K. Hirata
Nagoya Math. J.
Vol. 188 (2007), 1–18

# ESTIMATES FOR THE PRODUCTS OF THE GREEN FUNCTION AND THE MARTIN KERNEL

KENTARO HIRATA

**Abstract.** Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in \Omega$ be fixed. By $G_\Omega$ and $K_\Omega$ we denote the Green function and the Martin kernel for $\Omega$, respectively. Under a certain assumption on $\Omega$ near a boundary point $\xi$, we show that the product $G_\Omega(x, x_0)K_\Omega(x, \xi)$ is comparable to $|x - \xi|^{2-n}$ for $x$ in a nontangential cone with vertex at $\xi$. We also give an estimate for the product $K_\Omega(x, \xi)K_\Omega(x, \eta)$ in a uniform domain, where $\eta$ is another boundary point.

## §1. Introduction

The purpose of this paper is to show a relationship between the boundary decay of the Green function and the boundary growth of the Martin kernel. This is motivated by the results [9], [10], [11], [12], [15] concerned with the boundary decay of the Green function for a Lipschitz domain and the result [18] concerned with the boundary growth of the Martin kernel near singularity. Now, we denote a point in $\mathbb{R}^n$ by $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Theorem A. *Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\phi(0') = 0$, and let $\Phi = \{(x', x_n) : x_n > \phi(x')\}$. Denote by $G_\Phi(\,\cdot\,, e)$ and $K_\Phi(\,\cdot\,, o)$ the Green function for $\Phi$ with pole at $e = (0', 1)$ and the Martin kernel of $\Phi$ with pole at $o = (0', 0)$, respectively. Define*

$$I^+ = \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n}\, dx', \quad I^- = \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n}\, dx'.$$

*Then the following statements hold.*

(i) *If $I^+ < +\infty$ and $I^- = +\infty$, then*

$$\lim_{t \to 0+} \frac{G_\Phi(te, e)}{t} = +\infty \quad \text{and} \quad \lim_{t \to 0+} \frac{K_\Phi(te, o)}{t^{1-n}} = 0.$$

Received January 25, 2006.
Revised March 24, 2006, August 9, 2006.
2000 Mathematics Subject Classification: 31B25.

(ii) *If* $I^+ = +\infty$ *and* $I^- < +\infty$*, then*

$$\lim_{t\to 0+} \frac{G_\Phi(te,e)}{t} = 0 \quad \textit{and} \quad \lim_{t\to 0+} \frac{K_\Phi(te,o)}{t^{1-n}} = +\infty.$$

(iii) *If* $I^+ < +\infty$ *and* $I^- < +\infty$*, then* $\lim_{t\to 0+} G_\Phi(te,e)/t$ *and* $\lim_{t\to 0+} K_\Phi(te,o)/t^{1-n}$ *exist, and each of them is positive and finite.*

The proof of Theorem A was based on the convergence of $I^+$, $I^-$ and the minimal fine topology. The following question is natural: is the product $G_\Phi(te,e)K_\Phi(te,o)$ comparable to $t^{2-n}$ for $0 < t < 1/2$? We shall show such an estimate in more general domains. Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $\delta_\Omega(x)$ stand for the distance from $x$ to the boundary $\partial\Omega$. By $B(x,r)$ and $S(x,r)$, we denote the open ball and the sphere of center $x$ and radius $r$, respectively.

Definition 1.1. We say that $\xi \in \partial\Omega$ satisfies *a local carrot condition* (abbreviated to LCC) if there exist constants $\kappa \geq 2$, $r_\xi > 0$ and $A_\xi \geq 1$ with the following property: for each positive $r \leq r_\xi$, there is a point $y_r \in \Omega \cap S(\xi,r)$ with $\delta_\Omega(y_r) \geq r/A_\xi$ such that each $x \in \Omega \cap B(\xi, r/\kappa)$ can be connected to $y_r$ by a curve $\gamma$ in $\Omega \cap B(\xi,\kappa r)$ for which

$$\ell(\gamma(x,z)) \leq A_\xi \delta_\Omega(z) \quad \text{for all } z \in \gamma, \tag{1.1}$$

where $\ell(\gamma(x,z))$ denotes the length of the subarc $\gamma(x,z)$ of $\gamma$ from $x$ to $z$.

*Remark* 1.2. In the study of minimal Martin boundary points of a John domain, Aikawa, Lundh and the author introduced the notion "a system of local reference points" by using the quasi-hyperbolic metric instead of the stronger condition (1.1). See [4, Definition 2.1]. For the above question, we do not need to assume a global condition on $\Omega$, so we adopt (1.1) and the terminology "a local carrot condition".

Let $x_0 \in \Omega$ be fixed and $\alpha > 1$. A nontangential cone at $\xi \in \partial\Omega$ is denoted by

$$\Gamma_\alpha(\xi) = \{x \in \Omega \cap B(\xi, \delta_\Omega(x_0)/2) : |x - \xi| \leq \alpha\delta_\Omega(x)\}.$$

Note that $\Gamma_\alpha(\xi) \cap B(\xi,r)$ is nonempty for each $r > 0$ whenever (1.1) holds and $\alpha \geq A_\xi$. By the symbol $A$, we denote an absolute positive constant whose value is unimportant and may change from line to line. For two positive functions $f_1$ and $f_2$, we write $f_1 \approx f_2$ if there exists a constant $A \geq 1$ such that $f_1/A \leq f_2 \leq Af_1$. The constant $A$ will be called the constant of comparison. The LCC at $\xi$ implies that $\xi$ has a unique Martin kernel (see Lemma 2.5). By $G_\Omega(\,\cdot\,, x_0)$ and $K_\Omega(\,\cdot\,, \xi)$, we denote the Green function for $\Omega$ with pole at $x_0$ and the Martin kernel of $\Omega$ at $\xi$, respectively.

THEOREM 1.3. *Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$, $n \geq 3$, and suppose that $\xi \in \partial\Omega$ satisfies the LCC. Then*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \approx |x - \xi|^{2-n} \quad \text{for } x \in \Gamma_\alpha(\xi), \tag{1.2}$$

*where the constant of comparison depends only on $\alpha$, $\xi$ and $\Omega$.*

*Remark* 1.4. In Section 4, we give a bounded domain such that (1.2) fails to hold, which is also a simple counterexample to the 3G inequality.

We say that $\Omega$ is *a uniform domain* if there exists a constant $A_0 \geq 1$ such that each pair of points $x, y \in \overline{\Omega}$ can be connected by a curve $\gamma$ with $\gamma \setminus \{x, y\} \subset \Omega$ for which

$$\begin{aligned} &\ell(\gamma) \leq A_0|x - y|, \\ &\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A_0\delta_\Omega(z) \quad \text{for all } z \in \gamma. \end{aligned} \tag{1.3}$$

If $\Omega$ is a uniform domain, then all boundary points satisfy the LCC. Moreover, the constant of comparison in (1.2) can be taken independently of $\xi \in \partial\Omega$.

COROLLARY 1.5. *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$, $n \geq 3$. Then*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \approx |x - \xi|^{2-n} \quad \text{for } \xi \in \partial\Omega \text{ and } x \in \Gamma_\alpha(\xi),$$

*where the constant of comparison depends only on $\alpha$ and $\Omega$.*

Only the upper bound in Corollary 1.5 follows from the following 3G inequality. Let $\Omega$ be a bounded uniform domain in $\mathbb{R}^n$, $n \geq 3$. Then there exists a constant $A$ depending only on $\Omega$ such that

$$\frac{G_\Omega(x, y)G_\Omega(x, z)}{G_\Omega(y, z)} \leq A\left(|x - y|^{2-n} + |x - z|^{2-n}\right) \quad \text{for } x, y, z \in \Omega. \tag{1.4}$$

See Cranston-Fabes-Zhao [13] for Lipschitz domains and Aikawa-Lundh [5] for uniformly John domains, and also Bogdan [8] and Hansen [17] in which a certain global estimate for the Green function was obtained. If we let $z = x_0$ and let $y \to \xi \in \partial\Omega$, then for $x \in \Omega \cap B(\xi, \delta_\Omega(x_0)/2)$,

$$K_\Omega(x,\xi)G_\Omega(x,x_0) \le A(|x-\xi|^{2-n} + |x-x_0|^{2-n}) \le A|x-\xi|^{2-n}.$$

Corollary 1.5 asserts that the product $G_\Omega(\,\cdot\,,x_0)K_\Omega(\,\cdot\,,\xi)$ is bounded from below by the function $|\cdot - \xi|^{2-n}$ as well.

The 3G inequality in two dimensions was proved by Bass-Burdzy [7]: for any bounded domains $\Omega$ in $\mathbb{R}^2$, there exists a constant $A$ depending only on $\Omega$ such that

$$\frac{G_\Omega(x,y)G_\Omega(x,z)}{G_\Omega(y,z)} \le A\left(1 + \log^+ \frac{1}{|x-y|} + \log^+ \frac{1}{|x-z|}\right) \quad \text{for } x,y,z \in \Omega.$$

If $\Omega$ is a bounded uniform domain in $\mathbb{R}^2$, then the same reasoning as above gives that for $x \in \Omega$ close to $\xi \in \partial\Omega$,

$$K_\Omega(x,\xi)G_\Omega(x,x_0) \le A \log \frac{1}{|x-\xi|}.$$

When $\xi$ is an isolated boundary point (i.e. $B(\xi,\varepsilon) \setminus \{\xi\} \subset \Omega$ for some $\varepsilon > 0$), this is sharp. Indeed, letting $\delta = \min\{1, \varepsilon, |x_0 - \xi|\}/2$, we obtain by the Harnack inequality that for $x \in B(\xi,\delta) \setminus \{\xi\}$,

$$K_\Omega(x,\xi) = \frac{G_{\Omega\cup\{\xi\}}(x,\xi)}{G_{\Omega\cup\{\xi\}}(x_0,\xi)} \ge \frac{G_{B(\xi,2\delta)}(x,\xi)}{AG_\Omega(x_0,x)} \ge \frac{2\delta}{AG_\Omega(x,x_0)} \log \frac{1}{|x-\xi|}.$$

However, if $\Omega$ is the unit disc of $\mathbb{R}^2$, then $K_\Omega(r\xi,\xi)G_\Omega(r\xi,o) \approx 1$ for $\xi \in \partial\Omega$ and $1/2 < r < 1$. To obtain comparison estimate (1.2) for $n = 2$, we need some exterior condition. Let us define the Green capacity of a compact set $E$ in an open set $U$ by

$$\mathrm{Cap}_U(E) = \mu(U),$$

where $\mu$ is the associated Riesz measure of the regularized reduced function $\widehat{R}_1^E$ on $U$. We say that $\xi \in \partial\Omega$ satisfies *a capacity density condition* (abbreviated to CDC) if there exist constants $r'_\xi > 0$ and $A'_\xi > 0$ such that

$$\inf_{0<r<r'_\xi} \mathrm{Cap}_{B(\xi,2r)}(\overline{B(\xi,r)} \setminus \Omega) \ge A'_\xi.$$

THEOREM 1.6. *Let $\Omega$ be a proper subdomain of $\mathbb{R}^2$, and suppose that $\xi \in \partial\Omega$ satisfies the LCC and the CDC. Then*

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx 1 \quad \textit{for } x \in \Gamma_\alpha(\xi),$$

*where the constant of comparison depends only on $\alpha$, $\xi$ and $\Omega$.*

A uniform domain $\Omega$ is said to be *NTA* if there are constants $r_0 > 0$ and $A > 1$ such that for each $\xi \in \partial\Omega$ and $0 < r < r_0$, there is a ball $B(z, r/A)$ contained in $B(\xi, r) \setminus \Omega$. Observe that all boundary points of an NTA domain satisfy the CDC, and the constants $r'_\xi$ and $A'_\xi$ can be taken uniformly for $\xi \in \partial\Omega$.

COROLLARY 1.7. *Let $\Omega$ be an NTA domain in $\mathbb{R}^2$. Then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1 \quad \textit{for } \xi \in \partial\Omega \textit{ and } x \in \Gamma_\alpha(\xi),$$

*where the constant of comparison depends only on $\alpha$ and $\Omega$.*

*Remark* 1.8. Since the Green function and the Martin kernel are conformal invariant (cf. [14, Section 6.3]), it is easy to see that if $\Omega$ is a Jordan domain in $\mathbb{R}^2$ and $\xi \in \partial\Omega$, then $G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1$ for $x \in \psi^{-1}(\{(r, 0) : 1/2 < r < 1\})$, where $\psi$ is a conformal mapping from $\Omega$ onto the unit disc such that $\psi(x_0) = (0, 0)$ and $\psi(\xi) = (1, 0)$. In view of this, the LCC is not essential when $n = 2$. However $\partial\Omega$ does not need to be a Jordan curve and may have infinitely many components.

Without the assumptions on $I^+$, $I^-$ in Theorem A, we can obtain the following relationships as a consequence of Corollaries 1.5 and 1.7.

COROLLARY 1.9. *Let $\Phi$ be as in Theorem A and let $\alpha > 0$. Then the following hold*:

(i) $\displaystyle \liminf_{t \to 0} \frac{G_\Phi(te, e)}{t^\alpha} = 0$ *if and only if* $\displaystyle \limsup_{t \to 0} \frac{K_\Phi(te, o)}{t^{2-n-\alpha}} = +\infty$.

(ii) $\displaystyle \limsup_{t \to 0} \frac{G_\Phi(te, e)}{t^\alpha} = +\infty$ *if and only if* $\displaystyle \liminf_{t \to 0} \frac{K_\Phi(te, o)}{t^{2-n-\alpha}} = 0$.

Next, we give an estimate for the product of two Martin kernels with different singularities in a uniform domain. Let $\xi, \eta \in \partial\Omega$ and let $\gamma$ be a curve connecting $\xi$ and $\eta$ such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and (1.3) holds. We denote by $z_{\xi,\eta}$ the middle point of $\gamma$ so that $\ell(\gamma(\xi, z_{\xi,\eta})) = \ell(\gamma(z_{\xi,\eta}, \eta)) = \ell(\gamma)/2$, and define

$$g(\xi, \eta) = \max\left\{1, \frac{|\xi - \eta|^{2-n}}{G_\Omega(z_{\xi,\eta}, x_0)^2}\right\}.$$

Theorem 1.10. *Let $\Omega$ be a bounded uniform domain in $\mathbb{R}^n$, $n \geq 2$, and let $\xi, \eta \in \partial\Omega$ be distinct. Suppose that $\gamma$ is a curve connecting $\xi$ and $\eta$ such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and* (1.3) *holds. Then the following statements hold.*

(i) *If $n \geq 3$, then*

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx g(\xi,\eta)\big(|x-\xi|^{2-n}+|x-\eta|^{2-n}\big) \quad \text{for } x \in \gamma, \tag{1.5}$$

*where the constant of comparison depends only on $\Omega$.*

(ii) *If $n = 2$ and $\Omega$ is a bounded NTA domain, then* (1.5) *holds.*

Corollary 1.11. *Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$, and let $\xi, \eta \in \partial\Omega$ be distinct. Suppose that $\gamma$ is a curve connecting $\xi$ and $\eta$ such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and* (1.3) *holds. Then*

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx \frac{1}{|\xi-\eta|^n}\big(|x-\xi|^{2-n}+|x-\eta|^{2-n}\big) \quad \text{for } x \in \gamma,$$

*where the constant of comparison depends only on $\Omega$.*

## §2. Preparatory material

Throughout this section, we suppose that $\Omega$ is a proper subdomain of $\mathbb{R}^n$, $n \geq 2$. The quasi-hyperbolic metric on $\Omega$ is defined by

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curves $\gamma$ in $\Omega$ connecting $x$ and $y$, and $ds$ stands for the line element on $\gamma$. We say that $\{B(x_j, \delta_\Omega(x_j)/2)\}_{j=1}^N$ is a Harnack chain joining $x$ and $y$ in $\Omega$ if $x_1 = x$, $x_N = y$ and $x_{j+1} \in B(x_j, \delta_\Omega(x_j)/2)$ for $j = 1, \ldots, N-1$. The number $N$ is called the length of the Harnack chain. Observe that the shortest length of the Harnack chain joining $x$ and $y$ in $\Omega$ is comparable to $k_\Omega(x,y)+1$. The following Harnack inequality is valid.

Lemma 2.1. *There exists a constant $A > 1$ depending only on the dimension $n$ such that*

$$\exp(-A(k_\Omega(x,y)+1)) \leq \frac{h(x)}{h(y)} \leq \exp(A(k_\Omega(x,y)+1)) \quad \text{for } x, y \in \Omega,$$

*whenever $h$ is a positive harmonic function on $\Omega$.*

To apply Lemma 2.1 to the Green function, we need the following lemma (cf. [4, Lemma 7.2]).

LEMMA 2.2. *Let $z \in \Omega$. Then*

$$k_{\Omega\setminus\{z\}}(x,y) \le 3k_\Omega(x,y) + \pi \quad \textit{for } x, y \in \Omega \setminus B(z, \delta_\Omega(z)/2).$$

LEMMA 2.3. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Then there exists a constant $A$ depending only on $A_\xi$ such that if $0 < r < r_\xi$, then*

$$k_{\Omega\cap B(\xi,\kappa^3 r)}(x, y_r) \le A \log \frac{r}{\delta_\Omega(x)} + A \quad \textit{for } x \in \Omega \cap B(\xi, r/\kappa),$$

*where $y_r \in \Omega \cap S(\xi, r)$ is as in Definition* 1.1.

*Proof.* This follows from (1.1). ∎

LEMMA 2.4. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Let $0 < r < r_\xi$. If $z, w \in \Omega \setminus B(\xi, \kappa^3 r)$, then*

$$\frac{G_\Omega(x,z)}{G_\Omega(x,w)} \approx \frac{G_\Omega(y,z)}{G_\Omega(y,w)} \quad \textit{for } x, y \in \Omega \cap B(\xi, r/\kappa^3),$$

*where the constant of comparison depends only on $r_\xi$, $A_\xi$ and $\Omega$.*

*Proof.* This can be proved by the similar way as in [4], so we just sketch the proof. Note from Lemma 2.3 that $\xi$ has a system of local reference points $y_r$ of order 1 (see [4, Definition 2.1] for its definition). The existence of a curve with (1.1) shows that there is $\tau > 0$ such that $\int_{\Omega\cap B(\xi,r)} (r/\delta_\Omega(x))^\tau \, dx \le Ar^n$ for $0 < r < r_\xi$ (see [4, Lemma 4.1]). As in [4, Lemma 5.1], we can obtain the following Carleson estimate: for $x \in \Omega \cap S(\xi, r/\kappa^2)$ and $z \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$G_\Omega(x,z) \le AG_\Omega(y_r, z). \tag{2.1}$$

Let $\omega(x, E, U)$ denote the harmonic measure of a Borel set $E$ for an open set $U$ evaluated at $x$. Then the similar argument to [4, Lemma 6.1] gives that for $x \in \Omega \cap B(\xi, r/\kappa^3)$ and $w \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$\omega(x, \Omega \cap S(\xi, r/\kappa^2), \Omega \cap B(\xi, r/\kappa^2)) \le A \frac{G_\Omega(x,w)}{G_\Omega(y_r,w)}. \tag{2.2}$$

Therefore the maximum principle, together with (2.1) and (2.2), yields that for $x \in \Omega \cap B(\xi, r/\kappa^3)$ and $z, w \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$G_\Omega(x,z) \le A \frac{G_\Omega(y_r,z)}{G_\Omega(y_r,w)} G_\Omega(x,w).$$

Changing the roles of $z$ and $w$, we obtain the opposite inequality. Thus the lemma follows. ∎

Let $\xi \in \partial\Omega$ and let $\{y_j\}$ be a sequence in $\Omega$ converging to $\xi$. Observe that there is a subsequence $\{y_{j_k}\}$ such that $\{G_\Omega(\,\cdot\,, y_{j_k})/G_\Omega(x_0, y_{j_k})\}$ converges to a positive harmonic function on $\Omega$. We call such a limit function *the Martin kernel of* $\Omega$ (*with pole*) *at* $\xi$. A positive harmonic function $h$ is said to be *minimal* if every positive harmonic function less than or equal to $h$ coincides with a constant multiple of $h$.

LEMMA 2.5. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Then $\xi$ has a unique Martin kernel and it is minimal.*

*Proof.* This follows from Lemma 2.4 and the Martin representation theorem. ∎

## §3. Proofs of Theorems 1.3 and 1.6

*Proof of Theorem* 1.3. Suppose that $\xi \in \partial\Omega$ satisfies the LCC and put

$$A_1 = \max\left\{\kappa^3, \frac{\delta_\Omega(x_0)}{r_\xi}\right\}.$$

We may assume without loss of generality that $r_\xi \le \delta_\Omega(x_0)/2$. Let $x \in \Gamma_\alpha(\xi)$ and let $r = |x-\xi|/(\kappa^3 A_1)$. Then $\kappa^3 r < r_\xi$, since $|x-\xi| < \delta_\Omega(x_0) \le A_1 r_\xi$. Also, we have $|x-\xi| \ge \kappa^6 r$ and $|x_0-\xi| \ge \delta_\Omega(x_0) \ge |x-\xi| \ge \kappa^6 r$. Let $y_r \in \Omega \cap S(\xi, r)$ be such that $\delta_\Omega(y_r) \ge r/A_\xi$. Then Lemma 2.4 gives

$$\frac{G_\Omega(x,y)}{G_\Omega(x_0,y)} \approx \frac{G_\Omega(x,y_r)}{G_\Omega(x_0,y_r)} \quad \text{for } y \in \Omega \cap B(\xi, r).$$

Letting $y \to \xi$, we obtain

$$K_\Omega(x,\xi) \approx \frac{G_\Omega(x,y_r)}{G_\Omega(x_0,y_r)}. \tag{3.1}$$

We claim

$$G_\Omega(x_0, y_r) \approx G_\Omega(x_0, x). \tag{3.2}$$

To show this, we consider two cases.

**Case 1:** $\rho := \kappa|x-\xi| < r_\xi$. The LCC and Lemma 2.3 show that there is $y_\rho \in \Omega \cap S(\xi,\rho)$ with $\delta_\Omega(y_\rho) \ge \rho/A_\xi$ such that

$$k_\Omega(z, y_\rho) \le A \log \frac{\rho}{\delta_\Omega(z)} + A \quad \text{for } z \in \Omega \cap \overline{B(\xi, \rho/\kappa)}.$$

Observe that $x, y_r \in \Omega \cap \overline{B(\xi,\rho/\kappa)}$, $\delta_\Omega(x) \ge |x-\xi|/\alpha = \rho/(\alpha\kappa)$ and $\delta_\Omega(y_r) \ge \rho/(A_\xi A_1 \kappa^4)$. Therefore

$$k_\Omega(x, y_\rho) \le A \quad \text{and} \quad k_\Omega(y_r, y_\rho) \le A.$$

Since $x, y_r, y_\rho \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$, it follows from Lemmas 2.1 and 2.2 that

$$G_\Omega(x_0, y_r) \approx G_\Omega(x_0, y_\rho) \approx G_\Omega(x_0, x).$$

Thus (3.2) holds in this case.

**Case 2:** $\kappa|x-\xi| \ge r_\xi$. Since $r \ge r_\xi/(A_1\kappa^4)$, it follows from the Harnack inequality on the compact set $\Gamma_\alpha(\xi) \setminus B(\xi, r_\xi/(A_1\kappa^4))$ that $G_\Omega(x_0, y_r) \approx G_\Omega(x_0, x)$, where the constant of comparison depends only on $\xi$ and $\Omega$. Thus (3.2) follows.

We next claim

$$G_\Omega(x, y_r) \approx |x-\xi|^{2-n}. \tag{3.3}$$

Let $w \in S(y_r, \delta_\Omega(y_r)/2)$. Then the similar argument as above gives

$$G_\Omega(x, y_r) \approx G_\Omega(w, y_r) \approx |w - y_r|^{2-n}. \tag{3.4}$$

Since $|w-y_r| \approx r \approx |x-\xi|$, we obtain (3.3). Combining (3.1), (3.2) and (3.3), we complete the proof of Theorem 1.3. ∎

*Proof of Corollary* 1.5. If $\Omega$ is a uniform domain, then $\kappa$, $r_\xi$ and $A_\xi$ can be taken uniformly for $\xi \in \Omega$. Therefore (5.1) gives (3.2) and (3.3) with the comparison constant depending only on $\alpha$ and $\Omega$. ∎

*Proof of Theorem* 1.6. The proofs of (3.1), (3.2) and the first estimate in (3.4) are independent of the dimension. It is enough to show that $G_\Omega(w, y_r) \approx 1$ for $w \in S(y_r, \delta_\Omega(y_r)/2)$. This will be shown in Proposition 3.2 below. $\square$

LEMMA 3.1. *Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $z, w \in \Omega$ satisfy $|z - w| \leq \delta_\Omega(z)/4$. Suppose that $u$ is a subharmonic function on $B(z, \delta_\Omega(z)) \cup B(w, \delta_\Omega(w))$ such that $u \leq M$. If $u \leq (1-\theta)M$ on $B(z, \delta_\Omega(z)/8)$ for some $0 < \theta < 1$, then*

$$u \leq \Big(1 - \Big(\frac{4}{17}\Big)^n \theta\Big) M \quad \text{on } B(w, \delta_\Omega(w)/8).$$

*Proof.* Let $x \in B(w, \delta_\Omega(w)/8)$. Observe that

$$B(z, \delta_\Omega(z)/8) \subset B(x, 17\delta_\Omega(z)/32) \subset B(w, \delta_\Omega(w)).$$

Write $E_1 = B(x, 17\delta_\Omega(z)/32)$ and $E_2 = E_1 \setminus B(z, \delta_\Omega(z)/8)$. By the mean value inequality, we have

$$\begin{aligned} u(x) &\leq \frac{1}{|E_1|}\int_{E_1} u(y)\,dy \leq \frac{1}{|E_1|}\big((1-\theta)M|E_1 \setminus E_2| + M|E_2|\big) \\ &\leq M\Big(1 - \Big(\frac{4}{17}\Big)^n \theta\Big), \end{aligned}$$

where $|E|$ denotes the volume of a set $E$. Thus the lemma follows. $\square$

PROPOSITION 3.2. *Let $\Omega$ be a proper subdomain of $\mathbb{R}^2$ and suppose that $\xi \in \partial\Omega$ satisfies the LCC and the CDC. Then*

$$G_\Omega(x, y) \approx 1 \quad \text{for } x \in \Gamma_\alpha(\xi) \text{ and } y \in S(x, \delta_\Omega(x)/2),$$

*where the constant of comparison depends only on $\alpha$, $\xi$ and $\Omega$.*

*Proof.* Clearly, $G_\Omega(x, y) \geq G_{B(x,\delta_\Omega(x))}(x, y) \approx 1$ for $y \in S(x, \delta_\Omega(x)/2)$. Let us show

$$G_\Omega(x, y) \leq A \quad \text{for } x \in \Gamma_\alpha(\xi) \text{ and } y \in S(x, \delta_\Omega(x)/2). \tag{3.5}$$

The method is based on Aikawa [3, Proof of Lemma 2]. The CDC at $\xi$ implies that

$$\mathrm{Cap}_{B(\xi,2r)}(\overline{B(\xi, r)} \setminus \Omega) \geq A \quad \text{whenever } 0 < r < \delta_\Omega(x_0), \tag{3.6}$$

where $A > 0$ depends only on $r'_\xi$, $A'_\xi$ and $\delta_\Omega(x_0)$. Let $r = \delta_\Omega(x)/2$ and let $M = \sup_{S(x,r)} G_\Omega(x, \cdot)$. Then the maximum principle gives that for $z \in \Omega \cap B(\xi, r)$,

$$G_\Omega(x,z) \le M\omega(z, S(x,r), \Omega \setminus \overline{B(x,r)}) \le M\omega(z, S(\xi,r), B(\xi,r) \setminus E),$$

where $E = \overline{B(\xi, r/2)} \setminus \Omega$ and $\omega(z, F, U)$ is the harmonic measure of a set $F$ for an open set $U$ evaluated at $z$. By [1, Lemma 3] and (3.6), we have

$$\sup_{B(\xi,r/2)} \omega(\,\cdot\,, S(\xi,r), B(\xi,r) \setminus E) \le 1 - \frac{1}{A}\operatorname{Cap}_{B(\xi,r)}(E) \le 1 - \theta,$$

where $0 < \theta < 1$. Therefore

$$G_\Omega(x,z) \le M(1-\theta) \quad \text{for } z \in \Omega \cap B(\xi, r/2). \tag{3.7}$$

Fix $z \in \Omega \cap S(\xi, r/4)$ with $\delta_\Omega(z) \ge r/(4\alpha)$, and let $w \in S(x, 3r/2)$. Then $\delta_\Omega(w) \ge r/2$ and $|z - w| \le Ar$. We observe, as in the proof of Theorem 1.3, that

$$k_{\Omega\setminus\{x\}}(z,w) \le 3k_\Omega(z,w) + \pi \le A,$$

where $A$ depends only on $\alpha$, $\xi$ and $\Omega$. Therefore $z$ and $w$ can be joined by $\{B(w_j, \delta_{\Omega\setminus\{x\}}(w_j)/4)\}_{j=1}^N$ such that $w_1 = z$, $w_N = w$ and $w_{j+1} \in B(w_j, \delta_{\Omega\setminus\{x\}}(w_j)/4)$ for $j = 1, \dots, N-1$, where $N$ depends only on $\alpha$, $\xi$ and $\Omega$. Note from (3.7) that $G_\Omega(x, \cdot) \le M(1-\theta)$ on $B(w_1, \delta_{\Omega\setminus\{x\}}(w_1)/8)$. Apply Lemma 3.1 repeatedly. Then

$$G_\Omega(x,w) \le M\Big(1 - \Big(\frac{4}{17}\Big)^{nN}\theta\Big) \quad \text{for } w \in S\Big(x, \frac{3}{2}r\Big). \tag{3.8}$$

Observe that for $y \in B(x, 3r/2)$,

$$G_{B(x,3r/2)}(x,y) = G_\Omega(x,y) - R^{\Omega\setminus\overline{B(x,3r/2)}}_{G_\Omega(x,\,\cdot\,)}(y),$$

where $R^F_{G_\Omega(x,\,\cdot\,)}$ is the reduced function of $G_\Omega(x, \cdot)$ relative to a set $F$ in $\Omega$. By (3.8),

$$\sup_{S(x,r)} G_\Omega(x, \cdot) - M\Big(1 - \Big(\frac{4}{17}\Big)^{nN}\theta\Big) \le \sup_{S(x,r)} G_{B(x,3r/2)}(x, \cdot) = \log\frac{3}{2}.$$

Hence we obtain $M \le \log(3/2) \cdot (17/4)^{nN}/\theta$, and thus (3.5) holds. ∎

## §4. Counterexample

In this section, we give an example of a domain on which (1.2) fails to hold. Let us denote a point $x \in \mathbb{R}^n$ by $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and write $o = (0', 0)$.

EXAMPLE 4.1. Suppose that $n \geq 3$. Let $\Omega$ be the inverse of $\Omega^*$ with respect to $S(o, 1)$, where

$$\Omega^* = \{(x', x_n) : |x'| < 1/2,\ x_n > 0\} \setminus \overline{B(o, 1)}.$$

Let $x_0 = (0', 1/2)$. Then

$$\limsup_{x \to o,\, x \in E} \frac{G_\Omega(x, x_0) K_\Omega(x, o)}{|x|^{2-n}} = +\infty, \tag{4.1}$$

where $E = \{(0', x_n) : 0 < x_n < 1/4\}$.

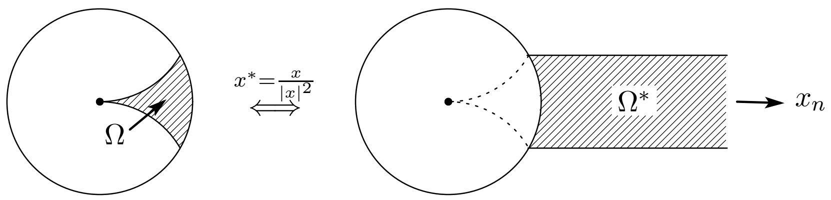


Figure 1: $\Omega$ and $\Omega^*$.

*Proof.* Suppose to the contrary that there is a constant $A$ such that

$$G_\Omega(x, x_0) K_\Omega(x, o) \leq A|x|^{2-n} \quad \text{for } x \in E.$$

Let $K_{\Omega^*}(\,\cdot\,, +\infty)$ denote the Martin kernel of $\Omega^*$ at $+\infty$, i.e. the limit function of $G_{\Omega^*}(\,\cdot\,, (y', y_n))/G_{\Omega^*}(x_0^*, (y', y_n))$ as $y_n \to +\infty$. Since $K_{\Omega^*}(x, +\infty) = (2/|x|)^{n-2} K_\Omega(x/|x|^2, o)$ and $G_{\Omega^*}(x, x_0^*) = (2|x|)^{2-n} G_\Omega(x/|x|^2, x_0)$ for $x \in \Omega^*$, it follows that for $x \in E^*$,

$$\begin{aligned} G_{\Omega^*}(x, x_0^*) K_{\Omega^*}(x, +\infty) &= |x|^{2(2-n)} G_\Omega(x/|x|^2, x_0) K_\Omega(x/|x|^2, o) \\ &\leq A|x|^{2-n}. \end{aligned} \tag{4.2}$$

Let $\omega = \{(x', x_n) : |x'| < 1/2,\ -\infty < x_n < +\infty\}$. Note that $\Omega^* \subset \omega$ and $\Omega^* \cap \{x_n > 1\} = \omega \cap \{x_n > 1\}$, and that the Martin kernels of $\omega$ at $+\infty$ and $-\infty$ are respectively of the form

$$K_\omega(x, +\infty) = e^{\tau x_n} f(x') \quad \text{and} \quad K_\omega(x, -\infty) = A e^{-\tau x_n} f(x'), \tag{4.3}$$

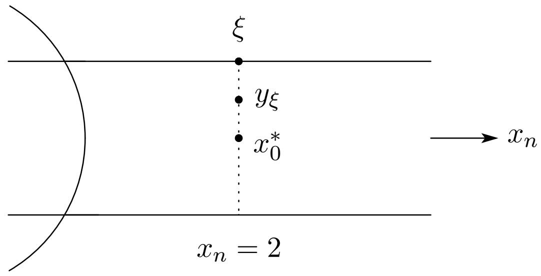


Figure 2: Positions of $\xi$ and $y_\xi$.

where $\tau > 0$ and $A > 0$ are constants and $f$ is a positive function on $\{x' \in \mathbb{R}^{n-1} : |x'| < 1/2\}$ vanishing continuously on $\{x' : |x'| = 1/2\}$. Let $\xi = (\xi', 2) \in \partial\omega$, and let $y_\xi$ be the point in the line segment $\overline{\xi x_0^*}$ such that $|y_\xi - \xi| = 1/4$. The boundary Harnack principle gives

$$\frac{G_{\Omega^*}(y, x_0^*)}{K_\omega(y, -\infty)} \approx \frac{G_{\Omega^*}(y_\xi, x_0^*)}{K_\omega(y_\xi, -\infty)} \quad \text{for } y = (y', 2) \in \omega \cap B(\xi, 1/4),$$

where the constant of comparison is independent of $y$, $y_\xi$ and $\xi$. Observe from the Harnack inequality that $G_{\Omega^*}(y, x_0^*) \geq A > 0$ and $K_\omega(y, -\infty) \approx K_\omega(x_0^*, -\infty) \approx 1$ for $y = (y', 2)$ with $\delta_\omega(y) \geq 1/4$. Therefore

$$K_\omega(y, -\infty) \leq A G_{\Omega^*}(y, x_0^*) \tag{4.4}$$

for $y = (y', 2) \in (\omega \cap B(\xi, 1/4)) \cup \{\delta_\omega(y) \geq 1/4\}$. The arbitrariness of $\xi = (\xi', 2) \in \partial\omega$ shows that (4.4) holds for all $y = (y', 2) \in \omega$, and so for all $y \in \{(y', y_n) \in \omega : y_n \geq 2\}$ by the maximum principle. It follows from (4.2) and (4.3) that for $x \in E^*$,

$$\frac{K_{\Omega^*}(x, +\infty)}{K_\omega(x, +\infty)} \approx K_\omega(x, -\infty) K_{\Omega^*}(x, +\infty) \leq A|x|^{2-n}.$$

As $x \in E^*$ and $x_n \to +\infty$, we have a contradiction, because

$$\limsup_{x_n \to +\infty} \frac{K_{\Omega^*}((0', x_n), +\infty)}{K_\omega((0', x_n), +\infty)} > 0 \tag{4.5}$$

(see Remark 4.2 below). Hence (4.1) holds. □

*Remark* 4.2. We see from [6, Theorems 9.2.6 and 9.3.3] that

$$\limsup_{x_n\to+\infty}\frac{K_{\Omega^*}((x',x_n),+\infty)}{K_\omega((x',x_n),+\infty)}>0.$$

As in the proof of Example 4.1, the boundary Harnack principle and the usual Harnack inequality give that for each $x_n \ge 2$,

$$\frac{K_{\Omega^*}((x',x_n),+\infty)}{K_\omega((x',x_n),+\infty)}\approx\frac{K_{\Omega^*}((0',x_n),+\infty)}{K_\omega((0',x_n),+\infty)}\quad\text{for } |x'|<1/2.$$

Thus (4.5) follows.

*Remark* 4.3. Aikawa and Lundh [5] constructed a bounded domain in $\mathbb{R}^n$, $n \ge 3$, such that 3G inequality (1.4) fails to hold. A domain $\Omega$ in Example 4.1 is also one of conterexamples to (1.4). Indeed, as stated in the introduction, (1.4) implies that $G_\Omega(x,x_0)K_\Omega(x,o) \le A|x|^{2-n}$ for $x \in \Omega$ close to $o$. But this contradicts (4.1).

## §5. Proof of Theorem 1.10

If $\Omega$ is a uniform domain, then the constants $\kappa$, $r_\xi$ and $A_\xi$ in (1.1) can be taken uniformly for $\xi \in \partial\Omega$. In this case, Lemma 2.4 is restated as follows: there is a constant $r_1 > 0$ depending only on $\Omega$ such that if $\xi \in \partial\Omega$ and $0 < r \le r_1$, then

$$\frac{G_\Omega(x,z)}{G_\Omega(x,w)}\approx\frac{G_\Omega(y,z)}{G_\Omega(y,w)}$$

for $x, y \in \Omega \cap \overline{B(\xi,r)}$ and $z, w \in \Omega \setminus B(\xi,\kappa^6 r)$, where the constant of comparison depends only on $\Omega$. This was indeed proved in [2] and is called *the uniform boundary Harnack principle* (abbreviated to UBHP). Recall that a uniform domain $\Omega$ is characterized in terms of the quasi-hyperbolic metric (cf. [16]):

$$k_\Omega(x,y)\le A\log\left(\frac{|x-y|}{\min\{\delta_\Omega(x),\delta_\Omega(y)\}}+1\right)+A\quad\text{for } x,y\in\Omega. \tag{5.1}$$

The following lemma is an elementary consequence of (5.1) and Lemma 2.1.

LEMMA 5.1. *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$, $n \ge 3$, or an NTA domain in $\mathbb{R}^2$. If $x, y \in \Omega$ satisfy $\delta_\Omega(y)/2 \le |x-y| \le A_2\min\{\delta_\Omega(x),\delta_\Omega(y)\}$ for some constant $A_2$, then*

$$G_\Omega(x,y)\approx|x-y|^{2-n},$$

*where the constant of comparison depends only on $A_2$ and $\Omega$.*

*Proof of Theorem* 1.10. We give a proof only when $n \ge 3$. We may assume without loss of generality that $\delta_\Omega(x_0) \ge (\kappa^6+2)A_0 r_1$, where $A_0$ is the constant in (1.3). Let $\xi, \eta \in \partial\Omega$ be distinct and let $\gamma$ be a curve connecting $\xi$ and $\eta$ such that $\gamma \setminus \{\xi,\eta\} \subset \Omega$ and (1.3) holds. Put $r = |\xi-\eta|/(\kappa^6+2)$. We consider two cases.

**Case 1:** $r \le r_1$. Let $x \in \gamma \cap \overline{B(\xi,r)}$. Then $x, x_0 \in \Omega \setminus B(\eta, \kappa^6 r)$. The UBHP gives

$$K_\Omega(x,\eta) \approx \frac{G_\Omega(x,w_\eta)}{G_\Omega(x_0,w_\eta)}, \tag{5.2}$$

where $w_\eta \in \gamma \cap S(\eta,r) \subset \Omega\setminus B(\xi,\kappa^6 r)$. We again apply the UBHP to obtain

$$\frac{G_\Omega(x,w_\eta)}{G_\Omega(x,x_0)} \approx \frac{G_\Omega(w_\xi,w_\eta)}{G_\Omega(w_\xi,x_0)}, \tag{5.3}$$

where $w_\xi \in \gamma \cap S(\xi,r)$. Note from (1.3) that $x \in \Gamma_{A_0}(\xi)$. Therefore (5.2), (5.3) and Corollary 1.5 give

$$K_\Omega(x,\eta) \approx \frac{G_\Omega(w_\xi,w_\eta)}{G_\Omega(w_\xi,x_0)G_\Omega(w_\eta,x_0)} \frac{|x-\xi|^{2-n}}{K_\Omega(x,\xi)}. \tag{5.4}$$

Let $z_{\xi,\eta}$ be the middle point of $\gamma$. Observe from (1.3) that $\delta_\Omega(w_\xi)$, $\delta_\Omega(w_\eta)$, $\delta_\Omega(z_{\xi,\eta})$ are greater than $r/A_0$, and that $|w_\xi - z_{\xi,\eta}|$, $|w_\eta - z_{\xi,\eta}|$ are bounded by $\ell(\gamma) \le A_0|\xi-\eta| = A_0(\kappa^6+2)r$. Therefore $k_\Omega(w_\xi, z_{\xi,\eta}) \le A$ and $k_\Omega(w_\eta, z_{\xi,\eta}) \le A$ by (5.1). Since $w_\xi, w_\eta, z_{\xi,\eta} \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$, it follows from Lemmas 2.1 and 2.2 that

$$G_\Omega(w_\xi,x_0) \approx G_\Omega(z_{\xi,\eta},x_0) \approx G_\Omega(w_\eta,x_0). \tag{5.5}$$

Also, we have by Lemma 5.1

$$G_\Omega(w_\xi,w_\eta) \approx |w_\xi - w_\eta|^{2-n} \approx r^{2-n} \approx |\xi-\eta|^{2-n}. \tag{5.6}$$

Combining (5.4), (5.5) and (5.6), we obtain

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx \frac{|\xi-\eta|^{2-n}}{G_\Omega(z_{\xi,\eta},x_0)^2}|x-\xi|^{2-n} \tag{5.7}$$

whenever $x \in \gamma \cap \overline{B(\xi,r)}$. If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi,r)$, then $|x-w_\xi| \le Ar \le A\delta_\Omega(x)$ by (1.3). Therefore Lemma 2.1 and (5.1) give

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx K_\Omega(w_\xi,\xi)K_\Omega(w_\xi,\eta).$$

Since $|x-\xi| \approx r = |w_\xi - \xi|$, it follows from (5.7) with $x = w_\xi$ that (5.7) holds for $x \in \gamma(\xi, z_{\xi,\eta})$. Observe that $|x-\xi|^{2-n} \approx |x-\xi|^{2-n} + |x-\eta|^{2-n}$ for $x \in \gamma(\xi, z_{\xi,\eta})$ and $|\xi-\eta|^{2-n}/G_\Omega(z_{\xi,\eta}, x_0)^2 \geq A(\Omega) > 0$. Hence we obtain

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx g(\xi,\eta)\big(|x-\xi|^{2-n} + |x-\eta|^{2-n}\big) \tag{5.8}$$

for $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we can obtain (5.8) for $x \in \gamma(z_{\xi,\eta}, \eta)$.

**Case 2:** $r > r_1$. Let $x \in \gamma \cap \overline{B(\xi, r_1)}$ and let $w_0 \in \gamma \cap S(\xi, r_1)$. Then

$$K_\Omega(w_0, \eta) \approx 1 \quad \text{and} \quad G_\Omega(w_0, x_0) \approx 1,$$

where the constants of comparisons depend on $r_1$, $\delta_\Omega(x_0)$ and $\operatorname{diam}(\Omega)$. Note that $|\xi - \eta| = (\kappa^6 + 2)r \geq \kappa^6 r_1$. By the UBHP and Corollary 1.5,

$$K_\Omega(x,\eta) \approx \frac{K_\Omega(w_0,\eta)}{G_\Omega(w_0,x_0)} G_\Omega(x,x_0) \approx \frac{|x-\xi|^{2-n}}{K_\Omega(x,\xi)} \approx \frac{|x-\xi|^{2-n} + |x-\eta|^{2-n}}{K_\Omega(x,\xi)}.$$

If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi, r_1)$, then $\delta_\Omega(x) \geq r_1/A_0$ by (1.3), and so

$$K_\Omega(x,\xi) \approx 1 \approx K_\Omega(x,\eta) \quad \text{and} \quad |x-\xi| \approx 1 \approx |x-\eta|,$$

where the constants of comparisons depend on $r_1/A_0$, $\delta_\Omega(x_0)$ and $\operatorname{diam}(\Omega)$. Since $|\xi-\eta|^{2-n}/G_\Omega(z_{\xi,\eta}, x_0)^2 \leq A(\Omega)$, we obtain $K_\Omega(x,\xi)K_\Omega(x,\eta) \approx g(\xi,\eta)$ $(|x-\xi|^{2-n} + |x-\eta|^{2-n})$ for $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we obtain this for $x \in \gamma(z_{\xi,\eta}, \eta)$. Thus the proof of Theorem 1.10 is complete. ∎

*Proof of Corollary* 1.11. Let $\gamma$ be a curve connecting $\xi$ and $\eta$ such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and (1.3) holds, and let $z_{\xi,\eta}$ be the middle point of $\gamma$. Then

$$\frac{1}{2A_0}|\xi-\eta| \leq \frac{1}{A_0}\ell(\gamma(\xi, z_{\xi,\eta})) \leq \delta_\Omega(z_{\xi,\eta}) \leq \ell(\gamma(\xi, z_{\xi,\eta})) \leq A_0|\xi-\eta|.$$

It is known that if $\Omega$ is a bounded $C^{1,1}$-domain, then $G_\Omega(z, x_0) \approx \delta_\Omega(z)$ for $z \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$. Hence Corollary 1.11 follows from Theorem 1.10.

∎

**Acknowledgement.** The author would like to thank Professor Hiroaki Aikawa for encouragement and valuable comments. Also, he is grateful to the referee for helpful suggestions.

## References


[1] H. Aikawa, *Norm estimate of Green operator, perturbation of Green function and integrability of superharmonic functions*, Math. Ann., **312** (1998), no. 2, 289–318.

[2] ———, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan, **53** (2001), no. 1, 119–145.

[3] ———, *Hölder continuity of the Dirichlet solution for a general domain*, Bull. London Math. Soc., **34** (2002), no. 6, 691–702.

[4] H. Aikawa, K. Hirata and T. Lundh, *Martin boundary points of a John domain and unions of convex sets*, J. Math. Soc. Japan, **58** (2006), no. 1, 247–274.

[5] H. Aikawa and T. Lundh, *The 3G inequality for a uniformly John domain*, Kodai Math. J., **28** (2005), no. 2, 209–219.

[6] D. H. Armitage and S. J. Gardiner, Classical potential theory, Springer-Verlag London Ltd., 2001.

[7] R. F. Bass and K. Burdzy, *Conditioned Brownian motion in planar domains*, Probab. Theory Related Fields, **101** (1995), no. 4, 479–493.

[8] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl., **243** (2000), no. 2, 326–337.

[9] K. Burdzy, *Brownian excursions and minimal thinness. Part III: Applications to the angular derivative problem*, Math. Z., **192** (1986), no. 1, 89–107.

[10] ———, Multidimensional Brownian excursions and potential theory, Pitman Research Notes in Mathematics Series, **164**, 1987.

[11] T. Carroll, *A classical proof of Burdzy's theorem on the angular derivative*, J. London Math. Soc. (2), **38** (1988), no. 3, 423–441.

[12] ———, *Boundary behaviour of positive harmonic functions on Lipschitz domains*, Ann. Acad. Sci. Fenn. Math., **27** (2002), no. 1, 231–236.

[13] M. Cranston, E. Fabes and Z. Zhao, *Conditional gauge and potential theory for the Schrödinger operator*, Trans. Amer. Math. Soc., **307** (1988), no. 1, 171–194.

[14] K. L. Chung and Z. Zhao, From Brownian motion to Schrödinger's equation, Springer-Verlag, 1995.

[15] S. J. Gardiner, *A short proof of Burdzy's theorem on the angular derivative*, Bull. London Math. Soc., **23** (1991), no. 6, 575–579.

[16] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math., **36** (1979), 50–74.

[17] W. Hansen, *Uniform boundary Harnack principle and generalized triangle property*, J. Funct. Anal., **226** (2005), no. 2, 452–484.

[18] K. Hirata, *Boundary behaviour of quotients of Martin kernels*, Proc. Edinb. Math. Soc. (2), **50** (2007), no. 2, 377–388.

*Faculty of Education and Human Studies*
*Akita University*
*Akita 010-8502*
*Japan*
hirata@math.akita-u.ac.jp