A NOTE ON THE BERGMAN METRIC OF
BOUNDED HOMOGENEOUS DOMAINS

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Abstract. We show that the Bergman metric of a bounded homogeneous domain has a potential function whose gradient has a constant norm with respect to the Bergman metric, and further that this constant is independent of the choice of such a potential function.

Let D be a bounded domain in $\mathbb{C}^n$ and let ds_D^2 be the Bergman metric on D. In function theory of several variables, the Bergman metric has played an important role as a canonically defined invariant metric.

Recently it was shown that ds_D^2 is complete if D is hyperconvex, i.e. if D admits a bounded plurisubharmonic exhaustion function (cf. [1], [3], [8]). This achievement reflects the developments of pluripotential theory on hyperconvex domains (cf. [2]).

On the other hand, it has long been known that some of the hyperconvex domains, strongly pseudoconvex domains and bounded symmetric domains for instance, share a property stronger than the Bergman completeness.

Namely, the Bergman metric on such domains is not only complete but also admits a potential function whose gradient is bounded, when it is measured with respect to ds_D^2. A remarkable consequence of this property is, according to a recent work of B.-Y. Chen [4], that the L^2 $\overline{\partial}$-cohomology groups of type (p, q) vanish if $p+q \neq n$ and that they are infinite dimensional if $p + q = n$. (See also [6] and [7].)

Therefore, given any domain D with a complete Bergman metric, it is interesting to decide whether or not D enjoys this property. Homogeneous domains are particularly interesting because the description of the actions of the group of biholomorphic automorphisms on the middle-degree L^2 $\overline{\partial}$-cohomology groups may then become a significant project which is very

likely to be profitable.

The purpose of this note is to report the following simple but indispensable observation towards this direction of research.

THEOREM 1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$. Then there exists a real analytic function φ on D satisfying*

$$ds_D^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta} dz_\alpha d\overline{z}_\beta$$

and

$$\left| \sum_{\alpha=1}^{n} \frac{\partial \varphi}{\partial z_\alpha} dz_\alpha \right|_{ds_D^2} = const.$$

Here $|\cdot|_{ds_D^2}$ stands for the length with respect to ds_D^2.

Proof. As was shown in [13], every bounded homogeneous domain say D is biholomorphically equivalent to a domain $\Omega \subset \mathbb{C}^n$ such that the group

$$\{u \in \mathrm{Aut}(\Omega) \mid u \text{ extends to an affine transformation on } \mathbb{C}^n\}$$

acts transitively on Ω.

Let $\sigma : D \to \Omega$ be a biholomorphism, and let $K_D(z)$ (resp. $K_\Omega(w)$) be the Bergman kernel function on D (resp. on Ω). Then we have

$$K_D(z) = K_\Omega(\sigma(z))\big|\det \partial\sigma(z)/\partial z\big|^2,$$
$$ds_D^2 = \sigma^* ds_\Omega^2$$

and

$$ds_D^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K_\Omega(\sigma(z))}{\partial z_\alpha \partial \overline{z}_\beta} dz_\alpha d\overline{z}_\beta.$$

What we want to show is that

$$\big| \partial \log K_\Omega(\sigma(z)) \big|_{ds_D^2} = const.$$

To see this, take any two points $\zeta, \xi \in D$ and let α be an affine transformation such that $\alpha(\Omega) = \Omega$ and $\sigma(\zeta) = \alpha(\sigma(\xi))$.

We note that

$$\partial \log K_\Omega(\alpha(w)) = \partial \log K_\Omega(w)$$

since α is affine.

Let $\tau = \sigma^{-1} \circ \alpha^{-1} \circ \sigma$. Then we have

$$
\begin{aligned}
\left|\left(\partial \log K_\Omega(\sigma(z))\right)\big|_{z=\zeta}\right|_{ds_D^2} &= \left|\left(\partial \log K_\Omega(\alpha(\sigma(\tau(z))))\right)\big|_{z=\zeta}\right|_{ds_D^2} \\
&= \left|\left(\partial \log K_\Omega(\sigma(\tau(z)))\right)\big|_{z=\zeta}\right|_{ds_D^2} \\
&= \left|\tau^*\left(\left(\partial \log K_\Omega(\sigma(z))\right)\big|_{z=\xi}\right)\right|_{ds_D^2} \\
&= \left|\left(\partial \log K_\Omega(\sigma(z))\right)\big|_{z=\xi}\right|_{ds_D^2}.
\end{aligned}
$$

The last equality holds because τ is an isometry with respect to the Bergman metric. $\qquad\square$

As is naturally expected, the following is true.

PROPOSITION. *For any bounded homogeneous domain D, the constant arising as the length of $\partial\varphi$ in Theorem 1 does not depend on the choice of φ.*

Proof. Clearly it suffices to consider Ω instead of D. Let ψ be a C^2 function on Ω satisfying $\partial\overline{\partial}\psi = \partial\overline{\partial}\log K_\Omega$ and $|\partial\psi| = \text{const.}$ We put $\omega = \partial\psi - \partial\log K_\Omega$. Then ω is a holomorphic 1-form on Ω satisfying

$$
d|\partial\log K_\Omega + \omega|^2 = 0.
$$

Let $w_0 \in \Omega$ be any point, let ξ be any tangent vector of Ω at w_0, and let α_t be a 1-parameter group of affine transformations of Ω generated by a vector field X such that $X(w_0) = \xi$.

Since α_t are all isometries with respect to ds_Ω^2, the Lie derivatives of ds_Ω^2 vanish with respect to X. Since α_t are all affine, the Lie derivatives of $\partial\log K_\Omega$ also vanish.

Therefore we obtain

$$
\begin{aligned}
\text{(1)} \qquad 0 &= \xi|\partial\log K_\Omega + \omega|^2 \\
&= \left(X|\partial\log K_\Omega + \omega|^2\right)_{w_0} \\
&= \langle(X\omega)_{w_0}, \omega_{w_0}\rangle + \langle\omega_{w_0}, (X\omega)_{w_0}\rangle.
\end{aligned}
$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product.

For the simplicity of notation we write $\xi\omega$ for $(X\omega)_{w_0}$ and write the result of (1) as

$$
\langle\xi\omega, \omega\rangle + \langle\omega, \xi\omega\rangle = 0.
$$

Similarly, denoting by J the complex structure of Ω, we have

$$\langle (J\xi)\omega, \omega \rangle + \langle \omega, (J\xi)\omega \rangle = 0.$$

Hence

$$\langle (\xi - iJ\xi)\omega, \omega \rangle + \langle \omega, (\xi + iJ\xi)\omega \rangle = 0.$$

Since ω is holomorphic we have

$$(\xi - iJ\xi)\omega = 0.$$

Therefore

$$\langle \omega, (\xi + iJ\xi)\omega \rangle = 0,$$

and hence

$$\langle \omega, \xi\omega \rangle = 0.$$

Since ξ was arbitrary, for any holomorphic vector field $\widetilde{\xi}$ on a neighbourhood of w_0 satisfying $\widetilde{\xi}(w_0) = \xi$ one has

$$\langle \omega, \widetilde{\xi}\omega \rangle = 0$$

near w_0.

Therefore, similarly as above we obtain

$$\langle \xi\omega, \xi\omega \rangle + \langle \omega, \xi(\widetilde{\xi}\omega) \rangle = 0,$$
$$\langle (J\xi)\omega, \xi\omega \rangle + \langle \omega, (J\xi)(\widetilde{\xi}\omega) \rangle = 0,$$
$$\langle (\xi + iJ\xi)\omega, \xi\omega \rangle = 0,$$

so that $\langle \xi\omega, \xi\omega \rangle = 0$.

Hence ω is holomorphic and parallel, which is impossible unless $\omega = 0$.

$\square$

That every bounded homogeneous domain D is equivalent to an affinely homogeneous domain Ω is a consequence of the fact that every bounded homogeneous domain is analytically equivalent to a Siegel domain of the 2nd kind. (See also [13].) Since K_Ω is not exhaustive on Ω (e.g. $K_\Omega(w) = \pi^{-1}(\operatorname{Re} w)^{-2}$ if $\Omega = \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$), it is not clear whether or not Ω (or D) is hyperconvex. Nevertheless, combining Theorem 1 with completeness of the Bergman metrics on (not necessarily homogeneous) Siegel domains of the 2nd kind as was proved by Nakajima [12], together with B.-Y. Chen's theorem, we obtain the following.

THEOREM 2.   *Let (D, ds_D^2) be a bounded homogeneous domain equipped with the Bergman metric and let $H_{(2)}^{p,q}(D)$ be the L^2 $\overline{\partial}$-cohomology group of D of type (p, q). Then*

$$\dim H_{(2)}^{p,q}(D) = \begin{cases} \infty & \text{if } p+q = n \\ 0 & \text{if } p+q \neq n \end{cases}$$

Moreover $H_{(2)}^{p,q}(D)$ are Hausdorff.

NOTE.   B.-Y. Chen and the referee kindly gave us the following comments after we finished writing the manuscript.

1. Completeness of the Bergman metric of D follows from the homogeneity of D.

2. H. Donnelly [5] has proved that $\partial \log K_D$ is bounded with respect to ds_D^2, which is already sufficient to conclude Theorem 2.

3. Any bounded homogeneous domain D is hyperconvex. To see this, one may use the fact that $\lim_{z \to \partial D} K_D(z) = \infty$ as was shown in [11, p. 31, Proposition 5.2]. Combining this with Donnelly's result on the boundedness of $\partial \log K_D$, we can conclude that $-(\log K_D + M)^{-1}$ is a plurisubharmonic bounded exhaustion function of D for sufficiently large M. Alternatively, one may also deduce the hyperconvexity of D by combining the facts that the Carathéodory distance is strongly complete on the Siegel domains of the 2nd kind (cf. [10, p. 178]) and that complex spaces with strongly complete Carathéodory distances are hyperconvex (cf. [10, p. 191, Theorem (4.4.12)]).

In spite of all this, Theorem 1 might still be of independent interest because of the

Question: How does the length of $\partial \varphi$ depend on D?

Remark.   After the submission of the paper, H. Ishi [9] proved that the Bergman metric of any Siegel domain admits a potential whose gradient is of constant length with respect to the Bergman metric. He even calculated the constant explicitly. However it is not yet clear whether or not the constant is independent of the choice of the potential when the domain is not homogeneous.

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