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FINITENESS OF ENTIRE FUNCTIONS SHARING A FINITE SET

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Abstract. For a finite set $S = \{a_1, \ldots, a_q\}$, consider the polynomial $P_S(w) = (w - a_1)(w - a_2) \cdots (w - a_q)$ and assume that $P'_S(w)$ has distinct k zeros. Suppose that $P_S(w)$ is a uniqueness polynomial for entire functions, namely that, for any nonconstant entire functions ϕ and ψ, the equality $P_S(\phi) = cP_S(\psi)$ implies $\phi = \psi$, where c is a nonzero constant which possibly depends on ϕ and ψ. Then, under the condition $q > k + 2$, we prove that, for any given nonconstant entire function g, there exist at most $(2q-2)/(q-k-2)$ nonconstant entire functions f with $f^*(S) = g^*(S)$, where $f^*(S)$ denotes the pull-back of S considered as a divisor. Moreover, we give some sufficient conditions of uniqueness polynomials for entire functions.

§1. Introduction

A finite subset S of $\mathbf{C}$ is called a uniqueness range set for meromorphic functions (or entire functions) if $f^*(S) = g^*(S)$ implies $f = g$ for arbitrary nonconstant meromorphic functions (or entire functions) f and g on $\mathbf{C}$, where $f^*(S)$ and $g^*(S)$ denote the pull-backs of S considered as a divisor, namely, the inverse images of S counted with multiplicities by f and g respectively. For $S := \{a_1, a_2, \ldots, a_q\}$, we consider the polynomial

$$P_S(w) := (w - a_1)(w - a_2) \cdots (w - a_q). \tag{1}$$

We call a nonconstant monic polynomial $P(w)$ a uniqueness polynomial for meromorphic functions (or entire functions) if, for any nonconstant meromorphic functions (or entire functions) ϕ and ψ on $\mathbf{C}$, the equation $P(\phi) = cP(\psi)$ implies $\phi = \psi$, where c is a nonzero constant which possibly depends on ϕ and ψ. Obviously, if S is a uniqueness range set for meromorphic functions (or entire functions), then $P_S(w)$ is a uniqueness polynomial for meromorphic functions (or entire functions).

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Assume that $P'_S(w)$ has k distinct zeros e_ℓ with multiplicities q_ℓ $(1 \leq \ell \leq k)$. In [1], the author gave some sufficient conditions for uniqueness range set under the condition

(H) $P_S(e_\ell) \neq P_S(e_m)$ *for* $1 \leq \ell < m \leq k$.

Main results in [1] are stated as follows.

THEOREM 1.1. *Let S be a finite subset of $\mathbf{C}$ such that $P_S(w)$ is a uniqueness polynomial for meromorphic functions* (*or entire functions*) *which satisfies the above condition* (H). *Assume that $k \geq 3$, or $k = 2$ and* $\min\{q_1, q_2\} \geq 2$. *If $q > 2k+6$* (*or $q > 2k+2$*), *then S is a uniqueness range set for meromorphic functions* (*or entire functions*).

We now introduce the following definition.

DEFINITION 1.2. A finite subset S of $\mathbf{C}$ is called a *finiteness range set for entire functions* if, for any given nonconstant entire function g, there exist only finitely many nonconstant entire functions f such that $f^*(S) = g^*(S)$.

The purpose of this paper is to give some sufficient conditions for a finiteness range set for entire functions. The main result is stated as follows.

THEOREM 1.3. *Take a finite set $S = \{a_1, a_2, \dots, a_q\}$ and assume that, for the polynomial $P_S(w)$ defined by* (1), *$P'_S(w)$ has distinct k zeros. If $P_S(w)$ is a uniqueness polynomial for entire functions and $q > k+2$, then S is a finiteness range set for entire functions. More precisely, for an arbitrarily given nonconstant entire function g, there exist at most* $(2q-2)/(q-k-2)$ *entire functions f such that $f^*(S) = g^*(S)$.*

The poof of Theorem 1.3 is given in the next section.

We give some sufficient conditions for uniqueness polynomials for entire functions in the last section. For example, the polynomial

$$P(w) = w^5 + \frac{5}{2}w^4 + \frac{5}{3}w^3 + c \quad \left(c \neq 0, \frac{1}{6}, \frac{1}{12}\right)$$

is a uniqueness polynomial for entire functions (cf. Theorem 3.4) which satisfies the condition $q = 5 > k + 2 = 4$, and so the set of zeros of $P(w)$ gives a finiteness range set for entire functions consisting of 5 values. In fact, $P(w)$ has no multiple zero by the condition $c \neq 0, 1/6$, and $P'(w)$ has two distinct zeros satisfying the conditions of Theorem 3.4. It is a very interesting problem to ask if there are smaller finiteness range sets for entire functions.

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§2. Proof of Main Theorem

We first introduce some notations. By a divisor we mean a map $\nu : \mathbf{C} \to \mathbf{Z}$ such that the set $\{z; \nu(z) \neq 0\}$ has no accumulation point. The counting function $N(r, \nu)$ of a divisor ν is defined by

$$N(r, \nu) = \int_0^r \left(\sum_{0<|z|\leq t} \nu(z) \right) \frac{dt}{t} + \nu(0) \log r,$$

and set $\bar{N}(r, \nu) := N(r, \min\{\nu, 1\})$.

In the following, a meromorphic function means a meromorphic function defined on $\mathbf{C}$. For a nonconstant meromorphic function f and another meromorphic function (possibly, a constant) α, we define the divisor ν_f^α by

$$\nu_f^\alpha(z) := \begin{cases} 0 & \text{if } f - \alpha \text{ does not vanish at } z \\ m & \text{if } f - \alpha \text{ has a zero of multiplicity } m \text{ at } z, \end{cases}$$

and $\nu_f^\infty := \nu_{1/f}^0$. As usual, by $T(r, f)$ and $m(r, f)$ we denote the order function and proximity function of f respectively, and $S(r, f)$ means a function of r satisfying the condition

$$S(r, f) = o(T(r, f)) \ \|,$$

where the notation $\|$ means that the inequality holds for every positive number r excluding a measurable set E with $\int_E dr < +\infty$.

The main tool for the proof of Theorem 1.3 is the truncated second main theorem for moving targets, which was proved by K. Yamanoi. A particular case of his result [3, Theorem 1] is stated as follows.

THEOREM 2.1. *Let f be a nonconstant meromorphic function and let $\alpha_1, \ldots, \alpha_q$ be mutually distinct meromorphic functions with $f \not\equiv \alpha_i$ $(1 \leq i \leq q)$. Then, for every $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that*

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{i=1}^q \bar{N}(r, \nu_f^{\alpha_i}) + C(\varepsilon) \left(\sum_{i=1}^q T(r, \alpha_i) \right) + O(1)$$

for any positive number r *excluding some set* $E \subset (1, +\infty)$ *with* $\int_E d \log \log r < +\infty$.

We now give the following;

DEFINITION 2.2. Let f be a nonconstant meromorphic function on $\mathbf{C}$. A meromorphic function $\alpha (\not\equiv f)$ is called *a small function with respect to* f if $T(r, \alpha) = S(r, f)$.

As an immediate consequence of Theorem 2.1, we have the following.

THEOREM 2.3. *Let* f *be a nonconstant meromorphic function and let* $\alpha_1, \alpha_2, \ldots, \alpha_q$ *be mutually distinct small functions with respect to* f. *Then, for every* $\varepsilon > 0$,

$$(q - 2 - \varepsilon) T(r, f) \leq \sum_{j=1}^{q} \bar{N}(r, \nu_f^{\alpha_j}) + O(1)$$

for any positive number r *excluding some set* $E \subset (1, +\infty)$ *with* $\int_E d \log \log r < +\infty$.

Now, we start the proof of Theorem 1.3. Assume that, for some N with $N > (2q - 2)/(q - k - 2)$, there exists a nonconstant entire function g such that $f_j^*(S) = g^*(S)$ for mutually distinct N nonconstant entire functions f_j $(1 \leq j \leq N)$, where we set $g = f_1$. As in §1, for $S = \{a_1, \ldots, a_q\}$, we consider the polynomial $P_S(w)$ defined by (1). By assumption, we can find entire functions α_j such that

$$P_S(g) = e^{\alpha_j} P_S(f_j) \quad (1 \leq j \leq N). \tag{2}$$

In this situation, we can show the following.

(2.4) *There are some positive numbers* K_1, K_2 *such that*

$$K_1 T(r, g) \leq T(r, f_j) \leq K_2 T(r, g) \|.$$

In fact, by the second main theorem and $f_j^{-1}(S) = g^{-1}(S)$,

$$\begin{aligned} (q-1) T(r, g) &\leq \sum_{i=1}^{q} \bar{N}(r, \nu_g^{a_i}) + S(r, g) \\ &= \sum_{i=1}^{q} \bar{N}(r, \nu_{f_j}^{a_i}) + S(r, g) \leq q T(r, f_j) + o(T(r, g)) \|, \end{aligned}$$

whence $T(r,g) = O(T(r,f_j)) \parallel$ and, similarly, $T(r,f_j) = O(T(r,g)) \parallel$.

By (2.4), a small function with respect to g is also a small function with respect to any f_j.

We take the logarithmic derivatives of the identities (2) and get

$$\frac{P_S'(g)g'}{P_S(g)} = \alpha_j' + \frac{P_S'(f_j)f_j'}{P_S(f_j)}. \tag{3}$$

Set $\varphi_j := P_S'(f_j)f_j'/P_S(f_j)$ and $\varphi = \varphi_1$. Then, we have the following assertion.

(2.5) *There exist some positive numbers K_1, K_2 such that*

$$K_1 T(r,g) \le T(r,\varphi_j) \le K_2 T(r,g) \| \quad (1 \le j \le N).$$

In fact, we get $T(r,\varphi_j) = O(T(r,g))\|$ by using the logarithmic derivative lemma. On the other hand, the second main theorem gives

$$\begin{aligned} (q-1)T(r,g) &\le \sum_{i=1}^{q} \bar{N}(r,\nu_g^{a_i}) + S(r,g) \\ &\le N(r,\nu_{\varphi_j}^{\infty}) + S(r,g) \le T(r,\varphi_j) + S(r,g). \end{aligned} \tag{4}$$

(2.6) *Each function α_j' is a small function with respect to φ.*

In fact, by the logarithmic derivative lemma, we have

$$m(r,\varphi_j) = S(r,P_S(f_j)) = S(r,f_j) = S(r,\varphi),$$

and so the identity (3) gives

$$T(r,\alpha_j') = m(r,\alpha_j') \le m(r,\varphi) + m(r,\varphi_j) + O(1) = S(r,\varphi).$$

(2.7) *The functions α_j' are mutually distinct.*

To see this, we assume that $\alpha_i' = \alpha_j'$ for some distinct i and j. Then, there is a constant c_0 with $\alpha_i = \alpha_j + c_0$ and hence

$$e^{c_0}P_S(f_i) = e^{\alpha_i - \alpha_j}P_S(f_i) = P_S(f_j).$$

This contradicts the assumption that $P_S(w)$ is a uniqueness polynomial for entire functions.

We now apply Theorem 2.3 to the function φ and small functions α'_j with respect to φ to show that, for any ε with $0 < \varepsilon < N - 2$,

$$(N - 2 - \varepsilon)T(r, \varphi) \leq \sum_{j=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_j}) + O(1)$$

for any positive number r excluding a set $E \subset (1, +\infty)$ with $\int_E d\log\log r < +\infty$.

By (3) we have

$$\bar{N}(r, \nu_{\varphi}^{\alpha'_j}) = \bar{N}(r, \nu_{\varphi_j}^{0}) \leq \bar{N}(r, \nu_{f'_j}^{0}) + \sum_{\ell=1}^{k} \bar{N}(r, \nu_{f_j}^{e_\ell}),$$

where $e_1, e_2, \ldots, e_k$ are all of distinct zeros of $P'_S(w)$. On the other hand, it holds that $\bar{N}(r, \nu_{f_j}^{e_\ell}) \leq T(r, f_j) + O(1)$ and

$$\begin{aligned}\bar{N}(r, \nu_{f'_j}^{0}) &\leq T(r, f'_j) + O(1) = m(r, f'_j) + O(1) \\ &\leq m(r, f_j) + m(r, f'_j/f_j) + O(1) \leq T(r, f_j) + S(r, f_j).\end{aligned}$$

Therefore,

$$\sum_{i=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_i}) \leq (k + 1) \sum_{j=1}^{N} \left(T(r, f_j) + S(r, f_j)\right).$$

Since $(q-1)T(r, f_j) \leq T(r, \varphi) + S(r, g)$ by the same reasoning as in deriving (4), we have

$$\begin{aligned}(N - 2 - \varepsilon)(q - 1)T(r, f_j) &\leq (N - 2 - \varepsilon)T(r, \varphi) + S(r, g) \\ &\leq \sum_{i=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_i}) + \tilde{S}(r, g) \\ &\leq (k + 1) \sum_{i=1}^{N} T(r, f_i) + \tilde{S}(r, g),\end{aligned}$$

where $\tilde{S}(r, g)$ denotes a term satisfying the condition that $\tilde{S}(r, g) = o(T(r, g)) + O(1)$ for any positive number r excluding a set $E \subset (1, +\infty)$ with $\int_E d\log\log r < +\infty$. Summing up these inequalities, we obtain

$$(N - 2 - \varepsilon)(q - 1) \sum_{j=1}^{N} T(r, f_j) \leq N(k + 1) \sum_{j=1}^{N} T(r, f_j) + \tilde{S}(r, g).$$

Dividing each term of this inequality by $\sum_{j=1}^{N} T(r, f_j)$ and letting $r \to +\infty$ outside some measurable set $E(\subset (1, +\infty)$ with $\int_E d\log\log r < +\infty$, we obtain

$$(N - 2 - \varepsilon)(q - 1) \leq N(k + 1).$$

Since we can take an arbitrarily small positive number ε, we can conclude $(N - 2)(q - 1) \leq N(k + 1)$ and hence

$$N \leq \frac{2q - 2}{q - k - 2}.$$

This contradicts the assumption. The proof of Theorem 1.3 is completed.

§3. Uniqueness polynomials for entire functions

We first discuss uniqueness polynomials for meromorphic functions (or entire functions) in a broad sense, which are defined as follows.

DEFINITION 3.1. A nonconstant monic polynomial $P(w)$ is called a *uniqueness polynomial for meromorphic functions (or entire functions) in a broad sense* if $P(f) = P(g)$ implies $f = g$ for two nonconstant meromorphic functions (or entire functions) f and g.

In [2], the author gave some sufficient conditions of uniqueness polynomials for meromorphic functions in a broad sense. Here, we study uniqueness polynomials for entire functions in a broad sense.

THEOREM 3.2. *Let $P(w)$ be a nonconstant monic polynomial without multiple zeros such that $P'(w)$ has distinct k zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively, and suppose that $P(w)$ satisfies the condition* (H). *If $k \geq 2$ and $q := \deg(P) \geq 4$, then $P(w)$ is a uniqueness polynomial for entire functions in a broad sense.*

Proof. Assume that there exist distinct entire functions f and g with $P(f) = P(g)$. Consider the polynomial $Q(z, w) := (P(z) - P(w))/(z - w)$ in z, w and the associated homogeneous polynomial

$$Q^*(u_0, u_1, u_2) := u_0^{q-1} Q\left(\frac{u_1}{u_0}, \frac{u_2}{u_0}\right)$$

in u_0, u_1, u_2, where $q = \deg P$. Define the algebraic curve

$$V : Q^*(u_0, u_1, u_2) = 0$$

in $P^2(\mathbf{C})$. As was shown in [2], V is irreducible. Consider the holomorphic map $\Phi := (1 : f : g) : \mathbf{C} \to P^2(\mathbf{C})$. Obviously, the image of Φ is included in V and omits the set $V \cap \{u_0 = 0\}$. Let $\mu : \tilde{V} \to V$ be the normalization of V. Then, $\mu^{-1}(V \cap \{u_0 = 0\})$ consists of at least $q-1$ points, because we can write

$$V : (u_1^{q-1} + u_1^{q-2}u_2 + \cdots + u_2^{q-1}) + u_0 R(u_0, u_1, u_2) = 0$$

with a homogeneous polynomial $R(u_0, u_1, u_2)$ of degree $q-2$ and the first term is factorized into distinct $q-1$ linear functions. Therefore, the associated map $\tilde{\Phi} : \mathbf{C} \to \tilde{V}$ with $\Phi = \mu \cdot \tilde{\Phi}$ omits $\geq q-1$ points. Since $q-1 \geq 3$ by the assumption, the universal covering surface of $\tilde{V} \setminus \mu^{-1}(\{u_0 = 0\})$ is biholomorphic to the unit disc in the complex plane. Therefore, the map $\tilde{\Phi}$, and so Φ, is a constant. This contradicts the assumption. The proof of Theorem 3.2 is completed.

Now, we inquire into uniqueness polynomials.

In [1], the author gave the following sufficient condition of uniqueness polynomials.

THEOREM 3.3. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q\prod_{\ell=1}^{k}(w - e_\ell)^{q_\ell}$ and assume that $P(w)$ satisfies the condition* (H). *If $k \geq 4$ and*

$$P(e_1) + P(e_2) + \cdots + P(e_k) \neq 0,$$

then $P(w)$ is a uniqueness polynomial for meromorphic functions.

As was shown in [1], any polynomial $P(w)$ with $k = 1$ is not a uniqueness polynomials for entire functions. We now study uniqueness polynomials for entire functions in the cases $k = 2$ and $k = 3$.

For the case $k = 2$, we have the following.

THEOREM 3.4. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q(w - e_1)^{q_1}(w - e_2)^{q_2}$ $(e_1 \neq e_2)$. If $q \geq 4$ and $P(e_1) \neq \pm P(e_2)$, then $P(w)$ is a uniqueness polynomial for entire functions.*

For the case $k = 3$, we can prove the following.

THEOREM 3.5. *Let $P(w)$ be a monic polynomial without multiple zero such that $P'(w)$ has distinct three zeros e_1, e_2, e_3 with multiplicities q_1, q_2, q_3, respectively, and suppose that $P(w)$ satisfies the conditions* (H). *Here, we choose indices so that $q_1 \leq q_2 \leq q_3$. Then, $P(w)$ is a uniqueness polynomial for entire functions except the cases*

(i) $q_1 = q_2 = q_3 = 1$,

(ii) $q_1 = 1, q_2 = q_3 \geq 2$ *and* $P(e_2) + P(e_3) = 0$ *and*

(iii) $q_1 = q_2 = q_3 \geq 2$ *and* $P(e_1) + P(e_2) + P(e_3) = 0$.

For the proof of Theorems 3.4 and 3.5, we show the following.

LEMMA 3.6. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q\prod_{\ell=1}^{k}(w - e_\ell)^{q_\ell}$. Assume that $P(w)$ satisfies the condition* (H) *and that there exist distinct nonconstant entire functions f, g such that $P(f) = cP(g)$ for a constant $c \neq 0, 1$. Set*

$$\Lambda := \{(\ell, m); P(e_\ell) = cP(e_m)\}.$$

Then,

(i) *If $(\ell_0, m) \notin \Lambda$ for any m or if $(m', \ell_0) \notin \Lambda$ for any m', then $q_{\ell_0} = 1$.*

(ii) *If $(\ell, m) \in \Lambda$, then $q_\ell = q_m$.*

Proof. Changing indices and exchanging the roles of f and g if necesary, we may assume that $(1, m) \notin \Lambda$ $(1 \leq m \leq k)$ for the proof of (i), and that $(1, 2) \in \Lambda$ and $q_2 \leq q_1$ for the proof of (ii). Consider the polynomials

$$Q(w) := P(w) - P(e_1),\ Q^*(w) := cP(w) - P(e_1)$$

and denote all distinct zeros of $Q(w)$ and of $Q^*(w)$ by $\alpha_1, \dots, \alpha_M$ and by $\beta_1, \dots, \beta_N$, respectively, where we may set $\alpha_1 = e_1$ and, furthermore, $\beta_1 = e_2$ if $(1, 2) \in \Lambda$. For convenience sake, we set $q^* := 0$ if $(1, m) \notin \Lambda$ for any $m(1 \leq m \leq k)$, and $q^* := q_2$ if $(1, 2) \in \Lambda$. As is easily seen, α_1 is a zero of $Q(w)$ with multiplicity $q_1 + 1$, and the other α_i's are its simple zeros because $P(w)$ has no multiple zero and satisfis the condition (H). Similarly, β_1 is a zero of $Q^*(w)$ with multiplicity $q^* + 1$ and the other β_j's are its simple zeros. Therefore, $M = q - q_1, N = q - q^*$. Now, we apply the second main theorem to obtain

$$(N - 1)T(r, g) \leq \sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) + S(r, g).$$

On the other hand, if $g(z_0) = \beta_j$ for some z_0, then $P(f(z_0)) = cP(g(z_0)) = cP(\beta_j) = P(e_1)$ and so $f(z_0) = \alpha_i$ for some i. Therefore,

$$\sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) \leq \sum_{i=1}^{M} \bar{N}(r, \nu_f^{\alpha_i}) \leq MT(r, f) + S(r, g). \tag{5}$$

Since $P(f) = cP(g)$ implies

$$qT(r, f) = T(r, P(f)) + O(1) = T(r, P(g)) + O(1) = qT(r, g) + O(1),$$

we can conclude

$$(N-1)T(r, g) \leq MT(r, g) + S(r, g).$$

By dividing this inequality by $T(r, g)$ and letting $r \to +\infty$ outside a set E with $\int_E dr < +\infty$, we see $N - 1 \leq M$, namely, $q - q^* - 1 \leq q - q_1$. For the proof of (i), we recall $q^* = 0$ and get $q_1 \leq 1$, which is the desired conclusion.

For the proof of (ii), we recall $q^* = q_2$. Then, we have $(q_2 \leq) q_1 \leq q_2 + 1$. Now, assume that $q_1 \neq q_2$, whence $q_1 = q_2 + 1$. Here, for any point z_0 with $f(z_0) = \alpha_1 (= e_1)$, we claim that $\nu^0_{g'}(z_0) \geq 2$. In this case, since $Q^*(g(z_0)) = cP(g(z_0)) - P(e_1) = cP(g(z_0)) - P(f(z_0)) = 0$, we have different kinds of two cases (a) $g(z_0) = \beta_1 (= e_2)$ and (b) $g(z_0) = \beta_j$ for $j \geq 2$. We first consider the case (a). Observe the identity $P'(f)f' = cP'(g)g'$ obtained from $P(f) = cP(g)$. Comparing the order of zeros z_0 of both sides, we obtain $(q_1 + 1)\nu_f^{e_1} - 1 = (q_2 + 1)\nu_g^{e_2} - 1$ at z_0. Since $q_2 < q_1$, we have $\nu_f^{e_1} < \nu_g^{e_2}$. Then,

$$\nu_f^{e_1} = (q_1 + 1)\nu_f^{e_1} - q_1 \nu_f^{e_1} = (q_2 + 1)\nu_g^{e_2} - q_1 \nu_f^{e_1} = q_1(\nu_g^{e_2} - \nu_f^{e_1})$$

at z_0. This implies $\nu_g^{e_2} > \nu_f^{e_1} \geq q_1 > q_2 \geq 1$ and so $\nu^0_{g'} \geq 2$ at z_0. We next consider the case (b). In this case, $P'(g(z_0)) \neq 0$, because $Q^*(e_j) \neq 0$ for $j > 2$ by the condition (H) and $(1, 2) \in \Lambda$. Therefore, $\nu^0_{g'} = (q_1 + 1)\nu_f^{e_1} - 1 \geq 2$ at z_0. In any case, $\nu^0_{g'}(z_0) \geq 2$. This implies that $\min(\nu_f^{\alpha_1}, 1) \leq (1/2)\nu^0_{g'}$ at z_0. Therefore, we can replace the first inequality of (5) by

$$\sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) \leq \frac{1}{2} N(r, \nu^0_{g'}) + \sum_{i=2}^{M} \bar{N}(r, \nu_f^{\alpha_i}),$$

and we have

$$(N-1)T(r, g) \leq \left(\frac{1}{2} + (M-1)\right) T(r, f) + S(r, g),$$

because $N(r,\nu^0_{g'}) \le T(r,g) + S(r,g) = T(r,f) + S(r,g)$. This implies that $q - q_2 \le q - q_1 + 1/2$ and so $q_1 \le q_2 + 1/2$, which is a contradiction. The proof of the assertion (ii) is completed.

We now start the proofs of Theorems 3.4 and 3.5. By Theorem 3.2, the given polynomial $P(w)$ is a uniqueness polynomial for entire functions in a broad sense. Assume that $P(w)$ is not a uniqueness polynomial for entire functions. Then, we can apply Lemma 3.6.

Proof of Theorem 3.4. By the assumption, we see $\max(q_1, q_2) \ge 2$, say $q_2 \ge 2$. By Lemma 3.6, (i), there is some ℓ with $(2,\ell) \in \Lambda$. Then, we have necessarily $\ell = 1$ because $c \ne 1$, and hence $q_1 = q_2 \ge 2$ by Lemma 3.6, (ii). We again apply Lemma 3.6, (i) to see $(1,2) \in \Lambda$. Therefore, we have $P(e_1)/P(e_2) = P(e_2)/P(e_1) = c$. This implies $P(e_1) = \pm P(e_2)$, which contradicts the assumption.

Proof of Theorem 3.5. Consider the case where $q_1 = q_2 = 1$. We may assume $q_3 \ge 2$, because otherwise we have the excluded case (i). Then, by Lemma 3.6, (i), there exists some ℓ with $(3,\ell) \in \Lambda$, which contradicts Lemma 3.6, (ii) because $q_3 \ne q_m$ for $m = 1, 2$. Next, consider the case where $q_1 = 1$ and $q_2 \ge 2$. Then, there are indices ℓ, m such that $(2,\ell), (3,m) \in \Lambda$ by Lemma 3.6, (i). Here, we have necessarily $\ell = 3$, $m = 2$ and $q_2 = q_3$ by Lemma 3.6, (ii). In this case, $P(e_2)/P(e_3) = P(e_3)/P(e_2) = c$, which implies the excluded case (ii). Lastly, consider the case where $q_1 \ge 2$. Then, by the assumption and Lemma 3.6, (i), there are indices ℓ_1, ℓ_2, ℓ_3 with $(1,\ell_1), (2,\ell_2), (3,\ell_3) \in \Lambda$. In this case, (ℓ_1, ℓ_2, ℓ_3) is a permutation of $(1, 2, 3)$ such that $\ell_m \ne m$ for every m by the condition (H). We then have $q_1 = q_2 = q_3$ and

$$\frac{P(e_1)}{P(e_{\ell_1})} = \frac{P(e_2)}{P(e_{\ell_2})} = \frac{P(e_3)}{P(e_{\ell_3})} = c(\ne 1)$$

by Lemma 3.6, (ii). We easily have $P(e_1) + P(e_2) + P(e_3) = 0$. The proof of Theorem 3.5 is completed.

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