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ON THE KOHNEN-ZAGIER FORMULA IN THE CASE OF '4 × GENERAL ODD' LEVEL

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Abstract. We study the Fourier coefficients of cusp forms of half integral weight and generalize the Kohnen-Zagier formula to the case of '4×general odd' level by using results of Ueda. As an application, we obtain a generalization of the result of Luo-Ramakrishnan [11] to the case of arbitrary odd level.

§1. Introduction

Let g be a Hecke eigen form of half-integral weight $k+1/2$ and f be the form of weight $2k$ associated to g by the Shimura correspondence. It was found by Waldspurger [20] that the square $c_g(r)^2$ of the r-th Fourier coefficient of g for squarefree r is proportional to the central value $L(f,\psi,k)$ of the L-function of f twisted by the Dirichlet character ψ. When ψ equals to $\left(\frac{D}{}\right)$ with a fundamental discriminant D, Kohnen-Zagier [6] (see also [4]) derived a remarkable explicit formula, by which $L(f,\psi,k)$ is expressed explicitly as the product of $c_g(|D|)^2$ with some elementary factors. Here g is the image of f under the D-th Shintani correspondence, which belongs to Kohnen's space.

Kohnen-Zagier's results have been generalized in several modular forms. Namely by Kohnen [5] to the case of primitive form f with arbitrary odd level and the trivial character, by Kojima-Tokuno [10] to the case of primitive form f with arbitrary odd level and arbitrary character, by Shimura [14] to the case of Hilbert modular forms g of half integral weight over totally real number fields, by Baruch-Mao to the case of primitive form f with squarefree odd level and arbitrary fundamental discriminant, by Kojima to the case of Maass wave forms g of half integral weight over imaginary quadratic fields, and also to the case of Jacobi forms g of integral weight by Gross-Kohnen-Zagier [2], Kojima [9].

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It should be noted that these results are all obtained under the condition that Kohnen's spaces satisfy the multiplicity one theorem. On the other hand, it is known that Kohnen's spaces do not necessarily have multiplicity one theorem for the general level N. This fact causes a serious difficulty when one wants to find an analogous result in the case g has an arbitrary level.

In [7] Kojima obtained a result when Kohnen's spaces have multiplicity *two* theorem, in the case of primitive form of squarefree odd level and general character, by embedding Kohnen's spaces into the space of Hilbert modular forms of half integral weight and developing Shimura's method on the latter.

The author, on the other hand, derived in [12] a similar formula in the case of primitive form of odd prime power level p^m by explicitly determining the multiplicity of Kohnen's space using a refinement of Shimura's trace formula by Ueda ([16], [17], [18]), and developing Kohnen-Zagier's method in several multiplicity cases.

The purpose of this paper is to go a step further toward a generalization of Kohnen-Zagier's formula. We treat here the case of primitive forms with *arbitrary* odd level. In such a case the result we obtain is not expressed as $|c_g(|D|)|^2$ for a single form g, but rather the average of them in the set of primitive forms g belonging to the same Hecke eigen system. Our proof is based on the complete determination of the multiplicity of Kohnen's spaces using Ueda's trace formula, and on the extension of Kohnen-Zagier's method with the basic identity of the Kernel function of Shimura-Shintani correspondence and with some basic properties of period integral valid in all multiplicity cases.

We shall also apply our formula to prove a generalization of the result of Luo-Ramakrishnan [11] to arbitrary odd level, which asserts that a Hecke eigen new form g is uniquely determined from the data $\{c_g(|D|)\}_D$ for fundamental discriminants D.

§2. Preliminaries

2.1. General notations

Throughout this paper, we use the following notations.

We denote the cardinality of a finite set A by $\#(A)$. The disjoint union of two sets A, B is denoted by $A + B$. For an integer m and a prime p, $\mathrm{ord}_p(m)$ means the p-adic additive valuation, that is, $p^{\mathrm{ord}_p(m)} \,||\, m$. If N is a positive integer, $\Pi(N)$ denotes the set of all prime divisors p of N such that $\mathrm{ord}_p(N) \geq 2$. The subset of $\Pi(N)$ consisting of p such that $\mathrm{ord}_p(N) \in 2\mathbf{Z}$

(resp. $\mathrm{ord}_p(N) \in 2\mathbf{Z}+1$) is denoted by $\Pi(N)_{\mathrm{even}}$ (resp. $\Pi(N)_{\mathrm{odd}}$), so that we have the decomposition $\Pi(N) = \Pi(N)_{\mathrm{even}} + \Pi(N)_{\mathrm{odd}}$. Furthermore we denote the set of all prime divisors p of N with $\mathrm{ord}_p(N) = 1$ by $I(N)$. We frequently write $\nu_1 = \#I(N)$, $\nu_2 = \#\Pi(N)_{\mathrm{even}}$, and $\nu_3 = \#\Pi(N)_{\mathrm{odd}}$ for simplicity.

For $c, d \in \mathbf{Z}$ with $d \neq 0$, $\left(\dfrac{c}{d}\right)$ denotes the quadratic residue symbol defined by Shimura (cf. Shimura [13]). If $z \in \mathbf{C}$ and $x \in \mathbf{C}$, we put $z^x = \exp(x\log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. We put $\mathrm{e}(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbf{C}$. The complex upper half plane is denoted by $\mathfrak{H}$. For a non-negative integer k, a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$ and $z \in \mathfrak{H}$, we define a function on $\mathfrak{H}$ by $f|[\alpha]_k(z) = (\det\alpha)^{k/2}(cz+d)^{-k}f(\alpha z)$.

For a positive integer m, we define an operator δ_m, the shift operator $U(m)$ and the twisting operator R_m on formal power series in $\mathrm{e}(z)$ by

$$\sum_{n\geq 1} a(n)\mathrm{e}(nz) \,|\, \delta_m = m^{k/2+1/4}\sum_{n\geq 1} a(n)\mathrm{e}(mnz),$$

$$\sum_{n\geq 1} a(n)\mathrm{e}(nz) \,|\, U(m) = \sum_{n\geq 1} a(mn)\mathrm{e}(nz),$$

and

$$\sum_{n\geq 1} a(n)\mathrm{e}(nz) \,|\, R_m = \sum_{n\geq 1} \left(\frac{n}{m}\right) a(n)\mathrm{e}(nz).$$

Let V be a finite-dimensional vector space over $\mathbf{C}$. We denote the trace of a linear operator T on V by $\mathrm{tr}(T\,;\,V)$.

Finally $\Gamma_0(N)$ denotes as usual the congruence subgroup of level N, that is,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

2.2. Modular forms of integral weight

Let k and N be positive integers.

We denote by $M_{2k}(N)$ (resp. $S_{2k}(N)$) the space of all holomorphic modular forms (resp. cusp forms) of weight $2k$ with trivial character on the congruence subgroup $\Gamma_0(N)$. Also we denote by $S_{2k}(N)^{\mathrm{new}}$ the subspace of $S_{2k}(N)$ spanned by new forms. Moreover we denote by $S_{2k}^*(N)$ the space of 'very new forms', that is, the orthogonal complement of the space $S_{2k}^2(N)$

generated by liftings of cusp forms of lower level ($< N$) with twisting operators R_p ($p \in \Pi(N)$) in $S_{2k}^{\mathrm{new}}(N)$ (cf. Ueda [17], [18], [19]).

Let $\alpha \in \mathrm{GL}_2^+(\mathbf{R})$. If $\Gamma_0(N)$ and $\alpha^{-1}\Gamma_0(N)\alpha$ are commensurable, we define a linear operator $[\Gamma_0(N)\alpha\Gamma_0(N)]_{2k}$ on $S_{2k}(N)$ by

$$f|[\Gamma_0(N)\alpha\Gamma_0(N)]_{2k} = (\det\alpha)^{k-1}\sum_{\alpha_\nu} f|[\alpha_\nu]_{2k}\,,$$

where α_ν runs over a system of representatives for $\Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N)$.

For a positive integer n with $(n, N) = 1$, we put

$$T_{2k,N}(n) = \sum_{ad=n}\left[\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\Gamma_0(N)\right]_{2k},$$

where the sum is extended over all pairs of integers (a, d) such that a, $d > 0$, $a|d$ and $ad = n$.

Let Q be a positive divisor of N such that $(Q, N/Q) = 1$. Take an element $\gamma_Q \in \mathrm{SL}_2(\mathbf{Z})$ which satisfies the conditions:

$$\gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} & (\mathrm{mod}\ Q)\,, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} & (\mathrm{mod}\ N/Q)\,. \end{cases}$$

Put $W_Q = \gamma_Q\begin{pmatrix} Q & 0 \\ 0 & 1\end{pmatrix}$. Then W_Q is a normalizer of $\Gamma_0(N)$. There it induces the Atkin-Lehner operator $[W_Q]_{2k}$ which is a $\mathbf{C}$-linear automorphism of order 2 on $S_{2k}(N)$; this operator is independent of a choice of an element γ_Q (cf. Ueda [15]).

Let $S_{2k}^{*,\tau}(N)$ be the subspace of $S_{2k}^*(N)$ generated by Hecke eigen forms which have eigen values $\tau(p)$ with respect to the Atkin-Lehner operators $W_{p^{\mathrm{ord}_p(N)}}$ ($p \in \Pi(N)$).

If f and g are cusp forms of weight $2k$ on a subgroup Γ of finite index in $\Gamma_0(1)$, we define their Petersson inner product $\langle f, g\rangle$ as follows:

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(1) : \Gamma]}\int_{\Gamma\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}y^{2k}\frac{dxdy}{y^2} \quad (\mathrm{Re}(\tau) = x,\ \mathrm{Im}(\tau) = y).$$

2.3. Modular forms of half-integral weight

Let k be a positive integer, M be a positive integer divisible by 4, and χ be an even Dirichlet character modulo M such that $\chi^2 = 1$.

Let $\mathfrak{G}(k+1/2)$ be the group consisting of pairs (α, φ), where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying

$$|\varphi(z)| = (\det \alpha)^{-k/2-1/4}|cz+d|^{k+1/2}$$

with the group law defined by $(\alpha, \varphi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \varphi(\beta z)\psi(z))$. The group algebra of $\mathfrak{G}(k+1/2)$ over $\mathbf{C}$ acts on complex-valued function f on $\mathfrak{H}$ by

$$f|\left(\sum_{\nu} m_\nu(\alpha_\nu, \varphi_\nu)\right) = \sum_{\nu} m_\nu \varphi_\nu(z)^{-1} f(\alpha_\nu z)\,.$$

For $M = 4N$ with $N \geq 1$ and an even Dirichlet character $\chi \pmod M$ we denote by $\Delta_0 = \Delta_0(M, \chi)_{k+1/2}$ the subgroup of $\mathfrak{G}(k+1/2)$ formed by elements (α, φ) with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$ and

$$\varphi(z) = \chi(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-k-1/2}(cz+d)^{k+1/2}\,.$$

We denote by $S_{k+1/2}(M, \chi)$ the space of cusp forms of weight $k+1/2$ with even Dirichlet character χ on $\Gamma_0(M)$, that is, the space of holomorphic functions on $\mathfrak{H}$ which satisfy $f|\xi = f$ for all $\xi \in \Delta_0(M, \chi)_{k+1/2}$ and vanish at all cusps of $\Gamma_0(M)$. Especially, if χ is the trivial character, we denote $S_{k+1/2}(M, \chi)$ simply by $S_{k+1/2}(M)$.

If f and g are cusp forms of weight $k+1/2$ on a subgroup Γ of finite index in $\Gamma_0(4)$, we define their Petersson inner product $\langle f, g\rangle$ as follows:

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(1):\Gamma]}\int_{\Gamma\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}y^{k+1/2}\frac{dxdy}{y^2} \quad (\mathrm{Re}(\tau) = x,\, \mathrm{Im}(\tau) = y).$$

Let $\xi \in \mathfrak{G}(k+1/2)$. If Δ_0 and $\xi^{-1}\Delta_0\xi$ are commensurable, we define a linear operator $[\Delta_0\xi\Delta_0]_{k+1/2}$ on $S_{k+1/2}(M, \chi)$ by

$$f|[\Delta_0\xi\Delta_0]_{k+1/2} = \sum_{\eta} f|\eta\,,$$

where η runs over a system of representatives for $\Delta_0\backslash\Delta_0\xi\Delta_0$.

For a positive integer n with $(n, M) = 1$, we put

$$\begin{aligned}\tilde{T}(n^2) &= \tilde{T}_{k+1/2,M,\chi}(n^2)\\ &= n^{k-3/2} \sum_{ad=n} a \left[\Delta_0 \left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{k+1/2} \right) \Delta_0 \right]_{k+1/2},\end{aligned}$$

where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a|d$ and $ad = n$.

These operators $\tilde{T}(n^2)$ $((n, M) = 1)$ are hermitian and commutative with each other on $S_{k+1/2}(M, \chi)$ and are called Hecke operators.

2.4. The Kohnen space

We keep to the notation in Section 2.3. Suppose that N is a positive odd integer. Then we have $\chi = \left(\frac{N_0}{\quad} \right)$ for some positive divisor N_0 of N. We define the Kohnen space $S^K_{k+1/2}(N, \chi)$ as follows:

$$\begin{aligned}&S^K_{k+1/2}(N, \chi)\\ &\quad = \left\{ g(z) = \sum_{n \geq 1} c_g(n) \mathrm{e}(nz) \in S_{k+1/2}(4N, \chi) \;\middle|\; \begin{array}{l} c_g(n) = 0 \ \text{for} \\ \chi_2(-1)(-1)^k n \\ \quad \equiv 2, 3 \pmod 4 \end{array} \right\},\end{aligned}$$

where χ_2 is the 2-primary component of χ. In particular we simply write $S^K_{k+1/2}(N, \chi) = S^K_{k+1/2}(N)$ if χ is the trivial character.

It is shown by Kohnen [3] that $S^K_{k+1/2}(N, \chi)$ is invariant under the action of the Hecke operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$ for all positive integers n with $(n, 2N) = 1$.

§3. Twisting operators and the decomposition of the Kohnen space

Let $M = 4N$ (N being an odd natural number) be, k be a positive integer, and χ be an even Dirichlet character modulo M such that $\chi^2 = 1$.

Denote by $\mathcal{H}_N$ the subalgebra of the Hecke algebra with respect to $\Gamma_0(N)$ and $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \pmod N,\ (a, N) = 1,\ ad - bc > 0 \right\}$, which is generated by the double cosets $\Gamma_0(N) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma_0(N)$ over $\mathbf{C}$, where $a, d > 0$, $a|d$ and $(d, N) = 1$.

Define a linear map $\tilde{R}$ from $\mathcal{H}_N$ to $\mathrm{End}_{\mathbf{C}}\Big(S^K_{k+1/2}(N,\chi)\Big)$ by requiring that

$$\tilde{R}\left(\mathit{\Gamma}_0(N)\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\mathit{\Gamma}_0(N)\right)$$
$$= a(ad)^{k-3/2}\left[\Delta_0(M,\chi)\left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2\end{pmatrix}, (d/a)^{k+1/2}\right)\Delta_0(M,\chi)\right]_{k+1/2}.$$

Then $\tilde{R}$ is a representation of $\mathcal{H}_N$. On the other hand, we have a representation $R : \mathcal{H}_N \to \mathrm{End}_{\mathbf{C}}(S_{2k}(N))$ defined by

$$R\left(\mathit{\Gamma}_0(N)\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\mathit{\Gamma}_0(N)\right) = (ad)^{2k-1}\left[\mathit{\Gamma}_0(N)\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\mathit{\Gamma}_0(N)\right]_{2k}.$$

Note that $\tilde{R}$ and R are semisimple. Therefore, if we can take two linear subspaces $\tilde{S}$ in $S^K_{k+1/2}(N,\chi)$ and S in $S_{2k}(N)$ satisfying

$$\mathrm{tr}(\tilde{R}(\xi)\,;\,\tilde{S}) = \mathrm{tr}(R(\xi)\,;\,S) \quad \text{for all } \xi \in \mathcal{H}_N,$$

the representations $\tilde{R}$ and R are equivalent. In other words, if the couple of two linear spaces $\tilde{S}$ and S satisfies that $\mathrm{tr}(\tilde{T}_{k+1/2,M,\chi}(n^2)\,;\,\tilde{S}) = \mathrm{tr}(T_{2k,N}(n)\,;\,S)$ for all natural numbers n with $(n,2N)=1$, $\tilde{S}$ and S are isomorphic as modules over the Hecke algebra.

When N is a squarefree integer, we have

$$\mathrm{tr}(\tilde{T}_{k+1/2,M,\chi}(n^2)\,;\,S^K_{k+1/2}(N,\chi)^{\mathrm{new}}) = \mathrm{tr}(T_{2k,N}(n)\,;\,S_{2k}(N)^{\mathrm{new}})$$

for all $n \in \mathbf{N}$ with $(n,2N)=1$, where $S^K_{k+1/2}(N,\chi)^{\mathrm{new}}$ is the subspace of $S^K_{k+1/2}(N,\chi)$ consisting of all new forms(cf. Kohnen [3]). Therefore we have the strong 'multiplicity one theorem' for $S^K_{k+1/2}(N)^{\mathrm{new}}$ in this case.

On the other hand, this trace relation does not necessarily hold when N has a square factor.

When N is a general odd natural number, Ueda [18] investigated the trace relation of $S^K_{k+1/2}(N)$ and $S^{\mathrm{new}}_{2k}(N)$ as module over the Hecke algebra and established a complete theory of new forms for $S^K_{k+1/2}(N)$. To be more precise, we define the space of new forms of $S^K_{k+1/2}(N)$ as follows.

Definition 1. (Ueda [18]) Let k be a positive integer with $k \geq 2$, and N be an odd positive integer. Define the space of old forms $\mathfrak{D}^K_{k+1/2}(N)$ in

$S^K_{k+1/2}(N)$ by

$$\mathfrak{D}^K_{k+1/2}(N) = \sum_{\substack{0<B|N \\ B\neq N}} \sum_{0<A|(N/B)} \sum_{\xi} S^K_{k+1/2}(B,\xi)|\delta_A$$
$$+ \sum_{\substack{0<B|N \\ B\neq N}} \sum_{0<A|(N/B)^2} \sum_{\xi} \sum_{\substack{(e_l)_{l\in\Pi(N)} \\ 0\leq e_l\leq 2}} S^K_{k+1/2}(B,\xi)|U(A) \prod_{l\in\Pi(N)} R_l^{e_l}.$$

Here, ξ runs over all even quadratic Dirichlet characters defined modulo $4B$ such that $\xi\left(\frac{A}{\cdot}\right) = 1$.

We denote the orthogonal complement of $\mathfrak{D}^K_{k+1/2}(N)$ in $S^K_{k+1/2}(N)$ by $\mathfrak{N}^K_{k+1/2}(N)$, and call it the space of new forms of $S^K_{k+1/2}(N)$.

These operators δ_A, $U(A)$ and R_l ($l \in \Pi(N)$) commute with the Hecke operators $\tilde{T}(n^2)$ (cf. Ueda [17]). Therefore, $\mathfrak{D}^K_{k+1/2}(N)$ and $\mathfrak{N}^K_{k+1/2}(N)$ are stable by them, so that one can decompose the latter into common eigen subspaces as follows:

$$\mathfrak{N}^K_{k+1/2}(N) = \bigoplus_{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})} \mathfrak{N}^{K,\kappa}_{k+1/2}(N),$$

where

$$\mathfrak{N}^{K,\kappa}_{k+1/2}(N) = \left\{ g \in \mathfrak{N}^K_{k+1/2}(N) \,\middle|\, g|R_p = \kappa(p)g \text{ for all } p \in \Pi(N) \right\}.$$

$\mathfrak{N}^{K,\kappa}_{k+1/2}(N)$ has an orthogonal $\mathbf{C}$-basis consisting of common eigen forms for all Hecke operators $\tilde{T}(p^2)_{k+1/2,M,1}$ (p : prime, $p \nmid N$) and Shift operators $U(p^2)$ (p : prime, $p|N$) (cf. Ueda [17]).

Let $k \geqq 2$, N be a positive odd integer and M_1 be the 'square-free part' of $M = 4N$ so that $M_1 = \prod_{p\in I(N)} p$. Then we have the following theorem.

THEOREM 1. *For each $\kappa \in \mathrm{Map}(\Pi(N),\{\pm1\})$, we have the following isomorphism as modules over the Hecke algebra*:

$$\mathfrak{N}^{K,\kappa}_{k+1/2}(N) \cong S^{*,\tau_\kappa}_{2k}(N) \oplus \bigoplus_{\substack{\Pi(N)_2=I+J+K \\ I+J\neq\phi,\, I,J\subseteq\Pi(N)^*_2}} \bigoplus_{\tilde{\tau}_\kappa} S^{*,\tilde{\tau}_\kappa}_{2k}\left(M_1 \prod_{l\in J} l \prod_{p\in\Pi(N)-(I+J)} p^{\mathrm{ord}_p(N)} \right) \Big| \prod_{p\in I+J} R_p,$$

where τ_κ *is the element of* $\mathrm{Map}(\Pi(N),\{\pm 1\})$ *whose value at* p *is* 1 *or* $\kappa(p)\left(\frac{-1}{p}\right)^k$ *according as* $p\in\Pi(N)_{\mathrm{even}}$ *or* $p\in\Pi(N)_{\mathrm{odd}}$, $\bigoplus_{\substack{\Pi(N)_2=I+J+K\\ I+J\neq\phi,\, I,J\subseteq\Pi(N)_2^*}}$ *is the direct sum extended over all partitions such that*

$$I+J+K=\Pi(N)_2=\{p\in\Pi(N)\mid \mathrm{ord}_p(N)=2\};$$
$$I+J\neq\phi,\quad I,J\subseteq\Pi(N)_2^*=\left\{p\in\Pi(N)_2\,\middle|\,\left(\frac{-1}{p}\right)=1\right\},$$

and $\tilde\tau_\kappa$ *runs over all elements of* $\mathrm{Map}(\Pi(N),\{\pm 1\})$ *whose value at* p *is* 1 *or* $\kappa(p)\times\left(\frac{-1}{p}\right)^k\prod_{q\in I+J}\left(\frac{p}{q}\right)$ *according as* $p\in\Pi(N)_{\mathrm{even}}$ *or* $p\in\Pi(N)_{\mathrm{odd}}$.

Proof. By the above argument, it suffices to show that

$$\begin{aligned}
&\mathrm{tr}(\tilde T(n^2)_{k+1/2,M,1}\,;\,\mathfrak{N}^{K,\kappa}_{k+1/2}(N))\\
&=\mathrm{tr}(T_{2k,N}(n)\,;\,S^{*,\tau_\kappa}_{2k}(N))+\sum_{\substack{\Pi(N)_2=I+J+K\\ I+J\neq\phi,\, I,J\subseteq\Pi(N)_2^*}}\sum_{\tilde\tau_\kappa}\\
&\qquad \mathrm{tr}\left(T_{2k,N}(n)\,;\,S^{*,\tilde\tau_\kappa}_{2k}\left(M_1\prod_{l\in J}l\prod_{p\in\Pi(N)-(I+J)}p^{\mathrm{ord}_p(N)}\right)\Big|\prod_{p\in I+J}R_p\right)
\end{aligned}$$

for all natural number n with $(n,M)=1$. From Ueda [18, Theorem 2], we have the following trace relation:

For all $n\in\mathbf{N}$ prime to $4N$,

$$\begin{aligned}
&\mathrm{tr}(\tilde T_{k+1/2,M,\chi}(n^2)\,;\,\mathfrak{N}^{K,\kappa}_{k+1/2}(N))\\
&=\sum_{\Pi(N)_2=I+J+K}\ \sum_{\substack{\tau\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \sigma\in\mathrm{Map}(\Pi(N)-(I+J),\{\pm1\})}}\Xi((\nu(I,J)_l),I+J,(\tau,\sigma))\\
&\qquad\times\mathrm{tr}\left(T_{2k,N}(n);S^{*,(\tau,\sigma)}_{2k}\left(M_1\prod_{l\in J}l\prod_{p\in\Pi(N)-(I+J)}p^{\mathrm{ord}_p(N)}\right)\right)\Big|\prod_{p\in I+J}R_p,
\end{aligned}$$

where $S^{*,(\tau,\sigma)}_{2k}(N)=\{f\in S^{*,\tau}_{2k}(N)\mid f|R_lW_l=\sigma(l)f|R_l\text{ for all }l\in\Pi(N)\}$ and $\Xi((\nu(I,J)_l),I+J,(\tau,\sigma))$ are the constants determined in Ueda [17, (2.22)]. On Ueda [17, (2.22)], we find that multiplicities $\Xi=\Xi((\nu(I,J)_l),I+J,(\tau,\sigma))$ as Hecke modules satisfy $\Xi\leq 1$ for all possible κ and τ. Also

χ is the trivial character, Ξ does not depend on eigen systems σ. Therefore, we can obtain the necessary conditions for $\Xi = 1$ by using the formula Ueda [17, (2.22)], that is,

$$(\mathit{CASE}\ 1) \qquad I = J = \phi \quad \text{and} \quad \tau = \tau_\kappa,$$

or

$$(\mathit{CASE}\ 2) \qquad I + J \neq \phi \quad \text{and} \quad \tau = \tilde{\tau}_\kappa.$$

The above trace relation in these situations implies the desired result. □

EXAMPLE 1. We further assume that $\Pi(N)_{\mathrm{odd}} = \phi$. Then we have the following isomorphism (as modules over the Hecke algebra) for each $\kappa \in \mathrm{Map}(\Pi(N), \{\pm 1\})$:

$$\mathfrak{N}_{k+1/2}^{K,\kappa}(N) \cong S_{2k}^{*,1}(N) \oplus \bigoplus_{\substack{\Pi(N)_2 = I+J+K \\ I+J\neq\phi,\ I,J\subseteq\Pi(N)_2^*}} \bigoplus_{\substack{\sigma(p)=1 \text{ for} \\ p\in\Pi(N)-(I+J)}} S_{2k}^{*,\sigma}\left(M_1 \prod_{l\in J} l \prod_{p\in\Pi(N)-(I+J)} p^{\mathrm{ord}_p(N)}\right) \Big| \prod_{p\in I+J} R_p.$$

Especially, in the case $N = p^{2m}$ with an odd prime number p, we have the following

$$\mathfrak{N}_{k+1/2}^{K,\pm1}(p^{2m}) \cong \begin{cases} S_{2k}^{\mathrm{new},+1}(p^{2m}) & \text{if } m \geq 2 \\ S_{2k}^{*,+1}(p^2) \oplus & \\ \quad \frac{1}{2}\left(1 + \left(\frac{-1}{p}\right)\right)(S_{2k}^{\mathrm{new}}(p)|R_p \oplus S_{2k}(1)|R_p) & \text{if } m = 1. \end{cases}$$

(Note that, for $m \geq 1$, we also have

$$\mathfrak{N}_{k+1/2}^{K,\pm1}(p^{2m+1}) \cong S_{2k}^{\mathrm{new},\pm\left(\frac{-1}{p}\right)^k}(p^m)$$

by Theorem 1 (cf. Ueda [16]).)

COROLLARY 1. *Let the notation be same as Theorem* 1. *Put* $\nu_2 = \#\Pi(N)_{\mathrm{even}}$ $(0 \leq \nu_2 \leq \#\Pi(N))$. *Let f be any primitive form in* $S_{2k}^{\mathrm{new}}(N)$. *Then, there are* 2^{ν_2} *number of common Hecke eigen forms* $\{g_\nu\}$ *in* $\mathfrak{N}_{k+1/2}^{K}(N)$ *corresponding to primitive form f via the Shimura correspondence* (*the Shimura correspondence will be referred to in the next section* 4).

Proof. There exist 2^{ν_2} mappings $\{\kappa\}$ corresponding to an eigen system τ_κ (resp. $\tilde{\tau}_\kappa$). Therefore $\mathfrak{N}^K_{k+1/2}(N)$ has 2^{ν_2} common Hecke eigen forms which have the same system of Hecke eigen values as a primitive form in

$$S^{*,\tau_\kappa}_{2k}(N) \quad \left(\text{resp. } S^{*,\sigma}_{2k}\left(M_1 \prod_{p\in J} p \prod_{p\in\Pi(N)-(I+J)} p^{\mathrm{ord}_p(N)}\right)\right)$$

by Theorem 1. Thus we have the above result. □

Remark 1. In Corollary 1, we can say about the situation that $\mathfrak{N}^K_{k+1/2}(N)$ satisfies the 'multiplicity 2^{ν_2}-condition'.

§4. Periods of cusp forms and the Shimura correspondence

Let N be a positive odd integer, D be a fundamental discriminant with $(-1)^k D > 0$ and $(D,N) = 1$, and Δ be any positive integer satisfying $\Delta \equiv 0, 1 \pmod 4$ and $D|\Delta$.

Denote by $\mathcal{Q}_{N,\Delta}$ the set of all integral binary quadratic forms $Q = [a,b,c](X,Y) = aX^2 + bXY + cY^2$ with $b^2 - 4ac = \Delta$ and $N|a$. The action of $\Gamma_0(N)$ to $[a,b,c] = [a,b,c](X,Y)$ is defined by

$$[a,b,c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (X,Y) = [a,b,c](\alpha X + \beta Y,\, \gamma X + \delta Y).$$

We denote by $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ the set of $\Gamma_0(N)$-equivalence classes of forms in $\mathcal{Q}_{N,\Delta}$. Furthermore we define the automorphisms $W_{N'}$ $(N' \,||\, N)$ on $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ by

$$Q \circ W_{N'} = \frac{1}{N'} Q \circ \begin{pmatrix} \alpha N' & \beta \\ \gamma N & \delta N' \end{pmatrix}, \text{ where } \alpha,\beta,\gamma,\delta \in \mathbf{Z},\ \alpha\delta N' - \beta\gamma\frac{N}{N'} = 1.$$

Put another way, this map is the action of the Atkin-Lehner operator to $Q = [a,b,c]$. The 'period' of a cusp form f in $S_{2k}(N)$ associated to $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ is defined by

$$r_{k,N}(f; D, \Delta) = \sum_{Q \in \mathcal{Q}_{N,\Delta}/\Gamma_0(N)} \omega_D(Q) \int_{C_Q} f(z) Q(z,1)^{k-1}\, dz,$$

where ω_D is the genus character given in Kohnen [4] and C_Q is the image in $\Gamma_0(N)\backslash\mathfrak{H} \cup \mathbf{P}^1(\mathbf{Q})$ of the semicircle $a|z|^2 + b\,\mathrm{Re}(z) + c = 0$ oriented from $(-b-\sqrt{\Delta})/2a$ to $(-b+\sqrt{\Delta})/2a$ if $a \neq 0$, or of the vertical line $b\ \mathrm{Re}(z) + c = 0$, oriented from $-c/b$ to $\sqrt{-1}\infty$ if $b>0$, and from $\sqrt{-1}\infty$ to $-c/b$ if $b<0$, if $a=0$.

Next we define the D-th Shimura correspondence $\mathcal{S}_{k,N,D}$ which maps $S_{k+1/2}^{K}(N)$ to $M_{2k}(N)$ (to $S_{2k}(N)$ if $k\geq 2$ or if N is cubefree) by

$$g|\mathcal{S}_{k,N,D}(\tau)=\sum_{n\geq 1}\left(\sum_{d|n,(d,N)=1}\left(\frac{D}{d}\right)d^{k-1}c_g\left(\frac{|D|n^2}{d^2}\right)\right)\mathrm{e}(n\tau)$$

for any $g(z)=\sum_{\substack{n\geq 1\\(-1)^k n\equiv 0,1 \pmod 4}} c_g(n)\mathrm{e}(n\tau)\in S_{k+1/2}^{K}(N)$. Then we have the following fact.

THEOREM 2. (Kohnen [4, Theorem 2]) *Let N be a positive odd integer and k be an integer satisfying $k\geq 2$. For any fundamental discriminant D with $(-1)^kD>0$ and $(D,N)=1$, the adjoint map of the Shimura correspondence $\mathcal{S}_{k,N,D}$ with respect to the Petersson inner product coincides with the D-th Shintani correspondence defined by*

$$f|\mathcal{S}^{*}_{k,N,D}(z):=(-1)^{[k/2]}2^k\sum_{\substack{m\geq 1\\(-1)^k m\equiv 0,1 \pmod 4}}\left(\sum_{t|N}\mu(t)\left(\frac{D}{t}\right)t^{k-1}r_{k,Nt}(f;D,(-1)^k mt^2)\right)\mathrm{e}(mz)$$

for any cusp form $f\in S_{2k}(N)$.

Especially, for new forms $f\in S_{2k}^{\mathrm{new}}(N)$ in which we will be interested, the Shintani correspondence is expressed as

$$f|\mathcal{S}^{*}_{k,N,D}(z)=(-1)^{[k/2]}2^k\sum_{\substack{m\geq 1\\(-1)^k m\equiv 0,1 \pmod 4}} r_{k,N}(f;D,(-1)^k m)\mathrm{e}(mz).$$

Remark 2. The 'lifting' maps $\mathcal{S}_{k,N,D}$ and $\mathcal{S}^{*}_{k,N,D}$ preserve old forms and new forms respectively and commute with all Hecke operators.

§5. Kohnen-Zagier's formula in the case of '$4\times$general odd' level

In this section we generalize the result of Kohnen-Zagier to the case of '$4 \times$ general odd' level by using Theorem 1.

Let $k \geq 2$ and the assumptions on N and D be the same as in Theorem 2. Suppose that $f(z) = \sum_{n\geq 1} a_f(n)\mathrm{e}(nz)$ is a primitive form in $S_{2k}^{\mathrm{new}}(N)$, and

$$g(\tau) = \sum_{\substack{n\geq 1 \\ (-1)^k n \equiv 0,1 \pmod 4}} c_g(n)\mathrm{e}(n\tau)$$

is a form in $\mathfrak{N}_{k+1/2}^K(N)$ corresponding to f under the Shimura correspondence $\mathcal{S}_{k,N,D}$.

Remark 3. We note that $g(\tau)$ belongs to $\mathfrak{N}_{k+1/2}^K(N; f)$, which follows from the compatibility of the Shimura correspondence with the Hecke operators. Here $\mathfrak{N}_{k+1/2}^K(N; f)$ is the subspace of $\mathfrak{N}_{k+1/2}^K(N)$ having the same Hecke eigen values as f for all Hecke operators $\tilde{T}(p^2)$ (p : prime, $\mathrm{ord}_p(N) = 0$) and $U(p^2)$ (p : prime, $\mathrm{ord}_p(N) \geq 1$), that is,

$$\mathfrak{N}_{k+1/2}^K(N; f) = \left\{ g(\tau) \,\middle|\, g|\tilde{T}(p^2) = a_f(p)g \text{ for all prime } p \text{ such that } p \nmid N \right\}.$$

We take two (disjoint) subsets I and J of $\Pi(N)_2^*$ (see Theorem 1). Suppose that f belongs to

$$S_{2k}^{*,\tau}\left(M_1 \prod_{p\in J} p \prod_{p\in\Pi(N)-(I+J)} p^{\mathrm{ord}_p(N)} \right) \Big| \prod_{p\in I+J} R_p,$$

which has the eigen value $\tau(p)$ for $W_{p^{\mathrm{ord}_p(N)}}$ such that $\tau(p) = 1$ for all $p \in \Pi(N)_{\mathrm{even}} - (I + J)$. If, in particular $I + J = \phi$, we assume that f belongs to $S_{2k}^{*,\tau}(N)$.

Remark 4. The eigen system τ for the Atkin-Lehner operator automatically satisfies the condition

$$\tau(p) = 1 \quad \text{for} \quad p \in I + J \ (\subseteq \Pi(N)_2^*),$$

because we have the following lemma which is proved by straightforward calculation.

LEMMA 1. *Let p be an odd prime and M be a positive integer prime to p. For any $f \in S_{2k}(pM)$, we have*

$$f|R_pW_{p^2} = \left(\frac{-1}{p}\right) f|R_p.$$

By Theorem 1 we have the following isomorphism as modules over the Hecke algebra:

$$\mathfrak{N}^{*,\kappa}_{k+1/2}(N) \cong S^{*,\tau}_{2k}\left(M_1 \prod_{p\in J} p \prod_{p\in\Pi(N)-(I+J)} p^{\mathrm{ord}_p(N)}\right) \Big| \prod_{p\in I+J} R_p\,,$$

where κ belongs to $\mathrm{Map}(\Pi(N),\{\pm1\})$ satisfying the following condition (parity condition):

$$\tau_{\mathrm{odd}} \quad : \quad \kappa(p) = \tau(p)\left(\frac{-1}{p}\right)^k \prod_{q\in I+J}\left(\frac{p}{q}\right) \text{ for all } p \in \Pi(N)_{\mathrm{odd}}$$

and $\mathfrak{N}^{*,\kappa}_{k+1/2}(N)$ is a certain subspace in $\mathfrak{N}^{K,\kappa}_{k+1/2}(N)$. Therefore, we have the orthogonal decomposition

$$\mathfrak{N}^{K}_{k+1/2}(N;f) = \bigoplus_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \mathfrak{N}^{K,\kappa}_{k+1/2}(N)\cap\mathfrak{N}^{K}_{k+1/2}(N;f).$$

From this, we obtain the following

$$g(\tau) = \sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} g_\kappa(\tau), \quad g_\kappa(\tau) \in \mathfrak{N}^{K,\kappa}_{k+1/2}(N).$$

Here, $g_\kappa(\tau)$ are all Hecke eigen forms which have the same Hecke eigen system as f. Furthermore, from the definition of f and the Shimura correspondence $\mathcal{S}_{k,N,D}$, we have the following relation:

$$c_{g_\kappa}(n^2|D|) = c_{g_\kappa}(|D|) \sum_{d|n,(d,N)=1} \mu(d)\left(\frac{D}{d}\right) d^{k-1} a_f(n/d)$$

among Fourier coefficients of g_κ and f. Therefore, we have $g_\kappa|\mathcal{S}_{k,N,D} = c_{g_\kappa}(|D|)f$ for each κ satisfying the parity condition. Namely, assuming $g_\kappa \neq 0$ for all κ satisfying the parity condition, we have

$$\sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \frac{\overline{c_{g_\kappa}(|D|)}}{\langle g_\kappa, g_\kappa\rangle} g_\kappa|\mathcal{S}_{k,N,D}(z) = \sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa\rangle} f(z).$$

Taking the Petersson inner product of each side with f, we obtain the following

$$\sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \frac{c_{g_\kappa}(|D|)}{\langle g_\kappa, g_\kappa\rangle}\langle f, g_\kappa|\mathcal{S}_{k,N,D}\rangle = \sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa\rangle}\langle f, f\rangle.$$

Observe that the left hand side is equal to the $|D|$-th Fourier coefficient of $f|\mathcal{S}^*_{k,N,D}$, because $\{g_\kappa\}_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}}$ form an orthogonal basis of $\mathfrak{N}^K_{k+1/2}(N;f)$. Therefore by the 'primitivity' of f and Theorem 2, we get

$$\sum_{\substack{\kappa\in\mathrm{Map}(\Pi(N),\{\pm1\})\\ \kappa \text{ with } (\tau_{\mathrm{odd}})}} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa\rangle}\langle f, f\rangle = (-1)^{[k/2]}2^k r_{k,N}(f;D,D).$$

We next calculate the period $r_{k,N}(f;D,D)$. For any positive integer M, we can show that the set

$$\Big\{[0,D,\mu]\circ W_t \,\Big|\, \mu\,(\mathrm{mod}\ D),\ t\ge 1 \text{ such that } t\,||\,M\Big\},$$

is a complete representative system of $\mathcal{Q}_{M,D^2}/\Gamma_0(M)$, because they are obtained by reducing to the representative system $\{[0,D,\mu]\,|\,\mu\,(\mathrm{mod}\ D)\}$ of $\mathcal{Q}_{1,D^2}$ (cf. Gross-Kohnen-Zagier [2, Chap. 1]). Putting $I(N)=\{p_{1,i}\}_{1\le i\le\nu_1}$, $\Pi(N)_{\mathrm{even}}=\{p_{e,j}\}_{1\le j\le\nu_2}$ and $\Pi(N)_{\mathrm{odd}}=\{p_{o,l}\}_{1\le l\le\nu_3}$, this set can be written in the set

$$\left\{[0,D,\mu]\circ W_{\prod_{1\le i\le\nu_1} p_{1,i}^{\alpha_i}}\circ W_{\prod_{1\le j\le\nu_2} p_{e,j}^{\beta_j}}\circ W_{\prod_{1\le l\le\nu_3} p_{o,l}^{\gamma_l}} \,\middle|\, \begin{array}{l}\alpha_i\in\{0,1\},\\ \beta_j\in\{0,\mathrm{ord}_{p_{e,j}}(N)\},\\ \gamma_l\in\{0,\mathrm{ord}_{p_{o,l}}(N)\}\end{array}\right\}.$$

Then, we get

$$\begin{aligned} r_{k,N}(f;D,D) = &\sum_{t||N}\sum_{\mu\,(\mathrm{mod}\ D)} \omega_D([0,D,\mu]\circ W_t)\\ &\times\int_{C_{[0,D,\mu]\circ W_t}} f(z)([0,D,\mu]\circ W_t(z,1))^{k-1}\,dz \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t||N} \sum_{\mu \,(\mathrm{mod}\ D)} \omega_D([0,D,\mu] \circ W_t) \\
&\quad \times \int_{C_{[0,D,\mu]}} f\big|W_t^{-1}(z)([0,D,\mu](z,1))^{k-1}\, dz \\
&= \sum_{(\alpha_i,\beta_j,\gamma_l)} \sum_{\mu \,(\mathrm{mod}\ D)} \\
&\quad \omega_D\Bigg([0,D,\mu] \circ W_{\prod_{1\le i\le \nu_1} p_{1,i}^{\alpha_i}} \circ W_{\prod_{1\le j\le \nu_2} p_{e,j}^{\beta_j}} \circ W_{\prod_{1\le l\le \nu_3} p_{o,l}^{\gamma_l}}\Bigg) \\
&\quad \times \int_{C_{[0,D,\mu]}} \prod_{1\le i\le \nu_1} \tau(p_{1,i})^{\delta(\alpha_i)} \prod_{1\le j\le \nu_2} \tau(p_{e,j})^{\delta(\beta_j)} \prod_{1\le l\le \nu_3} \tau(p_{o,l})^{\delta(\gamma_l)} \\
&\quad \times f(z)(Dz+\mu)^{k-1}\, dz,
\end{aligned}
$$

where $\delta(t) = 0, 1$ according as $t = 0$ or $t \neq 0$. On the other hand, by straightforward calculations we have

$$
\begin{aligned}
&\omega_D\Bigg([0,D,\mu] \circ W_{\prod_{1\le i\le \nu_1} p_{1,i}^{\alpha_i}} \circ W_{\prod_{1\le j\le \nu_2} p_{e,j}^{\beta_j}} \circ W_{\prod_{1\le l\le \nu_3} p_{o,l}^{\gamma_l}}\Bigg) \\
&\quad = \omega_D\left([0,D,\mu]\right) \prod_{1\le i\le \nu_1} \left(\frac{D}{p_{1,i}^{\alpha_i}}\right) \prod_{1\le l\le \nu_3} \left(\frac{D}{p_{o,l}^{\gamma_l}}\right) \\
&\quad = \left(\frac{D}{\mu}\right) \prod_{1\le i\le \nu_1} \left(\frac{D}{p_{1,i}}\right)^{\alpha_i} \prod_{1\le l\le \nu_3} \left(\frac{D}{p_{o,l}}\right)^{\gamma_l}.
\end{aligned}
$$

Therefore, from the definition of τ (or f), the period $r_{k,N}(f; D, D)$ is equal to

$$
\begin{aligned}
&2^{\nu_2}\left(\sum_{(\alpha_i)_{1\le i\le \nu_1}} \prod_{1\le i\le \nu_1} \left(\frac{D}{p_{1,i}}\right)^{\alpha_i} \tau(p_{1,i})^{\delta(\alpha_i)}\right) \\
&\quad \times \left(\sum_{(\gamma_l)_{1\le l\le \nu_3}} \prod_{1\le l\le \nu_3} \left(\frac{D}{p_{o,l}}\right)^{\gamma_l} \tau(p_{o,l})^{\delta(\gamma_l)}\right) \\
&\quad \times \sum_{\mu \,(\mathrm{mod}\ D)} \left(\frac{D}{\mu}\right) \int_{C_{[0,D,\mu]}} f(z)(Dz+\mu)^{k-1}\, dz.
\end{aligned}
$$

Especially, if we assume the additional condition $\tau(p) = \left(\frac{D}{p}\right)$ for all $p \in$

$I(N) + \Pi(N)_{\mathrm{odd}}$, then we see that this is equal to

$$2^{\nu_1+\nu_2+\nu_3} \sum_{\mu \,(\mathrm{mod}\ D)} \left(\frac{D}{\mu}\right) \int_{C_{[0,D,\mu]}} f(z)(Dz+\mu)^{k-1}\, dz.$$

Remark 5. If for some prime p of $I(N) + \Pi(N)_{\mathrm{odd}}$ we have $\tau(p) = -\left(\frac{D}{p}\right)$, then $r_{k,N}(f; D, D) = 0$. Actually, we can show that

$$\begin{aligned}
&\sum_{(\alpha_i)_{1\le i\le \nu_1}} \prod_{1\le i\le \nu_1} \left(\frac{D}{p_{1,i}}\right)^{\alpha_i} \tau(p_{1,i})^{\delta(\alpha_i)} \\
&= \sum_{(\alpha_i)_{1\le i\le \nu_1}} \prod_{\substack{1\le i\le \nu_1 \\ \tau(p_{1,i}) = -\left(\frac{D}{p_{1,i}}\right)}} (-1)^{\alpha_i} \prod_{\substack{1\le i\le \nu_1 \\ \tau(p_{1,i}) \ne -\left(\frac{D}{p_{1,i}}\right)}} \left(\frac{D}{p_{1,i}}\right)^{\alpha_i} \tau(p_{1,i})^{\alpha_i} = 0,
\end{aligned}$$

and

$$\sum_{(\gamma_l)_{1\le l\le \nu_3}} \prod_{1\le l\le \nu_3} \left(\frac{D}{p_{o,l}}\right)^{\gamma_l} \tau(p_{o,l})^{\delta(\gamma_l)} = 0$$

in the same way.

Furthermore, we have

$$\begin{aligned}
&\sum_{\mu \,(\mathrm{mod}\ D)} \left(\frac{D}{\mu}\right) \int_{C_{[0,D,\mu]}} f(z)(Dz+\mu)^{k-1}\, dz \\
&= \sum_{\mu \,(\mathrm{mod}\ D)} \left(\frac{D}{\mu}\right) \int_{-\mu/D}^{\infty\sqrt{-1}} f(z)(Dz+\mu)^{k-1}\, dz \\
&= (D\sqrt{-1})^{k-1}\sqrt{-1} \\
&\qquad \times \int_0^\infty \sum_{\mu \,(\mathrm{mod}\ D)} \left(\frac{D}{\mu}\right) \sum_{n\ge 1} a_f(n) t^{k-1} \mathrm{e}\left(\frac{n\mu}{|D|} + \sqrt{-1}nt\right) dt \\
&= (D\sqrt{-1})^{k-1}\sqrt{-1} \left(\frac{D}{-1}\right)^{1/2} |D|^{1/2} \int_0^\infty \sum_{n\ge 1} \left(\frac{D}{n}\right) a_f(n) e^{-2\pi n t} t^{k-1}\, dt \\
&= (-1)^{[k/2]} (2\pi)^{-k} |D|^{k-1/2} \Gamma(k) L(f, D, k),
\end{aligned}$$

where $L(f, D, s)$ is the twisted L-function

$$L(f, D, s) = \sum_{n\ge 1} \left(\frac{D}{n}\right) a_f(n) n^{-s} \quad (\mathrm{Re}(s) \gg 0),$$

which is extended to $\mathbf{C}$ as a holomorphic function.

From the above argument, we now have the following Theorem.

THEOREM 3. *In the same notation as above, assume that the eigen system τ for the Atkin-Lehner operator of f satisfies the following*

$$\tau(p) = \begin{cases} \left(\frac{D}{p}\right) & \text{if } p \in I(N) + \Pi(N)_{\text{odd}} \\ 1 & \text{if } p \in \Pi(N)_{\text{even}} - (I+J) \end{cases}$$

for two (disjoint) subsets I, J in $\Pi(N)_2^*$. Then we have

$$\sum_{\substack{\kappa \in \mathrm{Map}(\Pi(N),\{\pm 1\}) \\ \kappa \text{ with } (\tau_{\text{odd}})}} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f,D,k)}{\langle f, f\rangle},$$

where $\nu(N)$ denotes the number of different prime divisors of N.

Remark 6. This result is a generalization of Kohnen [5, Corollary].

Remark 7. By Remark 5, if for some prime p of $I(N) + \Pi(N)_{\text{odd}}$ we have $\tau(p) = -\left(\frac{D}{p}\right)$, then $r_{k,N}(f; D, D) = 0$ so that we have $c_{g_\kappa}(|D|) = 0$ as well for any κ satisfying the parity condition (τ_{odd}).

THEOREM 4. *Let the assumptions on f and τ be the same as in Theorem 3, and suppose we take a Hecke eigen form $g(\tau)$ in $\mathfrak{N}^K_{k+1/2}(N; f)$ satisfying the condition*

$$\sum_{S, S' \subseteq \Pi(N)_{\text{even}}} \kappa(S)\kappa(S')\langle g|R_S, g|R_{S'}\rangle \neq 0.$$

Then, we have

$$\left(\sum_{\substack{\kappa \in \mathrm{Map}(\Pi(N),\{\pm 1\}) \\ \kappa \text{ with } (\tau_{\text{odd}})}} \frac{\left| \sum_{S \subseteq \Pi(N)_{\text{even}}} \kappa(S) \prod_{p \in S} \left(\frac{|D|}{p}\right) \right|^2}{\sum_{S, S' \subseteq \Pi(N)_{\text{even}}} \kappa(S)\kappa(S')\langle g|R_S, g|R_{S'}\rangle} \right) |c_g(|D|)|^2$$
$$= 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f,D,k)}{\langle f, f\rangle},$$

where $\kappa(S) = \prod_{p \in S} \kappa(p)$, $\kappa(\phi) = 1$, $R_S = \prod_{p \in S} R_p$ and $R_\phi = 1$.

Proof. By straightforward calculations, we have

$$g_\kappa(\tau) = \frac{1}{2^{\nu_2}} \sum_{S \subseteq \Pi(N)_{\mathrm{even}}} \kappa(S) g|R_S(\tau)$$

for each κ satisfying the parity condition (τ_{odd}) and an arbitrary new form $g(\tau) \in \mathfrak{N}^K_{k+1/2}(N; f)$. Actually, we can check that the right hand side of this equation has the same eigen system as κ with respect to twisting operators R_p $(p \in \Pi(N))$. Therefore, we obtain the above result by putting it in Theorem 3. ∎

§6. Examples of Kohnen-Zagier's formula

In this section we give some examples of Kohnen-Zagier's formula by using Theorem 3 and Theorem 4. Thus the notation and assumptions are the same as in Section 5. Especially, we assume that $k \geq 2$, N an positive odd integer, D a fundamental discriminant with $(-1)^k D > 0$ and $(D, N) = 1$, f a primitive form of $S^{\mathrm{new}}_{2k}(N)$ and the eigen system τ of f for the Atkin-Lehner operator satisfies the same condition as in Theorem 3 and Theorem 4.

6.1. The case of $\Pi(N) = \phi$

Suppose that $\Pi(N)_{\mathrm{odd}} = \Pi(N)_{\mathrm{even}} = \phi$. Then, Theorem 4 induces Kohnen-Zagier's formula in the case of square-free level, that is,

$$\frac{|c_g(|D|)|^2}{\langle g, g\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f\rangle}$$

for an arbitrary new form $g(\tau) \in \mathfrak{N}^K_{k+1/2}(N; f)$ (cf. Kohnen-Zagier [6]; Kohnen [4, Corollary 1]).

6.2. The case of $\Pi(N)_{\mathrm{even}} = \phi$

Suppose that $\Pi(N)_{\mathrm{even}} = \phi$. Then κ satisfying the parity condition (τ_{odd}) is only one. Therefore, by using it, we have the following representation:

$$\frac{|c_g(|D|)|^2}{\langle g, g\rangle} \left(= \left(\frac{\left|\sum_{S \subseteq \Pi(N)_{\mathrm{even}}} \kappa(S) \prod_{p \in S} \left(\frac{|D|}{p}\right)\right|^2}{\sum_{S,\, S' \subseteq \Pi(N)_{\mathrm{even}}} \kappa(S)\kappa(S') \langle g|R_S, g|R_{S'}\rangle} \right) |c_g(|D|)|^2 \right)$$

$$= 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f\rangle}$$

for an arbitrary new form $g(\tau) \in \mathfrak{N}^K_{k+1/2}(N; f)$. Namely, Kohnen-Zagier's formula in this case has the same form as in the case of square-free level (cf. Sakata [12, Theorem 7]).

6.3. The case of $\Pi(N)_{\mathrm{odd}} = \phi$

Suppose that $\Pi(N)_{\mathrm{odd}} = \phi$. Then the parity condition does not make sense. Therefore, we have the following

$$\sum_{\kappa \in \mathrm{Map}(\Pi(N), \{\pm 1\})} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa \rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}$$

for an arbitrary new form $g(\tau) \in \mathfrak{N}^K_{k+1/2}(N; f)$. In other words, by changing D, we get the central value $L(f, D, k)$ in either one coefficients of the 2^{ν_2} different forms $\{g_\kappa\}_{\kappa \in \mathrm{Map}(\Pi(N), \{\pm 1\})}$ alternatively. In particular in the cases of $N = p^{2m}$ $(m \geq 2)$ or $N = p^2$ and $\left(\frac{-1}{p}\right) = 1$, the left hand side of this formula is expressed by the following

$$\begin{aligned}
&\sum_{\kappa \in \mathrm{Map}(\{p\}, \{\pm 1\})} \frac{|c_{g_\kappa}(|D|)|^2}{\langle g_\kappa, g_\kappa \rangle} \\
&= \left(\sum_{\kappa \in \mathrm{Map}(\{p\}, \{\pm 1\})} \frac{\left| \sum_{S \subseteq \{p\}} \kappa(S) \prod_{p \in S} \left(\frac{|D|}{p}\right) \right|^2}{\sum_{S, S' \subseteq \{p\}} \kappa(S)\kappa(S') \langle g|R_S, g|R_{S'} \rangle} \right) |c_g(|D|)|^2 \\
&= \left(\frac{\left| 1 + \left(\frac{|D|}{p}\right) \right|^2}{2\langle g, g \rangle + 2\langle g|R_p, g \rangle} + \frac{\left| 1 - \left(\frac{|D|}{p}\right) \right|^2}{2\langle g, g \rangle - 2\langle g|R_p, g \rangle} \right) |c_g(|D|)|^2 \\
&= \frac{2\Big(\langle g, g \rangle - \langle g|R_p, g \rangle \left(\frac{|D|}{p}\right) \Big)}{\langle g, g \rangle^2 - \langle g|R_p, g \rangle^2} |c_g(|D|)|^2
\end{aligned}$$

under the condition $g|R_p \neq \pm g$ and $\langle g, g \rangle \neq \pm \langle g|R_p, g \rangle$. Therefore we have Kohnen-Zagier's formula (in the case of level $4p^{2m}$)

$$\frac{\Big(\langle g, g \rangle - \langle g|R_p, g \rangle \left(\frac{|D|}{p}\right) \Big)}{\langle g, g \rangle^2 - \langle g|R_p, g \rangle^2} |c_g(|D|)|^2 = \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}.$$

(cf. Sakata [12, Theorem 6; Theorem 8]).

§7. Generalization of Luo-Ramakrishnan's result

Let k be an integer with $k \geq 2$, N be a positive odd integer, and τ be the eigen function defined in Theorem 3 in Section 5. The aim of this section is to establish the following theorem by using Theorem 3 and the same method as Luo-Ramakrishnan [11].

THEOREM 5. *Let κ be the eigen function with the parity condition (τ_{odd}). Furthermore, let $g_1(\tau)$, $g_2(\tau)$ be Hecke eigen forms in $\mathfrak{N}^{K,\kappa}_{k+1/2}(N)$, with n-th Fourier coefficients $c_{g_1}(n)$, $c_{g_2}(n)$ respectively. Suppose $c_{g_1}(|D|)^2 = c_{g_2}(|D|)^2$ for all (but a finite number of) fundamental discriminants D with $(-1)^k D > 0$. Then $g_1(\tau) = \pm g_2(\tau)$.*

This was given by Luo-Ramakrishnan (cf. [11, Theorem E]) in the case of N being an odd and 'square-free' integer. Our approach is essentially the same as [11].

Proof. For $i = 1, 2$, let $f_i(z)$ denote the primitive form in $S_{2k}(N)$ associated to $g_i(\tau)$ through the Shimura correspondence. Let τ_i denote the eigen system of $f_i(z)$ under the Atkin-Lehner operator. Then we can take a fundamental discriminant D_0 satisfying

$$(-1)^k D_0 > 0,\ (D_0, N) = 1 \text{ and } \tau_i(p) = \left(\frac{D_0}{p}\right) \text{ for all } p \in I(N) + \Pi(N)_{\mathrm{odd}}$$

by using the same argument as in [11]. For this D_0, we define the set $\mathfrak{D}$ consisting of positive integers D satisfying

$$\mu(D) \neq 0 \text{ and } D \equiv \nu^2 \pmod{4ND_0} \text{ for some } \nu \text{ coprime to } 4ND_0.$$

From the definition of $\mathfrak{D}$, we see that for every $D \in \mathfrak{D}$, DD_0 satisfies the following conditions:

$$(-1)^k DD_0 > 0,\ (DD_0, N) = 1 \text{ and}$$
$$\tau_1(p) = \tau_2(p) = \left(\frac{DD_0}{p}\right) \text{ for all } p \in I(N) + \Pi(N)_{\mathrm{odd}}.$$

On the other hand, $f_i \otimes \left(\frac{D_0}{}\right)$ is an primitive form (for each i) as $(D_0, N) = 1$. So the hypothesis of this theorem implies, thanks to Theorem 3 (with D replaced by DD_0), the following identity:

$$L(f_1 \otimes \left(\frac{D_0}{}\right), \left(\frac{D}{}\right), k) = \frac{\langle f_1, f_1\rangle \langle g_2, g_2\rangle}{\langle f_2, f_2\rangle \langle g_1, g_1\rangle} L(f_2 \otimes \left(\frac{D_0}{}\right), \left(\frac{D}{}\right), k)$$

for all D belonging to the set $\mathfrak{D}$.

By using the remark following Theorem B in [11], we conclude that $f_1 \otimes \left(\frac{D_0}{}\right) = f_2 \otimes \left(\frac{D_0}{}\right)$, which implies $f_1 = f_2$. Therefore we see that $g_1 = kg_2$ for some $k \in \mathbf{C}$ by using Theorem 1. But then, from hypothesis we deduce that $g_1 = \pm g_2$. $\square$

From Theorem 5, we obtain the following result immediately.

Corollary 2. *Let $g_1(\tau)$, $g_2(\tau)$ be Hecke eigen forms in $\mathfrak{N}^K_{k+1/2}(N)$ with n-th Fourier coefficients $c_{g_1}(n)$, $c_{g_2}(n)$ respectively. Suppose $c_{g_1}(|D|) = c_{g_2}(|D|)$ for all (but a finite number of) fundamental discriminants D with $(-1)^k D > 0$. Then $g_1(\tau) = g_2(\tau)$.*

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References

[1] E. M. Baruch and Z. Mao, *Central value of automorphic L-functions*, preprint (2003).

[2] B. Gross, W. Kohnen and D. Zagier, *Heegner Points and Derivatives of L-series*, Math. Ann., **278** (1987), 497–562.

[3] W. Kohnen, *New forms of half-integral weight*, J. reine und angew. Math., **333** (1982), 32–72.

[4] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann., **271** (1985), 237–268.

[5] W. Kohnen, *A Remark on the Shimura correspondence*, Glasgow Math. J., **30** (1988), 285–291.

[6] W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math., **64** (1981), 175–198.

[7] H. Kojima, *Remark on Fourier coefficients of modular forms of half-integral weight belonging to Kohnen's spaces II*, Kodai Math. J., **22** (1999), 99–115.

[8] H. Kojima, *On the Fourier coefficients of Maass wave forms of half integral weight over an imaginary quadratic field*, J. reine und angew. Math., **526** (2000), 155–179.

[9] H. Kojima, *On the Fourier coefficients of Jacobi forms of index N over totally real number fields*, preprint (2003).

[10] H. Kojima and Y. Tokuno, *On the Fourier coefficients of modular forms of half integral weight belonging to Kohnen's spaces and the critical values of zeta functions*, Tohoku Math. J., **56** (2004), 125–145.

[11] W. Luo and D. Ramakrishnan, *Determination of modular forms by twists of critical L-values*, Invent. Math., **130** (1997), 371–398.

[12] H. Sakata, *On the Kohnen-Zagier Formula in the case of level* $4p^m$, Math. Zeit., **250** (2005), 257–266.

[13] G. Shimura, *On modular forms of half integral weight*, Ann. of Math., **97** (1973), 440–481.

[14] G. Shimura, *On the Fourier coefficients of Hilbert modular forms of half-integral weight*, Duke Math. J., **71** (1993), 501–557.

[15] M. Ueda, *The Decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators*, J. Math. Kyoto Univ., **28** (1988), 505–558.

[16] M. Ueda, *New forms of half-integral weight and the twisting operators*, Proc. Japan. Acad., **66** (1990), 173–175.

[17] M. Ueda, *On twisting operators and New forms of half-integral weight*, Nagoya Math. J., **131** (1993), 135–205.

[18] M. Ueda, *On twisting operators and New forms of half-integral weight II: complete theory of new forms for Kohnen space*, Nagoya Math. J., **149** (1998), 117–171.

[19] M. Ueda, *On twisting operators and New forms of half-integral weight III: subspace corresponding to very new forms*, Comm. Math. Univ. Sancti Pauli, **50** (2001), 1–27.

[20] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demientier*, J. Math. Pure Appl., **60** (1981), 375–484.

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