersection $S = D'_0 \cap E'_0$ does not meet $\mathrm{Sing}(X)$. Hence S is smooth.

Case 2. By [M1], the canonical curve C is the complete intersection of two divisors in X of classes $|-K_X|$ and $|-\frac{1}{2}K_X|$, where X is a blowing-up of $\mathbb{P}^3$ at a one point. Then $|-\frac{1}{2}K_X|$ is very ample and therefore C is a hyperplane section of D. Since $|-\frac{1}{2}K_X| = |2h - e|$ is base point free, C has a smooth extension $\widetilde{D} \in |-K_X|$ by Lemma 2.5.

Case 3. Let α be a g^2_7 and $\beta = \omega_C \alpha^{-1}$ its Serre adjoint. By the Riemann-Roch theorem, β is also a g^2_7.

Case 3-1. If α is not isomorphic to β, then by [MI], the canonical curve C is the complete intersection of three divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegrees $(1,1)$, $(1,2)$ and $(2,1)$.

Let $W = (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^7$ be the unique divisor of bidegree $(1,1)$, and D_1, D_2 divisors of bidegrees $(1,2)$ and $(2,1)$ respectively such that $C = W \cap D_1 \cap D_2$. Then $|D_i - W|$ is base point free, $H^0(D_i - W) \neq 0$, and $H^1(D_1 - D_2) = 0$. Therefore, by the Lemma 2.5, there are divisors $\widetilde{D}_1$, $\widetilde{D}_2$ of bidegrees $(1,2)$ and $(2,1)$ such that $S := \widetilde{D}_1 \cap \widetilde{D}_2$ is smooth and $\widetilde{D}_1 \cap \widetilde{D}_2 \cap W = C$. Thus S is a smooth $K3$ extension of C.

Case 3-2. If α is isomorphic to β, then the canonical embedding factors through a weighted projective space

$$\mathbb{P}(1:1:1:2:2) = \mathbb{P}(1^3 : 2^2) = \operatorname{Proj} k[x_0, x_1, x_2, y_0, y_2],$$

where $\{x_0, x_1, x_2\}$ is a basis of $H^0(\alpha)$ and $\{y_0, y_1, \operatorname{Sym}^2(x)\}$ that of $H^0(\alpha^2) = H^0(\omega_C)$.

$$C \hookrightarrow \mathbb{P}(1^3 : 2^2) \hookrightarrow \mathbb{P}(2^6 : 2^2) \cong \mathbb{P}^7 = \mathbb{P}^* H^0(\omega_C).$$

THEOREM 4.4. ([MI]) *The canonical model C is the complete linear section of the weighted Grassmann $G := Gr(2, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})) \subset \mathbb{P}(1^3 : 2^6 : 3^1)$,*

$$[C \subset \mathbb{P}(1^3 : 2^2)] = [G \subset \mathbb{P}(1^3 : 2^6 : 3^1)] \cap \mathbb{P}(1^3 : 2^2).$$

Since C is smooth, its affine cone

$$\operatorname{Cone}(C) = \operatorname{Cone}(G) \cap \mathbb{A}(1:1:1:2:2) \subset \mathbb{A}(1^3 : 2^6 : 3^1),$$

is smooth away from the vertex. By the Bertini type lemma 2.6, there is a general 5-dimensional plane $\mathbb{P}(1:1:1:2:2:2)$ containing $\mathbb{P}(1:1:1:2:2)$ such that $S := G \cap \mathbb{P}(1:1:1:2:2)$ has at worst rational double points. Therefore C has an ample $K3$ extension.

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