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ENDOMORPHISMS OF DELIGNE-LUSZTIG VARIETIES

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Abstract. We study some conjectures on the endomorphism algebras of the cohomology of Deligne-Lusztig varieties which are a refinement of those of [BMi].

§1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group, defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic p. Let F be an isogeny on $\mathbf{G}$ such that some power F^δ is the Frobenius endomorphism attached to a split $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$ (where q is a real number such that q^δ is a power of p). The finite group $\mathbf{G}^F$ of fixed points under F is called a finite group of Lie type. When considering a simple group which is not a Ree or Suzuki group we may take F to be already a Frobenius endomorphism.

Let W be the Weyl group of $\mathbf{G}$ and let B (resp. B^+) be the corresponding braid group (resp. monoid). The canonical morphism of monoids $\beta : B^+ \to W$ has a section that we denote by $w \to \mathbf{w}$: it sends an element of W to the only positive braid $\mathbf{w}$ such that $\beta(\mathbf{w}) = w$ and such that the length of $\mathbf{w}$ in B^+ is the same as the Coxeter length of w; we write $\mathbf{W} = \{\mathbf{w} \mid w \in W\}$ and $\mathbf{S} = \{\mathbf{s} \mid s \in S\}$ where S is a set of Coxeter generators for W.

Let us recall how in [BMi] a "Deligne-Lusztig variety" is attached to each element of B^+. Let $\mathcal{B}$ be the variety of Borel subgroups of $\mathbf{G}$. The orbits of $\mathcal{B}\times\mathcal{B}$ are in natural bijection with W. Let $\mathcal{O}(w)$ be the orbit corresponding to $w \in W$. Let $\mathbf{b} \in B^+$ and let $\mathbf{b} = \mathbf{w}_1 \cdots \mathbf{w}_n$ be a decomposition of $\mathbf{b}$ as a product of elements of $\mathbf{W}$. To such a decomposition we attach the variety $\{(\mathbf{B}_1, \dots, \mathbf{B}_{n+1}) \mid (\mathbf{B}_i, \mathbf{B}_{i+1}) \in \mathcal{O}(w_i) \text{ and } \mathbf{B}_{n+1} = F(\mathbf{B}_1)\}$. It is shown in [De, p. 163] that the varieties attached to two such decompositions of $\mathbf{b}$ are canonically isomorphic. The projective limit of this system

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of isomorphisms defines what we call the Deligne-Lusztig variety $\mathbf{X}(\mathbf{b})$ attached to $\mathbf{b}$; it is the "usual" Deligne-Lusztig variety $\mathbf{X}(w)$ when we take $\mathbf{b} = \mathbf{w} \in \mathbf{W}$.

When W is an irreducible Coxeter group, the center of the pure braid group is cyclic. We denote by $\boldsymbol{\pi}$ its positive generator; we define $\boldsymbol{\pi}$ in general as the product of the corresponding elements for the irreducible components of W. Another way of constructing $\boldsymbol{\pi}$ is as $\mathbf{w}_0^2$ where w_0 is the longest element of W. The isogeny F acts naturally as a diagram automorphism, *i.e.*, an automorphism which preserves S (resp. $\mathbf{S}$), on W (resp. B^+); we still denote by F these diagram automorphisms. We call "F-root of order d of $\boldsymbol{\pi}$", an element $\mathbf{b} \in B^+$ such that $(\mathbf{b}F)^d = \boldsymbol{\pi} F^d$; in [BMi] it is proved that $\beta(\mathbf{b})$ is then a regular element of the coset WF (in the sense of [Sp]) for the eigenvalue $e^{2i\pi/d}$; when F acts trivially we just have a root of order d, *i.e.*, $\mathbf{b}^d = \boldsymbol{\pi}$. It is conjectured in *loc. cit.* that the $\mathbf{G}^F$-endomorphisms of the ℓ-adic cohomology complex of $\mathbf{X}(\mathbf{b})$ form a "cyclotomic Hecke algebra" attached to the complex reflection group $C_W(\beta(\mathbf{b})F)$. We will show below that for any F-root $\mathbf{b}$ of $\boldsymbol{\pi}$ except for $\boldsymbol{\pi}$ itself, there is another F-root $\mathbf{w} \in \mathbf{W}$ of $\boldsymbol{\pi}$ of the same order and an equivalence of étale sites $\mathbf{X}(\mathbf{b}) \simeq \mathbf{X}(\mathbf{w})$, thus the conjecture is about the variety $\mathbf{X}(\boldsymbol{\pi})$ and some "ordinary" Deligne-Lusztig varieties.

We shall make more specific this conjecture by replacing it by a set of conjectures that we shall study, and prove in some specific examples. In [DMR], of which this paper is a continuation, we already obtained some general results on some of the conjectures. We will get here further results for the element $\boldsymbol{\pi}$, and for all roots of $\boldsymbol{\pi}$ in the case of split groups of type A. We also study powers of Coxeter elements in type B and fourth roots of $\boldsymbol{\pi}$ in split type D_4.

§2. Conjectures

First, we should state that a guide for the following conjectures is that, using Lusztig's results in [Lu2], we will show in Section 4 that they all hold in the case of Coxeter elements.

We recall from [DMR, 2.1.1] that a possible presentation of B is

$$\langle \mathbf{w} \in \mathbf{W} \mid \mathbf{w}_1\mathbf{w}_2 = \mathbf{w}_3 \text{ when } w_1w_2 = w_3 \text{ and } l(w_1) + l(w_2) = l(w_3)\rangle.$$

We recall the action defined in [DMR, 5.1] of a submonoid of B^+ on $\mathbf{X}(\mathbf{w})$, and of the group it generates (which will be equal to $C_B(\mathbf{w}F)$ in every case we study) on $H^*_c(\mathbf{X}(\mathbf{w})) = \bigoplus_i H^i_c(\mathbf{X}(\mathbf{w}))$.

First, we recall the definition of the morphism $D_{\mathbf{t}} : \mathbf{X}(\mathbf{b}) \to \mathbf{X}(\mathbf{t}^{-1}\mathbf{b}F(\mathbf{t}))$ defined when $\mathbf{t}$ is a left divisor of $\mathbf{b}$: if $\mathbf{b} = \mathbf{t}\mathbf{t}'$, and if $\mathbf{t} = \mathbf{w}_1 \cdots \mathbf{w}_n$ and $\mathbf{t}' = \mathbf{w}'_1 \cdots \mathbf{w}'_{n'}$ are decompositions as products of elements of $\mathbf{W}$, it sends the element $(\mathbf{B}_1, \ldots, \mathbf{B}_{n+n'+1}) \in \mathbf{X}(\mathbf{b})$ attached to the decomposition $\mathbf{w}_1 \cdots \mathbf{w}_n \mathbf{w}'_1 \cdots \mathbf{w}'_{n'}$ to the element $(\mathbf{B}_{n+1}, \ldots, \mathbf{B}_{n+n'}, F(\mathbf{B}_1), \ldots, F(\mathbf{B}_{n+1}))$ attached to the decomposition $\mathbf{w}'_1 \cdots \mathbf{w}'_{n'} F(\mathbf{w}_1) \cdots F(\mathbf{w}_n)$.

Then we introduce categories as in [DMR, 5.1]: $\mathcal{B}$ is a category with objects the elements of B, and such that $\mathrm{Hom}_{\mathcal{B}}(\mathbf{b}, \mathbf{b}') = \{\mathbf{y} \in B \mid \mathbf{b}' = \mathbf{y}^{-1}\mathbf{b}F(\mathbf{y})\}$; composition of maps corresponds to the product in B; one has $\mathrm{End}_{\mathcal{B}}(\mathbf{b}) = C_B(\mathbf{b}F)$.

$\mathcal{D}^+$ is the smallest subcategory of $\mathcal{B}$ which contains the objects in B^+ and such that

$$\{\mathbf{y} \in B^+ \mid \mathbf{y} \preccurlyeq \mathbf{b},\, \mathbf{y}^{-1}\mathbf{b}F(\mathbf{y}) = \mathbf{b}'\} \subset \mathrm{Hom}_{\mathcal{D}^+}(\mathbf{b}, \mathbf{b}'),$$

where $\preccurlyeq$ denotes left divisibility in the braid monoid, and $\mathcal{D}$ is the smallest subcategory of $\mathcal{B}$ containing $\mathcal{D}^+$ and where all maps are invertible.

$\mathcal{C}^+$ is the category of quasi-projective varieties on $\mathbb{F}$, together with proper morphisms. $\mathcal{C}$ is the localized category by morphisms inducing equivalences of étale sites. An isomorphism in $\mathcal{C}$ induces a linear isomorphism in l-adic cohomology.

It is shown in [DMR, 5.2.1] that the map which sends the object $\mathbf{b}$ to $\mathbf{X}(\mathbf{b})$ and the map $\mathbf{t}$ to $D_{\mathbf{t}}$ extends to a functor $\mathcal{D}^+ \to \mathcal{C}^+$, which itself extends to a functor $\mathcal{D} \to \mathcal{C}$.

In the following we denote by $H_c^*(\mathbf{X})$ the graded vector space $\bigoplus_i H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$, *i.e.*, the ℓ-adic cohomology with compact support of the quasi-projective variety $\mathbf{X}$. With this notation, the monoid $\mathrm{End}_{\mathcal{D}^+}(\mathbf{b})$ acts on $\mathbf{X}(\mathbf{b})$ as a monoid of endomorphisms, and the group $\mathrm{End}_{\mathcal{D}}(\mathbf{b})$ acts linearly on $H_c^*(\mathbf{X}(\mathbf{b}))$.

Conjecture 2.1. *When* $\mathbf{b}$ *is an* F*-root of* $\boldsymbol{\pi}$ *we have* $\mathrm{End}_{\mathcal{D}}(\mathbf{b}) = \mathrm{End}_{\mathcal{B}}(\mathbf{b}) = C_B(\mathbf{b}F)$.

We will show this conjecture for $\boldsymbol{\pi}$, $\mathbf{w}_0$, Coxeter elements, all roots of $\boldsymbol{\pi}$ in types A and B, and 4-th roots of $\boldsymbol{\pi}$ in type D_4. We should note that in [DMR, 5.2.5] we have defined an action of $C_{B^+(\mathbf{b}F)}$ on $\mathbf{X}(\mathbf{b})$ which extends the action of $\mathrm{End}_{\mathcal{D}^+}(\mathbf{b})$, but we are not able to determine its image in $H_c^*(\mathbf{X}(\mathbf{b}))$ (except via Conjecture 2.1).

It is proved in [BMi, 6.8] that, except when $\mathbf{b}=\boldsymbol{\pi}$, of course, there is a morphism in $\mathcal{D}^+$ between any d-th F-root $\mathbf{b}$ of $\boldsymbol{\pi}$ and a "good" d-th F-root $\mathbf{w}$, "good" meaning that $(\mathbf{w}F)^i \in \mathbf{W}.F^i$ for $i \leq d/2$. Thus the variety $\mathbf{X}(\mathbf{b})$ is isomorphic in $\mathcal{C}$ to $\mathbf{X}(\mathbf{w})$, as was asserted above. We thus need only to consider a variety associated to a good F-root. We actually need only to consider one of them, according to the

Conjecture 2.2. *There is always a morphism in $\mathcal{D}^+$ between any two F-roots of $\boldsymbol{\pi}$ of the same order.*

This says in particular that two such roots are F-conjugate in B. The result of [BMi, 6.8] shows that it is sufficient to consider "good" F-roots of $\boldsymbol{\pi}$ in the above conjecture.

We will show this conjecture for $\mathbf{w}_0$, for Coxeter elements in split groups and n-th roots of $\boldsymbol{\pi}$ in split type A_n. It has now been proved in split type A as a consequence of a recent result of Birman, Gebhardt and Gonzales-Meneses (`ArXiv:math.GT/0606652`, Corollary 4.6 and Proposition 5.4) which states that in general there is a morphism in $\mathcal{D}^+$ between two conjugate roots, and of the theorem of Eilenberg [E] stating that in type A two roots of the same order are conjugate.

Since, when $\mathbf{w}$ is an F-root of $\boldsymbol{\pi}$, $wF = \beta(\mathbf{w})F$ is a regular element of WF (*cf.* [BMi, 6.6]), the group $C_W(wF)$ is naturally a complex reflection group. We will denote by $B(w)$ the corresponding braid group; it is shown in *e.g.*, [BDM] in the case $F = \mathrm{Id}$ and in Section 3 below in the general case that there is a natural map $\gamma : B(w) \to C_B(\mathbf{w}F)$ such that the image of $\beta \circ \gamma$ is $C_W(wF)$. We recall the following conjecture from [BDM, 0.1]

Conjecture 2.3. *γ is an isomorphism.*

The above conjecture is easy when $\mathbf{w} = \boldsymbol{\pi}$ or $\mathbf{w} = \mathbf{w}_0$. It has been proved in [BDM] for split types A and B. We prove it for Coxeter elements in split groups, and for 4th roots of $\boldsymbol{\pi}$ in type D_4. For this last case we use programs of N. Franco and J. Gonzales-Meneses which compute centralizers in Garside groups.

Assuming Conjecture 2.1, and since the operators $D_{\mathbf{t}}$ commute with the action of $\mathbf{G}^F$, we get an action of $B(w)$ as $\mathbf{G}^F$-endomorphisms of $H^*_c(\mathbf{X}(\mathbf{w}))$. The next conjecture states that this action factors through a cyclotomic Hecke algebra for $B(w)$. Let us recall their definition; the braid group $B(w)$ is generated by so-called "braid reflections" (see Definition 3.1) which form conjugacy classes in bijection with conjugacy classes of distinguished reflections in $C_W(wF)$ (see again Definition 3.1) (*cf.* [BMR, 2.15 and Appendix 1]). Let $\mathcal{C}$ be a set of representatives of conjugacy classes of distinguished reflections in $C_W(wF)$; for $s \in \mathcal{C}$ we choose a representative $\mathbf{s}$ of the corresponding class of braid reflections, and we denote by e_s the order of s. Let $A = \overline{\mathbb{Q}}_\ell[u_{s,j}]_{s,j}$ where $\{u_{s,j}\}_{\{s\in\mathcal{C},\, j=0,\ldots,e_s-1\}}$ are indeterminates. The generic Hecke algebra of $C_W(wF)$ over the ring A is defined as the quotient of $A[B(w)]$ by the ideal generated by $(\mathbf{s}-u_{s,0})\cdots(\mathbf{s}-u_{s,e_s-1})$. A d-cyclotomic Hecke algebra for $C_W(wF)$ is a "one-variable specialization" of the generic algebra which specializes to $\overline{\mathbb{Q}}_\ell[C_W(wF)]$ by the further specialization of the variable to $e^{2i\pi/d}$. To make this precise, we need some definitions: we choose an integer a and we denote by $e^{2i\pi/a|W|}$ a primitive $a|W|$-th root of unity in $\overline{\mathbb{Q}}_\ell$; we choose an indeterminate denoted by $x^{1/a}$. Then a d-cyclotomic Hecke algebra is a specialization of the generic Hecke algebra of the form $u_{s,j} \mapsto e^{2i\pi j/e_s}(e^{-2i\pi/ad}x^{1/a})^{n_{s,j}}$ for some integers $n_{s,j}$ (it is defined over $\overline{\mathbb{Q}}_\ell[x^{1/a}]$; it specializes to $\overline{\mathbb{Q}}_\ell[C_W(wF)]$ by the further specialization $x^{1/a} \mapsto e^{2i\pi/ad}$).

Conjecture 2.4. *If $\mathbf{w}$ is a d-th F-root of $\boldsymbol{\pi}$, the action of $B(w)$ on $H_c^*(\mathbf{X}(\mathbf{w}))$ factors through a specialization $x \mapsto q$ of a d-cyclotomic Hecke algebra $\mathcal{H}(w)$ for $C_W(wF)$.*

More precisely we have to state the specialization as $x^{1/a} \mapsto q^{1/a}$. This conjecture is proved for $\mathbf{w} = \mathbf{w}_0$ and $\mathbf{w} = \boldsymbol{\pi}$ in [DMR, 5.4.1] and [BMi, 2.7] respectively (see also [DMR, 5.3.4]). We will prove this conjecture for all roots of $\boldsymbol{\pi}$ in split type A for roots of even order in type B and for 4-th roots of $\boldsymbol{\pi}$ in type D_4.

Assuming Conjecture 2.4, let $\mathcal{H}_q(w)$ be the above specialization (the specialization for $x^{1/a} \mapsto q^{1/a}$ of $\mathcal{H}(w)$). We thus have a virtual representation ρ_w of $\mathcal{H}_q(w)$ on $\sum_i(-1)^i H_c^i(\mathbf{X}(\mathbf{w}))$. If we decompose $\sum_i(-1)^i H_c^i(\mathbf{X}(\mathbf{w})) = \sum_{\lambda\in\mathrm{Irr}(\mathbf{G}^F)} a_\lambda\lambda$ in the Grothendieck group of $\mathbf{G}^F$, we get thus for each λ a virtual character χ_λ of $\mathcal{H}_q(w)$ of dimension a_λ.

We call a representation special if its trace defines up to a scalar the canonical symmetrizing trace form on $\mathcal{H}_q(w)$ (see [BMM, 2.1] for the definition of the canonical trace form). The canonical trace form has not been proved to exist for all complex reflection groups; however it is known to exist for those groups that we will encounter in the present paper.

Conjecture 2.5. (i) *The χ_λ generate the Grothendieck group of $\mathcal{H}_q(w)$ and are irreducible up to sign.*

(ii) *The representation ρ_w is special.*

(i) above means that the image of $\mathcal{H}_q(w)$ by ρ_w is the "full $\mathbf{G}^F$-endomorphism algebra of $\sum_i(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$". We will be able to prove Conjecture 2.5 when $\mathbf{w}=\boldsymbol{\pi}$ and $\mathbf{G}$ is split of type A_n, G_2, E_6 and some small rank cases and also for the cases when we can prove the next conjecture.

Conjecture 2.6. *The groups $H^i_c(\mathbf{X}(\mathbf{w}))$ are disjoint from each other as $\mathbf{G}^F$-modules.*

Conjectures 2.5 and 2.6 thus imply that $\mathcal{H}_q(w)\simeq \operatorname{End}_{\mathbf{G}^F}(H^*_c(\mathbf{X}(\mathbf{w})))$. Conjecture 2.6 is the hardest in some sense, since it is very difficult to determine individually the cohomology groups of a Deligne-Lusztig variety, except when $\mathbf{w}$ is rather short. In addition to the known case of Coxeter elements, we will show Conjecture 2.6 for n-th roots of $\boldsymbol{\pi}$ in type A_n and 4-th roots of $\boldsymbol{\pi}$ in type D_4. Also, when $\mathbf{G}$ is of rank 2, Conjecture 2.6 follows from [DMR, 4.2.4, 4.2.9, 4.3.4, 4.3.5, 4.4.3 and 4.4.4] (with some indeterminacy left in type split G_2).

It should be pointed out that Conjectures 2.4 to 2.6 are consequences of a special case of the version for reductive groups of the Broué conjectures on blocks with abelian defect. They already have been formulated in a very similar form by Broué, see [Br], [BMa] and [BMi]. In particular, compared to [BMi, 5.7], we have only inverted the order of the assertions, in order to present them by order of increasing difficulty, and made the connection with the braid group a little bit more specific via Conjectures 2.1 to 2.3.

§3. Regular elements in braid groups

In this section we provide the needed background on braid monoids and groups.

We fix a complex vector space V of finite dimension. A *complex reflection* is an element of finite order of $\mathrm{GL}(V)$ whose fixed point space is a hyperplane. Let $W\in\mathrm{GL}(V)$ be a finite group generated by complex reflections, let $\mathcal{A}$ be the set of reflecting hyperplanes for reflections of W, and let $V^{\mathrm{reg}}=V-\bigcup_{H\in\mathcal{A}}H$. Choose $x\in V^{\mathrm{reg}}$ and let $\overline{x}\in V^{\mathrm{reg}}/W$ be its image. The group $\Pi_1(V^{\mathrm{reg}},x)$ is called the pure braid group of W and the group $B=\Pi_1(V^{\mathrm{reg}}/W,\overline{x})$ is called the braid group of W. The map

$V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ is an étale covering, so that it gives rise to the exact sequence

$$1 \longrightarrow \Pi_1(V^{\mathrm{reg}}, x) \longrightarrow B \xrightarrow{\beta} W \longrightarrow 1.$$

We denote by $N_W(E)$ (resp. $C_W(E)$) the stabilizer (resp. pointwise stabilizer) of a subset $E \subset V$ in W.

We choose distinguished generators of W and B as follows (see [BMR, 2.15]):

DEFINITION 3.1. (i) A reflection $s \in W$ of hyperplane H is distinguished if its only non trivial eigenvalue is $e^{2i\pi/e_H}$ where $e_H = |C_W(H)|$.

(ii) Let $H \in \mathcal{A}$ and let s_H be a distinguished reflection of hyperplane H. We call "braid reflection" associated to s_H an element of B of the form $\gamma \circ \overline{\lambda} \circ \gamma^{-1}$, where: γ is a path from $\overline{x}$ to a point $\overline{x_H}$ which is the image of a point $x_H \in V^{\mathrm{reg}}$ "close to H" in the sense that there is a ball around x_H which meets H and no other hyperplane, and contains the path $\lambda : t \mapsto \mathrm{proj}_H(x_H) + e^{2i\pi t/e_H}\,\mathrm{proj}_{H^\perp}(x_H)$ where proj means orthogonal projection (with respect to a chosen W-invariant inner product); and where $\overline{\lambda}$ is the image of λ.

It is clear from the definition that if $\mathbf{s}_H$ is a braid reflection associated to s_H, then $\beta(\mathbf{s}_H) = s_H$. Moreover it is proved in [BMR, 2.8] that braid reflections generate B and in [BMR, 2.14] that β induces a bijection from the conjugacy classes of braid reflections in B to the conjugacy classes of distinguished reflections in W.

We assume now given $\phi \in \mathrm{GL}(V)$ normalizing W; thus V^{reg} is ϕ-stable. We fix a d-regular element $w\phi \in W.\phi$ *i.e.*, an element which has an eigenvector in V^{reg} for the eigenvalue $\zeta = e^{2i\pi/d}$. We refer to [Sp] and [BMi, §3.B] for the properties of such elements. If V_ζ is the ζ-eigenspace of $w\phi$, then $C_W(w\phi) = N_W(V_\zeta)$; we denote this group by W_ζ. Its representation on V_ζ is faithful and makes it into a complex reflection group, with reflecting hyperplanes the traces on V_ζ of the reflecting hyperplanes of W.

To construct the braid group $B(w)$ of W_ζ we choose a base point x_ζ in $V_\zeta^{\mathrm{reg}} = V^{\mathrm{reg}} \cap V_\zeta$. Then $B(w) = \Pi_1(V_\zeta^{\mathrm{reg}}/W_\zeta, \overline{x}_\zeta)$, where $\overline{x}_\zeta$ is the image of x_ζ in $V_\zeta^{\mathrm{reg}}/W_\zeta$. Let $\gamma : B(w) \to B$ be the morphism induced by the injection $V_\zeta^{\mathrm{reg}}/W_\zeta \hookrightarrow V^{\mathrm{reg}}/W_\zeta$ composed with the quotient $V^{\mathrm{reg}}/W_\zeta \to V^{\mathrm{reg}}/W$. It is shown in [Be1, 1.2 (ii)] that γ factorizes through the isomorphism induced by a (unique) homeomorphism $V_\zeta^{\mathrm{reg}}/W_\zeta \simeq (V^{\mathrm{reg}}/W)^{\langle\zeta\rangle}$, where for a group

G acting on a vector space V we denote by V^G the the set of G-fixed points in V.

Let δ be the path $t \mapsto e^{2i\pi t/d} x_\zeta$ from x_ζ to ζx_ζ in V_ζ^{reg} and let $\overline{\delta}$ be its image in V^{reg}/W. Then the map $\mathbf{w}\boldsymbol{\phi} : \lambda \mapsto \overline{\delta} \circ (w\phi)(\lambda) \circ \overline{\delta}^{-1}$ is a lift to B of the action of $w\phi$ on W.

Remark 3.2. Note that we have not defined independently $\mathbf{w}$ and $\boldsymbol{\phi}$; if ϕ is 1-regular this can be done in the following way: let x be a fixed point of ϕ, and choose a path η_0 from x to x_ζ. Let η_1 be the path $\phi(\eta_0)^{-1} \circ \eta_0$, from $\phi(x_\zeta)$ to x_ζ. The image in V^{reg}/W of the path $\delta \circ w(\eta_1)$ from x_ζ to $w(x_\zeta)$ is an element $\mathbf{w}$ of B which by definition lifts $w \in W$.

Then if $\overline{\eta}_1$ is the image of η_1 in V^{reg}/W, the map $\boldsymbol{\phi} : \lambda \mapsto \overline{\eta}_1^{-1} \circ \phi(\lambda) \circ \overline{\eta}_1$ is an automorphism of B which lifts the action of ϕ on W.

Note also the following:

Remark 3.3. Let $\boldsymbol{\phi}$ and $\mathbf{w}$ be as in Remark 3.2, and let $\boldsymbol{\pi}$ be the loop $t \mapsto e^{2i\pi t} x_\zeta$ in V^{reg}; it is a generator of the center of the pure braid group. Then $(\mathbf{w}\boldsymbol{\phi})^d = \boldsymbol{\pi}\boldsymbol{\phi}^d$ in the semi-direct product $B \rtimes \langle \boldsymbol{\phi} \rangle$. Indeed the element $(\mathbf{w}\boldsymbol{\phi})^d \boldsymbol{\phi}^{-d}$ is represented by the path $\delta \circ w\phi(\delta) \circ (w\phi)^2(\delta) \circ \cdots \circ (w\phi)^{d-1}(\delta)$ which is equal to $\boldsymbol{\pi}$.

LEMMA 3.4. *We have $\gamma(B(w)) \subset C_B(\mathbf{w}\boldsymbol{\phi})$.*

Proof. It is easily checked that for any $\lambda \in B$ the path $\overline{\delta} \circ \lambda \circ \overline{\delta}^{-1}$ is homotopic to $\zeta^{-1}\lambda$, so $(\mathbf{w}\boldsymbol{\phi})(\lambda)$ is homotopic to $\zeta^{-1}(w\phi)(\lambda)$. If λ is the image of a path in V_ζ^{reg}, we have $\zeta^{-1}(w\phi)(\lambda) = \lambda$; so the action of $\mathbf{w}\boldsymbol{\phi}$ on the image of λ in B is trivial, as claimed. ∎

We now assume that W is a Coxeter group. The space V is the complexification of the real reflection representation of W and the real hyperplanes define chambers. We choose a fundamental chamber, which defines a Coxeter generating set S for W.

When ϕ is trivial, and W not of type F_4 or E_n, the morphism γ has been proved injective [Be1, 4.1]; if moreover W is of type A(split) or B it is proved to be an isomorphism onto $C_B(\mathbf{w}\boldsymbol{\phi})$ in [BDM].

We assume now that ϕ induces a diagram automorphism of W, *i.e.*, that it stabilizes the fundamental chamber. Then ϕ is 1-regular: in fact it has a fixed point in the fundamental chamber. We recall the following result from [VdL]:

PROPOSITION 3.5. *Assume that we have fixed a base point whose real part is in the fundamental chamber. Let* $\mathbf{W}$ *be the set of elements of* B *which can be represented by paths* λ *in* V^{reg}, *starting from the base point and satisfying the two following properties*:

(i) *The real part of* λ *meets each element of* $\mathcal{A}$ *at most once.*
(ii) *When the real part of* λ *meets* $H \in \mathcal{A}$, *the imaginary part of* λ *is on the same side of* H *as the fundamental chamber*;

then $\mathbf{W}$ *is in bijection with* W *via the map* $B \xrightarrow{\beta} W$. *Moreover if* $\mathbf{S} \subset \mathbf{W}$ *is such that* $\beta(\mathbf{S}) = S$ *then* B *has a presentation with generators* $\mathbf{S}$ *and relations the braid relations given by the Coxeter diagram of* W.

The elements of $\mathbf{S}$ are braid reflections.

COROLLARY 3.6. *Let* x *and* y *be two points with real parts in the fundamental chamber. If* γ *is a path from* x *to* y *with real part in the fundamental chamber, the isomorphism* $\Pi_1(V^{reg}/W, x) \to \Pi_1(V^{reg}/W, y)$ *which it defines is independent of* γ.

Proof. Two such isomorphisms differ by the inner automorphism of $\Pi_1(V^{\mathrm{reg}}/W, x)$ defined by a loop with real part in the fundamental chamber. An element defined by a loop is in the pure braid group. But by the Proposition 3.5 this element is also in $\mathbf{W}$, so it is trivial. ∎

DEFINITION 3.7. The braid monoid B^+ is defined to be the submonoid of B generated by $\mathbf{W}$. Its elements are called positive braids.

Recall that a d-regular element in $W.\phi$ is called a Springer element if it has a ζ-eigenvector with real part in the fundamental chamber (*cf.* [BMi, 3.10]). We choose as base point a fixed point of ϕ in the fundamental chamber. If $w\phi$ is a Springer element we can choose η_0 in Remark 3.2 such that $\mathrm{Re}(\eta_0)$ is in the fundamental chamber.

PROPOSITION 3.8. *Assume that* $d > 1$ *and that* $w\phi$ *is a Springer element. If* η_0 *in Remark* 3.2 *is chosen with real part in the fundamental chamber then the element* $\mathbf{w}$ *is independent of the choice of* η_0 *and is in* $\mathbf{W}$.

Proof. The element $\mathbf{w}$ is represented by $\delta \circ w(\eta_1)$. As $\mathrm{Re}(w(\eta_1))$ does not meet any reflecting hyperplane, the only possibility for the real part of this path to meet $H \in \mathcal{A}$ is that for some $t \in]0,1[$ we have $l_H(\mathrm{Re}(\delta(t))) = 0$, where l_H is a real linear form defining H. Write $x_\zeta = a+ib$ with a and b real and let $\theta = 2\pi t/d$; since $d > 1$ we have $\theta \in]0,\pi[$. We have $l_H(\mathrm{Re}(\delta(t))) = (\cos\theta) l_H(a) - (\sin\theta) l_H(b) = 0$. As this equation has only one solution in $]0,\pi[$, property 3.5 (i) above is satisfied. Moreover if θ is such a solution we have $l_H(\mathrm{Im}(\delta(t))) = l_H(a)\big(\sin\theta + \frac{\cos^2\theta}{\sin\theta}\big) = \frac{l_H(a)}{\sin\theta}$ which has the same sign as $l_H(a)$ since $\sin\theta > 0$. So property 3.5 (ii) is also satisfied. ∎

When $d = 1$ we still have that similarly $\boldsymbol{\pi}$ is independent of the choice of η_0 and is in B^+; a way to see this is to use that $\boldsymbol{\pi} = \mathbf{w}_0^2$ where $\mathbf{w}_0$ is the element $\mathbf{w}$ obtained for $d = 2$.

We show now how in some cases we can lift to B a distinguished reflection of $C_W(w\phi)$. We still assume that $w\phi$ is a Springer element. Recall that a *parabolic* subgroup of W is the stabilizer of a subset of V; and a *standard parabolic* subgroup of W is the subgroup generated by a subset of S (it is a parabolic subgroup).

First, to $H \in \mathcal{A}$ we associate the hyperplane $H \cap V_\zeta$ of V_ζ and the distinguished reflection t_H of W_ζ with reflecting hyperplane $H \cap V_\zeta$. Let $W_H = C_W(H \cap V_\zeta)$; it is a parabolic subgroup of W, thus a reflection subgroup. The element $w\phi$ normalizes W_H as it acts by ζ on $H \cap V_\zeta$; it is a regular element of $W_H.w\phi$ and we have $C_{W_H}(w\phi) = \langle t_H \rangle$. We can apply the constructions of this section to W_H: let $V^{\mathrm{reg}_H} = V - \bigcup_{\{H' \in \mathcal{A} \mid H' \supset H \cap V_\zeta\}} H'$ and let $V_\zeta^{\mathrm{reg}_H} = V_\zeta \cap V^{\mathrm{reg}_H}$. We get a morphism $\Pi_1(V_\zeta^{\mathrm{reg}_H}/C_{W_H}(w\phi), \overline{x}_\zeta) \to \Pi_1(V^{\mathrm{reg}_H}/W_H, \overline{x}_\zeta)$, whose image centralizes $\mathbf{w}\boldsymbol{\phi}$ as in Lemma 3.4. As x_ζ and x are both in the fundamental chamber of W_H, by Corollary 3.6 any path from x to x_ζ whose real part stays in this fundamental chamber defines a canonical isomorphism between $\Pi_1(V^{\mathrm{reg}_H}/W_H, \overline{x}_\zeta)$ and $\Pi_1(V^{\mathrm{reg}_H}/W_H, \overline{x})$ which commutes with $\mathbf{w}\boldsymbol{\phi}$. Let us denote by B_H this group. By composition with this isomorphism we get a morphism $\Pi_1(V_\zeta^{\mathrm{reg}_H}/C_{W_H}(w\phi), \overline{x}_\zeta) \to B_H$, whose image centralizes $\mathbf{w}\boldsymbol{\phi}$. Let $\mathbf{t}_H$ be the generator of the infinite cyclic group $\Pi_1(V_\zeta^{\mathrm{reg}_H}/C_{W_H}(w\phi))$ such that its image in $C_W(w\phi)$ is t_H; then $\mathbf{t}_H$ is a braid reflection in $B(w)$ and, if e_H is the order of t_H, then $\mathbf{t}_H^{e_H}$ is the loop $t \mapsto e^{2i\pi t} x_\zeta$ *i.e.*, the element π_H of B_H. If H is such that W_H is a standard parabolic subgroup generated by a subset I of S then $B_H = \Pi_1(V^{\mathrm{reg}_H}/W_H, \overline{x})$ is canonically embedded in B as the subgroup of B generated by the lift $\mathbf{I} \subset \mathbf{S}$ of I and if $\mathbf{s}_H \in B_H$ is the image of $\mathbf{t}_H$ by the above morphism we have $\mathbf{s}_H \in B_H \cap C_B(\mathbf{w}\boldsymbol{\phi})$.

§4. The case of Coxeter elements

We prove here, using the results of [BDM], [Be2] and [Lu2] that all our conjectures hold in the case of Coxeter elements for an untwisted quasi-simple reductive group $\mathbf{G}$ (the assumption of $\mathbf{G}$ being quasi-simple is equivalent to W being irreducible). Even though the results of Lusztig cover them, we are unable to handle twisted groups because the construction of a dual braid monoid when F is not trivial has not yet been carried out.

Let h be the Coxeter number of $\mathbf{G}$ and denote by n the semisimple rank of $\mathbf{G}$, which is also the Coxeter rank of W. We begin with

PROPOSITION 4.1. *The h-th roots of $\boldsymbol{\pi}$ are the lift to $\mathbf{W}$ of Coxeter elements of W. Conjecture 2.2 holds for h-th roots of $\boldsymbol{\pi}$, that is, there is always a morphism in $\mathcal{D}^+$ between two such roots.*

Proof. By *e.g.*, [BMi, 3.11] h-th roots of $\boldsymbol{\pi}$ exist. Such a root is an element of B^+ of length n, whose image in W is in the conjugacy class of Coxeter elements by [BMi, 3.12]. Since the minimal length of an element in this conjugacy class is n, which is attained exactly for Coxeter elements, we conclude that an h-th root of $\boldsymbol{\pi}$ is in $\mathbf{W}$, and its image is a Coxeter element.

Now, by [Bou, Chap. V §6, Proposition 1], any two such elements are connected by a morphism in $\mathcal{D}^+$ (it is easy to identify the conjugating process used in *loc. cit.* to morphisms in $\mathcal{D}^+$). ∎

We now show

PROPOSITION 4.2. *Let $\mathbf{c}$ be the lift in $\mathbf{W}$ of a Coxeter element. Then $C_B(\mathbf{c})$ is the cyclic group generated by $\mathbf{c}$.*

Proof. Here we use the results of [BDM] and [Be2]. By [Be2, 2.3.2], B admits a Garside structure where $\mathbf{c}$ is a Garside element. By [BDM, 2.26], the centralizer of $\mathbf{c}$ is generated by the lcm of orbits of atoms under the action of $\mathbf{c}$. Such an element is a $\mathbf{c}$-stable simple element of the dual braid monoid. By [Be2, 1.4.3], it is the lift to $\mathbf{W}$ of an element c_1 in the Coxeter class of a parabolic subgroup of W. But the centralizer of c in W is the cyclic subgroup generated by c, in particular a simple element centralizes $\mathbf{c}$ only if its image in W is a power of c. Thus, we have to show that no power c^k of c

with $1 < k < h$ is the image of a simple element, or equivalently that no such power divides c for the reflection length. By [Be2, 1.2.1] this is equivalent to showing that the equality $\dim\ker(c^k-1)+\dim\ker(c^{1-k}-1)=n$ cannot hold for such k. But, by [Sp, 4.2], the eigenvalues of c are ζ^{1-d_i} where $\zeta = e^{2i\pi/h}$ and where d_i are the reflection degrees of W. Thus the equality to study becomes $|\{i \mid (1-d_i)k \equiv 0 \pmod h\}| + |\{i \mid (1-d_i)(1-k) \equiv 0 \pmod h\}| = n$. Both conditions cannot occur simultaneously since this would imply $d_i \equiv 1 \pmod h$, which is impossible since the irreducibility of W implies that $1 < d_i \leq h$. Thus it is sufficient to exhibit a d_i which satisfies neither condition. But $d_i = h$ itself is such a d_i, whence the result. ∎

It follows immediately from Proposition 4.2 that Conjecture 2.1 holds for Coxeter elements, that is $\operatorname{End}_{\mathcal{D}}(\mathbf{c}) = C_B(\mathbf{c}) = \langle\mathbf{c}\rangle$, since by definition $\mathbf{c} \in \operatorname{End}_{\mathcal{D}}(\mathbf{c})$.

Let us now prove that Conjecture 2.3 holds. The space V_ζ is one-dimensional, and for any $x_\zeta \in V_\zeta^{\mathrm{reg}} = V_\zeta - \{0\}$ the group $\Pi_1(V_\zeta/C_W(c), x_\zeta)$ is cyclic, generated by the loop $\mathbf{b} = t \mapsto e^{2i\pi t/h}$. Doing if necessary a conjugation in B, we may take any h-th root of $\boldsymbol{\pi}$ to prove Conjecture 2.3. We will choose a Springer element, so we may assume that $\mathrm{Re}(x_\zeta)$ is in the real fundamental chamber of W. Then, by Proposition 3.8, the image $\gamma(\mathbf{b})$ is in $\mathbf{W}$, and since its image in W is c it is equal to $\mathbf{c}$. We have shown that γ is an isomorphism.

Conjectures 2.4, 2.5 and 2.6 will follow from the following proposition from [Lu2]:

Proposition 4.3. *Let c be a Coxeter element. Then*

(i) *F is a semisimple automorphism of $\bigoplus_i H^i_c(\mathbf{X}(c))$; it has h distinct eigenvalues; the corresponding eigenspaces are mutually non-isomorphic irreducible $\mathbf{G}^F$-modules.*

(ii) *For $s = 1, \ldots, h-1$, the endomorphism F^s has no fixed points on $\mathbf{X}(\mathbf{c})$.*

(iii) *The eigenvalues of F are monomials in q which, under the specialization $q \mapsto e^{2i\pi/h}$, specialize to $1, \zeta, \zeta^2, \ldots, \zeta^{h-1}$ where $\zeta = e^{2i\pi/h}$.*

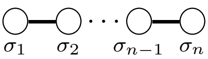
Proof. (i) is [Lu2, 6.1 (i)]; (ii) is 6.1.2 of *loc. cit.*; (iii) results from the tables pages 146–147 of *loc. cit.* ∎

We show now how this implies Conjectures 2.4, 2.5 and 2.6. Conjecture 2.6 is immediate from Proposition 4.3 (i). The generic Hecke algebra $\mathcal{H}(c)$ of the cyclic group $C_W(c)$ of order h is generated by one element T with the relation $(T-u_0)\cdots(T-u_{h-1})=0$. The map which sends $\mathbf{c}$ to $D_{\mathbf{c}}=F$ is thus a representation of this algebra, specialized to $u_i\mapsto\lambda_i$ where $\lambda_0,\ldots,\lambda_{h-1}$ are the eigenvalues of F on $\bigoplus_i H^i_c(\mathbf{X}(c))$. By Proposition 4.3 (iii) this is indeed an h-cyclotomic algebra for $C_W(c)$. It remains to see that the virtual representation $\sum_i(-1)^iH^i_c(\mathbf{X}(c))$ of $\mathcal{H}(c)$ is special. But, by *e.g.*, [BMa, 2.2], the symmetrizing trace on $\mathcal{H}(c)$ is characterized by its vanishing on T^i for $i=1,\ldots,h-1$. By the Lefschetz trace formula, one has $\sum_i(-1)^i\operatorname{Trace}(F^s\mid H^i_c(\mathbf{X}(c),\overline{\mathbb{Q}}_\ell))=|\mathbf{X}(c)^{F^s}|$, so this vanishing is a consequence of Proposition 4.3 (ii).

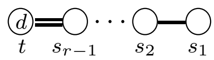
§5. Regular elements in type A

We prove Conjecture 2.1 for roots of $\boldsymbol{\pi}$ when W is of type A and F acts trivially on W. Here we assume that $d>1$. The case of $\boldsymbol{\pi}$ will be treated in Section 7.

Let W be a Coxeter group of type A_{n-1} and let B be the associated braid group. There are d-regular elements in W for any d dividing n or $n-1$. To handle the case $d|n-1$, it will be simpler to embed W as a standard parabolic subgroup of a group W' of type A_n, to embed B as the corresponding subgroup of the associated braid group B' and to consider d-regular elements for $d|n$ in W'.

We denote by $\mathbf{S}=\{\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_n\}$ the set of generators of B', the generators of B being $\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_{n-1}$, where the Coxeter diagram of W' is 

. We denote by $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ respectively the positive generators of the centers of the pure braid groups respectively associated to W and W'.

Let $\mathbf{c}=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{n-1}$, the lift in $\mathbf{W}$ of a Coxeter element. Let r and d be two integers such that $rd=n$ and let $\mathbf{w}=\mathbf{c}^r$; it is a d-th root of $\boldsymbol{\pi}$ (*cf.* Proposition 4.1). The image w of $\mathbf{w}$ in W is a regular element of order d and its centralizer $C_W(w)$ is isomorphic to the complex reflection group $G(d,1,r)$ which has a presentation given by the diagram: 

.

We denote by $B(d,1,r)$ the braid group associated to $G(d,1,r)$; it has a presentation given by the same diagram (deleting the relations giving the orders of the generators).

Conjecture 2.3 holds in our case. Indeed, Bessis [Be1, 4.1] has proved that the morphism $\gamma : B(w) \to C_B(\mathbf{w})$ is injective and in [BDM], this morphism is proved to be bijective. More precisely Bessis proves that $\gamma(\mathbf{s}_i) = \prod_{j=0}^{d-1} \boldsymbol{\sigma}_{i+rj}$ and $\gamma(\prod_{i=1}^{r-1} \mathbf{s}_i\mathbf{t}) = \mathbf{c}$, where $\mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_{r-1}$ are the generators of $B(w)$ given in [BMR, 3.6]: they are braid reflections which satisfy the braid relations given by the above diagram for $G(d,1,r)$. We shall identify $B(w)$ with its image, so that we shall identify $\mathbf{s}_i$ and $\mathbf{t}$ with the elements given by the above formulas.

Let now $\mathbf{c}' = \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_n\boldsymbol{\sigma}_n$, a n-th root of $\boldsymbol{\pi}'$ in B', *cf.* [BMi, A1.1]. Let $\mathbf{w}' = \mathbf{c}'^r$; it is a d-th root of $\boldsymbol{\pi}'$ and if w' is its image in W', the centralizer $C_{W'}(w')$ is also isomorphic to $G(d,1,r)$. It has been proved in [BDM, 5.2] that $C_B(\mathbf{w}) \simeq C_{B'}(\mathbf{w}')$. More precisely, let X_n be the configuration space of n distinct points in $\mathbb{C}$, let μ_n be the set of n-th roots of 1, and let $\nu_{n+1} = \mu_n \cup \{0\}$. We have $B' = \Pi_1(X_{n+1}, \mu_{n+1})$. Let X_n^* be the configuration space of n non-zero distinct points in $\mathbb{C}$; we have morphisms $\Pi_1(X_{n+1}, \nu_{n+1}) \xleftarrow{\Psi} \Pi_1(X_n^*, \mu_n) \xrightarrow{\Theta} \Pi_1(X_n, \mu_n)$, (these morphisms are called A and B in *loc. cit.*). One gets the map Ψ by adjoining to a braid a constant string at 0.

If we choose an isotopy from ν_{n+1} to μ_{n+1} we get an isomorphism of $\Pi_1(X_{n+1}, \nu_{n+1})$ with B'; this can be done by bringing 0 along a path ending at $e^{2i\pi\frac{n}{n+1}}$.

Similarly, if we choose an isotopy mapping the n first $n+1$-th roots of 1 to μ_n, we get an isomorphism $\alpha : \Pi_1(X_n, \mu_n) \xrightarrow{\sim} B$.

It is shown in *loc. cit.* that the map $\alpha \circ \Theta$ has a section Θ' above $C_B(\mathbf{w})$.

Let ψ be the restriction to $C_B(\mathbf{w})$ of $\Psi \circ \Theta'$; it is an isomorphism from $C_B(\mathbf{w})$ to $C_{B'}(\mathbf{w}')$. We have $\psi(\mathbf{c}) = \mathbf{c}'$ and $\psi(\mathbf{s}_i) = \mathbf{s}_i$. Let $\mathbf{t}' = \psi(\mathbf{t})$; it satisfies $\prod_{i=1}^{r-1} \mathbf{s}_i\mathbf{t}' = \mathbf{c}'$.

The following theorem proves Conjecture 2.1 in type A. Note that if we have a standard parabolic subgroup W_1 of a Coxeter group W_2, and if B_1 and B_2 are the corresponding braid groups, the category $\mathcal{D}_1$ (resp. $\mathcal{D}_1^+$) associated to B_1 as in Section 1 is a full subcategory of the category $\mathcal{D}_2$ (resp. $\mathcal{D}_2^+$) associated to B_2. This allows us in the following theorem to state the results in term of the categories associated to B'. We will denote these categories by $\mathcal{D}$ and $\mathcal{D}^+$.

Theorem 5.1. *One has* $\mathbf{s}_i \in \operatorname{End}_{\mathcal{D}^+}(\mathbf{w}) \cap \operatorname{End}_{\mathcal{D}^+}(\mathbf{w}')$, $\mathbf{t} \in \operatorname{End}_{\mathcal{D}}(\mathbf{w})$ *and* $\mathbf{t}' \in \operatorname{End}_{\mathcal{D}}(\mathbf{w}')$, *so that* $\operatorname{End}_{\mathcal{D}}(\mathbf{w}) = C_B(\mathbf{w}) \simeq B(d,1,r)$ *and* $\operatorname{End}_{\mathcal{D}}(\mathbf{w}') = C_{B'}(\mathbf{w}') \simeq B(d,1,r)$.

The end of this section is devoted to the proof of that theorem.

In the next lemma $\preccurlyeq$ (resp. $\succcurlyeq$) denotes left divisibility (resp. right divisibility) in the braid monoid.

LEMMA 5.2. *Let B be the braid group of an arbitrary finite Coxeter group; let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in B^+$ be such that $\mathbf{x}\mathbf{y}^{-1}\mathbf{z} \in B^+$. Then there exist $\mathbf{x}_1, \mathbf{z}_1 \in B^+$ such that $\mathbf{x} \succcurlyeq \mathbf{x}_1$, $\mathbf{z}_1 \preccurlyeq \mathbf{z}$ and $\mathbf{y} = \mathbf{z}_1\mathbf{x}_1$.*

Proof. Let $\mathbf{b} = \mathbf{x}\mathbf{y}^{-1}\mathbf{z}$, so that $\mathbf{y}^{-1}\mathbf{z} = \mathbf{x}^{-1}\mathbf{b}$. By [Mi, 3.2], if we denote by $\mathbf{z}_1$ the left gcd of $\mathbf{y}$ and $\mathbf{z}$, and by $\mathbf{x}_2$ the left gcd of $\mathbf{x}$ and $\mathbf{b}$, we have $\mathbf{z}_1^{-1}\mathbf{y} = \mathbf{x}_2^{-1}\mathbf{x}$, whence the result, putting $\mathbf{x}_1 = \mathbf{x}_2^{-1}\mathbf{x}$. ∎

LEMMA 5.3. *For $i = 1, \ldots, n$, let $\mathbf{c}_i = \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_i$.*

(i) *We have $\mathbf{c}_k\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+1}\mathbf{c}_k$ for $i < k$.*
(i') *We have $\mathbf{c}'\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+1}\mathbf{c}'$ for $i < n-1$.*
(ii) *We have ${}^{\mathbf{c}^2}\boldsymbol{\sigma}_{n-1} = \boldsymbol{\sigma}_1$.*
(ii') *We have ${}^{\mathbf{c}'^2}\boldsymbol{\sigma}_{n-1} = \boldsymbol{\sigma}_1$.*
(iii) *For $\mathbf{x} \in B^+$, one has $\boldsymbol{\sigma}_{i+1} \preccurlyeq \mathbf{c}_k\mathbf{x} \Leftrightarrow \boldsymbol{\sigma}_i \preccurlyeq \mathbf{x}$ for $i < k$.*
(iv) *For $j \leq k$, we have $\{i \mid \boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}_k^j\} = \{1, \ldots, j\}$.*
(iv') *For $j \leq n$ we have $\{i \mid \boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}'^j\} = \{1, \ldots, j\}$.*
(v) *$\mathbf{c}'^j = \mathbf{c}_n^j\boldsymbol{\sigma}_{n-j+1} \cdots \boldsymbol{\sigma}_n$ for $1 \leq j \leq n$.*

Proof. Let us prove (i) and (i'). We get $\mathbf{c}_k\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+1}\mathbf{c}_k$ by commuting $\boldsymbol{\sigma}_i$ (resp. $\boldsymbol{\sigma}_{i+1}$) with the factors of $\mathbf{c}_k$ and applying once the braid relation between $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_{i+1}$. Moreover, as $\mathbf{c}' = \mathbf{c}_n\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_n$ commutes with $\boldsymbol{\sigma}_i$ for $i < n-1$, we get also (i').

From (i), by induction on j we get (v).

Let us prove (ii). For proving ${}^{\mathbf{c}^2}\boldsymbol{\sigma}_{n-1} = \boldsymbol{\sigma}_1$, we use (i) to get $\mathbf{c}\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_{n-2} = \boldsymbol{\sigma}_2 \cdots \boldsymbol{\sigma}_{n-1}\mathbf{c}$, then we multiply both sides on the right by $\boldsymbol{\sigma}_{n-1}$ and on the left by $\boldsymbol{\sigma}_1$ to get $\boldsymbol{\sigma}_1\mathbf{c}^2 = \mathbf{c}^2\boldsymbol{\sigma}_{n-1}$.

We deduce (ii'): we have ${}^{\mathbf{c}'}\boldsymbol{\sigma}_{n-1} = {}^{\mathbf{c}\boldsymbol{\sigma}_n}\boldsymbol{\sigma}_{n-1} = {}^{\mathbf{c}\boldsymbol{\sigma}_{n-1}^{-1}}\boldsymbol{\sigma}_n = {}^{\boldsymbol{\sigma}_n^{-1}\mathbf{c}}\boldsymbol{\sigma}_n$, so that ${}^{\mathbf{c}'^2}\boldsymbol{\sigma}_{n-1} = {}^{\mathbf{c}\boldsymbol{\sigma}_n\boldsymbol{\sigma}_n^{-1}\mathbf{c}}\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_1$ by (ii).

Let us prove (iii). We have $\boldsymbol{\sigma}_{i+1} \preccurlyeq \mathbf{c}_k\mathbf{x} \Leftrightarrow \boldsymbol{\sigma}_{i+1}^{-1}\mathbf{c}_k\mathbf{x} \in B^+$; by (i) $\boldsymbol{\sigma}_{i+1}^{-1}\mathbf{c}_k\mathbf{x} = \mathbf{c}_k\boldsymbol{\sigma}_i^{-1}\mathbf{x}$ and by Lemma 5.2 this implies that either $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{x}$ or $\mathbf{c}_k \succcurlyeq \boldsymbol{\sigma}_i$. But, as no braid relation can be applied in $\mathbf{c}_k$, the only j such that $\mathbf{c}_k \succcurlyeq \boldsymbol{\sigma}_j$ is k. So we are in the case $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{x}$, whence the implication from left to right. The converse implication comes from the fact that $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{x} \Rightarrow \mathbf{c}_k\boldsymbol{\sigma}_i^{-1}\mathbf{x} \in B^+$.

Let us prove (iv). If $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}_k^j$ we cannot have $i > j$ otherwise applying j times (iii), we get $\boldsymbol{\sigma}_{i-j} \preccurlyeq 1$ which is false. Conversely, if $i \leq j$, by applying $i-1$ times (iii) we get $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}_k^j \Leftrightarrow \boldsymbol{\sigma}_1 \preccurlyeq \mathbf{c}_k^{j-i+1}$ which is true.

(v) is a direct computation using (i).

Let us prove (iv'). By (iv) and (v), we have $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}'^j$ for $i \leq j$. On the other hand by (iii) and (v) we see that if $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{c}'^j$ with $i > j$, then $\boldsymbol{\sigma}_{i-j} \preccurlyeq \boldsymbol{\sigma}_{n-j+1} \cdots \boldsymbol{\sigma}_n$ which is impossible. ∎

In the next lemma, as in [DMR, 5.2.7], for $I \subset S$ we denote by B_I^+ the submonoid of B^+ generated by $\mathbf{I} = \{\mathbf{s} \in \mathbf{S} \mid s \in I\}$, and by $\mathcal{D}_I^+$ the "parabolic" subcategory of $\mathcal{D}^+$ where we keep only the maps coming from elements of B_I^+.

LEMMA 5.4. *Assume $1 \leq i < r$ and let $\mathbf{I}_i = \{\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+r}, \ldots, \boldsymbol{\sigma}_{i+(d-1)r}\}$. Then*

(i) $\mathbf{s}_i \in \operatorname{End}_{\mathcal{D}_{I_i}^+}(\mathbf{w}) \cap \operatorname{End}_{\mathcal{D}_{I_i}^+}(\mathbf{w}')$. *In particular* $\mathbf{s}_i \in \operatorname{End}_{\mathcal{D}^+}(\mathbf{w}) \cap \operatorname{End}_{\mathcal{D}^+}(\mathbf{w}')$.

(ii) *The conjugation by either* $\mathbf{w}$ *or* $\mathbf{w}'$ *stabilizes* $\mathbf{I}_i$ *and induces on this set the cyclic permutation* $\boldsymbol{\sigma}_{i+jr} \mapsto \boldsymbol{\sigma}_{i+(j+1)r \pmod n}$.

Proof. We can assume $r \geq 2$ otherwise there is nothing to prove. Let $\mathbf{y}_j = \boldsymbol{\sigma}_{i+r(j-1)}$ for $j = 1, \ldots, d$, and, following the notation of [DMR, 5.1.1 (i)] let $\mathbf{w}_1 = \mathbf{w}$ and $\mathbf{w}_{j+1} = \mathbf{y}_1^{-1}\mathbf{w}\mathbf{y}_j$ for $j = 1, \ldots, d$. We have $\mathbf{w}_j \in B^+$ as $\mathbf{y}_1 = \boldsymbol{\sigma}_i$ divides $\mathbf{w} = \mathbf{c}_{n-1}^r$ by Lemma 5.3 (iv). We have $\mathbf{w}_{j+1} = \mathbf{y}_j^{-1}\mathbf{w}_j\mathbf{y}_j$ by using ${}^{\mathbf{w}}\mathbf{y}_{j-1} = \mathbf{y}_j$, which is a consequence of Lemma 5.3 (i), and the fact that $\mathbf{y}_j$ and $\mathbf{y}_1$ commute as $r \geq 2$; from Lemma 5.3 (ii) we have ${}^{\mathbf{w}}\mathbf{y}_d = \mathbf{y}_1$, whence $\mathbf{w}_{d+1} = \mathbf{w}$ so that (i) is proved for $\mathbf{w}$. We get (i) for $\mathbf{w}'$ by the same computation, replacing $\mathbf{w}$ by $\mathbf{w}'$ and using Lemma 5.3 (i'), (ii') and (iv') instead of Lemma 5.3 (i), (ii) and (iv). We have also got (ii) along the way. ∎

To prove that $\mathbf{t} \in \operatorname{End}_{\mathcal{D}}(\mathbf{w})$, we shall find $\mathbf{y} \in B^+$ such that $\mathbf{y}\mathbf{w}\mathbf{y}^{-1} \in B^+$, $\mathbf{y} \in \operatorname{Hom}_{\mathcal{D}^+}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1}, \mathbf{w})$ and $\mathbf{y}\mathbf{t}\mathbf{y}^{-1} \in \operatorname{End}_{\mathcal{D}^+}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1})$. Then, as $\mathcal{D}$ is a groupoid, we will get $\mathbf{t} \in \operatorname{End}_{\mathcal{D}}(\mathbf{w})$.

We will follow the same lines to prove that $\mathbf{t}' \in \operatorname{End}_{\mathcal{D}}(\mathbf{w}')$.

LEMMA 5.5. *When $j \leq k$, let $\boldsymbol{\sigma}_{j,k} = \boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1}\cdots\boldsymbol{\sigma}_k$ and let $\mathbf{x}_i = \boldsymbol{\sigma}_{i,i+r-2}$. With this notation, let $\mathbf{y} = \prod_{i=1}^{d-1}\mathbf{y}_i$ where $\mathbf{y}_i = \prod_{j=d}^{i+1}\mathbf{x}_{i(r-1)+j}$. Then $\mathbf{y}\mathbf{w}\mathbf{y}^{-1} \in B^+$ and $\mathbf{y} \in \operatorname{Hom}_{\mathcal{D}^+}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1}, \mathbf{w})$.*

Proof. We set $\mathbf{w}_d = \mathbf{w}$, and then by decreasing induction on i we define $\mathbf{y}_i^{-1}\mathbf{w}_i\mathbf{y}_i = \mathbf{w}_{i+1}$. It is enough to show that $\mathbf{y}_i^{-1}\mathbf{w}_i \in B^+$; we will get this by proving by induction that $\mathbf{w}_i = \mathbf{y}_i\mathbf{c}^{r-1}\mathbf{c}_{i(r-1)+d-1}$. This formula is true for $i = d$ (here $\mathbf{y}_d = 1$). Let us assume it true for $i+1$ and let us prove it for i. We have

$$\begin{aligned}
\mathbf{w}_{i+1} &= \mathbf{y}_{i+1}\mathbf{c}^{r-1}\mathbf{c}_{(i+1)(r-1)+d-1}\\
&= \mathbf{x}_{(i+1)(r-1)+d}\cdots\mathbf{x}_{(i+1)(r-1)+i+1}\mathbf{c}^{r-1}\mathbf{c}_{(i+1)(r-1)+d-1}\\
&= \mathbf{c}^{r-1}\mathbf{x}_{i(r-1)+d}\cdots\mathbf{x}_{i(r-1)+i+1}\mathbf{c}_{(i+1)(r-1)+d-1} \quad \text{(by 5.3 (i))}\\
&= \mathbf{c}^{r-1}\mathbf{c}_{(i+1)(r-1)+d-1}\mathbf{x}_{i(r-1)+d-1}\cdots\mathbf{x}_{i(r-1)+i} \quad \text{(by 5.3 (i))}\\
&= \mathbf{c}^{r-1}\mathbf{c}_{i(r-1)+d-1}\mathbf{x}_{i(r-1)+d}\mathbf{x}_{i(r-1)+d-1}\cdots\mathbf{x}_{i(r-1)+i}\\
&= \mathbf{c}^{r-1}\mathbf{c}_{i(r-1)+d-1}\mathbf{y}_i
\end{aligned}$$

which, conjugating by $\mathbf{y}_i$, gives the stated value for $\mathbf{w}_i$. ∎

We note that in particular we have $\mathbf{y}\mathbf{w}\mathbf{y}^{-1} = \mathbf{w}_1 = \mathbf{c}^{r-1}\mathbf{c}_{d-1}\mathbf{y}_0 = \boldsymbol{\sigma}_{r,r+d-2}\mathbf{c}^{r-1}\mathbf{x}_d\cdots\mathbf{x}_1$.

We prove now the analogous lemma for $\mathbf{w}'$. Let $\mathbf{c}'_i = \mathbf{c}_i\boldsymbol{\sigma}_i$.

Lemma 5.6. *Let* $\mathbf{y}' = \prod_{i=1}^{d-1}\mathbf{y}'_i$ *where* $\mathbf{y}'_i = \prod_{j=d+1}^{i+1}\mathbf{x}_{i(r-1)+j}$. *Then* $\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1} \in B^+$ *and* $\mathbf{y}' \in \operatorname{Hom}_{\mathcal{D}^+}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1}, \mathbf{w}')$.

Proof. We can assume $r \geq 2$ since for $r = 1$ we have $\mathbf{y}' = 1$. We set $\mathbf{w}'_d = \mathbf{w}'$, and then by decreasing induction on i, we define $\mathbf{w}'_i = \mathbf{y}'_i\mathbf{w}'_{i+1}\mathbf{y}'^{-1}_i$. We will show that $\mathbf{y}'^{-1}_i\mathbf{w}'_i \in B^+$ by proving by decreasing induction on i that $\mathbf{w}'_i = \mathbf{y}'_i\mathbf{c}^{r-1}_{rd}\mathbf{c}'_{(r-1)i+d}$. This equality has no meaning for $i = d$, as $\mathbf{x}_{dr+1}$ does not make sense. So we rewrite it as $\mathbf{w}'_i = \mathbf{c}^{r-1}_{rd}\delta_{r-1}(\mathbf{y}'_i)\mathbf{c}'_{(r-1)i+d}$, where we define formally $\delta_i(\mathbf{x}_j) = \mathbf{x}_{j-i}$ which makes sense when $j - i \leq n$. Now the formula for $i = d$ becomes $\mathbf{w}' = \mathbf{c}^r_{rd}\boldsymbol{\sigma}_{rd-r+1,rd}$, which holds by Lemma 5.3 (v). This is the starting point of the induction. Let us assume the formula true for i and let us prove it for $i-1$. We have

$$\begin{aligned}
\mathbf{w}'_i &= \mathbf{c}^{r-1}_{rd}\delta_{r-1}(\mathbf{y}'_i)\mathbf{c}'_{(r-1)i+d}\\
&= \mathbf{c}^{r-1}_{rd}\delta_{r-1}(\mathbf{y}'_i)\mathbf{c}_{(r-1)i+d}\boldsymbol{\sigma}_{(r-1)i+d}\\
&= \mathbf{c}^{r-1}_{rd}\mathbf{c}_{(r-1)i+d}\delta_{r-2}(\mathbf{y}'_i)\boldsymbol{\sigma}_{(r-1)i+d}
\end{aligned}$$

the last equality as $r \geq 2$ implies that the largest j such that $\boldsymbol{\sigma}_j$ occurs in $\delta_{r-1}(\mathbf{y}'_i)$ is at most $(r-1)i+d$,

$$\begin{aligned}&= \mathbf{c}_{rd}^{r-1}\mathbf{c}_{(r-1)(i-1)+d}\mathbf{y}'_{i-1}\boldsymbol{\sigma}_{(r-1)i+d}\\&= \mathbf{c}_{rd}^{r-1}\mathbf{c}'_{(r-1)(i-1)+d}\mathbf{y}'_{i-1},\end{aligned}$$

where we get the last line by commuting $\boldsymbol{\sigma}_{(r-1)i+d}$ and $\mathbf{x}_{(i-1)(r-1)+j}$ for $j < d$, and applying formula $\mathbf{x}_{a+1}\mathbf{x}_a\boldsymbol{\sigma}_{a+r-1} = \boldsymbol{\sigma}_a\mathbf{x}_{a+1}\mathbf{x}_a$ to the two first terms of $\mathbf{y}'_{i-1}$. ∎

From Lemma 5.6 we get $\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1} = \mathbf{w}'_1 = \mathbf{c}_{rd}^{r-1}\mathbf{c}'_d\mathbf{y}'_0 = \boldsymbol{\sigma}_{r,r+d-1}\boldsymbol{\sigma}_{r+d-1}$ $\mathbf{c}_{rd}^{r-1}\mathbf{x}_{d+1}\cdots\mathbf{x}_1$.

Let us now prove

Lemma 5.7. (i) *We have* $\mathbf{y}\mathbf{t}\mathbf{y}^{-1} = \boldsymbol{\sigma}_{r,r+d-2}$.

(ii) *We have* $\mathbf{y}'\mathbf{t}'\mathbf{y}'^{-1} = \boldsymbol{\sigma}_{r,r+d-1}\boldsymbol{\sigma}_{r+d-1}$.

Proof. Let us prove (i). We show that $\mathbf{t} = \prod_{j=1}^{d-1}\mathbf{a}_{jr,(j+1)r-1}$ where $\mathbf{a}_{k,l} = \boldsymbol{\sigma}_{k,l}\boldsymbol{\sigma}_{k,l-1}^{-1}$. For this, replace each $\mathbf{a}_{k,l}$ by its value to get $\prod_{i=1}^{d-1}\boldsymbol{\sigma}_{ir,(i+1)r-1}$ $\prod_{i=1}^{d-1}\boldsymbol{\sigma}_{ir,(i+1)r-2}^{-1}$, which is equal to $\boldsymbol{\sigma}_{r,dr-1}\prod_{i=1}^{d-1}\boldsymbol{\sigma}_{ir,(i+1)r-2}^{-1}$, in turn equal to $\mathbf{c}_{r-1}^{-1}\mathbf{c}\prod_{i=1}^{d-1}\boldsymbol{\sigma}_{ir,(i+1)r-2}^{-1}$. We can put $\mathbf{c}$ on the right of this product if we replace $\boldsymbol{\sigma}_{k,l}$ by $\boldsymbol{\sigma}_{k+1,l+1}$ for all k, l. We get $\prod_{i=0}^{d-1}\boldsymbol{\sigma}_{ir+1,(i+1)r-1}^{-1}\mathbf{c}$, which is equal to $\mathbf{t} = (\prod_{i=1}^{r-1}\mathbf{s}_i)^{-1}\mathbf{c}$, as wanted, since $\prod_{i=1}^{r-1}\mathbf{s}_i = \prod_{j=0}^{d-1}\mathbf{x}_{jr+1}$ where all factors commute.

By decreasing induction on i, we find $\prod_{j=i}^{d-1}\mathbf{y}_j = \prod_{j=d}^{i+1}\boldsymbol{\sigma}_{ir+j-i,jr-1}$, so, setting $\mathbf{z}_i = \boldsymbol{\sigma}_{i+r,(i+1)r-1}$, we have $\mathbf{y} = \prod_{i=d-1}^{1}\mathbf{z}_i$. But $\mathbf{z}_j$ conjugates $\mathbf{a}_{jr,(j+1)r-1}$ into $\mathbf{a}_{j+r,(j+1)r-1}$ and $\mathbf{a}_{j+r-1,(j+1)r-1}$ into $\boldsymbol{\sigma}_{j+r-1}$. It commutes with $\boldsymbol{\sigma}_i$ for $i < j+r-1$ and with $\mathbf{a}_{ir,(i+1)r-1}$ for $i > j$. So by induction on j we see that $\prod_{i=j}^{1}\mathbf{z}_i$ conjugates $\mathbf{t}$ into $(\prod_{i=1}^{j}\boldsymbol{\sigma}_{i+r-1})\mathbf{a}_{j+r,(j+1)r-1}$ $(\prod_{i=j+1}^{d-1}\mathbf{a}_{ir,(i+1)r-1})$, whence (i).

We now prove (ii). We claim that $\mathbf{y}' = \boldsymbol{\sigma}_{d+r,dr}\mathbf{y}$: indeed $\mathbf{y}'_i = \mathbf{x}_{i(r-1)+d+1}\mathbf{y}_i$ and $\mathbf{x}_{i(r-1)+d+1}$ commutes with $\mathbf{y}_k$ for $k < i$; this gives the claim as $\prod_{i=1}^{d-1}\mathbf{x}_{i(r-1)+d+1} = \boldsymbol{\sigma}_{d+r,dr}$.

Let us now conjugate $\mathbf{t}' = \mathbf{t}\boldsymbol{\sigma}_{dr}^2$ by $\mathbf{y}' = \boldsymbol{\sigma}_{d+r,dr}\mathbf{y}$. By (i), $\mathbf{y}'$ conjugates $\mathbf{t}$ into $\boldsymbol{\sigma}_{r,r+d-2}$ as $\boldsymbol{\sigma}_{d+r,dr}$ commutes with $\boldsymbol{\sigma}_{r,r+d-2}$. On the other hand conjugation by $\mathbf{y}'$ has the same effect on $\boldsymbol{\sigma}_{dr}$ as conjugation by $\boldsymbol{\sigma}_{d+r,dr}\mathbf{z}_{d-1} = \boldsymbol{\sigma}_{d+r,dr}\boldsymbol{\sigma}_{d+r-1,dr-1}$ as $\mathbf{z}_i$ for $i < d-1$ commutes with $\boldsymbol{\sigma}_{dr}$.

It remains to see that $\boldsymbol{\sigma}_{d+r,dr}\boldsymbol{\sigma}_{d+r-1,dr-1}$ conjugates $\boldsymbol{\sigma}_{dr}$ into $\boldsymbol{\sigma}_{d+r-1}$ *i.e.*, $\boldsymbol{\sigma}_{d+r,dr}\boldsymbol{\sigma}_{d+r-1,dr-1}\boldsymbol{\sigma}_{dr} = \boldsymbol{\sigma}_{d+r-1}\boldsymbol{\sigma}_{d+r,dr}\boldsymbol{\sigma}_{d+r-1,dr-1}$; this can be written $\boldsymbol{\sigma}_{d+r,dr}\boldsymbol{\sigma}_{d+r-1,dr} = \boldsymbol{\sigma}_{d+r-1,dr}\boldsymbol{\sigma}_{d+r-1,dr-1}$, which is true. ∎

COROLLARY 5.8. *We have* $\mathbf{t} \in \operatorname{End}_{\mathcal{D}}(\mathbf{w})$ *and* $\mathbf{t}' \in \operatorname{End}_{\mathcal{D}}(\mathbf{w}')$.

Proof. If $\mathbf{I} = \{\boldsymbol{\sigma}_r, \ldots, \boldsymbol{\sigma}_{r+d-2}\}$, then $\boldsymbol{\sigma}_{r,r+d-2} \in \operatorname{End}_{\mathcal{D}^+_{\mathbf{I}}}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1})$, as $\boldsymbol{\sigma}_{r,r+d-2} \preccurlyeq \mathbf{y}\mathbf{w}\mathbf{y}^{-1}$. By Lemma 5.5 and the remarks made above that lemma, we get the first assertion.

Similarly, if $\mathbf{I}' = \{\boldsymbol{\sigma}_r, \ldots, \boldsymbol{\sigma}_{r+d-1}\}$, then $\boldsymbol{\sigma}_{r,r+d-1}\boldsymbol{\sigma}_{r+d-1} \in \operatorname{End}_{\mathcal{D}^+_{\mathbf{I}'}}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1})$ as $\boldsymbol{\sigma}_{r,r+d-1}\boldsymbol{\sigma}_{r+d-1} \prec \mathbf{y}\mathbf{w}\mathbf{y}^{-1}$. By Lemma 5.6, we get the second assertion. ∎

§6. Regular elements in type B

We now prove Conjecture 2.1 for roots of $\boldsymbol{\pi}$ when W is of type B_n. We will see W as the centralizer in a Coxeter group W' of type A_{2n-1} of the longest element w_0. Let V' be the vector space which affords the reflection representation of W'. As w_0 is a 2-regular element, the eigenspace V'_{-1} of w_0 affords the reflection representation of W, and if we choose a base point in V'^{reg}_{-1} we get an embedding $\Pi_1(V^{\mathrm{reg}}/W) \hookrightarrow \Pi_1(V'^{\mathrm{reg}}/W')$ of the braid group B of type B_n into the braid group B' of type A_{2n-1}. If $\mathbf{S}' = \{\boldsymbol{\sigma}'_1, \ldots, \boldsymbol{\sigma}'_{2n-1}\}$ is the generating set of B' then it is known (see [Mi, 4.4]) that $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}'_n$ and $\boldsymbol{\sigma}_{n+1-i} = \boldsymbol{\sigma}'_i\boldsymbol{\sigma}'_{2n-i}$ for $1 \leq i < n$ are generators of B such that the relations are given by the Coxeter diagram $\underset{\boldsymbol{\sigma}_1}{\bigcirc}\!=\!\underset{\boldsymbol{\sigma}_2}{\bigcirc}\!-\!\underset{\boldsymbol{\sigma}_3}{\bigcirc}\cdots\underset{\boldsymbol{\sigma}_n}{\bigcirc}$.

Let w be a d-regular element of W for some d, and let $\zeta = e^{2i\pi/d}$. With notation as above 3.2, we have $V^{\mathrm{reg}}_{\zeta}/C_W(w) \simeq (V^{\mathrm{reg}}/W)^{\langle\zeta\rangle}$ and $V^{\mathrm{reg}}/W = V'^{\mathrm{reg}}_{-1}/C_{W'}(w_0) \simeq (V'^{\mathrm{reg}}/W')^{\langle -1\rangle}$, so that $V^{\mathrm{reg}}_{\zeta}/C_W(w) \simeq (V'^{\mathrm{reg}}/W')^{\langle -1,\zeta\rangle}$. This is equal to $(V'^{\mathrm{reg}}/W')^{\langle\zeta\rangle}$ if d is even and to $(V'^{\mathrm{reg}}/W')^{\langle\zeta'\rangle}$ with $\zeta' = e^{i\pi/d}$ if d is odd. As $C_W(w) = C_{W'}(w, w_0)$, we see that if w_0 is a power of w, which implies that d is even, then $C_W(w) = C_{W'}(w)$. In this case, as $V^{\mathrm{reg}}_{\zeta} = V'^{\mathrm{reg}}_{\zeta}$ we see that the map γ from $B(w)$ to $C_{B'}(\mathbf{w})$ is the composition of the map which we still denote by γ from $B(w)$ to $C_B(\mathbf{w})$ and of the embedding $B \to B'$.

We make a specific choice of a regular element. Let $\mathbf{c} = \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_n$. It is a $2n$-th root of $\boldsymbol{\pi}$ (*cf.* Proposition 4.1). Let r and d be two integers such that $rd = 2n$ and let $\mathbf{w} = \mathbf{c}^r$; it is a d-th root of $\boldsymbol{\pi}$ and its image $w \in W$ is a good regular element.

If d is odd, we have $C_B(\mathbf{w}) = C_B(\mathbf{c}^r) = C_B(\mathbf{c}^{pgcd(n,r)}) = C_B(\mathbf{c}^{r/2})$ and $C_W(w) = C_W(c^{r/2})$ since $\mathbf{w}_0 = \mathbf{c}^n$ is central. So we are reduced to study $C_B(\mathbf{w})$ when d is even and $C_B(\mathbf{w}^2)$ when d is even and $d/2$ odd (see Theorem 6.3 (ii), below).

When d is even we have seen above that $C_W(w) = C_{W'}(w)$, and this group is a complex reflection group of type $G(d,1,r)$ (see also [BMi, A.1.2]).

We have $\mathbf{c} = \boldsymbol{\sigma}'_n\boldsymbol{\sigma}'_{n-1}\boldsymbol{\sigma}'_{n+1}\cdots\boldsymbol{\sigma}'_1\boldsymbol{\sigma}'_{2n-1}$. In order to apply the results of Section 5, we use a conjugation by $\mathbf{v}^{-1}$ where $\mathbf{v}$ is the canonical lift of the longest element of the standard parabolic subgroup of W' generated by $\sigma'_1,\ldots,\sigma'_{n-1}$: indeed we have $\mathbf{c} = \mathbf{v}^{-1}\boldsymbol{\sigma}'_1\cdots\boldsymbol{\sigma}'_{2n-1}\mathbf{v}$.

This proves that the generators of $C_B(\mathbf{w})$ are the elements $\mathbf{s}_i = \mathbf{v}^{-1}(\prod_{j=0}^{d-1}\boldsymbol{\sigma}'_{i-1+rj})\mathbf{v} = \prod_{k=0}^{d/2-1}\boldsymbol{\sigma}_{i+kr}$ for $i = 2,\ldots,r$ and the element $\mathbf{t}$ such that $\mathbf{t}\prod_{i=2}^{r}\mathbf{s}_i = \mathbf{c}$; note that we have not chosen for the element $\mathbf{t}$ the conjugate by $\mathbf{v}^{-1}$ of the element $\mathbf{t}$ of Section 5 but we have applied a further conjugation by $\mathbf{c}$ in order to simplify the computations. We have $\mathbf{t} = (\hat{\boldsymbol{\sigma}}_2\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_{r+1}\hat{\boldsymbol{\sigma}}_{r+2}\cdots\boldsymbol{\sigma}_{(d/2-1)r+1})^{-1}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{(d/2-1)r+1}$, where $\hat{\boldsymbol{\sigma}}_i$ means deleting $\boldsymbol{\sigma}_i$ from the product: we have deleted all $\boldsymbol{\sigma}_i$ such that $i \equiv 2 \pmod r$.

We first prove a lemma analogous to Lemma 5.3.

Lemma 6.1. (i) *We have* $\mathbf{c}\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+1}\mathbf{c}$ *for* $2 \le i < n-1$.

(ii) *We have* ${}^{\mathbf{c}^2}\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_2$.

(iii) *For* $\mathbf{x} \in B^+$ *and* $2 \le i < n$ *we have* $\boldsymbol{\sigma}_i \prec \mathbf{x} \Leftrightarrow \boldsymbol{\sigma}_{i+1} \prec \mathbf{c}\mathbf{x}$.

(iv) *We have* $\{i \mid \boldsymbol{\sigma}_i \prec \mathbf{c}^j\} = \{1,\ldots,j\}$ *for* $j \le n$.

Proof. Statements (i) and (iii) have the same proof as the corresponding statements in Lemma 5.3.

Let us prove (ii). We have

$$\begin{aligned}\mathbf{c}^2 &= \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_n)\mathbf{c} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\mathbf{c}(\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{n-1})\\ &= \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_n)(\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{n-1})\\ &= \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_n)(\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{n-1})\\ &= \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_n)\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdots\boldsymbol{\sigma}_{n-1})\\ &= \boldsymbol{\sigma}_2\mathbf{c}\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{n-1},\end{aligned}$$

whence $\mathbf{c}^2\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_2\mathbf{c}^2$.

The proof of (iv) is similar to that of Lemma 5.3 (iv): it uses similarly the fact that $\boldsymbol{\sigma}_i \prec \mathbf{c}$ if and only if $i = 1$, but it also needs the fact that $\boldsymbol{\sigma}_2 \prec \mathbf{c}^2$, which we have seen in the proof of (ii). ∎

LEMMA 6.2. *The group $C_B(\mathbf{w})$ has a presentation with generators $\mathbf{t}$, $\mathbf{s}_2,\ldots,\mathbf{s}_r$, the relations being the braid relations given by the diagram* $\underset{\mathbf{t}}{\bigcirc}\!=\!\underset{\mathbf{s}_2}{\bigcirc}\!-\!\underset{\mathbf{s}_3}{\bigcirc}\cdots\underset{\mathbf{s}_r}{\bigcirc}$.

Proof. We already know that $C_B(\mathbf{w})$ has a presentation with generators $\mathbf{s}_2,\ldots,\mathbf{s}_r$, $\mathbf{t}'$ and relations the braid relations given by $\underset{\mathbf{s}_2}{\bigcirc}\!-\!\underset{\mathbf{s}_3}{\bigcirc}\cdots\underset{\mathbf{s}_r}{\bigcirc}\!=\!\underset{\mathbf{t}'}{\bigcirc}$, where $\mathbf{t}'$ is the element conjugate by $\mathbf{v}^{-1}$ of the element $\mathbf{t}$ of Section 5. We have $\mathbf{t}=\mathbf{c}\mathbf{t}'\mathbf{c}^{-1}=(\mathbf{s}_2\cdots\mathbf{s}_r)\mathbf{t}'(\mathbf{s}_2\cdots\mathbf{s}_r)^{-1}$. The commutation $\mathbf{t}'$ with $\mathbf{s}_i$ for $i<r$ is equivalent by 6.1 (i) to the commutation of $\mathbf{t}$ with $\mathbf{s}_i$ for $i>2$. It remains to see that the braid relation between $\mathbf{t}'$ and $\mathbf{s}_r$ is equivalent to the braid relation between $\mathbf{t}$ and $\mathbf{s}_2$. This is proved by decreasing induction on i using the following fact that is the result of a simple computation: if $\mathbf{s}_{i-1}$, $\mathbf{s}_i$ and $\mathbf{u}'$ are elements of a group and if $\mathbf{u}=\mathbf{s}_i\mathbf{u}'\mathbf{s}_i^{-1}$ then the braid relations given by $\underset{\mathbf{s}_{i-1}}{\bigcirc}\!-\!\underset{\mathbf{s}_i}{\bigcirc}\!=\!\underset{\mathbf{u}'}{\bigcirc}$ imply $\mathbf{s}_{i-1}\mathbf{u}\mathbf{s}_{i-1}\mathbf{u}=\mathbf{u}\mathbf{s}_{i-1}\mathbf{u}\mathbf{s}_{i-1}$. Conjugating by $\mathbf{s}_2$ the relation $\mathbf{s}_2{}^{\mathbf{s}_3\cdots\mathbf{s}_r}\mathbf{t}'\mathbf{s}_2{}^{\mathbf{s}_3\cdots\mathbf{s}_r}\mathbf{t}'={}^{\mathbf{s}_3\cdots\mathbf{s}_r}\mathbf{t}'\mathbf{s}_2{}^{\mathbf{s}_3\cdots\mathbf{s}_r}\mathbf{t}'\mathbf{s}_2$, which we get at the end of the induction, gives the braid relation which we want. The converse is similar. ∎

THEOREM 6.3. *Assume d even; then*

(i) *We have $\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+}(\mathbf{w})$ $(i=1,\ldots,r-1)$ and $\mathbf{t}\in\operatorname{End}_{\mathcal{D}}(\mathbf{w})$, so that $\operatorname{End}_{\mathcal{D}}(\mathbf{w})=C_B(\mathbf{w})\simeq B(d,1,r)$.*

(ii) *If $d/2$ odd, we have $\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+}(\mathbf{w}^2)$ $(i=1,\ldots,r-1)$ and $\mathbf{t}\in\operatorname{End}_{\mathcal{D}}(\mathbf{w}^2)$, so that $\operatorname{End}_{\mathcal{D}}(\mathbf{w}^2)=C_B(\mathbf{w}^2)\simeq B(d,1,r)$.*

Proof. In the following lemma, the statements about $\mathbf{w}^2$ assume that $d/2$ is odd. The proof follows the same lines as that of Lemma 5.4, using Lemma 6.1 instead of Lemma 5.3.

LEMMA 6.4. *Assume $2\le i\le r$ and let $\mathbf{I}_i=\{\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_{i+r},\ldots,\boldsymbol{\sigma}_{i+(d/2-1)r}\}$. Then*

(i) *$\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+_{I_i}}(\mathbf{w})$ and $\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+_{I_i}}(\mathbf{w}^2)$. In particular $\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+}(\mathbf{w})$ and $\mathbf{s}_i\in\operatorname{End}_{\mathcal{D}^+}(\mathbf{w}^2)$.*

(ii) *The conjugation by $\mathbf{w}$ (resp. $\mathbf{w}^2$) stabilizes $\mathbf{I}_i$ and induces the cyclic permutation $\boldsymbol{\sigma}_{i+jr}\mapsto\boldsymbol{\sigma}_{i+(j+1)r\ (\mathrm{mod}\ n)}$ $\big($resp. $\boldsymbol{\sigma}_{i+jr}\mapsto\boldsymbol{\sigma}_{i+(j+2)r\ (\mathrm{mod}\ n)}\big)$.*

LEMMA 6.5. *For $i\le j$ we set $\boldsymbol{\sigma}_{i,j}=\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}\cdots\boldsymbol{\sigma}_j$, and we set $\mathbf{x}_i=\boldsymbol{\sigma}_{i+1,i+r-1}$. Let $\mathbf{y}=\prod_{i=1}^{d/2-1}\mathbf{y}_i$ where $\mathbf{y}_i=\prod_{k=1}^{d/2-i}\mathbf{x}_{(i-1)(r-1)+d/2-k+1}$. Then $\mathbf{y}\mathbf{w}\mathbf{y}^{-1}\in B^+$ and $\mathbf{y}\in\operatorname{Hom}_{\mathcal{D}^+}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1},\mathbf{w})$.*

Proof. We set $\mathbf{c}_i=\boldsymbol{\sigma}_{1,i}$. Let $\mathbf{w}_{d/2}=\mathbf{w}$, and by decreasing induction on i define $\mathbf{y}_i^{-1}\mathbf{w}_i\mathbf{y}_i=\mathbf{w}_{i+1}$. We claim that $\mathbf{y}_i^{-1}\mathbf{w}_i\in B^+$, which implies the result: in fact we prove by induction that $\mathbf{w}_i=\mathbf{y}_i\mathbf{c}_{i(r-1)+d/2}\mathbf{c}^{r-1}$. This equality is clearly true for $i=d/2$ (with $\mathbf{y}_{d/2}=1$). Let us assume it to be true for $i+1$ and let us prove it for i. We have

$$\begin{aligned}
\mathbf{w}_{i+1}&=\mathbf{y}_{i+1}\mathbf{c}_{(i+1)(r-1)+d/2}\mathbf{c}^{r-1}\\
&=\prod_{k=1}^{d/2-i-1}\mathbf{x}_{i(r-1)+d/2-k+1}\mathbf{c}_{(i+1)(r-1)+d/2}\mathbf{c}^{r-1}\\
&=\mathbf{c}_{(i+1)(r-1)+d/2}\prod_{k=1}^{d/2-i-1}\mathbf{x}_{i(r-1)+d/2-k}\mathbf{c}^{r-1}\quad\text{(by 6.1 (i))}\\
&=\mathbf{c}_{i(r-1)+d/2}\prod_{k=0}^{d/2-i-1}\mathbf{x}_{i(r-1)+d/2-k}\mathbf{c}^{r-1}\\
&=\mathbf{c}_{i(r-1)+d/2}\mathbf{c}^{r-1}\prod_{k=0}^{d/2-i-1}\mathbf{x}_{(i-1)(r-1)+d/2-k}\quad\text{(by 6.1 (i))}\\
&=\mathbf{c}_{i(r-1)+d/2}\mathbf{c}^{r-1}\mathbf{y}_i
\end{aligned}$$

which, conjugating by $\mathbf{y}_i$ gives the equality for $\mathbf{w}_i$. ∎

We note that the above proof shows that

$$\begin{aligned}
\mathbf{ywy}^{-1}=\mathbf{y}_1\mathbf{c}_{r-1+d/2}\mathbf{c}^{r-1}&=\prod_{i=d/2}^{2}\mathbf{x}_i\prod_{i=1}^{r-1+d/2}\boldsymbol{\sigma}_i\mathbf{c}^{r-1}\\
&=\boldsymbol{\sigma}_{1,r-1+d/2}\prod_{i=d/2-1}^{1}\mathbf{x}_i\mathbf{c}^{r-1}
\end{aligned}$$

the last equality by Lemma 6.1 (i), so $\mathbf{ywy}^{-1}=\boldsymbol{\sigma}_{1,d/2}\prod_{i=d/2}^{1}\mathbf{x}_i\mathbf{c}^{r-1}$.

Lemma 6.6. *We have* $\mathbf{yty}^{-1}=\boldsymbol{\sigma}_{1,d/2}$.

Proof. For $i=1,\ldots,d/2-1$, let $\mathbf{t}_i=(\hat{\boldsymbol{\sigma}}_2\boldsymbol{\sigma}_3\cdots\boldsymbol{\sigma}_{r+1}\hat{\boldsymbol{\sigma}}_{r+2}\cdots\boldsymbol{\sigma}_{ir+1})^{-1}$ $\boldsymbol{\sigma}_{1,i(r-1)+d/2}$. We have $\mathbf{t}=\mathbf{t}_{d/2-1}$ and $\mathbf{t}_0=\boldsymbol{\sigma}_{1,d/2}$. We prove by induction that $\mathbf{y}_i$ (*cf.* Lemma 6.5) conjugates $\mathbf{t}_i$ into $\mathbf{t}_{i-1}$, which proves the lemma. Keeping the notation of Lemma 6.5, we have $\mathbf{t}_i=(\mathbf{x}_2\mathbf{x}_{r+2}\cdots\mathbf{x}_{(i-1)r+2})^{-1}$

$\boldsymbol{\sigma}_{1,i(r-1)+d/2}$. By definition $\mathbf{y}_i = \prod_{k=1}^{d/2-i} \mathbf{x}_{(i-1)(r-1)+d/2-k+1}$. This product commutes with $\mathbf{x}_2\mathbf{x}_{r+2}\cdots\mathbf{x}_{(i-2)r+2}$. As the factor indexed by $k = d/2 - i$ in $\mathbf{y}_i$ is equal to $\mathbf{x}_{(i-1)r+2}$, we get

$$\begin{aligned}\mathbf{y}_i\mathbf{t}_i\mathbf{y}_i^{-1} = (&\mathbf{x}_2\mathbf{x}_{r+2}\cdots\mathbf{x}_{(i-2)r+2})^{-1}\\ &\times \prod_{k=1}^{d/2-i-1} \mathbf{x}_{(i-1)(r-1)+d/2-k+1}\boldsymbol{\sigma}_{1,i(r-1)+d/2}\mathbf{y}_i^{-1}.\end{aligned}$$

We use the fact that conjugation by $\boldsymbol{\sigma}_{1,i(r-1)+d/2}$ of the factor $\mathbf{x}_{(i-1)(r-1)+d/2-k+1}$ in $\mathbf{y}_i^{-1}$ for $k > 1$ changes k into $k-1$: this allows to simplify the product and we get

$$\begin{aligned}\mathbf{y}_i\mathbf{t}_i\mathbf{y}_i^{-1} &= (\mathbf{x}_2\mathbf{x}_{r+2}\ldots\mathbf{x}_{(i-2)r+2})^{-1}\boldsymbol{\sigma}_{1,i(r-1)+d/2}\mathbf{x}^{-1}_{(i-1)(r-1)+d/2}\\ &= (\mathbf{x}_2\mathbf{x}_{r+2}\ldots\mathbf{x}_{(i-2)r+2})^{-1}\boldsymbol{\sigma}_{1,(i-1)(r-1)+d/2} = \mathbf{t}_{i-1}.\end{aligned}$$

∎

Let $\mathbf{I} = \{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{d/2}\}$; we have $\boldsymbol{\sigma}_{i,d/2} \in \operatorname{End}_{\mathcal{D}^+_{\mathbf{I}}}(\mathbf{ywy}^{-1})$, by the remark following the proof of Lemma 6.5. This, together with Lemmas 6.5 and 6.6 proves the statements about $\mathbf{t}$ in the theorem. ∎

Note added in proof. The conjugacy of two roots of the same order of $\boldsymbol{\pi}$ in type B can be deduced from the result of Eilenberg in type A by using the embedding of B_n in A_{2n-1}; thus, as in type A, Conjecture 2.2 follows from the result of Birman, Gebhardt and Gonzales-Menes mentioned in the introduction.

§7. The elements $\boldsymbol{\pi}$ and $\mathbf{w}_0$

We consider here the order one F-root of $\boldsymbol{\pi}$, which is $\mathbf{y} = \boldsymbol{\pi}$, and the order two F-root $\mathbf{y} = \mathbf{w}_0$. For $\mathbf{w}_0$, we will just show how Conjectures 2.1 to 2.4 follow from known results. For $\boldsymbol{\pi}$, we will in addition prove Conjecture 2.5 in a certain number of cases, including split type A in general.

We recall that (*cf.* [DMR, Proposition 2.1.6]) the group $C_W(F)$ (resp. $C_W(w_0F)$) is a Coxeter group with Coxeter generators the elements w_0^I for I an element of the set of orbits S/F (resp. $I \in S/w_0F$). The corresponding braid groups $C_B(\boldsymbol{\pi}F)$ (resp. $C_B(\mathbf{w}_0F)$) have as generators the corresponding elements $\mathbf{w}_0^I$.

Since the generators $\mathbf{w}_0^I$ divide $\boldsymbol{\pi}$ (resp. $\mathbf{w}_0$) in B^+, Conjecture 2.1 is trivial. For $\boldsymbol{\pi}$, Conjecture 2.2 is also trivial. For $\mathbf{w}_0$ it results from the remark below 2.2 and the fact that $\mathbf{w}_0$ is the only "good" square F-root of $\boldsymbol{\pi}$ since it is the only element of $\mathbf{W}$ of its length.

Since $\boldsymbol{\pi}F$ (resp. $\mathbf{w}_0F$) acts as a diagram automorphism, Conjecture 2.3 is [Mi, Corollary 4.4].

Conjecture 2.4 holds for the cases $\mathbf{y}=\boldsymbol{\pi}$ and $\mathbf{y}=\mathbf{w}_0$ by the following results:

Proposition 7.1. ([DMR, 5.3.4]) *The map $\mathbf{t}\mapsto D_{\mathbf{t}}$ from $C_B(F)$ to the $\mathbf{G}^F$-endomorphisms of $H_c^*(\mathbf{X}(\boldsymbol{\pi}))$ factors through the specialization $x\mapsto q$ of a 1-cyclotomic Hecke algebra for $C_W(F)$ which is the specialization $u_{\mathbf{w}_0^I,0}\mapsto x^{l(w_0^I)}$, $u_{\mathbf{w}_0^I,1}\mapsto -1$ of the generic Hecke algebra of $C_W(F)$.*

Proposition 7.2. ([DMR, 5.4.1]) *The map $\mathbf{t}\mapsto D_{\mathbf{t}}$ from $C_B(w_0F)$ to the $\mathbf{G}^F$-endomorphisms of $H_c^*(\mathbf{X}(\mathbf{w}_0))$ factors through the specialization $x\mapsto q$ of a 2-cyclotomic Hecke algebra for $C_W(w_0F)$ which is the specialization $u_{\mathbf{w}_0^I,\varepsilon}\mapsto x^{l(w_0^I)}$, $u_{\mathbf{w}_0^I,1-\varepsilon}\mapsto(-1)^{1+l(\mathbf{w}_0^I)}$, where $\varepsilon=0$ if $l(w_0^I)$ is even and 1 otherwise, of the generic Hecke algebra of $C_W(w_0F)$ (here the names $u_{\mathbf{w}_0^I,0}$ and $u_{\mathbf{w}_0^I,1}$ are given in accordance with the definition of a cyclotomic Hecke algebra in Section 2).*

We will now consider Conjecture 2.5 for $\mathbf{y}=\boldsymbol{\pi}$. Let $\mathcal{H}$ be the cyclotomic algebra of Proposition 7.1. Since some characters of $\mathcal{H}$ are only defined over $\overline{\mathbb{Q}}_\ell[x^{1/2}]$ (for W irreducible, this happens only for the characters of degree 512 of $W(E_7)$ and those of degree 4096 of $W(E_8)$), we need to take the integer a defined above 2.4 equal to 2 and thus consider the specialization $f: x^{1/2}\mapsto q^{1/2}$ of the algebra with parameters $u_{\mathbf{w}_0^I,0}\mapsto((-x^{1/2})^{2l(w_0^I)})$ and $u_{\mathbf{w}_0^I,1}\mapsto -1$. In the terms of [DMR, 5.3.1] this corresponds to the specialization $x^{1/2}\mapsto -q^{1/2}$ of $\mathcal{H}_x(W,F)$. We recall from [DMR, 5.3.2] that if f' is the specialization $x^{1/2}\mapsto -q^{1/2}$ (which corresponds to the specialization $x^{1/2}\mapsto q^{1/2}$ of $\mathcal{H}_x(W,F)$), and if we fix an F-stable Borel subgroup $\mathbf{B}$, then $\mathcal{H}\otimes_{f'}\overline{\mathbb{Q}}_\ell\simeq\operatorname{End}_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F}(\operatorname{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}\operatorname{Id})$.

Let σ be the semi-linear automorphism, coming from $\operatorname{Gal}(\overline{\mathbb{Q}}_\ell(x^{1/2})/\overline{\mathbb{Q}}_\ell(x))$, given by $x^{1/2}\mapsto -x^{1/2}$ of $\mathcal{H}$; thus $f\circ\sigma=f'$. Let $\mathcal{H}_q=\mathcal{H}\otimes_{f'}\overline{\mathbb{Q}}_\ell$ and let $\chi\mapsto\chi_q$ be the bijection between characters of $\mathcal{H}$ and $\mathcal{H}_q$ obtained via f'; we then denote by $\chi_q\mapsto\rho_\chi$ the bijection between characters of $\mathcal{H}_q$ and characters of $\mathbf{G}^F$ occurring in $\operatorname{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}\operatorname{Id}$ coming from [DMR, 5.3.2].

Let us recall that the representation $\operatorname{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}\operatorname{Id}$ of $\mathcal{H}\otimes_{f'}\overline{\mathbb{Q}}_\ell$ is special (*cf.* 2.5 (ii)); this follows from the fact that the image of any non-trivial $\mathbf{w}\in\mathbf{W}$

has zero trace in this representation, which characterizes the canonical trace form for Hecke algebras of Coxeter groups.

It follows that Conjecture 2.5 is implied by the

CONJECTURE 7.3. *We have an equality of virtual* $\mathbf{G}^F \times \mathcal{H}_q$*-modules*

$$\sum_i (-1)^i H^i_c(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell) = \sum_{\chi \in \operatorname{Irr}(W^F)} \rho_\chi \otimes \sigma(\chi)_q.$$

In the remaining part of this section we will prove the following theorem:

THEOREM 7.4. *Conjecture* 7.3 *holds if the characteristic is almost good for* $\mathbf{G}$ *and if* (W, F) *is irreducible of type untwisted* A_n, B_2, B_3, B_4, D_4, D_5, D_6, D_7, G_2 *or* E_6.

Recall that the characteristic is almost good for $\mathbf{G}$ if it is good for each simple component of exceptional type of $\mathbf{G}$.

Proof. We have to prove that the virtual character of $\mathcal{H}_q$ appearing in the ρ_χ-isotypic component of $\sum_i (-1)^i H^i_c(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell)$ is equal to $\sigma(\chi)_q$. This is equivalent to proving that for any $\mathbf{x} \in C_{B^+}(F)$ and any $\chi \in \operatorname{Irr}(W^F)$, we have:

$$\left\langle g \mapsto \sum_i (-1)^i \operatorname{Trace}(gD_\mathbf{x} \mid H^i_c(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell)), \rho_\chi \right\rangle_{\mathbf{G}^F} = \sigma(\chi)_q(T_\mathbf{x}), \tag{1}$$

where $T_\mathbf{x}$ denotes the image of $\mathbf{x}$ in $\mathcal{H}_q$. We will prove this equality for sufficiently many elements of $C_{B^+}(F)$ to deduce it for all elements for groups in the list of Theorem 7.4.

The lemmas used in the proof have a larger scope of validity than the theorem. The proof proceeds by induction on the semi-simple rank of $\mathbf{G}$ because of the next lemma.

LEMMA 7.5. *Let* $\mathbf{G}$ *be an arbitrary reductive group as in Section* 1. *If Conjecture* 7.3 *holds for any reductive group with semi-simple rank less than that of* $\mathbf{G}$ *then* (1) *holds for any* $\mathbf{x} \in C_{B^+_\mathbf{I}}(F)$ *for any* F*-stable proper subset* I *of* S.

Proof. If $\mathbf{L}_I$ is the standard Levi subgroup of $\mathbf{G}$ corresponding to I, then by [DMR, Théorème 5.2.10], for $\mathbf{x} \in C_{B^+_\mathbf{I}}(F)$ we have:

$$\begin{aligned}&\left\langle g \mapsto \sum_i (-1)^i \operatorname{Trace}(gD_\mathbf{x} \mid H^i_c(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell)), \rho_\chi \right\rangle_{\mathbf{G}^F}\\ &\quad = \left\langle l \mapsto \sum_i (-1)^i \operatorname{Trace}(lD_\mathbf{x} \mid H^i_c(\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\pi}_\mathbf{I}), \overline{\mathbb{Q}}_\ell)), {}^*R^{\mathbf{G}^F}_{\mathbf{L}_I^F} \rho_\chi \right\rangle_{\mathbf{L}_I^F},\end{aligned}$$

which by assumption is equal to

$$\sum_{\varphi\in\operatorname{Irr}(W_I^F)} \sigma(\varphi)_q(T_{\mathbf{x}})\langle\rho_\varphi, {}^*R_{\mathbf{L}_I^F}^{\mathbf{G}^F}\rho_\chi\rangle_{\mathbf{L}_I^F}.$$

As $\langle\rho_\varphi, {}^*R_{\mathbf{L}_I^F}^{\mathbf{G}^F}\rho_\chi\rangle_{\mathbf{L}_I^F} = \langle\varphi, \operatorname{Res}_{W_I^F}^{W^F}\chi\rangle_{W_I^F}$, *cf.* [CR, Theorem 70.24], we get (1) for $\mathbf{x}$. ∎

Lemma 7.6. *Here again* $\mathbf{G}$ *is arbitrary. Equality* (1) *holds if* $\mathbf{x} = \boldsymbol{\pi}^n$, *with* n *multiple of* δ.

Proof. As $\boldsymbol{\pi}^n$ acts by F^n on $\mathbf{X}_{\boldsymbol{\pi}}$, we have

$$\sum_i (-1)^i \operatorname{Trace}(gD_{\boldsymbol{\pi}^n} \mid H_c^i(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell)) = |\mathbf{X}(\boldsymbol{\pi})^{gF^n}|$$

by the Lefschetz trace formula. We shall use the same methods and notation as in [BMi, §2.B and §6.D]. Proposition [DMR, 3.3.7] shows that

$$|\mathbf{X}(\boldsymbol{\pi})^{gF^n}| = \sum_{\rho\in\operatorname{Irr}(\mathbf{G}^F)} \rho(g) \sum_{\chi\in\operatorname{Irr}(W)^F} \langle\rho, R_{\tilde\chi}\rangle\tilde\chi_q(T_\pi^n F).$$

We have $\tilde\chi_q(T_\pi^n F) = q^{n(2N-a_\chi-A_\chi)}\tilde\chi(F)$ [BMi, Proposition 6.11] whence we get as in the proof of [BMi, Proposition 2.5]

$$\begin{aligned}|\mathbf{X}(\boldsymbol{\pi})^{gF^n}| &= \sum_{\rho\in\operatorname{Irr}(\mathbf{G}^F)} \rho(g)\langle\rho, \operatorname{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}\operatorname{Id}\rangle q^{n(2N-a_\rho-A_\rho)}\\ &= \sum_{\chi\in\operatorname{Irr}(W^F)} \chi(1)q^{n(2N-a_\chi-A_\chi)}\rho_\chi(g).\end{aligned}$$

We now use $\chi_q(T_{\boldsymbol{\pi}}^n) = \sigma(\chi)_q(T_{\boldsymbol{\pi}}^n) = \chi(1)q^{n(2N-a_\chi-A_\chi)}$ [BMi, Corollaire 4.21], which gives the result. ∎

Lemma 7.7. *If the characteristic is almost good for the split irreducible group* $\mathbf{G}$, *equality* (1) *holds when* $\mathbf{x}$ *is a root of* $\boldsymbol{\pi}$.

Proof. Lemma 7.6 shows the result for $\mathbf{x} = \boldsymbol{\pi}$; we thus assume that $\mathbf{x}$ is a d-th root of $\boldsymbol{\pi}$ with $d \geq 2$. By [DMR, 5.2.2 (i)] the endomorphism $D_{\mathbf{w}}$ of $\mathbf{X}(\boldsymbol{\pi})$ satisfies the trace formula so that

$$\sum_i (-1)^i \operatorname{Trace}(gD_{\mathbf{w}} \mid H_c^i(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell)) = |\mathbf{X}(\boldsymbol{\pi})^{gD_{\mathbf{w}}}|.$$

Moreover, by [DMR, 5.2.2 (ii)], we have $(g \mapsto |\mathbf{X}(\boldsymbol{\pi})^{gD_{\mathbf{w}}}|) = \mathrm{Sh}^d(g \mapsto \mathrm{Trace}(gT_{\mathbf{w}} \mid \mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F} \mathrm{Id}))$ where Sh is the twisting operator on $\mathbf{G}^F$-class functions. So we have to prove

$$\mathrm{Sh}^d(g \mapsto \mathrm{Trace}(gT_{\mathbf{w}} \mid \mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F} \mathrm{Id})) = \sum_{\chi_q \in \mathrm{Irr}(\mathcal{H}_q(W,F))} \sigma(\chi)_q(T_{\mathbf{w}})\rho_\chi,$$

which is equivalent to

$$(1') \qquad \sum_{\chi_q \in \mathrm{Irr}(\mathcal{H}_q(W,F))} \chi_q(T_{\mathbf{w}}) \,\mathrm{Sh}^d \rho_\chi = \sum_{\chi_q \in \mathrm{Irr}(\mathcal{H}_q(W,F))} \sigma(\chi)_q(T_{\mathbf{w}})\rho_\chi.$$

To prove this, we may replace $\mathbf{w}$ by a conjugate in B so we may assume that $\mathbf{w}$ is a "good" root, in particular that $\mathbf{w} \in \mathbf{W}$. As usual we set $w = \beta(\mathbf{w})$.

We have $\chi_q(T_{\mathbf{w}}) = \chi(w) q^{\frac{2N - a_{\rho_\chi} - A_{\rho_\chi}}{d}}$: one gets this by applying [BMM, 6.15 (2)] to $\mathcal{H}$; with the notation of *loc. cit.*, it is a principal algebra (see *loc. cit.* 6.3), with $\theta_0(\mathbf{w}_0^{\mathbf{I}}) = x^{l(w_0^I)}$; we have $D_0 = 2N$ and if we take $P(q) = |(\mathbf{G}/\mathbf{B})^F|$ the degree $\mathrm{Deg}_\chi^{(P)}$ identifies with the generic degree of ρ_χ.

As $\mathcal{H}$ is split over $\mathbb{Z}[x^{1/2}, x^{-1/2}]$, there exists a sign $\varepsilon_{d,\chi}$ depending only on $(a_{\rho_\chi} + A_{\rho_\chi})/d$ such that $\sigma(\chi)_q(T_{\mathbf{w}}) = \varepsilon_{d,\chi}\chi_q(T_{\mathbf{w}})$. This sign is equal to -1 if and only if $(a_{\rho_\chi} + A_{\rho_\chi})/d \in \mathbb{Z} + 1/2$ and $\chi(w) \neq 0$. Equation $(1')$ becomes then

$$(1'') \qquad \sum_{\chi_q \in \mathrm{Irr}(\mathcal{H}_q(W,F))} q^{\frac{2N - a_{\rho_\chi} - A_{\rho_\chi}}{d}} \chi(w) \,\mathrm{Sh}^d \rho_\chi = \sum_{\chi_q \in \mathrm{Irr}(\mathcal{H}_q(W,F))} \varepsilon_{d,\chi} q^{\frac{2N - a_{\rho_\chi} - A_{\rho_\chi}}{d}} \chi(w)\rho_\chi.$$

For computing Sh^d, we shall use Shoji's results on the identification of character sheaves with almost characters. Here we need the assumption that the characteristic is almost good. We recall these results: unipotent characters of $\mathbf{G}^F$ have been divided by Lusztig into families. Unipotent character sheaves have also been divided into families which are in one-to-one correspondence with the families of unipotent characters. In [Sh2, 3.2 and 4.1] Shoji proves that the transition matrix from the unipotent characters to the characteristic functions of the unipotent character sheaves is block diagonal according to the families, and in [Sh1, 3.3] he proves that the characteristic functions of the character sheaves are eigenvectors of Sh. From this we see that $(1'')$ is equivalent to the set of its projections on each family. Moreover a_ρ and A_ρ are constant when ρ runs over a family of unipotent characters. So $(1'')$ is equivalent to the set of equations

$$\sum_{\rho_\chi\in\mathcal{F}} \chi(w)\operatorname{Sh}^d \rho_\chi = \varepsilon_{d,\mathcal{F}} \sum_{\rho_\chi\in\mathcal{F}} \chi(w)\rho_\chi, \tag{$1'''$}$$

where $\mathcal{F}$ runs over the families. We have written $\varepsilon_{d,\mathcal{F}}$ instead of $\varepsilon_{d,\chi}$ because this sign depends only on the family of ρ_χ.

We can also assume that $\mathbf{G}$ is adjoint as the unipotent characters factorize through the adjoint group and Sh is compatible with this factorization.

Lusztig defined in [Lu3, 4.24.1] almost characters R_ρ indexed by unipotent characters. If R_w is the Deligne-Lusztig character given by the virtual representation $\sum_{i\geq 0}(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$, we have $R_w = \sum_{\chi\in\operatorname{Irr}(W)}\chi(w)R_{\rho_\chi}$; for any unipotent character ρ we also have $\langle\rho, R_{\rho_\chi}\rangle_{\mathbf{G}^F} = \Delta_\rho\langle R_\rho,\rho_\chi\rangle_{\mathbf{G}^F}$ for a sign Δ_ρ defined in [Lu3]. Almost characters being an orthonormal basis of the space of unipotent class functions, we get $\sum_{\rho_\chi\in\mathcal{F}}\chi(w)\rho_\chi = \sum_{\rho\in\mathcal{F}}\langle R_w,\rho\rangle_{\mathbf{G}^F}\Delta_\rho R_\rho$.

In [Lu4, 23.1] Lusztig has defined a bijection $\rho\mapsto A_\rho$ from the set of unipotent characters to the set of unipotent character sheaves, compatible with the partition into families. Shoji, in ([Sh1] and [Sh2]) proved that the almost character R_ρ is a multiple of the characteristic function of χ_{A_ρ} relative to the Frobenius endomorphism F of the character sheaf A_ρ and that (*cf.* [Sh1, 3.6 and 3.8]) $\operatorname{Sh}(\chi_{A_\rho}) = \lambda_\rho\chi_{A_\rho}$ where λ_ρ is as in [DMR, 3.3.4].

Using this, we see that $(1''')$ is equivalent to:

$$\text{if } \langle R_w,\rho\rangle_{\mathbf{G}^F}\neq 0 \text{ then } \lambda_\rho^d = \varepsilon_{d,\mathcal{F}} \tag{$1''''$}$$

This would be a consequence of conjecture [BMi, 5.13]. We prove it by a case by case analysis.

If $\mathbf{G}$ is classical, we have always $\varepsilon_{d,\mathcal{F}} = 1$ and $\lambda_\rho = \pm 1$, so $(1'''')$ holds if d is even. Assume d odd; one checks that in a Coxeter group of type A_n, B_n or D_n, any odd order element lies in a parabolic subgroup of type A. Let us denote by $\mathbf{L}$ the corresponding Levi subgroup of $\mathbf{G}$, which is an F-stable Levi subgroup of an F-stable parabolic subgroup. We have $R_w = R^{\mathbf{G}}_{\mathbf{L}}(R^{\mathbf{L}}_w)$ where $R^{\mathbf{L}}_w$ is the Deligne-Lusztig character of $\mathbf{L}^F$ associated to w. As λ_ρ is constant in a Harish-Chandra series and is equal to 1 for a group of type A, we get the result in this case.

If $\mathbf{G}$ is of exceptional type we can check the result, using the explicit description of the coefficients $\langle R_w, \rho\rangle_{\mathbf{G}^{F'}}$ and of λ_ρ in [Lu3]. The most complicated case to check is when for some d we have $\varepsilon_{d,\mathcal{F}} = -1$. In type E_7 there is exactly one such family; it contains 4 unipotent characters. Two of them are some ρ_χ for a χ such that $a_\chi + A_\chi = 63$. In type E_8 there are two such families, each with 4 unipotent characters. In each of these families there are two ρ_χ with respectively $a_\chi + A_\chi = 105$ and $a_\chi + A_\chi = 135$. So in all cases we have $\varepsilon_{d,\mathcal{F}} = -1$ if and only if $d \equiv 2 \pmod 4$. In each case for the two other unipotent characters of the family one has $\lambda_\rho = \pm i$. One checks that if $\rho \in \mathcal{F}$ and $\langle \rho, R_w\rangle_{\mathbf{G}^F} \neq 0$ then if $d \not\equiv 2 \pmod 4$ one has $\lambda_\rho^d \neq 1$ whence the result in this case; and if $d \equiv 2 \pmod 4$ we have $\lambda_\rho = \pm i$, thus $\lambda_\rho^d = -1$ and we also get the result in that case. ∎

Let Φ be the class function on $\mathcal{H}_q$ with values in the Grothendieck group of $\mathbf{G}^F$ given by

$$\Phi(T_{\mathbf{x}}) = \Big(g \mapsto \sum_i (-1)^i \operatorname{Trace}(gD_{\mathbf{x}} \mid H^i_c(\mathbf{X}(\boldsymbol{\pi}), \overline{\mathbb{Q}}_\ell))\Big) - \sum_\chi \sigma(\chi)_q(T_{\mathbf{x}})\rho_\chi.$$

To prove Theorem 7.4 we have to prove that $\Phi = 0$. By Lemmas 7.5, 7.6 and 7.7 respectively we know that

(a) $\Phi(T_{\mathbf{x}}) = 0$ for $\mathbf{x} \in B_{\mathbf{I}}$ for any proper subset I of S.
(b) $\Phi(T_{\boldsymbol{\pi}}^n) = 0$ for $n > 0$.
(c) We have $\Phi(T_{\mathbf{x}}) = 0$ if $\mathbf{x}$ is a root of $\boldsymbol{\pi}$ and the characteristic is almost good.

We shall prove that in any of the cases considered in Theorem 7.4 a class function on $\mathcal{H}_q$ which satisfies these three properties is zero. Such a class function, can be written $\sum_\chi \lambda_\chi \chi_q$. We show that the three above properties imply $\lambda_\chi = 0$ for all χ. Let us translate each of these properties into a property of $(\lambda_\chi)_\chi$.

Lemma 7.8. *Property* (a) *means that $(\lambda_\chi)_\chi$ is linearly spanned by vectors $(\chi(w))_\chi$ with $w \in W$* cuspidal (*i.e., the conjugacy class of w has no representative in a proper parabolic subgroup of W*).

Proof. Consider the scalar product on the Grothendieck group of $\mathcal{H}_q$ such that the χ_q form an orthonormal basis (which corresponds to the usual scalar product on the vectors $(\lambda_\chi)_\chi$); then the $\chi_{c,q} = \sum_{\chi \in \operatorname{Irr}(W^F)} \chi(c)\chi$

are pairwise orthogonal when c runs over a set of representatives of the conjugacy classes in W. The statement to prove is that a class function satisfies property (a) if and only if it is orthogonal to the $\chi_{c,q}$ with c non cuspidal.

With our choice of scalar product, restriction and induction satisfy Frobenius reciprocity, as the scalar product is compatible with the specialization to W, as are restriction and induction. So for $I \subset S$, a class function is zero on $\mathcal{H}_q(W_I)$ if and only if it is orthogonal to any $\operatorname{Ind}_{\mathcal{H}_q(W_I)}^{\mathcal{H}_q} \phi$; but the $\chi_{c,q}$ with c non-cuspidal span the same subspace as the $\operatorname{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \phi$ with $I \subsetneq S$, so we get the result. ∎

Lemma 7.9. *If* $\mathbf{x}$ *is a* d*-th root of* $\boldsymbol{\pi}$*, property* (c) *is equivalent to* $\sum_\chi \lambda_\chi \chi(x) q^{\frac{2N-a_{\rho_\chi}-A_{\rho_\chi}}{d}} = 0$.

Proof. This is a simple translation of (c), using the value of $\chi_q(T_{\mathbf{x}})$. ∎

We now prove the theorem when $\mathbf{G}$ is split of type A_n. The only cuspidal class is the class of a Coxeter element c. So by Lemma 7.8 $(\lambda_\chi)_\chi$ has to be equal to $a(\chi(c))_\chi$ for some $a \in \overline{\mathbb{Q}}_\ell$. Lemma 7.9 then gives $a\sum_\chi \chi(c)^2 q^{\frac{2N-a_{\rho_\chi}-A_{\rho_\chi}}{d}} = 0$, so that $a = 0$, as all summands are non-negative and at least one is non-zero.

For the other types we need property (b).

Lemma 7.10. *Property* (b) *is equivalent to*: *for all* i*, we have*

$$\sum_{\{\chi \mid a_{\rho_\chi}+A_{\rho_\chi}=i\}} \lambda_\chi \chi(1) = 0.$$

Proof. Using the value of $\chi_q(T_{\boldsymbol{\pi}}^n)$ property (b) is equivalent to the fact that for all n we have

$$\sum_i q^{n(2N-i)} \sum_{\{\chi \mid a_{\rho_\chi}+A_{\rho_\chi}=i\}} \lambda_\chi \chi(1) = 0.$$

We get the result using the linear independence of the characters of $\mathbb{Z}$. ∎

The proof of Theorem 7.4 in the remaining types is obtained by a computer calculation which shows that the vectors given by Lemmas 7.10 and 7.9 span for any q the space given by Lemma 7.8 (note that only the vectors given by Lemma 7.9 depend on q). ∎

§8. Pieces of the Deligne-Lusztig varieties

In this section, we introduce a technique inspired by [Lu2], which will allow us to compute Harish-Chandra restrictions of the cohomology of some Deligne-Lusztig varieties; we will also find a criterion for irreducibility of a generalized Deligne-Lusztig variety (see Proposition 8.4).

The technique is intersecting with Bruhat cells. Let $\mathcal{B}$ be the variety of Borel subgroups of $\mathbf{G}$. We recall from [DMR, 2.2.18] that, given a decomposition $\mathbf{t} = \mathbf{w}_1 \cdots \mathbf{w}_k$ of $\mathbf{t} \in B^+$ as a product of elements of $\mathbf{W}$, and denoting by $\mathcal{O}(w)$ the $\mathbf{G}$-orbit in $\mathcal{B} \times \mathcal{B}$ indexed by $w \in W$, the variety $\mathcal{O}(\mathbf{w}_1, \dots, \mathbf{w}_k) = \{(\mathbf{B}_1, \dots, \mathbf{B}_{k+1}) \mid (\mathbf{B}_i, \mathbf{B}_{i+1}) \in \mathcal{O}(w_i)\}$ depends only on $\mathbf{t}$ and not on the chosen decomposition; it affords two canonical projections $p'(\mathbf{B}_1, \dots, \mathbf{B}_{k+1}) = \mathbf{B}_1$ and $p''(\mathbf{B}_1, \dots, \mathbf{B}_{k+1}) = \mathbf{B}_{k+1}$ and the Deligne-Lusztig variety is $\mathbf{X}(\mathbf{t}) = \{x \in \mathcal{O}(\mathbf{t}) \mid p''(x) = F(p'(x))\}$. We fix an F-stable Borel subgroup $\mathbf{B} = \mathbf{TU} \subset \mathbf{G}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is an F-stable maximal torus. We identify W with $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We put $\mathbf{B}^- = {}^{w_0}\mathbf{B} = \mathbf{TU}^-$. For $v \in W$, we define the piece $\mathbf{X}^v(\mathbf{t}) = \{x \in \mathbf{X}(\mathbf{t}) \mid (\mathbf{B}, p'(x)) \in \mathcal{O}(v)\}$. We have $\mathbf{X}(\mathbf{t}) = \coprod_{v \in W} \mathbf{X}^v(\mathbf{t})$, and the action of $\mathbf{G}^F$ on $\mathbf{X}(\mathbf{t})$ restricts to an action of $\mathbf{B}^F$ on each piece.

Remark 8.1. If $\mathbf{t} = \mathbf{w}_1 \cdots \mathbf{w}_k$ is a decomposition of $\mathbf{t} \in B$ as a product of elements of $\mathbf{W}$, we recall from [DMR, 2.2.12 et 2.3.2] that

$$\mathbf{X}(\mathbf{t}) = \{(g_1\mathbf{B}, g_2\mathbf{B}, \dots, g_k\mathbf{B}) \mid g_i^{-1} g_{i+1} \in \mathbf{B} w_i \mathbf{B}, \text{ for } i = 1, \dots, k-1 \\ \text{and } g_k^{-1}{}^F g_1 \in \mathbf{B} w_k \mathbf{B}\}.$$

In this model we get $\mathbf{X}^v(\mathbf{t})$ by adding the condition $g_1 \in \mathbf{B} v \mathbf{B}$.

Let $\mathcal{H}(W)$ be the generic Hecke algebra of W over $\mathbb{C}[x]$. This is the quotient of the group algebra $\mathbb{C}[x]B$ by the relations $(\mathbf{s}+1)(\mathbf{s}-x) = 0$ for $\mathbf{s} \in \mathbf{S}$. We denote by $T_{\mathbf{b}}$ the image of $\mathbf{b} \in B^+$ in $\mathcal{H}(W)$. The algebra $\mathcal{H}(W)$ has a basis $\{T_{\mathbf{w}} \mid \mathbf{w} \in \mathbf{W}\}$. We will also sometimes denote by T_w the elements of this basis. We will denote by $\mathcal{H}_q(W)$ the specialized algebra by the specialization $x \mapsto q$ and keep the notation $T_{\mathbf{w}}$ for the basis of this algebra (trying to make clear by the context which algebra is meant). Finally we note $A|T_v$ the coefficient of the element $A \in \mathcal{H}(W)$ on the basis element T_v. We recall that the canonical symmetrizing form is $T_v \mapsto T_v|1$. Since T_v and $q^{-l(v)}T_{v^{-1}}$ are dual bases for this form we have $A|T_v = q^{-l(v)} A T_{v^{-1}}|1$. With this notation, we have

Proposition 8.2. *Let* $\mathbf{t} \in B^+$ *and* $v \in W$; *for any* m *multiple of* δ *we have* $\left|\left(\mathbf{U}^F \backslash \mathbf{X}^v(\mathbf{t})\right)^{F^m}\right| = T_v T_{\mathbf{t}} | T_{{}^F v}$, *where the elements on the right-hand side are taken in the Hecke algebra* $\mathcal{H}_{q^m}(W)$.

Proof. We may assume $\mathbf{t} \in \mathbf{W}$. Indeed, by [DMR, 2.3.3], if $\mathbf{t} = \mathbf{w}_1 \cdots \mathbf{w}_k$ is a decomposition as a product of elements of $\mathbf{W}$, and if F_1 is the isogeny on $\mathbf{G}^k$ defined by $F_1(g_1, \ldots, g_k) = (g_2, \ldots, g_k, F(g_1))$, then $\mathbf{X}(\mathbf{t}) \simeq \mathbf{X}_{\mathbf{G}^k}((w_1, \ldots, w_k), F_1)$ and this isomorphism restricts to

$$\mathbf{X}^v(\mathbf{t}) \simeq \coprod_{v_2, \ldots, v_k \in W} (\mathbf{X}_{\mathbf{G}^k}^{(v, v_2, \ldots, v_k)}((w_1, \ldots, w_k), F_1)).$$

Thus

$$(\mathbf{U}^F \backslash \mathbf{X}^v(\mathbf{t}))^{F^m} \simeq \coprod_{v_2, \ldots, v_k} \Big((\mathbf{U}^k)^{F_1} \backslash (\mathbf{X}_{\mathbf{G}^k}((w_1, \ldots, w_k), F_1))^{(v, v_2, \ldots, v_k)}\Big)^{F_1^{km}},$$

and it is also clear that:

$$\begin{aligned}
&\sum_{v_2, \ldots, v_k} T_{(v, v_2, \ldots, v_k)} T_{(w_1, \ldots, w_k)} | T_{F_1(v, v_2, \ldots, v_k)} \\
&\quad = \sum_{v_2, \ldots, v_k} (T_v T_{w_1} | T_{v_2})(T_{v_2} T_{w_2} | T_{v_3}) \cdots (T_{v_k} T_{w_k} | T_{{}^F v}) = T_v T_{\mathbf{t}} | T_{{}^F v}.
\end{aligned}$$

As $F_1^{k\delta}$ is the smallest power of F_1 which is a split Frobenius, we are indeed reduced to the same statement for $\mathbf{G}^k$, F_1, $(w_1, \ldots, w_k)$, $(v, v_2, \ldots, v_k)$.

We then assume $\mathbf{t} \in \mathbf{W}$. Thus $\mathbf{X}^v(t) = \{g\mathbf{B} \mid g \in \mathbf{B}v\mathbf{B},\ g^{-1}\,{}^F g \in \mathbf{B}t\mathbf{B}\}$. The map $u \mapsto uv\mathbf{B}$ then fibers $\{u \in \mathbf{U} \mid (uv)^{-1}\,{}^F(uv) \in \mathbf{B}t\mathbf{B}\}$ on $\mathbf{X}^v(t)$ with fibers isomorphic to $\mathbf{U} \cap {}^v\mathbf{U}$, since $\mathbf{B}v\mathbf{B} = \mathbf{U}_v v\mathbf{B}$, where for $v \in W$ we set $\mathbf{U}_v = \mathbf{U} \cap {}^v\mathbf{U}^-$. This fibration is $\mathbf{U}^F$-equivariant for the action of $\mathbf{U}^F$ by left multiplication on both spaces. The quotient by $\mathbf{U}^F$ is thus obtained by $u \mapsto u^{-1}.{}^F u$ which maps the above variety to $\mathbf{U} \cap v\mathbf{B}t\mathbf{B}\,{}^F v^{-1}$. As the fibers $\mathbf{U} \cap {}^v\mathbf{U}$ are connected and have $q^{ml(w_0 v)}$ fixed points under F^m, the cardinality we seek is thus $q^{-ml(w_0 v)}|\mathbf{U}^{F^m} \cap v\mathbf{B}t\mathbf{B}\,{}^F v^{-1}|$. Thus the proposition results from the following lemma, applied with F replaced by F^m and w by ${}^F v$:

Lemma 8.3. *Assume* F *split. For* $v, t, w \in W$ *and* $T_v, T_t, T_w \in \mathcal{H}_q(W)$ *we have*

$$T_v T_t | T_w = q^{-l(w_0 v)} |(\mathbf{U} \cap v\mathbf{B}t\mathbf{B}w^{-1})^F|.$$

Proof. We have $T_vT_t|T_w = T_{t^{-1}}T_{v^{-1}}|T_{w^{-1}} = q^{-l(w)}T_{t^{-1}}T_{v^{-1}}T_w|1 = q^{l(t)-l(w)}T_{v^{-1}}T_w|T_t$. We recall that $\mathcal{H}_q(W)$ may be realized as a subalgebra of $\mathbb{C}[\mathbf{G}^F]$ via the isomorphism $\mathcal{H}_q(W) \simeq \operatorname{End}_{\mathbf{G}^F} \operatorname{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F} \mathbb{C}$. By this isomorphism, T_w corresponds to $q^{l(w)}e_{\mathbf{B}}\dot{w}e_{\mathbf{B}}$ where $\dot{w}$ is a representative of w in $N(\mathbf{T})^F$ and where $e_{\mathbf{B}}$ is the idempotent $|\mathbf{B}^F|^{-1}\sum_{b\in\mathbf{B}^F} b$. Thus:

$$\begin{aligned}T_{v^{-1}}T_w|T_t &= q^{l(v)+l(w)-l(t)}e_{\mathbf{B}}v^{-1}e_{\mathbf{B}}we_{\mathbf{B}} \mid e_{\mathbf{B}}te_{\mathbf{B}}\\ &= |\mathbf{B}^F|^{-1}q^{l(v)+l(w)-l(t)}|v^{-1}\mathbf{B}^F w \cap \mathbf{B}^F t\mathbf{B}^F|\\ &= q^{-l(w_0)+l(v)+l(w)-l(t)}|\mathbf{U}^F \cap v\mathbf{B}^F t\mathbf{B}^F w^{-1}|.\end{aligned}$$

The lemma follows, since $(\mathbf{U}\cap v\mathbf{B}t\mathbf{B}w^{-1})^F = \mathbf{U}^F\cap v\mathbf{B}^F t\mathbf{B}^F w^{-1}$ which may be seen by using the uniqueness properties of the Bruhat decomposition. ∎ ∎

For $\mathbf{t} \in B^+$, we call support of $\mathbf{t}$ the set of $\mathbf{s} \in \mathbf{S}$ which appear in a decomposition of $\mathbf{t}$ as a product of elements of $\mathbf{S}$: this set does not depend on the decomposition as it is not changed by a braid relation. With this notation, we have

PROPOSITION 8.4. *Let $\mathbf{t} \in B^+$. The variety $\mathbf{X}(\mathbf{t})$ is irreducible (in particular, with the convention of* [DMR, 3.3.5], *$H_c^{2l(\mathbf{t})}(\mathbf{X}(\mathbf{t}))$ is given by $h^{2l(t)}t^{l(\mathbf{t})}\,\mathrm{Id}$) if and only if the support of $\mathbf{t}$ meets every orbit of F on $\mathbf{S}$ (i.e., if the group $\mathbf{L}_I$ of* [DMR, 2.3.8] *cannot be taken different from $\mathbf{G}$).*

Proof. We adapt the proof of [Lu1, 3.10 (d)] to our case. From Proposition 8.2 we get that if m is a multiple of δ, $|(\mathbf{U}^F\backslash\mathbf{X}^v(\mathbf{t}))^{F^m}|$ is the coefficient $T_vT_{\mathbf{t}}|T_{Fv}$ in $\mathcal{H}_{q^m}(W)$; this coefficient is equal to $T_{\mathbf{t}}T_{Fv}|T_v$ thus $|(\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t}))^{F^m}|$ is the trace of the endomorphism $x \mapsto T_{\mathbf{t}}{}^F x$ of $\mathcal{H}_{q^m}(W)$.
The next lemma generalizes [Lu5, Lemma 10.4 (c)].

LEMMA 8.5. *Let $x_1,\dots,x_n \in W$; then the coefficient $T_{x_1}T_{x_2}\cdots T_{x_n}|1$ is a polynomial in q^m of degree less than or equal to*

$$\sum_{i=1}^n l(x_i) - \sup\{l(x_1),\dots,l(x_n)\}.$$

Proof of the lemma. As the coefficient $T_{x_1}T_{x_2}\cdots T_{x_n}|1$ is invariant under cyclic permutation of the factors, it is sufficient to prove that its degree is at most $l(x_1)+\cdots+l(x_{n-1})$. We prove it by induction on n and for a fixed

n by induction on $l(x_1)$. If $n=0$ then $1|1=1$ has indeed degree 0. Assume now $n>0$. If $l(x_1)=0$ we get the result by induction on n. Assume now $l(x_1)>0$. If there exists $x_1''\neq 1$ such that $x_1=x_1'x_1''$ with $l(x_1)=l(x_1')+l(x_1'')$ and $l(x_1'')+l(x_2)=l(x_1''x_2)$, then we get the result by induction on $l(x_1)$, as $l(x_1)+l(x_2)+\cdots+l(x_{n-1})=l(x_1')+l(x_1''x_2)+l(x_3)+\cdots+l(x_{n-1})$ and $T_{x_1}T_{x_2}\cdots T_{x_n}|1=T_{x_1'}T_{x_1''x_2}\cdots T_{x_n}|1$. Otherwise there exists $s\in S$, $y_1,y_2\in W$ such that $x_1=y_1s$, $x_2=sy_2$ and $l(y_i)=l(x_i)-1$ for $i=1,2$. We then have $T_{x_1}T_{x_2}\cdots T_{x_n}|1=q^m(T_{y_1}T_{y_2}\cdots T_{x_n}|1)+(q^m-1)(T_{y_1}T_{x_2}\cdots T_{x_n}|1)$, and we get the result by induction on $l(x_1)$ as $1+l(y_1)+l(x_2)+\cdots+l(x_{n-1})=l(x_1)+\cdots+l(x_{n-1})$. ∎

LEMMA 8.6. *Let* $\mathbf{t}\in B^+$; *then* $\operatorname{Trace}(x\mapsto T_{\mathbf{t}}{}^Fx\mid\mathcal{H}_{q^m}(W))$ *is a polynomial in* q^m *of degree* $l(\mathbf{t})$ *and the coefficient of* $q^{ml(\mathbf{t})}$ *in this polynomial is the number of* $\mathbf{v}\in\mathbf{W}^F$ *who are right multiples of all elements of the support of* $\mathbf{t}$.

Proof. As we have $T_{\mathbf{t}}T_{^Fv}|T_v=q^{-ml(v)}(T_{\mathbf{t}}T_{^Fv}T_{v^{-1}}|1)$, to show the lemma, it is enough to show that $T_{\mathbf{t}}T_{^Fv}T_{v^{-1}}|1$ is a polynomial in q^m of degree $<l(\mathbf{t})+l(v)$ except if $v={}^Fv$ and all $\mathbf{s}$ in the support of $\mathbf{t}$ divide $\mathbf{v}$ on the left, and that in this last case it is a unitary polynomial of degree $l(\mathbf{t})+l(v)$. Let us write $\mathbf{t}=\mathbf{t}'\mathbf{s}$ where $\mathbf{s}\in\mathbf{S}$. If $l(s^Fv)>l(^Fv)$, then by Lemma 8.5 the degree of $T_{\mathbf{t}}T_{^Fv}T_{v^{-1}}|1=T_{\mathbf{t}'}T_{s^Fv}T_{v^{-1}}|1$ is less than $l(\mathbf{t})+l(v)$. Otherwise, $T_{\mathbf{t}}T_{^Fv}T_{v^{-1}}|1=q^m(T_{\mathbf{t}'}T_{^Fv}T_{v^{-1}}|1)+(q^m-1)(T_{\mathbf{t}'}T_{s^Fv}T_{v^{-1}}|1)$. By Lemma 8.5, we see that only the first summand can contribute to $q^{m(l(\mathbf{t})+l(v))}$; and by induction on $l(\mathbf{t}')$, we see that the contribution to $q^{m(l(\mathbf{t})+l(v))}$ of $T_{\mathbf{t}}T_{^Fv}T_{v^{-1}}|1$ is $T_{^Fv}T_{v^{-1}}|1$ if all $\mathbf{s}$ in the support of $\mathbf{t}$ divide on the left ${}^Fv$, and is 0 otherwise. The result follows. ∎

From the last lemma, $|(\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t}))^{F^m}|$ is a polynomial of degree $l(\mathbf{t})$ in q^m, unitary if and only if the support of $\mathbf{t}$ meets every F-orbit in $\mathbf{S}$. As all irreducible components of $\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t})$ have the same dimension, since $\mathbf{X}(\mathbf{t})$ is the transverse intersection of the graph of F with the smooth irreducible variety $\mathcal{O}(\mathbf{t})$, this variety is irreducible if and only if $|(\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t}))^{F^m}|$ is a unitary polynomial in q^m.

To prove the proposition it remains to check that $\mathbf{X}(\mathbf{t})$ is irreducible if and only if $\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t})$ is. The "only if" part is clear; to see the "only" part, we may follow the arguments of [Lu2, 4.8]: if $\mathbf{U}^F\backslash\mathbf{X}(\mathbf{t})$ is irreducible, the set π of irreducible components of $\mathbf{X}(\mathbf{t})$ is a single orbit under $\mathbf{U}^F$, so its cardinality is a power of p. The set π is in bijection with the set of irreducible components of the (smooth) compactification $\mathbf{X}(\underline{s}_1,\ldots,\underline{s}_r)$ (see [DMR, 2.3.4]). But the $\mathbf{G}^F$-stabilizer of $(\mathbf{B},\ldots,\mathbf{B})\in\mathbf{X}(\underline{s}_1,\ldots,\underline{s}_r)$ is $\mathbf{B}^F$, thus the orbit of $(\mathbf{B},\ldots,\mathbf{B})$ (and a fortiori the number of irreducible components of $\mathbf{X}(\underline{s}_1,\ldots,\underline{s}_r)$) has cardinality a divisor of $|\mathbf{G}^F/\mathbf{B}^F|$, which is prime to p, whence the result. ∎

By Proposition 8.2, we see that the variety $\mathbf{X}^v(\mathbf{w})$ is non-empty if and only if $T_vT_{\mathbf{w}}|T_{^Fv}\neq 0$. We shall study this condition, especially when v is F-stable. In what follows, we will denote by $\leq$ the Bruhat order on W.

Proposition 8.7. *Assume that $\mathbf{w}\in B^+$ is of the form $\mathbf{w}=\mathbf{w}_1\cdots\mathbf{w}_k$ where $\mathbf{w}_i\in\mathbf{W}$ have mutually disjoint support. Then $T_vT_{\mathbf{w}}|T_v\neq 0$ is equivalent to $T_vT_{\mathbf{w}_i}|T_v\neq 0$ for all i.*

Proof. By induction on k, it is enough to show the case $k=2$ of the proposition. By the isomorphism of the Hecke algebra with a subalgebra of the group algebra of $\mathbf{G}^F$, we have $T_vT_{\mathbf{w}}|T_v\neq 0$ if and only if $\mathbf{B}v\mathbf{B}w_1\mathbf{B}w_2\mathbf{B}\supset\mathbf{B}v\mathbf{B}$. We then use the following lemma:

Lemma 8.8. *For $w,w'\in W$ we have: $\mathbf{B}w\mathbf{B}w'\mathbf{B}\subset(\coprod_{v'\leq w'}\mathbf{B}wv'\mathbf{B})\cap(\coprod_{v\leq w}\mathbf{B}vw'\mathbf{B})$.*

Proof. The inclusion in *e.g.*, the left union is an easy induction on $l(w')$, using the exchange lemma. ∎

Thus $\mathbf{B}v\mathbf{B}w_1\mathbf{B}w_2\mathbf{B}=\coprod_{v_1}\mathbf{B}vv_1\mathbf{B}w_2\mathbf{B}$ for some $v_1\leq w_1$, and in turn this last union is a union of double cosets of the form $\mathbf{B}vv_1v_2\mathbf{B}$, where $v_2\leq w_2$; now the assumption on supports implies that $vv_1v_2=v$ if and only if $v_1=v_2=1$. Since $v_1=1$ occurring is equivalent to $\mathbf{B}v\mathbf{B}w_1\mathbf{B}\supset\mathbf{B}v\mathbf{B}$ and then in turn $v_2=1$ occurring is equivalent to $\mathbf{B}v\mathbf{B}w_2\mathbf{B}\supset\mathbf{B}v\mathbf{B}$, we get the proposition. ∎

Lemma 8.9. *If $v,w,x\in W$ and if $T_vT_w|T_x\neq 0$, then $x\geq vw$.*

Proof. The condition is equivalent to $T_wT_{x^{-1}}|T_{v^{-1}}\neq 0$. Applying Lemma 8.8, this implies $v^{-1}=wx'$ with $x'\leq x^{-1}$, *i.e.*, $vw=x'^{-1}$ with $x'^{-1}\leq x$. ∎

Lemma 8.10. *Let $v,t\in W$ where t is a reflection of root $\alpha>0$. Then $T_vT_t|T_v\neq 0$ if and only if $v\alpha<0$.*

Proof. By [Dy, 1.2 and 1.12], $v\alpha < 0$ if and only if $vt < v$. If $T_vT_t|T_v \neq 0$, then by Lemma 8.9 we have $v > vt$ thus $v\alpha < 0$. Conversely, if $v\alpha < 0$, we will show by induction on $l(t)$ that $T_vT_t|T_v \neq 0$. If $l(t) = 1$, then $v = v't$ with $l(v) = l(v')+1$. We have then $T_vT_t = (q-1)T_v + qT_{vt}$, thus $T_vT_t|T_v = q-1 \neq 0$. Otherwise, by [Dy, 1.4], we may write $t = at'a$ where $a \in S$ and $l(t) = l(t') + 2$. We have: $T_vT_{at'a}|T_v = q^{-l(v)}T_vT_aT_{t'}T_aT_{v^{-1}}|1$; when $va > v$ this is equal to $qT_{va}T_{t'}|T_{va}$. Otherwise, it is equal to $q^{-l(v)}(q^2(T_{va}T_{t'}T_{av^{-1}}|1)+(q-1)^2(T_vT_{t'}T_{v^{-1}}|1)+q(q-1)(T_{va}T_{t'}T_{v^{-1}}|1)+q(q-1)(T_vT_{t'}T_{av^{-1}}|1))$ whose first term is equal to $q^3T_{va}T_{t'}|T_{va}$. Since the structure constants of the Hecke algebra are polynomials which positive highest coefficient, we see in both cases that $T_vT_t|T_v$ will be non-zero if $T_{va}T_{t'}|T_{va} \neq 0$ is non-zero. Since t' is a reflection of root $a\alpha$, we see by induction, that this coefficient is non-zero if $va(a\alpha) < 0$, *i.e.*, $v\alpha < 0$, whence the result. ∎

Recall that an element $w \in W$ is reduced-I, with $I \subset S$, if it is of minimal length in its coset wW_I. To continue our study, we define $E_W(w) = \{w_0v \in W \mid T_vT_w|T_v \neq 0\}$. With this notation, we have

LEMMA 8.11. *Let $I \subset S$ be F-stable. Assume that $w \in W$ is of the form $w = sw'$ with $s \in S - I$ and and $w' \in W_I$. Then $E_W(w)$ consists of the products v_1v_2 where $v_2 \in E_{W_I}(w')$ and where v_1 is a reduced-I element such that $l(v_1v_2s) > l(v_1v_2)$.*

Proof. By Proposition 8.7, $T_vT_w|T_v \neq 0$ if and only if $T_vT_s|T_v \neq 0$ and $T_vT_{w'}|T_v \neq 0$. Let $w_0v = v_1v_2$ where $v_2 \in W_I$ and v_1 is reduced-I. Then we have $v = (w_0v_1w_0^I).(w_0^Iv_2)$. Note that $w_0v_1w_0^I$ is still reduced-I; it follows that $T_v = T_{w_0v_1w_0^I}T_{w_0^Iv_2}$ and that the condition $T_{w_0v_1w_0^I}T_{w_0^Iv_2}T_{w'}| T_{w_0v_1w_0^I}T_{w_0^Iv_2} \neq 0$ is equivalent to $T_{w_0^Iv_2}T_{w'}|T_{w_0^Iv_2} \neq 0$. It remains to express the condition $T_{w_0v_1v_2}T_s|T_{w_0v_1v_2} \neq 0$; this condition is equivalent to $l(w_0v_1v_2s) < l(w_0v_1v_2)$, which in turn is equivalent to $l(v_1v_2s) > l(v_1v_2)$. ∎

Note that $v_1 = 1$ and v_2 arbitrary in $E_{W_I}(w')$ satisfy the above condition, so that $E_W(w) \supset E_{W_I}(w')$.

We will apply Lemma 8.11 in a more specific situation where the following holds:

PROPOSITION 8.12. *Under the assumptions of Lemma* 8.11*, assume in addition that $S = I \cup \{s\}$, that there is a unique $s' \in I$ which does not* *commute with s and that $ss's = s'ss'$. Assume also that any $v \in E_{W_I}(w')$ whose support contains s' is such that s' is not in the support of $s'v$. Then $E_W(w) = E_{W_I}(w') \cup \{sv \mid v \in E_{W_I}(w')$ and $s'v < v\}$.*

Proof. We take $v_1v_2 \in E_W(w)$ as in Lemma 8.11; if $v_1 \neq 1$, then $l(v_1s) < l(v_1)$ since v_1 is reduced-I. It follows that v_2 does not commute with s, thus the support of v_2 must contain s'. We claim that $v_1 = s$; otherwise, as v_1 is reduced-I, it would end with $s's$ since $s''s = ss''$ for $s'' \neq s'$; but then $s'sv_2s$ would not be reduced since by the assumption of the proposition v_2s has a reduced expression starting with $s's$ as s commutes with all terms of a reduced expression for v_2 excepted s'. ∎

For $I \subset S$, we denote by Φ_I the corresponding parabolic root subsystem, and we denote by $\mathbf{L}_I$ the Levi subgroup generated by $\mathbf{T}$ and $\{\mathbf{U}_\alpha\}_{\alpha\in\Phi_I}$. We denote $\mathbf{B}_I$ (resp. $\mathbf{B}_I^-$, W_I, $\mathbf{U}_I$, $\mathbf{U}_I^-$) the intersection with $\mathbf{L}_I$ of $\mathbf{B}$ (resp. $\mathbf{B}^-$, W, $\mathbf{U}$, $\mathbf{U}^-$), by $\mathbf{P}_I$ the parabolic subgroup $\mathbf{L}_I\mathbf{B}$ and by $\mathbf{U}_{\mathbf{P}_I}$ its unipotent radical. We will use the following proposition in the proof of Proposition 8.17:

Proposition 8.13. *Let $I_1, \ldots, I_k$ be mutually disjoint subsets of S and let $x_i \in \mathbf{L}_{I_i}$. Then the condition $x_1 \cdots x_k \in \mathbf{U}^-\mathbf{B}$ is equivalent to $x_i \in \mathbf{U}_{I_i}^-\mathbf{B}_{I_i}$ for all i.*

Proof. If $k = 1$, let us write $x_1 = u\dot{v}b$ with $u \in \mathbf{U}_{I_1}^-$, with $\dot{v}$ a representative of $v \in W_{I_1}$ and $b \in \mathbf{B}_{I_1}$. As $\mathbf{U}_{I_1}^- \subset \mathbf{U}^-$ and $\mathbf{B}_{I_1}^- \subset \mathbf{B}$, the existence of the Bruhat decomposition with respect to the pair of Borel subgroups $(\mathbf{B}^-, \mathbf{B})$, which is obtained by multiplying on the left by w_0 the classical Bruhat decomposition, implies that $v = 1$.

By induction on k it is enough to prove the statement for $k = 2$. Let $I = I_1 \cup I_2$; we have $\mathbf{U}_I^- = \mathbf{U}_{I_1}^-\mathbf{U}_{I_2}^-$ and $\mathbf{B}_I = \mathbf{B}_{I_1}\mathbf{B}_{I_2}$. From the case $k = 1$ we get that $x_1x_2 \in \mathbf{U}^-\mathbf{B}$ is equivalent to $x_1x_2 \in \mathbf{U}_{I_1}^-\mathbf{U}_{I_2}^-\mathbf{B}_{I_1}\mathbf{B}_{I_2}$. As $\mathbf{B}_{I_1}$ normalizes $\mathbf{U}_{I_2}^-$, this is equivalent to $x_1x_2 \in (\mathbf{U}_{I_1}^-\mathbf{B}_{I_1}).(\mathbf{U}_{I_2}^-\mathbf{B}_{I_2})$. Let us write $x_1x_2 = y_1y_2$ according to this decomposition; as $\mathbf{L}_{I_1} \cap \mathbf{L}_{I_2} = \mathbf{T}$ we get $x_i \in y_i\mathbf{T} \subset \mathbf{U}_{I_i}^-\mathbf{B}_{I_i}$ for $i = 1, 2$. ∎

Lemma 8.14. *Let $w = v_1 \cdots v_k \in W$ be such that $l(v_1) + \cdots + l(v_k) = l(w)$ and let $\dot{w}, \dot{v}_1, \ldots, \dot{v}_k$ be representatives in $N_{\mathbf{G}}(\mathbf{T})$ such that $\dot{w} = \dot{v}_1 \cdots \dot{v}_k$. Then $\mathbf{U}_w\dot{w} = \mathbf{U}_{v_1}\dot{v}_1 \cdots \mathbf{U}_{v_k}\dot{v}_k$.*

Proof. We have $\mathbf{U}_{v_1}\dot v_1\cdots\mathbf{U}_{v_k}\dot v_k=\mathbf{U}_{v_1}{}^{v_1}\mathbf{U}_{v_2}\cdots{}^{v_1\cdots v_{k-1}}\mathbf{U}_{v_k}\dot w$, and we have $\mathbf{U}_w=\prod_{\alpha\in N(w^{-1})}\mathbf{U}_\alpha$, where $N(w)=\{\alpha>0\mid {}^w\alpha<0\}$. Let $\coprod$ represent disjoint union. The lemma is thus a consequence of $N(w^{-1})=\coprod_i v_1\cdots v_{i-1}(N(v_i^{-1}))$, which itself is obtained by iterating the well known formula: $l(x)+l(y)=l(xy)\Leftrightarrow N(xy)=y^{-1}(N(x))\coprod N(y)$. ∎

Corollary 8.15. *Let $I_1,\ldots,I_k$ be disjoint parts of S, and let $v_i\in W_{I_i}$. Then $\mathbf{B}v_1\cdots v_k\mathbf{B}\cap\mathbf{U}^-=\prod_i((\mathbf{B}_{I_i}v_i\mathbf{B}_{I_i})\cap\mathbf{U}^-)$.*

Proof. As in Proposition 8.13, it is enough to prove the result for $k=2$. By Lemma 8.14 we have $\mathbf{B}v_1v_2\mathbf{B}=\mathbf{U}_{v_1v_2}v_1v_2\mathbf{B}=\mathbf{U}_{v_1}\dot v_1\mathbf{U}_{v_2}\dot v_2\mathbf{B}$. Let $x\in\mathbf{B}v_1v_2\mathbf{B}\cap\mathbf{U}^-$ and write accordingly $x=x_1x_2b$ where $x_1\in\mathbf{U}_{v_1}\dot v_1$, $x_2\in\mathbf{U}_{v_2}\dot v_2$ and $b\in\mathbf{B}$. We have $x_1x_2\in\mathbf{U}^-\mathbf{B}$ thus by Proposition 8.13 $x_1=u_1b_1\in\mathbf{U}^-_{I_1}\mathbf{B}$. We have $u_1=x_1b_1^{-1}\in\mathbf{U}_{v_1}\dot v_1\mathbf{B}\cap\mathbf{U}^-=\mathbf{B}v_1\mathbf{B}\cap\mathbf{U}^-$. As $x\in\mathbf{U}^-$ we have also $b_1x_2b\in\mathbf{U}^-$, thus $b_1x_2b\in\mathbf{B}v_2\mathbf{B}\cap\mathbf{U}^-$. ∎

Proposition 8.16. *Let $\mathbf{w}\in B^+$, and let I be an F-stable subset of S. Then for any $v\in W$, the left multiplication action of $\mathbf{P}_I^F$ on $\mathbf{X}(\mathbf{w})$ stabilizes $\coprod_{v'\in W_Iv}\mathbf{X}^{v'}(\mathbf{w})$.*

Proof. An element $(g_1\mathbf{B},\ldots,g_r\mathbf{B})\in\mathbf{X}(\mathbf{w})$ is in $\mathbf{X}^v(\mathbf{w})$ if and only if $g_1\in\mathbf{B}v\mathbf{B}$. If $p\in\mathbf{B}w'\mathbf{B}$ with $w'\in W_I$, then by Lemma 8.8 we have $pg_1\in\mathbf{B}w''v\mathbf{B}$ with $w''\leq w'$, thus $w''\in W_I$, whence the result. ∎

In the next proposition, for $I\subset S$ we denote by B_I^+ the submonoid of B^+ generated by $\mathbf{I}=\{\mathbf{s}\in\mathbf{S}\mid s\in I\}$.

Proposition 8.17. *Under the assumptions of Proposition 8.16, assume in addition that ${}^{\mathbf{w}_0}\mathbf{w}=\mathbf{s}\mathbf{w}'$ where $\mathbf{s}\notin\mathbf{I}$ and where $\mathbf{w}'\in B_I^+$. Let $\mathbf{U}_{\mathbf{P}_I}$ be the unipotent radical of $\mathbf{P}_I$. Then, if $\mathbf{w}_0^I\in\mathbf{W}$ lifts w_0^I, for any I we have an isomorphism of $\mathbf{L}_I^F\times\langle F^\delta\rangle$-modules*

$$H^i_c\Big(\Big(\coprod_{v\in W_Iw_0}\mathbf{X}^v(\mathbf{w})\Big)/\mathbf{U}^F_{\mathbf{P}_I}\Big)\xrightarrow{\sim}H^{i-2}_c(\mathbf{X}_{\mathbf{L}_I}({}^{\mathbf{w}_0^I}\mathbf{w}'))(-1)\oplus H^{i-1}_c(\mathbf{X}_{\mathbf{L}_I}({}^{\mathbf{w}_0^I}\mathbf{w}')).$$

Proof. To simplify the notation we write just $\mathbf{L}$, $\mathbf{P}$ for $\mathbf{L}_I$, $\mathbf{P}_I$, and we set $\mathbf{Y}=\coprod_{v\in W_Iw_0}\mathbf{X}^v(\mathbf{w})$. Let us see first that the proposition follows from its special case where $\mathbf{w}\in\mathbf{W}$. If $\mathbf{w}=\mathbf{w}_1\cdots\mathbf{w}_k$ is a decomposition of $\mathbf{w}$ as a product of elements of $\mathbf{W}$, we have $\mathbf{w}'=\mathbf{w}_1'{}^{\mathbf{w}_0}\mathbf{w}_2\cdots{}^{\mathbf{w}_0}\mathbf{w}_k$, where ${}^{\mathbf{w}_0}\mathbf{w}_1=\mathbf{s}\mathbf{w}_1'$. Using [DMR, 2.3.3] as in the beginning of the proof of Proposition 8.2, we apply the proposition with $\mathbf{G}^k$, F_1, $\mathbf{L}^k$, $(\mathbf{w}_1,\dots,\mathbf{w}_k)$ and $(\mathbf{s},1,\dots,1)$ replacing respectively $\mathbf{G}$, F, $\mathbf{L}$, $\mathbf{w}$ and $\mathbf{s}$. If we set $\mathbf{Y}'=\coprod_{v_1,\dots,v_k\in W_Iw_0}\left(\mathbf{X}_{\mathbf{G}^k}^{(v_1,\dots,v_k)}((\mathbf{w}_1,\dots,\mathbf{w}_k),F_1)\right)$ we obtain that the cohomology groups of $\mathbf{Y}'/(\mathbf{U}_\mathbf{P}^k)^{F_1}$ are sums of those of $\mathbf{X}_{\mathbf{L}^k}(({}^{\mathbf{w}_0^I}\mathbf{w}'_1,\mathbf{w}_2,\dots,\mathbf{w}_k),F_1)$. This last variety is isomorphic to $\mathbf{X}_\mathbf{L}({}^{\mathbf{w}_0^I}\mathbf{w}')$. On the other hand $(\mathbf{U}_\mathbf{P}^k)^{F_1}\simeq\mathbf{U}_\mathbf{P}^F$ and $\mathbf{Y}'$ is formed from pieces from $\mathbf{Y}$. Indeed, by the beginning of the proof of Proposition 8.2, we have

$$\mathbf{Y}=\coprod_{v\in W_I,\,v_2,\dots,v_k\in W}(\mathbf{X}_{\mathbf{G}^k}^{(v,v_2,\dots,v_k)}((w_1,\dots,w_k),F_1)).$$

We show that the only non-empty pieces of $\mathbf{Y}$ are those such that $v_i\in W_I$ for all i, *i.e.*, those of $\mathbf{Y}'$: the $(v,v_2,\dots,v_k)$ piece is non-empty if and only if $T_{(w_1,\dots,w_k)}T_{(v_2^{-1},v_3^{-1},\dots,{}^Fv^{-1})}|T_{(v^{-1},v_2^{-1},\dots,v_k^{-1})}\neq 0$. As $w_2,\dots,w_k\in {}^{w_0}W_I$ and $v\in W_Iw_0$ the non-vanishing of this coefficient implies that $v_2,\dots,v_k\in W_Iw_0$; indeed, proceeding by induction on i, if $v_{i+1}\in W_Iw_0$ with $i\geq 2$, then the product $T_{w_i}T_{v_{i+1}^{-1}}$ involves only T_y for $y\in W_Iw_0$, thus $v_i\in W_Iw_0$.

We thus assume now $\mathbf{w}\in\mathbf{W}$. We use then the model 8.1 of $\mathbf{X}^v(\mathbf{w})$ taking $k=2$, $w_1={}^{w_0}s$ and $w_2={}^{w_0}w'$:

$$\mathbf{X}^v(\mathbf{w})=\{(g_1\mathbf{B},g_2\mathbf{B})\mid g_1^{-1}g_2\in\mathbf{B}^{w_0}s\mathbf{B},\ g_2^{-1}{}^Fg_1\in\mathbf{B}^{w_0}w'\mathbf{B},\\ \text{and } g_1\in\mathbf{B}v\mathbf{B}\}.$$

Let $\dot w_0$ be a rational representative of w_0. Taking $g_1\dot w_0$ and $g_2\dot w_0$ as variables we get

$$\mathbf{X}^v(\mathbf{w})=\{(g_1\mathbf{B}^-,g_2\mathbf{B}^-)\mid g_1\in\mathbf{B}vw_0\mathbf{B}^-,\ g_1^{-1}.g_2\in\mathbf{B}^-s\mathbf{B}^-,\\ g_2^{-1}.{}^Fg_1\in\mathbf{B}^-w'\mathbf{B}^-\}.$$

By arguing as in the proof of Proposition 8.16 we see that

$$\bigcup_{v\in W_Iw_0}\mathbf{B}vw_0\mathbf{B}^-=\mathbf{P}\mathbf{B}^-.$$

Thus

$$(1)\quad \mathbf{Y}=\{(g_1\mathbf{B}^-,g_2\mathbf{B}^-)\mid g_1\in\mathbf{P}\mathbf{B}^-,\ g_2\in\mathbf{G},\ g_1^{-1}.g_2\in\mathbf{B}^-s\mathbf{B}^-,\\ g_2^{-1}.{}^Fg_1\in\mathbf{B}^-w'\mathbf{B}^-\}.$$

The action of $\mathbf{P}^F$ is by left multiplication. In the following, we fix F^δ-stable representatives, denoted $\dot s$ and $\dot w'$ of s and w'. For $v\in W$, let $\mathbf{U}_v^-=\mathbf{U}^-\cap{}^v\mathbf{U}$.

Lemma 8.18. *The variety* $\mathbf{X}=\{p\in\mathbf{P}\mid p^{-1}{}^Fp\in\mathbf{B}^-sw'\mathbf{B}^-\}$ *admits natural actions of* $\mathbf{B}^-$ *by right multiplication and of* $\mathbf{P}^F$ *by left multiplication. The map* $p\mapsto(p\mathbf{B}^-,pu_p\dot s\mathbf{B}^-)$ *where* u_p *is the unique element of* $\mathbf{U}_s^-$ *such that* $p^{-1}{}^Fp\in u_p\dot s\mathbf{B}^-w'\mathbf{B}^-$ *defines a* $\mathbf{P}^F$ *and* F^δ *equivariant isomorphism of varieties* $\mathbf{X}/\mathbf{B}_I^-\xrightarrow{\sim}\mathbf{Y}$.

Proof. The existence and uniqueness of u_p come from Lemma 8.14: we have

$$\mathbf{B}^-sw'\mathbf{B}^-=\mathbf{U}^-_{sw'}\dot s\dot w'\mathbf{B}^-=\mathbf{U}^-_s\dot s\mathbf{U}^-_{w'}\dot w'\mathbf{B}^-=\mathbf{U}^-_s\dot s\mathbf{B}^-w'\mathbf{B}^-$$

where the $\mathbf{U}_s^-$ part is unique. The image of the map $p\mapsto(p\mathbf{B}^-,pu_p\dot s\mathbf{B}^-)$ is easily checked to be in the model (1) of $\mathbf{Y}$. If $b\in\mathbf{B}_I^-$, the element u_{pb} is determined by $b^{-1}u_p\dot s\mathbf{B}^-=u_{pb}\dot s\mathbf{B}^-$. We have thus $pu_p\dot s\mathbf{B}^-=pbu_{pb}\dot s\mathbf{B}^-$, which shows that p and pb have the same image in $\mathbf{Y}$.

Conversely, given $(g_1\mathbf{B}^-,g_2\mathbf{B}^-)\in\mathbf{Y}$, the equality $g_1\mathbf{B}^-=p\mathbf{B}^-$ defines $p\in\mathbf{X}$ up to right translation by $\mathbf{B}_I^-$. We must check that $g_2\mathbf{B}^-=pu_p\dot s\mathbf{B}^-$. By definition of $\mathbf{Y}$, we have $g_2\in{}^Fg_1\mathbf{B}^-w'^{-1}\mathbf{B}^-\cap g_1\mathbf{B}^-s\mathbf{B}^-={}^Fp\mathbf{B}^-w'^{-1}\mathbf{B}^-\cap p\mathbf{B}^-s\mathbf{B}^-$ whence $p^{-1}g_2\in p^{-1}{}^Fp\mathbf{B}^-w'^{-1}\mathbf{B}^-\cap\mathbf{B}^-s\mathbf{B}^-\subset u_p\dot s\mathbf{B}^-w'\mathbf{B}^-w'^{-1}\mathbf{B}^-\cap\mathbf{B}^-s\mathbf{B}^-$; but $\mathbf{B}^-w'\mathbf{B}^-w'^{-1}\mathbf{B}^-$ is a union of double cosets of the form $\mathbf{B}^-v\mathbf{B}^-$, where $v\in W_I$. Thus $u_p\dot s\mathbf{B}^-\dot w'\mathbf{B}^-w'^{-1}\mathbf{B}^-\cap\mathbf{B}^-s\mathbf{B}^-=u_p\dot s\mathbf{B}^-$, whence the result. ∎

Let us decompose $p\in\mathbf{X}$ as ul, with $u\in\mathbf{U_P}$ and $l\in\mathbf{L}$. The action of $\mathbf{B}_I^-$ does not change the component u thus the quotient of $\mathbf{X}/\mathbf{B}_I^-$ by $\mathbf{U}_{\mathbf{P}^F}$ is realized by the Lang map $(u,l)\mapsto(u^{-1}.{}^Fu,l)$. If we take ${}^{l^{-1}}(u^{-1}.{}^Fu)$ and l as variables we get

$$\mathbf{Y}/\mathbf{U}_\mathbf{P}^F\simeq\{u\in\mathbf{U_P},\,l\in\mathbf{L}\mid ul^{-1}.{}^Fl\in\mathbf{B}^-sw'\mathbf{B}^-\}/\mathbf{B}_I^-,$$

where the action of $b\in\mathbf{B}_I^-$ is by conjugation by b^{-1} on u and by right multiplication on l, and where the action of $\mathbf{L}^F$ is by left multiplication on l.

Lemma 8.19. *For* $u\in\mathbf{U_P}$, $l\in\mathbf{L}$, *the condition* $ul\in\mathbf{B}^-sw'\mathbf{B}^-$ *is equivalent to* $u\in\mathbf{B}^-_{\{s\}}s\mathbf{B}^-_{\{s\}}$ *and* $l\in\mathbf{B}_I^-w'\mathbf{B}_I^-$.

Proof. We have $\mathbf{B}^-sw'\mathbf{B}^-=\mathbf{U}_s^-\dot s\mathbf{U}_w^-\dot w\mathbf{B}_I^-\mathbf{U}_\mathbf{P}^-=\mathbf{U}_s^-\dot s\mathbf{B}_I^-w\mathbf{B}_I^-\mathbf{U}_\mathbf{P}^-$. Thus using that l normalizes $\mathbf{U}_\mathbf{P}^-$, we see that there exists $u'\in\mathbf{U}_\mathbf{P}^-$ such that $uu'\in\mathbf{U}_s^-\dot s\mathbf{B}_I^-w\mathbf{B}_I^-l^{-1}$. Thus there exists $l_s\in\mathbf{U}_s^-\dot s\subset\mathbf{L}_{\{s\}}$, $l'\in$

$\mathbf{B}_I^- w\mathbf{B}_I^- l^{-1} \subset \mathbf{L}$ such that $uu' = l_s l'$. We may then apply Proposition 8.13 with $k = 2$, $I_1 = \{s\}$ and $I_2 = I$ (exchanging the roles of $\mathbf{B}$ and of $\mathbf{B}^-$), and we get $l_s \in \mathbf{U}_s\mathbf{B}^-_{\{s\}}$. Thus there exists $u_s \in \mathbf{U}_s$ such that $l_s \in u_s\mathbf{B}^-_{\{s\}}$. We have $u_s^{-1}u \in \mathbf{B}^-_{\{s\}}\mathbf{L}u'^{-1} \subset \mathbf{L}\mathbf{U}^-_{\mathbf{P}}$ thus $u_s^{-1}u \in \mathbf{U}_{\mathbf{P}} \cap \mathbf{L}\mathbf{U}^-_{\mathbf{P}} = \{1\}$ thus $u = u_s \in \mathbf{U}_s$, and since $l_s \in \mathbf{U}_s^-\dot{s}$ we even have $u \in \mathbf{U}_s^-\dot{s}\mathbf{B}^-_{\{s\}} = \mathbf{B}^-_{\{s\}}s\mathbf{B}^-_{\{s\}}$. The condition $uu' = l_s l'$ becomes thus $u' = b_s l'$ for some $b_s \in \mathbf{B}^-_{\{s\}}$; as $\mathbf{T}\mathbf{U}^-_{\mathbf{P}} \cap \mathbf{L} = \mathbf{T}$, this implies $l' \in \mathbf{T}$ thus $l \in \mathbf{B}_I^- w'\mathbf{B}_I^-$. ∎

As we have $\mathbf{U} \cap \mathbf{B}^-_{\{s\}}s\mathbf{B}^-_{\{s\}} = \mathbf{U}_s^* \subset \mathbf{U}_{\mathbf{P}}$, we get thus $\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F \simeq \{u \in \mathbf{U}_s^*,\, l \in \mathbf{L} \mid l^{-1}\,{}^F l \in \mathbf{B}_I^- w'\mathbf{B}_I^-\}/\mathbf{B}_I^-$ where the action of $\mathbf{L}^F$ is by left multiplication on l and the action of $b \in \mathbf{B}_I^-$ is by right multiplication on l and conjugation by b^{-1} on u. Note that, as $\mathbf{U}_I^-$ centralizes $\mathbf{U}_s$ since no root in Φ_I^- can add to the simple root corresponding to s, the action of $\mathbf{B}_I^-$ on u is through $\mathbf{T}$.

We have $\mathbf{X}_{\mathbf{L}_I}({}^{w_0^I}w') = \{l \in \mathbf{L} \mid l^{-1}\,{}^F l \in \mathbf{B}_I^- w'\mathbf{B}_I^-\}/\mathbf{B}_I^-$. We conclude arguing as in [DMR, 3.2.10]. To simplify the notation we write w'' for ${}^{w_0^I}w'$. Let $\tilde{\mathbf{Y}}$ be the variety $\{u \in \mathbf{U}_s,\, l \in \mathbf{L} \mid l^{-1}\,{}^F l \in \mathbf{B}_I^- w'\mathbf{B}_I^-\}/\mathbf{B}_I^-$; the projection $\pi : \tilde{\mathbf{Y}} \to \mathbf{X}_{\mathbf{L}_I}(w'')$ defined by $(u,l) \mapsto l$ is a fibration by affine lines, and π restricted to $\tilde{\mathbf{Y}} - \mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F$ is an isomorphism. Let i be the closed inclusion $\tilde{\mathbf{Y}} - \mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F \hookrightarrow \tilde{\mathbf{Y}}$ and j the open inclusion $\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F \hookrightarrow \tilde{\mathbf{Y}}$. If we make $\mathbf{L}^F$ act by left multiplication on $\tilde{\mathbf{Y}}$ then i, j and π are $\mathbf{L}^F$-equivariant. Let $\Lambda_{\mathbf{X}}$ be the constant sheaf $\overline{\mathbb{Q}}_\ell$ on a variety $\mathbf{X}$; we have an exact sequence $0 \to j_!\Lambda_{\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F} \to \Lambda_{\tilde{\mathbf{Y}}} \to i_!\Lambda_{\tilde{\mathbf{Y}}-\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F} \to 0$. Using that $R\pi_!\Lambda_{\tilde{\mathbf{Y}}} \simeq \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}[-2](-1)$ and that $R\pi_! i_!\Lambda_{\tilde{\mathbf{Y}}-\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F} \simeq \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}$, we deduce a distinguished triangle $R\pi_! j_!\Lambda_{\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F} \to \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}[-2](-1) \xrightarrow{\partial} \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')} \rightsquigarrow$ where $\partial \in \operatorname{Ext}^2(\Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}, \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}) = H^2(\mathbf{X}_{\mathbf{L}_I}(w''), \overline{\mathbb{Q}}_\ell)$. All maps being $\mathbf{G}^F$-equivariant, we even have $\partial \in H^2(\mathbf{X}_{\mathbf{L}_I}(w''), \overline{\mathbb{Q}}_\ell)^{\mathbf{G}^F} \simeq H_c^{2l-2}(\mathbf{X}_{\mathbf{L}_I}(w''), \overline{\mathbb{Q}}_\ell)^{\mathbf{G}^F}$ where the isomorphism comes from the smoothness of the variety $\mathbf{X}_{\mathbf{L}_I}(w'')$. But by [DMR, 3.3.14] we have $H_c^{2l-2}(\mathbf{X}_{\mathbf{L}_I}(w''), \overline{\mathbb{Q}}_\ell)^{\mathbf{G}^F} = 0$, thus $\partial = 0$ and the distinguished triangle gives an isomorphism $R\pi_! j_!\Lambda_{\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F} \simeq \Lambda_{\mathbf{X}_{\mathbf{L}_I}(w'')}[-2](-1) \oplus \Lambda_{\mathbf{X}_{(\mathbf{L}_I)}(w'')}[-1]$ whence $H_c^i(\mathbf{Y}/\mathbf{U}_{\mathbf{P}}^F) \simeq H_c^{i-2}(\mathbf{X}_{\mathbf{L}_I}(w''))(-1) \oplus H_c^{i-1}(\mathbf{X}_{\mathbf{L}_I}(w''))$ as wanted. ∎

In the proof of Proposition 8.21 we will use the next lemma.

LEMMA 8.20. *Let $\dot{w}_1, \ldots, \dot{w}_k$ be representatives in $N_{\mathbf{G}}(\mathbf{T})$ of $w_1, \ldots, w_k \in W$; for any $u_1, \ldots, u_k \in \mathbf{U}$, there exist unique $u_i' \in \mathbf{U}_{w_i}$ such that* *for all i we have $u_1\dot w_1\cdots u_i\dot w_i\in u'_1\dot w_1\cdots u'_i\dot w_i\mathbf{U}$. This defines a morphism $\mathbf{U}^k\to\prod_{i=1}^k\mathbf{U}_{w_i}$.*

Proof. It is known that for $v\in W$, the equality $u\dot v=u'\dot v u''$ with $u\in\mathbf{U}$, $u'\in\mathbf{U}_v$ and $u''\in\mathbf{U}\cap{}^v\mathbf{U}$ defines an isomorphism $\mathbf{U}\xrightarrow{\sim}\mathbf{U}_v\times\mathbf{U}\cap{}^v\mathbf{U}$. The lemma is a consequence of this fact by induction on k. ∎

The elements w that we will handle in this paper will have $E_W(w)$ of the form $W_I w_0\cup\{v\}$, where v satisfies the assumptions of the next proposition.

Proposition 8.21. *Let $\mathbf{w}=\mathbf{w}_1\cdots\mathbf{w}_k$ be a decomposition of $\mathbf{w}$ with $\mathbf{w}_i\in\mathbf{W}$; let $v\in W^F$, and let I be an F-stable subset of S; we write $\mathbf{P}$ for $\mathbf{P}_I$. Then*

(i) *For all i we have an isomorphism of $\langle F\rangle$-modules:*

$$H^i_c(\mathbf{X}^v(\mathbf{w}))^{\mathbf{U}^F_{\mathbf{P}}}\simeq H_c^{i+2l(w_0v)}(\mathbf{Z}^v_{\mathbf{w}})(l(w_0v)),$$

where $(l(w_0v))$ is a "Tate twist" and

$$\mathbf{Z}^v_{\mathbf{w}}=\Big\{(y,x,u_1,\dots,u_k)\in\mathbf{U}_{\mathbf{P}}\times\mathbf{U}_I\times\prod_i\mathbf{U}_{w_i}\;\Big|\; yx^{-1}\,{}^F x\in\dot v u_1\dot w_1\cdots u_k\dot w_k\mathbf{B}v^{-1}\Big\}.$$

(ii) *Let $\overline{\mathbf{Z}}^v_{\mathbf{w}}$ be the variety $\{(y,x,u_1,\dots,u_k)\in\mathbf{Z}^v_{\mathbf{w}}\mid y\in\mathbf{U}_{\mathbf{P}}\cap{}^v\mathbf{U}^-\}$. The map $\mathbf{Z}^v_{\mathbf{w}}\to\overline{\mathbf{Z}}^v_{\mathbf{w}}$ given by $(y,x,u_1,\dots,u_k)\mapsto(y_2,x,u'_1,\dots,u'_k)$ where y_2 is defined by $y=y_1y_2$ with $y_1\in\mathbf{U}_{\mathbf{P}}\cap{}^v\mathbf{U}$ and $y_2\in\mathbf{U}_{\mathbf{P}}\cap{}^v\mathbf{U}^-$ and where the u'_i are defined by ${}^{\dot v^{-1}}y_1^{-1}u_1\dot w_1\cdots u_i\dot w_i\in u'_1\dot w_1\cdots u'_i\dot w_i\mathbf{U}$ for any i (cf. Lemma 8.20), is a fibration whose fibers are all isomorphic to ${}^{v^{-1}}\mathbf{U}_{\mathbf{P}}\cap\mathbf{U}$.*

(iii) *If in addition v is the unique element of W_Iv such that $\mathbf{X}^v(\mathbf{w})$ is nonempty, there is an action of $\mathbf{L}_I^F$ on $\mathbf{Z}^v_{\mathbf{w}}$ such that the isomorphism of 8.21 (i) is $\mathbf{L}_I^F$-equivariant (for the action of $\mathbf{L}_I^F$ on $\mathbf{X}^v(\mathbf{w})$ given by Proposition 8.16).*

(iv) *If in addition ${}^{v^{-1}}\mathbf{U}_I\subset\mathbf{U}^-$ and $\operatorname{proj}_{\mathbf{U}_I}(\mathbf{U}\cap v\mathbf{B}w_1\mathbf{B}\cdots w_k\mathbf{B}v^{-1})=\prod_{s\in I}\mathbf{U}_s^*$ (where we have denoted $\operatorname{proj}_{\mathbf{U}_I}$ the natural projection $\mathbf{U}\to\mathbf{U}_I$), then the projection $\pi:(y,x,u_1,\dots,u_k)\mapsto x$ is an epimorphism $\mathbf{Z}^v_{\mathbf{w}}\to\mathbf{X}_{\mathbf{L}_I}(c)$, where c is a Coxeter element of W_I, which is $\mathbf{L}_I^F$-equivariant (for the action of $\mathbf{L}_I^F$ on $\mathbf{Z}^v_{\mathbf{w}}$ given in 8.21 (iii)).*

Proof.

LEMMA 8.22. *Let $\dot v, \dot w_1, \ldots, \dot w_k$ be representatives in $N_{\mathbf{G}}(\mathbf{T})$ of v, $w_1, \ldots, w_k \in W$ and let*

$$\mathcal{O}^v(w_1,\ldots,w_k) = \{(x_1\mathbf{B},\ldots,x_k\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^k \mid x_i^{-1}x_{i+1} \in \mathbf{B}w_i\mathbf{B} \text{ and } x_1 \in \mathbf{B}v\mathbf{B}\};$$

there exist unique $u \in \mathbf{U}_v$ and $u_i \in \mathbf{U}_{w_i}$ such that $x_1 \in uv\mathbf{B}$ and that for all i we have $x_{i+1} \in u\dot v u_1 \dot w_1 \cdots u_i \dot w_i \mathbf{B}$. This defines a morphism

$$\mathcal{O}^v(w_1,\ldots,w_k) \longrightarrow \mathbf{U}_v \times \left(\prod_{i=1}^k \mathbf{U}_{w_i}\right).$$

Proof. The proof is similar to that of Lemma 8.20. ∎

Consider the map $\Psi : (u, g_2\mathbf{B}, \ldots, g_k\mathbf{B}) \mapsto (uv\mathbf{B}, g_2\mathbf{B}, \ldots, g_k\mathbf{B})$ from the variety

$$\mathbf{Z} = \{(u, g_2\mathbf{B}, \ldots, g_k\mathbf{B}) \mid u \in \mathbf{U},\ (uv)^{-1}g_2 \in \mathbf{B}w_1\mathbf{B},\ g_i^{-1}g_{i+1} \in \mathbf{B}w_i\mathbf{B},\ g_k^{-1}{}^F(uv) \in \mathbf{B}w_k\mathbf{B}\}$$

to $\mathbf{X}^v(\mathbf{w})$. As $\mathbf{B}v\mathbf{B} = \mathbf{U}_v v\mathbf{B}$, it is a fibration whose fibers are isomorphic to $\mathbf{U} \cap {}^v\mathbf{U}$, an affine space of dimension $l(w_0 v)$. This fibration is $\mathbf{U}^F$ (thus $\mathbf{U}_{\mathbf{P}}^F$)-equivariant, for the action by left multiplication of all components. Applying Lemma 8.22 with $x_i = (uv)^{-1}g_{i+1}$ we get $\mathbf{Z} \simeq \{(u, u_1, \ldots, u_k) \mid u \in \mathbf{U}, u_i \in \mathbf{U}_{w_i},\ u^{-1}{}^F u \in \dot v u_1 \dot w_1 \cdots u_k \dot w_k \mathbf{B} v^{-1}\}$. The map $u = x_{\mathbf{P}}x \mapsto (y = {}^{x^{-1}}(x_{\mathbf{P}}^{-1}{}^F x_{\mathbf{P}}), x)$, where $x_{\mathbf{P}} \in \mathbf{U}_{\mathbf{P}}$ and $x \in \mathbf{U}_I$, defines thus an isomorphism between $\mathbf{Z}/\mathbf{U}_{\mathbf{P}}^F$ and $\mathbf{Z}_{\mathbf{w}}^v$; 8.21 (i) results immediately from this isomorphism and the isomorphism of cohomology implied by Ψ.

Let us prove 8.21 (ii). It is clear that the image of the map is in $\overline{\mathbf{Z}}_{\mathbf{w}}^v$. Consider now the fiber of $(y_2, x, u_1', \ldots, u_k')$. Let y_1 in $\mathbf{U}_{\mathbf{P}} \cap {}^v\mathbf{U}$ be arbitrary; the formulas ${}^{\dot v^{-1}}y_1^{-1} u_1 \dot w_1 \cdots u_i \dot w_i \in u_1' \dot w_1 \cdots u_i' \dot w_i \mathbf{B}$ for any i define unique u_i (*cf.* Lemma 8.20), and the element thus obtained $(y_1 y_2, x, u_1, \ldots, u_k)$ is in the fiber and we get thus all the fiber.

We prove now 8.21 (iii). If v is the unique element of $W_I v$ such that $\mathbf{X}^v(\mathbf{w})$ is non-empty, we will define an action of $\mathbf{P}^F$ on $\mathbf{Z}$ such that Ψ is $\mathbf{P}^F$-equivariant. Under this assumption, by Proposition 8.16 $\mathbf{X}^v(\mathbf{w})$ is $\mathbf{P}^F$-stable, *i.e.*, if $p \in \mathbf{P}^F$ and $(uv\mathbf{B}, g_2\mathbf{B}, \ldots, g_k\mathbf{B}) \in \mathbf{X}^v(\mathbf{w})$ where $u \in \mathbf{U}_v$, then $pu \in \mathbf{U}_v v\mathbf{B}$. The action of $p \in \mathbf{P}^F$ on $\mathbf{X}^v(\mathbf{w})$ is thus given by

$$(uv\mathbf{B}, g_2\mathbf{B}, \ldots, g_k\mathbf{B}) \longmapsto (\overline{u}v\mathbf{B}, pg_2\mathbf{B}, \ldots, pg_k\mathbf{B})$$

where $\overline{u} \in \mathbf{U}_v$ is defined by $pu = \overline{u}b$ with $b \in {}^v\mathbf{B}$. Let $z \in \mathbf{Z}$ have image $(uv\mathbf{B}, g_2\mathbf{B}, \ldots, g_k\mathbf{B})$ in $\mathbf{X}^v(\mathbf{w})$, *i.e.*, $z = (uu', g_2\mathbf{B}, \ldots, g_k\mathbf{B})$ where $u' \in \mathbf{U} \cap {}^v\mathbf{U}$. We want to define the action of p by the map

$$z \longmapsto (\overline{u}.\operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}(bu'), pg_2\mathbf{B}, \ldots, pg_k\mathbf{B})$$

where $pu = \overline{u}b$ as above and where $\operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}$ is the projection of ${}^v\mathbf{B}$ on $\mathbf{U}\cap{}^v\mathbf{U}$ according to the decomposition ${}^v\mathbf{B} = (\mathbf{U}\cap{}^v\mathbf{U}).(\mathbf{U}^-\cap{}^v\mathbf{U}).\mathbf{T}$. This map clearly commutes with Ψ, but it is not obvious that it defines an action. Let $p' \in \mathbf{P}^F$; let us write $p'\overline{u} = \overline{\overline{u}}b'$ where $\overline{\overline{u}} \in \mathbf{U}_v$ and $b' \in {}^v\mathbf{B}$. We must check that the action of pp' is the composition of that of p and that of p'; this is equivalent to $\operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}(b'bu') = \operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}(b'\operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}(bu'))$, and this last equality is easy to check. This action of $\mathbf{P}^F$ gives after quotienting $\mathbf{Z}$ by $\mathbf{U}_\mathbf{P}^F$ an action of $\mathbf{L}_I^F$ on $\mathbf{Z}_\mathbf{w}^v$, for which 8.21 (iii) holds.

Let us now prove 8.21 (iv). By [Lu2, 2.5] the variety $\mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$ has a single piece $\mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c})$. As $\mathbf{c} \in \mathbf{W}$, the model 8.1 gives $\mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c}) \simeq \{g\mathbf{B}_I \in \mathbf{L}_I/\mathbf{B}_I \mid g^{-1}{}^Fg \in \mathbf{B}_I\mathbf{c}\mathbf{B}_I$ and $g \in \mathbf{B}_I w_0^I \mathbf{B}_I\}$. Defining $u \in \mathbf{U}_I$ by $g \in uw_0^I\mathbf{B}_I$ we get $\mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c}) \simeq \{u \in \mathbf{U}_I \mid {}^{w_0^I}(u^{-1}{}^Fu) \in \mathbf{B}_I\mathbf{c}\mathbf{B}_I\}$. As $\mathbf{B}_I\mathbf{c}\mathbf{B}_I \cap \mathbf{U}_I^- = \prod_{s\in I}\mathbf{U}_s^{-*}$ (*cf.* [Lu2, 2.6]) and ${}^{w_0^I}(\prod_{s\in I}\mathbf{U}_s^*) = \prod_{s\in I}\mathbf{U}_s^{-*}$, we get $\mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c}) \simeq \{u \in \mathbf{U}_I \mid u^{-1}{}^Fu \in \prod_{s\in I}\mathbf{U}_s^*\}$, on which $t \in \mathbf{T}^F$ acts by $u \mapsto {}^tu$ and $u_+ \in \mathbf{U}_I^F$ acts by $u \mapsto u_+u$; the action of $u_- \in \mathbf{U}_I^{-F}$ maps u on the element u' of $\mathbf{U}_I$ such that $u_-u \in u'\mathbf{B}_I^-$: such an element exists by uniqueness of the piece $\mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c})$. On the other hand the projection on the first two components of $\mathbf{Z}^v(\mathbf{w})$ is surjective on couples $(y,x) \in \mathbf{U}_\mathbf{P}\times\mathbf{U}_I$ which satisfy $yx^{-1}.{}^Fx \in \mathbf{U}\cap v\mathbf{B}w_1\mathbf{B}\cdots w_k\mathbf{B}v^{-1}$ and the projection of $\mathbf{U}\cap v\mathbf{B}w_1\mathbf{B}\cdots w_k\mathbf{B}v^{-1}$ on $\mathbf{U}_I$ is equal to $\prod_{s\in I}\mathbf{U}_s^*$ by assumption. Thus we see that π indeed defines an epimorphism $\mathbf{Z}_\mathbf{w}^v \to \mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$.

It remains to check that this epimorphism is $\mathbf{L}_I^F$-equivariant. For this, it is enough to check that the above action and that on the component x of $(y, x, u_1, \ldots, u_k) \in \mathbf{Z}_\mathbf{w}^v$, which results itself of the action on the projection on $\mathbf{U}_I$ of the first component of an element of $\mathbf{Z}$, coincide. For this it is enough to check separately that the actions of $\mathbf{B}_I^F$ and $\mathbf{U}_I^{-F}$ coincide.

Let $u \in \mathbf{U}_v$ and $u' \in \mathbf{U}\cap{}^v\mathbf{U}$; the first component of the image of $z = (uu', g_2\mathbf{B}, \ldots) \in \mathbf{Z}$ by the action of $xt \in \mathbf{B}_I^F$, with $t \in \mathbf{T}^F$ and $x \in \mathbf{U}_I^F$, is $x{}^tu\operatorname{proj}_{\mathbf{U}\cap{}^v\mathbf{U}}(tu') = x{}^tu{}^tu'$, since $x \in \mathbf{U}_v$ and $u' \in \mathbf{U}\cap{}^v\mathbf{U}$. Since $\mathbf{U}_I \subset \mathbf{U}_v$, the action of xt on the projection u_I of uu' is thus $u_I \mapsto x{}^tu_I$, which coincides with the action on $\mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$.

Similarly, the action of $y \in \mathbf{U}_I^{-F}$ maps u_I on $\operatorname{proj}_{\mathbf{U}_I}(\overline{u})$ where $yu \in \overline{u}{}^v\mathbf{B}$

with $\overline{u} \in \mathbf{U}_v$. On the other hand, the image by y of $u_I \in \mathbf{X}_{\mathbf{L}_I}^{w_0^I}(\mathbf{c})$ is u'_I such that $yu_I \in u'_I\mathbf{B}_I^- \subset u'_I{}^v\mathbf{B}$. Let us write $u = u_\mathbf{P}u_I$ with $u_\mathbf{P} \in \mathbf{U}_\mathbf{P}$; thus $yu_\mathbf{P}u_I = {}^yu_\mathbf{P}yu_I \in {}^yu_\mathbf{P}u'_I{}^v\mathbf{B} = u'_I({}^{u_I'^{-1}y}u_\mathbf{P})^v\mathbf{B} = u'_I({}^{u_I'^{-1}y}u_\mathbf{P})_-{}^v\mathbf{B} = {}^{u'_I}({}^{u_I'^{-1}y}u_\mathbf{P})_-u'_I{}^v\mathbf{B}$, where we have denoted by x_- the projection on $\mathbf{U}_v$ of an element $x \in \mathbf{U} = \mathbf{U}_v.(\mathbf{U}\cap{}^v\mathbf{U})$. But we have ${}^{u'_I}({}^{u_I'^{-1}y}u_\mathbf{P})_- \in \mathbf{U}_\mathbf{P}\cap{}^v\mathbf{U}^-$ since $u'_I \in \mathbf{U}_I \cap {}^v\mathbf{U}^-$, thus $\overline{u} = {}^{u'_I}({}^{u_I'^{-1}y}u_\mathbf{P})_-u'_I$ and its component in $\mathbf{U}_I$ is u'_I as required. ∎

The map π of 8.21 (iv) is the composition of the fibration of 8.21 (ii) and the projection

$$\overline{\pi} : (y, x, u_1, \ldots, u_k) \longmapsto x$$

which is an epimorphism $\overline{\mathbf{Z}}_\mathbf{w}^v \to \mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$. All fibers of $\mathbf{Z}_\mathbf{w}^v \to \overline{\mathbf{Z}}_\mathbf{w}^v$ are affine spaces of dimension $l(w_0v)$. It will be easier to compute the fibers of $\overline{\pi}$ than those of π. In the case where we will apply Proposition 8.21, we will be in the situation of one of the next two propositions:

PROPOSITION 8.23. *Assume under the assumptions of* 8.21 (iv) *that the fibers of* $\overline{\pi}$ *are affine spaces of dimension* d. *Then for every* i *we have an isomorphism of* $\mathbf{L}_I^F \times \langle F\rangle$*-modules*: $H_c^i(\mathbf{X}^v(\mathbf{w}))^{\mathbf{U}_{\mathbf{P}_I}^F} \simeq H_c^{i-2d}(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-d)$.

Proof.

LEMMA 8.24. *With the same notation as above Proposition* 8.13, *let* $\mathbf{V}$ *be a variety given with an action of* $\mathbf{P}_I^F$, *let* $\mathbf{V}'$ *be a variety given with an action of* $\mathbf{L}_I^F$ *and let* $\pi : \mathbf{V}/\mathbf{U}_{\mathbf{P}_I}^F \to \mathbf{V}'$ *be an* $\mathbf{L}_I^F$*-equivariant epimorphism whose fibers are all affine spaces of dimension* d. *Then for all* i *we have an isomorphism* $H_c^i(\mathbf{V})^{\mathbf{U}_{\mathbf{P}_I}^F} \simeq H_c^{i-2d}(\mathbf{V}')(-d)$ *of* $\mathbf{L}_I^F \times \langle F\rangle$*-modules.*

Proof. This lemma results from standard properties of ℓ-adic cohomology , see *e.g.*, [DM, 10.10 and 10.12]. ∎

The proposition results immediately from Proposition 8.21 and Lemma 8.24, taking in account the "Tate twist" induced by the quotient $\mathbf{Z}_\mathbf{w}^v \to \overline{\mathbf{Z}}_\mathbf{w}^v$. ∎

In the next proposition we assume $\mathbf{w} \in \mathbf{W}$ and write $\mathbf{X}(w)$ for $\mathbf{X}(\mathbf{w})$.

PROPOSITION 8.25. *Assume that in addition to w, there is another element $w' < w$ satisfying the assumptions of* 8.21 (iv) *with the same v; let $\overline{\pi}' : \overline{\mathbf{Z}}^v_{w'} \to \mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$ be the epimorphism analogous to $\overline{\pi}$ and assume that the fibers of $\overline{\pi}'$ are affine lines and that the fibers of*

$$\overline{\pi} \coprod \overline{\pi}' : \overline{\mathbf{Z}}^v_w \coprod \overline{\mathbf{Z}}^v_{w'} \longrightarrow \mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$$

are affine planes, the above union being taken in $(\mathbf{U}_{\mathbf{P}_I} \cap {}^v\mathbf{U}^-) \times \mathbf{U}_I \times \mathbf{U}$. Then for any i we have an isomorphism of $\mathbf{L}_I^F \times \langle F \rangle$-modules:

$$H^i_c(\mathbf{X}^v(w))^{\mathbf{U}^F_{\mathbf{P}_I}} \simeq H^{i-3}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-1) \oplus H^{i-4}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-2).$$

Proof. Proposition 8.23 and Lemma 8.24 give respectively the isomorphisms of $\mathbf{L}_I^F \times \langle F \rangle$-modules $H^i_c(\mathbf{X}^v(w'))^{\mathbf{U}^F_{\mathbf{P}_I}} \simeq H^{i-2}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-1)$ and $H^i_c(\mathbf{X}^v(w) \coprod \mathbf{X}^v(w'))^{\mathbf{U}^F_{\mathbf{P}_I}} \simeq H^{i-4}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-2)$. The assumption $w' < w$ implies that $\mathbf{X}^v(w)$ is open in $\mathbf{X}^v(w) \coprod \mathbf{X}^v(w')$. We deduce a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{i-3}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-1) &\longrightarrow H^i_c(\mathbf{X}^v(w))^{\mathbf{U}^F_{\mathbf{P}_I}} \\ &\longrightarrow H^{i-4}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-2) \longrightarrow H^{i-2}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-1) \longrightarrow \cdots . \end{aligned}$$

We deduce the proposition by observing that a morphism of $\mathbf{L}_I^F$-modules from $H^{i-4}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-2)$ to $H^{i-2}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{c}))(-1)$ must be 0. Indeed cohomology groups of different degrees of $\mathbf{X}_{\mathbf{L}_I}(\mathbf{c})$ are disjoint as $\mathbf{L}_I^F$-modules. ∎

§9. The n-th roots of $\boldsymbol{\pi}$ in type A_n

In this section we compute the cohomology groups $H^i_c(\mathbf{X}(\mathbf{w}))$ as $\mathbf{G}^F \times \langle F \rangle$-modules when $\mathbf{G}$ is a split group of type A_n ($n \geq 1$) and $\mathbf{w}$ an n-th root of $\boldsymbol{\pi}$ and we show Conjectures 2.1 to 2.6 for this case. The Coxeter presentation of W is given by the diagram $\underset{s_1}{\bigcirc}\!\!-\!\!\underset{s_2}{\bigcirc}\cdots\underset{s_n}{\bigcirc}$, and we denote $\mathbf{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ the corresponding generating set of B.

Conjecture 2.1 holds by Theorem 5.1, and Conjecture 2.3 holds as remarked in Section 5. As noticed in Section 2, Conjecture 2.2 follows from a result of Eilenberg and a recent result of Birman, Gebhardt and Gonzales-Meneses. However, we will give a simple proof of it in our case.

PROPOSITION 9.1. *There is a morphism in $\mathcal{D}^+$ between any two of n-th roots of $\boldsymbol{\pi}$ in B^+.*

Proof. We will show the proposition by showing that any two roots $\mathbf{b}$ and $\mathbf{b}'$ are equivalent by the transitive closure of the relation on B^+ given by $\mathbf{xy} \sim \mathbf{yx}$.

By [BMi, 3.12 and A.1.1] the image in w, identified with the symmetric group $\mathfrak{S}_{n+1}$, of any n-th root of $\boldsymbol{\pi}$ is an n-cycle. So it is enough to see that if $\mathbf{b} \in B^+$ has length $n+1$ and support $\mathbf{S}$, and is such that $\beta(\mathbf{b})$ is an n-cycle then there exists a morphism from $\mathbf{b}$ to $\mathbf{s}_1 \cdots \mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_n$ in $\mathcal{D}^+$.

We first show that there exists a morphism in $\mathcal{D}^+$ from $\mathbf{b}$ to an element of the form $\mathbf{x}\mathbf{s}_n\mathbf{s}_n\mathbf{y}$. By assumption, in the decomposition of $\mathbf{b}$ as a product of elements of $\mathbf{S}$, exactly one, say $\mathbf{s}_i$ is present twice. We write $\mathbf{b} = \mathbf{x}\mathbf{s}_i\mathbf{y}\mathbf{s}_i\mathbf{z}$ where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in B^+$ do not have $\mathbf{s}_i$ in their support. There are three cases:

(a) The support of $\mathbf{y}$ contains neither $\mathbf{s}_{i+1}$ nor $\mathbf{s}_{i-1}$. Then $\mathbf{b} = \mathbf{x}\mathbf{y}\mathbf{s}_i\mathbf{s}_i\mathbf{z}$, so that $\beta(\mathbf{b}) = \beta(\mathbf{xyz})$ is an element of length $n-1$ of support $S - \{s_i\}$. Such an element can be an n-cycle only if $i = 1$ or $i = n$. In the latter case $\mathbf{b}$ is as desired. If $i = 1$, as $\mathbf{s}_1$ commutes with all elements of $\mathbf{S} - \{\mathbf{s}_2\}$, we see that $\mathbf{b}$ has the form $\mathbf{x}\mathbf{s}_1\mathbf{s}_1\mathbf{s}_2\mathbf{y}$ or $\mathbf{x}\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1\mathbf{y}$. In both cases $\mathbf{b}$ is equivalent to $\mathbf{x}\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{y} = \mathbf{x}\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{y}$ and we are reduced to case (c) below with $i = 2$.

(b) The support of $\mathbf{y}$ contains $\mathbf{s}_{i+1}$ and $\mathbf{s}_{i-1}$. Then we use that $\mathbf{b}$ is equivalent to $\mathbf{y}\mathbf{s}_i\mathbf{z}\mathbf{x}\mathbf{s}_i$, and the support of $\mathbf{zx}$ contains neither $\mathbf{s}_{i+1}$ nor $\mathbf{s}_{i-1}$. We are back to case (a).

(c) The support of $\mathbf{y}$ contains one of $\mathbf{s}_{i+1}$ and $\mathbf{s}_{i-1}$. Then $\mathbf{s}_i$ commutes with either $\mathbf{x}$ or $\mathbf{z}$ and if $i = n$ then $\mathbf{b}$ is equivalent to an element of the form we want. Otherwise replacing if needed $\mathbf{b}$ by the equivalent element $\mathbf{y}\mathbf{s}_i\mathbf{z}\mathbf{x}\mathbf{s}_i$, we may assume that $\mathbf{y}$ involves $\mathbf{s}_{i+1}$. Then $\mathbf{b}$ can be written $\mathbf{x}\mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_i\mathbf{y}'\mathbf{z} = \mathbf{x}\mathbf{s}_{i+1}\mathbf{s}_i\mathbf{s}_{i+1}\mathbf{y}'\mathbf{z}$. By induction on i we are reduced to $i = n$ whence the result.

We prove now that any pair of elements $\mathbf{x}\mathbf{s}_n\mathbf{s}_n\mathbf{y}$ of length $n+1$ with support $\mathbf{S}$ are connected by a morphism in $\mathcal{D}^+$. We use [Bou, Chap. V §6, Lemma 1] which says "If X is a finite forest and if $x \mapsto g_x$ is a mapping from X to a group Γ such that g_x and g_y commute if x and y are not connected in X, then the elements of Γ which are the products of all the g_x in some order are conjugate by cyclic permutation", where conjugation by cyclic permutation is the transitive closure of $g_{x_1} \cdots g_{x_k} \mapsto g_{x_2} \cdots g_{x_k} g_{x_1}$ ([Bou] does not state that the conjugation is by cyclic permutation but it is established in the proof). We apply this result to the map from the Coxeter diagram of type A_n which maps the i-th vertex to $\mathbf{s}_i \in B$, with the exception of the n-th vertex which is mapped onto $\mathbf{s}_n\mathbf{s}_n$. This gives the result. ∎

We denote by $\rho_b^{(n)}$ the unipotent representation of $\mathbf{G}^F$ which corresponds to the partition $1,\ldots,1,2,b$ of $n+1$. Let $\mathrm{St}^{(n)}$ be the Steinberg representation and $\mathrm{Id}^{(n)}$ be the identity representation of $\mathbf{G}^F$. We will deduce Conjectures 2.4 to 2.6 from the following theorem. In this theorem, we adopt the conventions of [DMR, 3.3.5] to describe the cohomology of a variety $\mathbf{X}(\mathbf{w})$ as a $\mathbf{G}^F\times\langle F\rangle$-module; we describe the cohomology as a 2-variable polynomial with coefficients in the Grothendieck group of $\mathbf{G}^F$, where the degree in the variable h represents the degree of the cohomology group, and where the degree in t encodes the eigenvalues of F: by a theorem of Lusztig, given a unipotent character ρ, the eigenvalues of F on the ρ-isotypic part of a cohomology group $H^j_c(\mathbf{X}(\mathbf{w}))$ are of the form $q^i\lambda_\rho$ where λ_ρ is a complex number of module 1 or $q^{1/2}$ which depends only on ρ and neither on j nor on $\mathbf{w}$. We encode such an eigenvalue by t^i.

Theorem 9.2. *Let $\mathbf{w}\in B^+$ be an n-th root of $\boldsymbol{\pi}$; then we have as $\mathbf{G}^F\times\langle F\rangle$-modules:*

$$\sum_i h^i H^i_c(\mathbf{X}(\mathbf{w})) = \mathrm{St}^{(n)}\,h^{n+1} + \sum_{b=2}^{n-1}\rho_b^{(n)}t^b h^{n+b} + \mathrm{Id}^{(n)}\,t^{n+1}h^{2(n+1)}.$$

Proof. We prove the theorem by induction on n. If $n=1$ we have $\mathbf{w}=\boldsymbol{\pi}$. Then the only unipotent representations of $\mathbf{G}^F$ are $\mathrm{St}^{(1)}$ and $\mathrm{Id}^{(1)}$ and the result is given by [DMR, 3.3.14] and [DMR, 3.3.15]. If $n\geq 2$, by Proposition 9.1 and [DMR, 3.1.6], it is sufficient to prove the result for a fixed root of $\boldsymbol{\pi}$.

We choose $\mathbf{w}=\mathbf{s}_1\cdots\mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_n$. We shall prove the theorem using the results of Section 8. Let $I=\{s_1,\ldots,s_{n-1}\}$.

Lemma 9.3. *The variety $\mathbf{X}^v(\mathbf{w})$ is not empty if and only if v is the longest element in its coset vW_I.*

Proof. We apply Proposition 8.2: $\mathbf{X}^v(\mathbf{w})$ is not empty if and only if $T_vT_{\mathbf{w}}|T_v$ is not equal to zero. In the Hecke algebra $T_{\mathbf{w}}=(q-1)T_{s_1\cdots s_n}+qT_{s_1\cdots s_{n-1}}$, so if $T_vT_{\mathbf{w}}|T_v\neq 0$ then $T_vT_{s_1\cdots s_n}|T_v$ or $T_vT_{s_1\cdots s_{n-1}}|T_v$ is not zero. By [Lu2, 2.5] or Proposition 8.7, the only v such that $T_vT_{s_1\cdots s_n}|T_v\neq 0$ is w_0. Let us write $v=xy$ with x reduced-I and $y\in W_I$; then $T_vT_{s_1\cdots s_{n-1}}|T_v\neq 0$

if and only if $T_y T_{s_1\cdots s_{n-1}}|T_y \neq 0$; by the same result as above, applied in W_I, this coefficient is not zero if and only if $y = w_0^I$. So v has to be the longest element in its coset, and we have shown that this is equivalent to the non-vanishing of $T_v T_w|T_v$ except possibly for $v = w_0$. In this last case $T_{w_0} T_{s_1\cdots s_n}|T_{w_0} = (q-1)^n$ and $T_{w_0} T_{s_1\cdots s_{n-1}}|T_{w_0} = (q-1)^{n-1}$ and the sum of these two coefficients is again non-zero. ∎

The reduced-I elements are $s_i s_{i+1}\cdots s_n$, for $i \leq n$; their number is $n+1 = |W/W_I|$. The elements v of maximal length in their cosets vW_I are then $s_i s_{i+1}\cdots s_n w_0^I = s_1\cdots s_{i-1} w_0$. They are in the coset $W_I w_0$, except $s_1\cdots s_n w_0 = w_0^I$. Let $\mathbf{P}$ denote the parabolic subgroup $\mathbf{P}_I$; by Proposition 8.16 $\mathbf{X}(\mathbf{w})$ is the union of two $\mathbf{P}^F$-stable pieces: $\bigcup_{v\in W_I} \mathbf{X}^{vw_0}(\mathbf{w})$, which is an open subvariety as $\bigcup_{v\in W_I} \mathbf{B}vw_0\mathbf{B}$ is open in $\bigcup_{v\in W_I} \mathbf{B}vw_0\mathbf{B} \cup \mathbf{B}w_0^I\mathbf{B}$, and the closed subvariety $\mathbf{X}^{w_0^I}(\mathbf{w})$. As $H^i_c(\mathbf{X}(\mathbf{w})/\mathbf{U}_\mathbf{P}^F) = {}^*R^\mathbf{G}_\mathbf{L}(H^i_c(\mathbf{X}(\mathbf{w})))$ (*cf. e.g.*, [DM, 10.10]), we get, setting $\mathbf{X}_1 = (\bigcup_{v\in W_I} \mathbf{X}^{vw_0}(w))/\mathbf{U}_\mathbf{P}^F$ and $\mathbf{X}_2 = \mathbf{X}^{w_0^I}(w)/\mathbf{U}_\mathbf{P}^F$, the following long exact sequence of $\mathbf{L}^F \times \langle F\rangle$-modules, where $\mathbf{L}$ denotes $\mathbf{L}_I$:

$$(1)\quad \cdots \longrightarrow H^i_c(\mathbf{X}_1) \longrightarrow {}^*R^\mathbf{G}_\mathbf{L}(H^i_c(\mathbf{X}(w))) \longrightarrow H^i_c(\mathbf{X}_2) \longrightarrow H^{i+1}_c(\mathbf{X}_1) \longrightarrow \cdots$$

We now apply Proposition 8.17 with $s = s_1$ (indeed ${}^{\mathbf{w}_0}\mathbf{w} = \mathbf{s}_n\mathbf{w}'$ where $\mathbf{w}' = \mathbf{s}_{n-1}\cdots\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1 \in B_I^+$), whence

$$H^i_c(\mathbf{X}_1) \simeq H^{i-2}_c(\mathbf{X}_\mathbf{L}({}^{w_0^I}\mathbf{w}'))(-1) \oplus H^{i-1}_c(\mathbf{X}_\mathbf{L}({}^{w_0^I}\mathbf{w}')).$$

The element ${}^{w_0^I}\mathbf{w}' = \mathbf{s}_1\cdots\mathbf{s}_{n-2}\mathbf{s}_{n-1}\mathbf{s}_{n-1}$ is the element of $\mathbf{L}$ analogous to $\mathbf{w}$. So by induction on n we get the equality of $\mathbf{L}^F \times \langle F\rangle$-modules:
(1′)

$$\sum_i h^i H^i_c(\mathbf{X}_1) = (th^2+h)\Big(\mathrm{St}^{(n-1)}\, h^n + \sum_{b=2}^{n-2} \rho_b^{(n-1)} t^b h^{n+b-1} + \mathrm{Id}^{(n-1)}\, t^n h^{2n}\Big).$$

To compute the $\mathbf{L}^F \times \langle F\rangle$-module $\sum_i h^i H^i_c(\mathbf{X}_2)$, we apply Propositions 8.21 and 8.23 with $v = w_0^I$, with $k = n+1$ and with $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n, \mathbf{s}_n$ for $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Let us check the assumptions: the assumption (iii) on v holds by Lemma 9.3; assumption (iv) holds since ${}^{v^{-1}}\mathbf{U}_I = \mathbf{U}_I^- \subset \mathbf{U}^-$; the other assumptions to check are

$$(2)\qquad \mathrm{proj}_{\mathbf{U}_I}(\mathbf{U} \cap v\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}v^{-1}) = \prod_{\alpha\in I} \mathbf{U}^*_\alpha$$

and that the fibers of $\overline{\pi}$ are affine lines. We have

$$\begin{aligned}&\operatorname{proj}_{\mathbf{U}_I}(\mathbf{U}\cap v\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}v^{-1})\\ &\quad={}^v(\operatorname{proj}_{\mathbf{U}_I^-}({}^{v^{-1}}\mathbf{U}\cap\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}))\\ &\quad={}^v(\operatorname{proj}_{\mathbf{U}_I^-}((\mathbf{U}_I^-\cdot\mathbf{U}_\mathbf{P})\cap\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}))\\ &\quad={}^v(\mathbf{U}_I^-\cap\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}),\end{aligned}$$

the last equality as, since $\mathbf{U}_\mathbf{P}\subset\mathbf{B}$ we have

$$\begin{aligned}&(\mathbf{U}_I^-\cdot\mathbf{U}_\mathbf{P})\cap\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}\\ &\quad=(\mathbf{U}_I^-\cap\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B})\cdot\mathbf{U}_\mathbf{P}.\end{aligned}$$

But we have $\mathbf{B}s_1\mathbf{B}\cdots\mathbf{B}s_{n-1}\mathbf{B}s_n\mathbf{B}s_n\mathbf{B}=\mathbf{B}s_1\cdots s_{n-1}s_n\mathbf{B}\cup\mathbf{B}s_1\cdots s_{n-1}\mathbf{B}$. By Corollary 8.15 we have $\mathbf{B}s_1\cdots s_{n-1}\mathbf{B}\cap\mathbf{U}^-=\prod_{i=1}^{i=n-1}\mathbf{U}^*_{-\alpha_i}\subset\mathbf{U}_I^-$, and in the same way (or by [Lu2, 2.2]) we have $\mathbf{B}s_1\cdots s_n\mathbf{B}\cap\mathbf{U}^-=\prod_{i=1}^{i=n}\mathbf{U}^*_{-\alpha_i}$ which has empty intersection with $\mathbf{U}_I^-$. So (2) is proved.

Let us compute the fibers of $\overline{\pi}$. We have $\mathbf{U}_\mathbf{P}\cap{}^v\mathbf{U}^-=\mathbf{U}_\mathbf{P}\cap(\mathbf{U}_I.\mathbf{U}_\mathbf{P}^-)=1$, so

$$\begin{aligned}\overline{\mathbf{Z}^v_\mathbf{w}}\simeq\{(x,u_1,\ldots,u_{n+1})\mid x\in\mathbf{U}_I,\ u_i\in\mathbf{U}_{s_i}\ (i=1,\ldots,n),\ u_{n+1}\in\mathbf{U}_{s_n},&\\ x^{-1}\cdot{}^Fx\in\dot v u_1\dot s_1\cdots u_{n-1}\dot s_{n-1}u_n\dot s_n u_{n+1}\dot s_n\mathbf{B}v^{-1}\}.&\end{aligned}$$

Using the above description of the projection we see that the fibers of $\overline{\pi}$ are those of the map from

$$\begin{aligned}\{(u_1,\ldots,u_{n+1})\mid u_i\in\mathbf{U}_{s_i}\ (i=1,\ldots,n),\ u_{n+1}\in\mathbf{U}_{s_n},&\\ u_1\dot s_1\cdots u_{n-1}\dot s_{n-1}u_n\dot s_n u_{n+1}\dot s_n\mathbf{B}\in\mathbf{B}s_1\cdots s_{n-1}\mathbf{B}\}&\end{aligned}$$

to $\mathbf{G}/\mathbf{B}$ given by $(u_1,\ldots,u_{n+1})\mapsto u_1\dot s_1\cdots u_n\dot s_n u_{n+1}\dot s_n\mathbf{B}$. The condition on the u_i implies $u_{n+1}=1$ and that the image of an n-tuple $(u_1,\ldots,u_{n-1},u_n)$ does not depend on u_n and is injective on $(u_1,\ldots,u_{n-1})$. So the fibers are indeed affine lines.

So we may apply Proposition 8.23: let us write $\gamma_b^{(n)}$ for the unipotent character corresponding to the partition $1,\ldots,1,b$ of a split group of type A_n; multiplying by th^2 the two variable polynomial which encodes the cohomology of the Coxeter variety $\mathbf{L}$, we get that the cohomology of $\mathbf{X}_2$ is given by $\sum_{b=1}^n t^bh^{n+b}\gamma_b^{(n-1)}$ as a $\mathbf{L}^F\times\langle F\rangle$-module.

We now compute the t^b-isotypic part of the exact sequence (1). For $2 \le b \le n-1$ we get the exact sequence

$$0 \longrightarrow \rho_b^{(n-1)} + \rho_{b-1}^{(n-1)} \longrightarrow ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{n+b}(\mathbf{X}(w)))_{t^b} \longrightarrow \gamma_b^{(n-1)} \longrightarrow 0$$

which gives ${}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{n+b}(\mathbf{X}(w)) = \rho_b^{(n-1)} + \rho_{b-1}^{(n-1)} + \gamma_b^{(n-1)}$. The Littlewood-Richardson formula allows to compute ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ of any unipotent character. In particular it shows that the only characters of $\mathbf{G}^F$ whose ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ contains $\gamma_b^{(n-1)}$ are $\rho_b^{(n)}$, $\gamma_{b+1}^{(n)}$ and $\gamma_b^{(n)}$; so $H_c^{n+b}(\mathbf{X}(w))$ contains one of these three characters. But ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\gamma_{b+1}^{(n)})$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\gamma_b^{(n)})$ contain characters different from $\rho_b^{(n-1)}$, $\rho_{b-1}^{(n-1)}$ and $\gamma_b^{(n-1)}$. So $H_c^{n+b}(\mathbf{X}(w))$ contains $\rho_b^{(n)}$. But ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_b^{(n)}) = \rho_b^{(n-1)} + \rho_{b-1}^{(n-1)} + \gamma_b^{(n-1)}$. So $H_c^{n+b}(\mathbf{X}(w)) = \rho_b^{(n)}$ as a $\mathbf{G}^F$-module.

For $b = 0, 1, n, n+1$ we get respectively the exact sequences:

$$0 \longrightarrow \mathrm{St}^{(n-1)} \longrightarrow ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{n+1}(\mathbf{X}(w)))_{t^0} \longrightarrow 0$$

$$0 \to ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{n+1}(\mathbf{X}(w)))_t \to \mathrm{St}^{(n-1)} \to \mathrm{St}^{(n-1)} \to ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{n+2}(\mathbf{X}(w)))_t \to 0$$

$$0 \to ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{2n}(\mathbf{X}(w)))_{t^n} \to \mathrm{Id}^{(n-1)} \to \mathrm{Id}^{(n-1)} \to ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{2n+1}(\mathbf{X}(w)))_{t^n} \to 0$$

$$0 \longrightarrow \mathrm{Id}^{(n-1)} \longrightarrow ({}^*R_{\mathbf{L}}^{\mathbf{G}} H_c^{2n+2}(\mathbf{X}(w)))_{t^{n+1}} \longrightarrow 0$$

We know that the only character χ of $\mathbf{G}^F$ such that ${}^*R_{\mathbf{L}}^{\mathbf{G}}\chi$ is $\mathrm{St}^{(n-1)}$-isotypic is $\mathrm{St}^{(n)}$, and that the only character χ such that ${}^*R_{\mathbf{L}}^{\mathbf{G}}\chi$ is $\mathrm{Id}^{(n-1)}$-isotypic is $\mathrm{Id}^{(n)}$. So we see that the $(H^i(\mathbf{X}(w)))_{t^b}$ in the above exact sequences are $\mathrm{Id}^{(n)}$-isotypic or $\mathrm{St}^{(n)}$-isotypic. The exact sequence for $b = 0$ (resp. $b = n+1$) gives $H^{n+1}(\mathbf{X}(w))$ (resp. $H^{2n+2}(\mathbf{X}(w))$). For $b = 1$ or $b = n$, applying Propositions [DMR, 3.3.14] and [DMR, 3.3.15] we see that the $(H^i(\mathbf{X}(w)))_{t^b}$ in the above exact sequences must be zero and the arrows $\mathrm{St}^{(n-1)} \to \mathrm{St}^{(n-1)}$ and $\mathrm{Id}^{(n-1)} \to \mathrm{Id}^{(n-1)}$ must be isomorphisms. This completes the proof of the theorem. ∎

Let us explain now why Theorem 9.2 implies Conjectures 2.4 to 2.6; Conjecture 2.6 is immediate. Let us show Conjecture 2.4. By Section 5, the centralizer $C_B(\mathbf{w})$ is cyclic, generated by $\mathbf{w}$, and $C_W(w) = G(1,1,n)$. The endomorphism $D_{\mathbf{w}}$ acts as F on $\mathbf{X}(\mathbf{w})$. Thus the value of the eigenvalues of F given in Theorem 9.2 shows that the representation $\mathbf{w} \mapsto D_{\mathbf{w}}$ of $B(1,1,n)$ on $\mathrm{End}_{\mathbf{G}^F}(\bigoplus_i H_c^i(\mathbf{X}(\mathbf{w})))$ factors through an n-cyclotomic Hecke algebra $\mathcal{H}(w)$ with parameters $(1, x^2, x^3, \ldots, x^{n-1}, x^n)$. To show Conjecture 2.5, it remains to see that the virtual representation $\sum_i (-1)^i H_c^i(\mathbf{X}(\mathbf{w}))$ of $\mathcal{H}(w)$ is special. Proceeding as in Section 4, it is enough to show that $|\mathbf{X}(\mathbf{w})^{F^i}| = 0$ for $i = 1, \ldots, n-1$. But this is exactly the statement [BMi, 5.2].

§10. Conjecture 2.4 in type A

We consider a group $\mathbf{G}$ of type A_{n-1} and a group $\mathbf{G}'$ of type A_n. We keep the notation of Section 5: W (resp. W') is the Weyl group of $\mathbf{G}$ (resp. $\mathbf{G}'$) and we consider $\mathbf{w} = \mathbf{c}^r$ (resp. $\mathbf{w}' = \mathbf{c}'^r$), a d-th root of $\boldsymbol{\pi}$ in the braid group B (resp. B') of W (resp. W'). We have $dr = n$ with $d \geq 2$.

In Section 5 we had two incarnations of the braid group $B(d,1,r)$: one as $C_B(\mathbf{w})$, with generators $\mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_{r-1}$ and one as $C_{B'}(\mathbf{w}')$ with generators $\mathbf{t}', \mathbf{s}_1, \ldots, \mathbf{s}_{r-1}$; in both cases these generators correspond to braid reflections of $B(d,1,r)$.

The group $G(d,1,r)$ has two orbits of reflecting hyperplanes corresponding to reflections of order d and 2 respectively, so that for a choice of indeterminates $\mathbf{u} = (u_{\mathbf{t},0}, \ldots, u_{\mathbf{t},d-1}; u_{\mathbf{s}_1,0}, u_{\mathbf{s}_1,1})$ the generic Hecke algebra $\mathcal{H}_\mathbf{u}$ of $G(d,1,r)$ is the quotient of $\overline{\mathbb{Q}}_\ell[\mathbf{u}]B(d,1,r)$ by the relations $(\mathbf{t} - u_{\mathbf{t},0}) \cdots (\mathbf{t} - u_{\mathbf{t},d-1}) = 0$ and $(\mathbf{s}_1 - u_{\mathbf{s}_1,0})(\mathbf{s}_1 - u_{\mathbf{s}_1,1}) = 0$; we will write the first relation as $(\mathbf{t}' - u_{\mathbf{t},0}) \cdots (\mathbf{t}' - u_{\mathbf{t},d-1}) = 0$ when considering the other incarnation of $B(d,1,r)$.

Next theorem proves that 2.4 holds.

Theorem 10.1.

- *The map* $\mathbf{x} \mapsto D_\mathbf{x}$ *from* $C_B(\mathbf{w})$ *to* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H^i_c(\mathbf{X}(\mathbf{w})))$ *factors through the specialization* $x \mapsto q$ *of a* d*-cyclotomic Hecke algebra* $\mathcal{H}$ *for* $B(d,1,r)$ *with parameters* $(1, x, x^2, \ldots, x^{d-1}; x^d, -1)$.
- *The map* $\mathbf{x} \mapsto D_\mathbf{x}$ *from* $C_{B'}(\mathbf{w}')$ *to* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H^i_c(\mathbf{X}(\mathbf{w}')))$ *factors through the specialization* $x \mapsto q$ *of a* d*-cyclotomic Hecke algebra* $\mathcal{H}'$ *for* $B(d,1,r)$ *with parameters* $(1, x^2, x^3, \ldots, x^{d-1}, x^{d+1}; x^d, -1)$.

With the notation of Section 5, we have to show that the operators induced on the cohomology by $D_{\mathbf{s}_i}$, $D_\mathbf{t}$ and $D_{\mathbf{t}'}$ satisfy the expected polynomial relations. The end of this section is devoted to the proof of this theorem.

The next lemma will allow us to compute by induction the relations satisfied by the $D_{\mathbf{s}_i}$, using [DMR, 5.2.9].

Lemma 10.2. *For* $i = 1, \ldots, r-1$ *let* $\mathbf{I}_i = \{\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+r}, \ldots, \boldsymbol{\sigma}_{i+(d-1)r}\}$ *as in Lemma* 5.4; *then we have* $\alpha_{\mathbf{I}_i}(\mathbf{w}) = \alpha_{\mathbf{I}_i}(\mathbf{w}') = \boldsymbol{\sigma}_i^2$.

Proof. As the elements of $\mathbf{I}_i$ commute pairwise and as, by Lemma 5.3 (iii), $\boldsymbol{\sigma}_i$ is the only divisor of $\mathbf{w}$ in $\mathbf{I}_i$, we have $\alpha_{\mathbf{I}_i}(\mathbf{w}) = \boldsymbol{\sigma}_i^k$ for some k. Now

$$\boldsymbol{\sigma}_i^k \preccurlyeq \mathbf{w} \Leftrightarrow \boldsymbol{\sigma}_i^{-k}\mathbf{w} \in B^+ \Leftrightarrow \mathbf{c}^{i-1}\boldsymbol{\sigma}_1^{-k}\mathbf{c}^{r-i+1} \in B^+.$$

But $\boldsymbol{\sigma}_1$ does not divide $\mathbf{c}^{i-1}$ on the right for $i < n$ by the "right-side version" of Lemma 5.3 (iv) so, by Lemma 5.2, $\boldsymbol{\sigma}_i^k \preccurlyeq \mathbf{w}$ if and only if $\boldsymbol{\sigma}_1^k \preccurlyeq \mathbf{c}^{r-i+1}$. To prove the assertion about $\alpha_{\mathbf{I}_i}(\mathbf{w})$ it is sufficient to show that $\boldsymbol{\sigma}_1^2 \preccurlyeq \mathbf{c}^2$ and $\boldsymbol{\sigma}_1^3 \not\preccurlyeq \mathbf{c}^r$. The former statement follows from Lemma 5.3 (ii) which gives $\boldsymbol{\sigma}_1^{-2}\mathbf{c}^2 = \boldsymbol{\sigma}_1^{-1}\mathbf{c}^2\boldsymbol{\sigma}_{n-1}^{-1} \in B^+$. To get the latter we write $\boldsymbol{\sigma}_1^{-3}\mathbf{c}^r = (\boldsymbol{\sigma}_1^{-2}\mathbf{c}^2)\boldsymbol{\sigma}_{n-1}^{-1}\mathbf{c}^{r-2}$ and, as $\boldsymbol{\sigma}_{n-1} \not\preccurlyeq \mathbf{c}^{r-2}$, we have to see by Lemma 5.2 that $\boldsymbol{\sigma}_1^{-2}\mathbf{c}^2 \not\succcurlyeq \boldsymbol{\sigma}_{n-1}$, which is equivalent to $\boldsymbol{\sigma}_1^{-2}\mathbf{c}^2\boldsymbol{\sigma}_{n-1}^{-1} = \boldsymbol{\sigma}_1^{-3}\mathbf{c}^2 \notin B^+$. But $\boldsymbol{\sigma}_1^3 \preccurlyeq \mathbf{c}^2$ is impossible by [Mi, 4.8] as $\nu(\boldsymbol{\sigma}_1^3) = 3$ and $\nu(\mathbf{c}^2) \leq 2$ (recall that $\nu(\mathbf{b}) = \inf\{k \in \mathbb{N} \mid \mathbf{b} \preccurlyeq \mathbf{w}_0^k\}$ for $\mathbf{b} \in B^+$).

The proof of the assertion about $\alpha_{\mathbf{I}_i}(\mathbf{w}')$ follows the same lines, using Lemma 5.3 (iv') instead of Lemma 5.3 (iv) and Lemma 5.3 (ii') instead of Lemma 5.3 (ii). At the end we have to see that $\boldsymbol{\sigma}_1^3 \not\preccurlyeq \mathbf{c}'^2$. But by Lemma 5.3 (v), we have $\mathbf{c}'^2 = \mathbf{c}^2\boldsymbol{\sigma}_{n-1}\boldsymbol{\sigma}_n$, which can be written $\mathbf{c}'^2 = (\mathbf{c}\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_{n-1})(\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n-1}\boldsymbol{\sigma}_n)$. The two factors are in $\mathbf{W}$, so that $\nu(\mathbf{c}'^2) = 2$, and we conclude as in the $\mathbf{w}$ case. ∎

Proposition 10.3. *For $i \in \{1, \ldots, r-1\}$, the image $T_{\mathbf{s}_i}$ of $D_{\mathbf{s}_i}$ in either $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_j H_c^j(\mathbf{X}(\mathbf{w})))$ or $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_j H_c^j(\mathbf{X}(\mathbf{w}')))$ satisfies $(T_{\mathbf{s}_i} + 1)(T_{\mathbf{s}_i} - q^d) = 0$.*

Proof. We prove the statement for $\mathbf{w}$, the proof for $\mathbf{w}'$ being exactly the same. By [DMR, 5.2.9], which can be applied by Lemma 5.4 (i), the $T_{\mathbf{s}_i}$ satisfy the same relations as the $D_{\mathbf{s}_i}$ on $\bigoplus_j H_c^j(\mathbf{X}_{\mathbf{L}_{\mathbf{I}_i}}(\alpha_{\mathbf{I}_i}(\mathbf{w}), \omega_{\mathbf{I}_i}(\mathbf{w})F))$. By the above lemma we have $\alpha_{\mathbf{I}_i}(\mathbf{w}) = \boldsymbol{\sigma}_i^2$. The group $\mathbf{L}_{\mathbf{I}_i}$ has type A_1^d where the components are permuted cyclically by $\omega_{\mathbf{I}_i}(\mathbf{w})F$. If $\mathbf{s}$ denotes the positive generator of the braid group of type A_1, through the isomorphism with A_1^d we have $\boldsymbol{\sigma}_i^2 \leftrightarrow (\mathbf{s}^2, 1, \ldots, 1)$, $\mathbf{s}_i \leftrightarrow (\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s})$ and $\omega_{\mathbf{I}_i}(\mathbf{w})F$ corresponds to $(x_1, \ldots, x_d) \mapsto ({}^F x_2, \ldots, {}^F x_d, {}^F x_1)$. The variety we have to study is $\mathbf{X}_{\mathbf{L}_{\mathbf{I}_i}}(\mathbf{s}^2, 1, \ldots, 1)$, which can be identified with the three-term sequences of d-tuples of Borel subgroups of $\mathbf{L}_{I_i}$ of the form

$$(\mathbf{B}_1, \ldots, \mathbf{B}_d) \xrightarrow{(s,1,\ldots,1)} (\mathbf{B}'_1, \ldots, \mathbf{B}'_d) \xrightarrow{(s,1,\ldots,1)} ({}^F\mathbf{B}_2, \ldots, {}^F\mathbf{B}_d, {}^F\mathbf{B}_1),$$

where for two Borel subgroups $\mathbf{B}$ and $\mathbf{B}'$, we write $\mathbf{B} \xrightarrow{v} \mathbf{B}'$ to say that $(\mathbf{B}, \mathbf{B}') \in \mathcal{O}(v)$ (we say that $\mathbf{B}$ and $\mathbf{B}'$ are in relative position v). The conditions on the relative positions imply $\mathbf{B}'_2 = \mathbf{B}_2, \ldots, \mathbf{B}'_d = \mathbf{B}_d$ and ${}^F\mathbf{B}_3 = \mathbf{B}_2, \ldots, {}^F\mathbf{B}_d = \mathbf{B}_{d-1}, {}^F\mathbf{B}_1 = \mathbf{B}_d$; so that $\mathbf{X}_{\mathbf{L}_{\mathbf{I}_i}}(\mathbf{s}^2, 1, \ldots, 1)$ identifies with the three-term sequences $\mathbf{B}_1 \xrightarrow{s} \mathbf{B}'_1 \xrightarrow{s} {}^F\mathbf{B}_2 = {}^{F^d}\mathbf{B}_1$, *i.e.*, to the variety $\mathbf{X}(\mathbf{s}^2)$ of a group of type A_1 with Frobenius endomorphism

F^d. Let us put $\mathbf{s}_{(i)} = (\mathbf{s}, 1, \ldots, 1, \mathbf{s}, 1, \ldots, 1)$ where the second $\mathbf{s}$ is at the place i with $i > 1$. In the same way as above we identify the variety $\mathbf{X}_{\mathbf{L}_{\mathbf{I}_i}}(\mathbf{s}_{(i)})$ with the variety $\mathbf{X}(\mathbf{s}^2)$ by identifying a sequence $(\mathbf{B}_1, \ldots, \mathbf{B}_d)$ such that $(\mathbf{B}_1, \ldots, \mathbf{B}_d) \xrightarrow{s_{(i)}} ({}^F\mathbf{B}_2, \ldots, {}^F\mathbf{B}_{d-1}, {}^F\mathbf{B}_d)$ with the three-term sequence $\mathbf{B}_1 \xrightarrow{s} {}^{F^{i-1}}\mathbf{B}_i \xrightarrow{s} {}^{F^d}\mathbf{B}_1$. We can decompose the morphism $D_{\mathbf{s},\ldots,\mathbf{s}}$ as $D_{\mathbf{s}^{(2)}} \circ \cdots \circ D_{\mathbf{s}^{(d)}} \circ D_{\mathbf{s}^{(1)}}$, where $\mathbf{s}^{(i)} = (1, \ldots, 1, \mathbf{s}, 1, \ldots, 1)$, with $\mathbf{s}$ at the ith place. Then $D_{\mathbf{s}^{(1)}}$ sends $\mathbf{X}(\mathbf{s}^2, 1, \ldots, 1)$ to $\mathbf{X}(\mathbf{s}_{(d)})$ and $D_{\mathbf{s}^{(i)}}$ sends $\mathbf{X}(\mathbf{s}_{(i)})$ to $\mathbf{X}(\mathbf{s}_{(i-1)})$ for $i > 1$. With the above identifications, one checks that $D_{\mathbf{s}^{(1)}}$ sends $\mathbf{B}_1 \xrightarrow{s} \mathbf{B}'_1 \xrightarrow{s} {}^{F^d}\mathbf{B}_1$ to $\mathbf{B}'_1 \xrightarrow{s} {}^{F^d}\mathbf{B}_1 \xrightarrow{s} {}^{F^d}\mathbf{B}'_1$ and that $D_{\mathbf{s}^{(i)}}$ sends $\mathbf{B}_1 \xrightarrow{s} {}^{F^{i-1}}\mathbf{B}_i \xrightarrow{s} {}^{F^d}\mathbf{B}_1$ to $\mathbf{B}_1 \xrightarrow{s} {}^{F^{i-2}}({}^F\mathbf{B}_i) \xrightarrow{s} {}^{F^d}\mathbf{B}_1$ for $i > 1$. So that $D_{(\mathbf{s},\ldots,\mathbf{s})}$ identifies with the operator $D_\mathbf{s}$ on the variety $\mathbf{X}(\mathbf{s}^2)$ of a group of type A_1 with Frobenius endomorphism F^d. This is a particular case of [DMR, 5.3.4], where it is proved that the operator $T_\mathbf{s}$ induced by $D_\mathbf{s}$ on $H^*_c(\mathbf{X}(\mathbf{s}^2))$ satisfies $(T_\mathbf{s} - q^d)(T_\mathbf{s} + 1) = 0$, whence the proposition. ∎

We will now prove that the operators induced on the cohomology of $\mathbf{X}(\mathbf{w})$ and $\mathbf{X}(\mathbf{w}')$ respectively by $D_\mathbf{t}$ and $D_{\mathbf{t}'}$ satisfy the claimed polynomial relation.

LEMMA 10.4. *Let* $\mathbf{I} = \{\boldsymbol{\sigma}_r, \ldots, \boldsymbol{\sigma}_{r+d-2}\}$ *and* $\mathbf{I}' = \{\boldsymbol{\sigma}_r, \ldots, \boldsymbol{\sigma}_{r+d-1}\}$, *as in Corollary* 5.8; *we have*

(i) $\alpha_\mathbf{I}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1}) = \mathbf{y}\mathbf{t}\mathbf{y}^{-1}$,

(i') $\alpha_{\mathbf{I}'}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1}) = \mathbf{y}'\mathbf{t}'\mathbf{y}'^{-1}$.

(ii) $\omega_\mathbf{I}(\mathbf{y}\mathbf{w}\mathbf{y}^{-1})$ *commutes with* $\boldsymbol{\sigma}_{r+i}$ *for* $0 \le i \le d-2$,

(ii') $\omega_{\mathbf{I}'}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1})$ *commutes with* $\boldsymbol{\sigma}_{r+i}$ *for* $0 \le i \le d-1$.

Proof. Let us prove (i). As $\mathbf{y}\mathbf{t}\mathbf{y}^{-1} = \boldsymbol{\sigma}_{r,r+d-2} \in B^+_I$ and $\mathbf{y}\mathbf{w}\mathbf{y}^{-1} = \mathbf{y}\mathbf{t}\mathbf{y}^{-1}\mathbf{c}^{r-1}\mathbf{x}_d \cdots \mathbf{x}_1$ by the remark which follows the proof of Lemma 5.5, it is enough to see that $\alpha_\mathbf{I}(\mathbf{c}^{r-1}\mathbf{x}_d \cdots \mathbf{x}_1) = 1$, *i.e.*, that for $i \in 0, \ldots, d-2$ we have $\boldsymbol{\sigma}_{r+i} \not\preccurlyeq \mathbf{c}^{r-1}\mathbf{x}_d \cdots \mathbf{x}_1$. By Lemma 5.2, this amounts to prove that for $i \in 1, \ldots, d-1$ we have $\boldsymbol{\sigma}_i \not\preccurlyeq \mathbf{x}_d \cdots \mathbf{x}_1$. But $\boldsymbol{\sigma}_i$ commutes with $\mathbf{x}_j$ for $j > i+1$, so that $\boldsymbol{\sigma}_i^{-1}\mathbf{x}_d \cdots \mathbf{x}_1 = \mathbf{x}_d \cdots \mathbf{x}_{i+2}\boldsymbol{\sigma}_i^{-1}\mathbf{x}_{i+1} \cdots \mathbf{x}_1$. As $\boldsymbol{\sigma}_i$ is not in the support of $\mathbf{x}_d \cdots \mathbf{x}_{i+2}$, we have $\mathbf{x}_d \cdots \mathbf{x}_{i+2} \not\succcurlyeq \boldsymbol{\sigma}_i$, so by Lemma 5.2 it is enough to see that $\boldsymbol{\sigma}_i \not\preccurlyeq \mathbf{x}_{i+1} \cdots \mathbf{x}_1$. But $\boldsymbol{\sigma}_i^{-1}\mathbf{x}_{i+1}\mathbf{x}_i = \mathbf{x}_{i+1}\mathbf{x}_i\boldsymbol{\sigma}_{i+r-1}^{-1}$: indeed we can write this equality as $\boldsymbol{\sigma}_{i+1,i+r-1}\boldsymbol{\sigma}_{i,i+r-1} = \boldsymbol{\sigma}_{i,i+r-1}\boldsymbol{\sigma}_{i,i+r-2}$, using Lemma 5.3 (i) in the parabolic subgroup generated by $\boldsymbol{\sigma}_i, \ldots, \boldsymbol{\sigma}_{i+r-1}$. So, as $\boldsymbol{\sigma}_{i+r-1} \not\preccurlyeq \mathbf{x}_{i-1} \cdots \mathbf{x}_1$ because $\boldsymbol{\sigma}_{i+r-1}$ is not in the support of this element, again by Lemma 5.2 it is sufficient to see that $\mathbf{x}_{i+1}\mathbf{x}_i \not\succcurlyeq \boldsymbol{\sigma}_{i+r-1}$. But this is the exact analogue in the parabolic subgroup generated by $\boldsymbol{\sigma}_i, \ldots, \boldsymbol{\sigma}_{i+r-1}$ of what we have proved at the end of Lemma 10.2, as $\boldsymbol{\sigma}_i^{-2}\boldsymbol{\sigma}_{i,i+r-1}^2 = \mathbf{x}_{i+1}\mathbf{x}_i$.

The proof of (i') is along the same lines: we start with $\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1} = \mathbf{y}'\mathbf{t}'\mathbf{y}'^{-1}\mathbf{c}_{rd}^{r-1}\mathbf{x}_{d+1}\cdots\mathbf{x}_1$ as noticed in the remark following Lemma 5.6. We have to see that $\alpha_{\mathbf{I}'}(\mathbf{c}_{rd}^{r-1}\mathbf{x}_{d+1}\cdots\mathbf{x}_1) = 1$, *i.e.*, that $\boldsymbol{\sigma}_{r+i} \not\preccurlyeq \mathbf{c}_{rd}^{r-1}\mathbf{x}_{d+1}\cdots\mathbf{x}_1$ for $i \in 0, \ldots, d-1$. By Lemma 5.2, this amounts to prove that for $1, \ldots, d$ we have $\boldsymbol{\sigma}_i \not\preccurlyeq \mathbf{x}_{d+1}\cdots\mathbf{x}_1$. But $\boldsymbol{\sigma}_i$ commutes with $\mathbf{x}_j$ for $j > i+1$, and as in the proof of (i) we are reduced to prove that $\boldsymbol{\sigma}_i \not\preccurlyeq \mathbf{x}_{i+1}\cdots\mathbf{x}_1$. Then we finish exactly as in (i).

Let us prove (ii). We have to see that $\boldsymbol{\sigma}_{r+i}\mathbf{c}^{r-1}\mathbf{x}_d\cdots\mathbf{x}_1 = \mathbf{c}^{r-1}\mathbf{x}_d\cdots\mathbf{x}_1$ $\boldsymbol{\sigma}_{r+i}$. This can be written

$$\mathbf{c}^{r-1}\mathbf{x}_d\cdots\mathbf{x}_{i+3}\boldsymbol{\sigma}_{i+1}\mathbf{x}_{i+2}\mathbf{x}_{i+1}\cdots\mathbf{x}_1 = \mathbf{c}^{r-1}\mathbf{x}_d\cdots\mathbf{x}_{i+2}\mathbf{x}_{i+1}\boldsymbol{\sigma}_{r+i}\mathbf{x}_i\cdots\mathbf{x}_1.$$

So we have to prove that $\boldsymbol{\sigma}_{i+1}\mathbf{x}_{i+2}\mathbf{x}_{i+1} = \mathbf{x}_{i+2}\mathbf{x}_{i+1}\boldsymbol{\sigma}_{r+i}$, *i.e.*, $\boldsymbol{\sigma}_{i+1,i+r}$ $\boldsymbol{\sigma}_{i+1,i+r-1} = \boldsymbol{\sigma}_{i+2,i+r}\boldsymbol{\sigma}_{i+1,r+i}$. This last equality is a consequence of Lemma 5.3 (i) applied in the parabolic subgroup generated by $\{\boldsymbol{\sigma}_{i+1}, \ldots, \boldsymbol{\sigma}_{i+r}\}$.

The proof of (ii') is exactly the same, replacing $\mathbf{I}$ by $\mathbf{I}'$, $\mathbf{c}$ by $\mathbf{c}_{rd}$ and $\mathbf{x}_d\cdots\mathbf{x}_1$ by $\mathbf{x}_{d+1}\cdots\mathbf{x}_1$. ∎

Corollary 10.5.

- *The image* $T_{\mathbf{t}}$ *of* $D_{\mathbf{t}}$ *in* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H_c^i(\mathbf{X}(\mathbf{w})))$ *satisfies* $(T_{\mathbf{t}}-1)(T_{\mathbf{t}}-q)\cdots(T_{\mathbf{t}}-q^{d-1}) = 0$.
- *The image* $T_{\mathbf{t}'}$ *of* $D_{\mathbf{t}'}$ *in* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H_c^i(\mathbf{X}(\mathbf{w}')))$ *satisfies* $(T_{\mathbf{t}'}-1)(T_{\mathbf{t}'}-q^2)(T_{\mathbf{t}'}-q^3)\cdots(T_{\mathbf{t}'}-q^{d-1})(T_{\mathbf{t}'}-q^{d+1}) = 0$.

Proof. We prove first the result for $\mathbf{t}$. Using conjugation by $D_{\mathbf{y}}$ we see that it is equivalent to prove that the image of $D_{\mathbf{yty}^{-1}}$ in $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H_c^i(\mathbf{X}(\mathbf{ywy}^{-1})))$ satisfies the same relation. By [DMR, 5.2.9] and Lemma 10.4 this operator satisfies the same relations as $D_{\mathbf{yty}^{-1}}$ on $\bigoplus_i H_c^i(\mathbf{X}_{\mathbf{L}_I}(\alpha_{\mathbf{I}}(\mathbf{ywy}^{-1}), \omega_{\mathbf{I}}(\mathbf{ywy}^{-1})F)) = \bigoplus_i H_c^i(\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\sigma}_{r,r+d-2}))$. The element $\boldsymbol{\sigma}_{r,r+d-2}$ is the lift of a Coxeter element in the braid group of $W_{\mathbf{I}}$. The group $\mathbf{L}_I$ has type split A_{d-1}, and $D_{\mathbf{yty}^{-1}}$ acts as F on $\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\sigma}_{r,r+d-2})$. The eigenvalues of F in the case of a Coxeter element are given in [Lu2] and gives the relation we want for $T_{\mathbf{t}}$.

We follow the same lines for proving the assertion on $\mathbf{t}'$. We are reduced to compute the relation satisfied by $D_{\mathbf{y}'\mathbf{t}'\mathbf{y}'^{-1}}$ on $\bigoplus_i H_c^i(\mathbf{X}_{\mathbf{L}_{I'}}(\alpha_{\mathbf{I}'}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1}), \omega_{\mathbf{I}'}(\mathbf{y}'\mathbf{w}'\mathbf{y}'^{-1})F)) = \bigoplus_i H_c^i(\mathbf{X}_{\mathbf{L}_{I'}}(\boldsymbol{\sigma}_{r,r+d-1}\boldsymbol{\sigma}_{r+d-1}))$. The element $\boldsymbol{\sigma}_{r,r+d-1}$

$\boldsymbol{\sigma}_{r+d-1}$ relatively to the group $\mathbf{L}_{I'}$ is an element as studied in Section 9. The group $\mathbf{L}_{I'}$ is split of type A_d and $D_{\mathbf{y}'\mathbf{t}'\mathbf{y}'^{-1}}$ acts as F on $\mathbf{X}_{\mathbf{L}_{I'}}(\boldsymbol{\sigma}_{r,r+d-1}\,\boldsymbol{\sigma}_{r+d-1})$. We end the proof by noticing, as in the end of Section 9, that the eigenvalues of F given in Theorem 9.2 give the relation we want for $T_{\mathbf{t}'}$. ∎

§11. Conjecture 2.4 in type B

In this section we keep the notation of Section 6; we prove Conjecture 2.4 in a group $\mathbf{G}$ of type B_n for a d-regular element with d even. So we have $\mathbf{w} = \mathbf{c}^r$ with $dr = 2n$ and d even.

THEOREM 11.1. *The map* $\mathbf{x} \mapsto D_\mathbf{x}$ *from* $C_B(\mathbf{w})$ *to* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H^i_c(\mathbf{X}(\mathbf{w})))$ *factors through the specialization* $x \mapsto q$ *of a* d*-cyclotomic Hecke algebra* $\mathcal{H}$ *for* $B(d,1,r)$ *with parameters*

$$(1, x, x^2, \dots, x^{d/2-1}, x^{d/2}, -x, -x^2, \dots, -x^{d/2-1}; x^{d/2}, -1).$$

With the notation of Section 6, we have to show that the operators induced on the cohomology by $D_{\mathbf{s}_i}$ and $D_\mathbf{t}$ satisfy the expected polynomial relations.

The end of this section is devoted to the proof of this theorem. The next lemma is the analogue of Lemma 10.2.

LEMMA 11.2. *For* $i = 2, \dots, r$ *let* $\mathbf{I}_i = \{\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+r}, \dots, \boldsymbol{\sigma}_{i+(d/2-1)r}\}$ *as in Lemma* 6.4; *then we have* $\alpha_{\mathbf{I}_i}(\mathbf{w}) = \boldsymbol{\sigma}_i$.

Proof. As the elements of $\mathbf{I}_i$ commute pairwise and as, by Lemma 6.1 (iv), $\boldsymbol{\sigma}_i$ is the only divisor of $\mathbf{w}$ in $\mathbf{I}_i$, we have $\alpha_{\mathbf{I}_i}(\mathbf{w}) = \boldsymbol{\sigma}_i^k$ for some k. But $\mathbf{c}$ is a good root of $\boldsymbol{\pi}$, so for any $r \le n$ we have $\nu(\mathbf{c}^r) = 1$. As $\nu(\boldsymbol{\sigma}_i^2) = 2$ we cannot have $\boldsymbol{\sigma}_i^2 \preccurlyeq \mathbf{w}$, whence the result. ∎

COROLLARY 11.3. *For* $i \in \{1, \dots, r-1\}$ *the image* $T_{\mathbf{s}_i}$ *of* $D_{\mathbf{s}_i}$ *in* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_j H^j_c(\mathbf{X}(\mathbf{w})))$ *satisfies* $(T_{\mathbf{s}_i}+1)(T_{\mathbf{s}_i} - q^{d/2}) = 0$.

Proof. The proof is similar to that of Proposition 10.3, except that here $D_{\mathbf{s}_i}$ eventually identifies with the operator $D_\mathbf{s}$ on the variety $\mathbf{X}(\mathbf{s})$ for a group of type A_1 with Frobenius endomorphism $F^{d/2}$, whence the result. ∎

LEMMA 11.4. *Let* $\mathbf{I} = \{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{d/2}\}$ *be as at the end of Section* 6; *we have*

(i) $\alpha_{\mathbf{I}}(\mathbf{ywy}^{-1}) = \mathbf{yty}^{-1}$.
(ii) $\omega_{\mathbf{I}}(\mathbf{ywy}^{-1})$ *commutes with* $\boldsymbol{\sigma}_i$ *for* $1 \le i \le d/2$.

Proof. Let us prove (i). As $\mathbf{yty}^{-1} = \boldsymbol{\sigma}_{1,d/2}$ and $\mathbf{ywy}^{-1} = \mathbf{yty}^{-1} \prod_{i=d/2}^{1} \mathbf{x}_i \mathbf{c}^{r-1}$ by the remark which follows the proof of Lemma 6.5, it is enough to see that $\boldsymbol{\sigma}_i \not\preccurlyeq \prod_{i=d/2}^{1} \mathbf{x}_i \mathbf{c}^{r-1}$ for $i = 0, \ldots, d/2$. We use the following lemma which is an immediate consequence of Lemma 5.2:

LEMMA 11.5. *Assume that* $\mathbf{a}, \mathbf{b} \in B^+$*, that* $\boldsymbol{\sigma}_i \preccurlyeq \mathbf{ab}$*, that* $\boldsymbol{\sigma}_i \mathbf{a} = \mathbf{a} \boldsymbol{\sigma}_j$ *and that* $\boldsymbol{\sigma}_i \not\preccurlyeq \mathbf{a}$*; then* $\boldsymbol{\sigma}_j \preccurlyeq \mathbf{b}$.

By this lemma, for $i > 1$ we have $\boldsymbol{\sigma}_i \preccurlyeq \prod_{j=d/2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \Leftrightarrow \boldsymbol{\sigma}_i \preccurlyeq \prod_{j=i}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \Leftrightarrow \boldsymbol{\sigma}_{i+r-1} \preccurlyeq \prod_{j=i-2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \Leftrightarrow \boldsymbol{\sigma}_{i+r-1} \preccurlyeq \mathbf{c}^{r-1}$, the second equivalence as, by the proof of Lemma 10.4, we have $\boldsymbol{\sigma}_i^{-1} \mathbf{x}_i \mathbf{x}_{i-1} = \mathbf{x}_i \mathbf{x}_{i-1} \boldsymbol{\sigma}_{i+r-1}^{-1}$. But $\boldsymbol{\sigma}_{i+r-1} \preccurlyeq \mathbf{c}^{r-1}$ is false by Lemma 6.1 (iv).

For $i = 1$ we get $\boldsymbol{\sigma}_1 \preccurlyeq \prod_{j=d/2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \Leftrightarrow \boldsymbol{\sigma}_1 \preccurlyeq \mathbf{x}_1 \mathbf{c}^{r-1} \Leftrightarrow \boldsymbol{\sigma}_1^2 \preccurlyeq \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_r \mathbf{c}^{r-1} \Rightarrow \boldsymbol{\sigma}_1^2 \preccurlyeq \mathbf{c}^r$, the last equivalence as $\boldsymbol{\sigma}_{r+1} \cdots \boldsymbol{\sigma}_n \mathbf{c}^{r-1} = \mathbf{c}^{r-1} \boldsymbol{\sigma}_2 \cdots \boldsymbol{\sigma}_{n-r+1}$. But $\boldsymbol{\sigma}_1^2 \preccurlyeq \mathbf{c}^r$ is impossible as $\nu(\mathbf{c}^r) = 1$.

Let us prove (ii). By (i) we have $\omega_{\mathbf{I}}(\mathbf{ywy}^{-1}) = \prod_{i=d/2}^{1} \mathbf{x}_i \mathbf{c}^{r-1}$. For $i > 1$, we have

$$\begin{aligned}\prod_{j=d/2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \boldsymbol{\sigma}_i &= \prod_{j=d/2}^{1} \mathbf{x}_j \boldsymbol{\sigma}_{i+r-1} \mathbf{c}^{r-1} = \prod_{j=d/2}^{i+1} \mathbf{x}_i \mathbf{x}_{i-1} \boldsymbol{\sigma}_{i+r-1} \prod_{j=i-2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} \\ &= \prod_{j=d/2}^{i+1} \boldsymbol{\sigma}_i \mathbf{x}_i \mathbf{x}_{i-1} \prod_{j=i-2}^{1} \mathbf{x}_j \mathbf{c}^{r-1} = \boldsymbol{\sigma}_i \prod_{j=d/2}^{1} \mathbf{x}_j \mathbf{c}^{r-1}.\end{aligned}$$

For $i = 1$, we prove by induction on i that $\boldsymbol{\sigma}_2 \cdots \boldsymbol{\sigma}_{i+1} \mathbf{c}^i \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_{i+1} \mathbf{c}^i$; this equality for $i = r-1$ gives $\mathbf{x}_1 \mathbf{c}^{r-1} \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \mathbf{x}_1 \mathbf{c}^{r-1}$; then we use that $\boldsymbol{\sigma}_1$ commutes with $\mathbf{x}_i$ for $i > 1$. ∎

As for Corollary 10.5, we deduce the following corollary which ends the proof of the theorem.

COROLLARY 11.6. *Let* $T_{\mathbf{t}}$ *be the image of* $D_{\mathbf{t}}$ *in* $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H_c^i(\mathbf{X}(\mathbf{w})))$*; then we have* $(T_{\mathbf{t}} - 1)(T_{\mathbf{t}} - q) \cdots (T_{\mathbf{t}} - q^{d/2})(T_{\mathbf{t}} + q)(T_{\mathbf{t}} + q^2) \cdots (T_{\mathbf{t}} + q^{d/2-1}) = 0$.

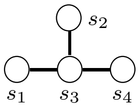
Proof. Using conjugation by $D_{\mathbf{y}}$, we see that it is equivalent to prove that the image of $D_{\mathbf{yty}^{-1}}$ in $\operatorname{End}_{\mathbf{G}^F}(\bigoplus_i H^i_c(\mathbf{X}(\mathbf{ywy}^{-1})))$ satisfies the same relation. By [DMR, 5.2.9] and Lemma 11.4 this operator satisfies the same relation as $D_{\mathbf{yty}^{-1}}$ on

$$\bigoplus_i H^i_c(\mathbf{X}_{\mathbf{L}_I}(\alpha_{\mathbf{I}}(\mathbf{ywy}^{-1}),\omega_{\mathbf{I}}(\mathbf{ywy}^{-1})F)) = \bigoplus_i H^i_c(\mathbf{X}_{\mathbf{L}_I}(\textstyle\prod_{i=1}^{d/2}\boldsymbol{\sigma}_i)).$$

The element $\prod_{i=1}^{d/2}\boldsymbol{\sigma}_i$ is the lift of a Coxeter element in the braid group of $W_{\mathbf{I}}$. The group $\mathbf{L}_I$ has type $B_{d/2}$ and $D_{\mathbf{yty}^{-1}}$ acts as F on $\mathbf{X}_{\mathbf{L}_I}(\prod_{i=1}^{d/2}\boldsymbol{\sigma}_i)$. The eigenvalues of F in the case of a Coxeter element are given in [Lu2] and give the relation which we want for $T_{\mathbf{t}}$. ∎

§12. The 4-th roots of $\boldsymbol{\pi}$ in D_4

We use the same numbering for the reflections of $W = W(D_4)$ as in Bourbaki: we will study the 4-th root of $\boldsymbol{\pi}$ given by $\mathbf{w} = \mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3$.



The centralizer of w is the complex reflection group $G(4,2,2)$. We show Conjectures 2.1 to 2.6 for w.

Proposition 12.1. *The element* $\mathbf{w}$ *verifies Conjectures* 2.1 *and* 2.2. *In particular, the map which sends the standard generators of the braid group* $B(4,2,2)$ *to* $\mathbf{b}_1 = (\mathbf{s}_1\mathbf{s}_2)^{\mathbf{s}_3}$, $\mathbf{b}_2 = \mathbf{s}_1\mathbf{s}_4$ *and* $\mathbf{b}_3 = (\mathbf{s}_2\mathbf{s}_4)^{\mathbf{s}_3\mathbf{s}_4}$ *is an isomorphism* $B(4,2,2) \simeq C_B(\mathbf{w})$.

Proof. A 4-th root of π is an element of length 6 whose image is a 4-regular element. As all twelve 4-regular elements of W are of length 6, all 4-th roots of $\boldsymbol{\pi}$ are in $\mathbf{W}$. There are thus 12 good roots of $\boldsymbol{\pi}$, and a direct check shows that there are morphisms in $\mathcal{D}^+$ between any two of them, thus proving Conjecture 2.2 in this case. We may show Conjecture 2.1 using the programs written by Franco and Gonzalez-Meneses to compute centralizers in braid groups. They give that $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ generate $C_B(\mathbf{w})$. However, one sees that the elements $\mathbf{wb}_1 = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_4\mathbf{s}_1\mathbf{s}_3$, $\mathbf{b}_2$ and $\mathbf{wb}_3 = \mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_3$ are in $\operatorname{End}_{\mathcal{D}^+}(\mathbf{w})$ by using the decompositions $\mathbf{wb}_1 = (\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1)(\mathbf{s}_2\mathbf{s}_4)(\mathbf{s}_1\mathbf{s}_3)$, $\mathbf{b}_2 = (\mathbf{s}_1)(\mathbf{s}_4)$ and $\mathbf{wb}_3 = (\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1)(\mathbf{s}_4)(\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4)(\mathbf{s}_3)$. We get thus that $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ are in $\operatorname{End}_{\mathcal{D}}(\mathbf{w})$ whence Conjecture 2.1.

Let us show that the elements $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are the image of braid reflections around hyperplanes of $C_W(w)$. We start with $\mathbf{b}_2$; let V be the reflection representation of W, and let V_i be the eigenspace of w in V for the eigenvalue i. This is a 2-dimensional space, and the set of fixed points of s_1s_4 on this space has dimension 1 and is of the form $H\cap V_i$ for some reflecting hyperplane of W. We use now the constructions at the end of Section 3: the group B_H is the parabolic subgroup of B generated by $\mathbf{s}_1$ and $\mathbf{s}_4$, and the element $\mathbf{s}_H$ is a square root of the element $\boldsymbol{\pi}$ of this parabolic subgroup, so is equal to $\mathbf{s}_1\mathbf{s}_4$; thus this element is indeed the image in B of the braid reflection $\mathbf{t}_H$ of $B(w)$. To handle the other elements we use the triality automorphism of $\mathbf{G}$. This automorphism corresponds to an automorphism of V which induces on B the automorphism given by $\mathbf{s}_2 \mapsto \mathbf{s}_1 \mapsto \mathbf{s}_4 \mapsto \mathbf{s}_2$ and the corresponding automorphism on W. Let $\boldsymbol{\psi}$ be the automorphism of B which is the triality followed by conjugation by $\mathbf{s}_2\mathbf{s}_3$ and let ψ be the corresponding automorphism of V. Then $\boldsymbol{\psi}$ fixes $\mathbf{w}$ and does the permutation $\mathbf{b}_1 \mapsto \mathbf{b}_2 \mapsto \mathbf{b}_3 \mapsto \mathbf{b}_1$. Thus ψ induces an automorphism of V_i, and the braid reflections which are images of $\mathbf{t}_H$ by the powers of this automorphism have images $\mathbf{b}_1$ and $\mathbf{b}_3$.

The fact that the images of these braid reflections generate $C_B(\mathbf{w})$ implies that they generate $B(w)$; thus Conjecture 2.3 holds. They satisfy the defining braid relations for the braid group of $G(4,2,2)$, that is $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 = \mathbf{b}_2\mathbf{b}_3\mathbf{b}_1 = \mathbf{b}_3\mathbf{b}_1\mathbf{b}_2$ (all three products are easily checked to be equal to $\mathbf{w}$). ∎

The next proposition shows Conjecture 2.4.

PROPOSITION 12.2. *The map $\mathbf{b}_i \mapsto D_{\mathbf{b}_i}$ factors through a representation $\mathcal{H}_q(w) \to \mathrm{End}(\bigoplus_i H^i_c(\mathbf{X}(w)))$ of the specialization $x \mapsto q$ of a 4-cyclotomic Hecke algebra $\mathcal{H}(w)$ for $G(4,2,2)$ with parameters 1 and x^2.*

We note that the above algebra is indeed a 4-cyclotomic algebra since the parameters specialize to 1 and -1 by the specialization $x \mapsto i$.

Proof. It is enough to prove the quadratic relations $(D_{\mathbf{b}_i}-1)(D_{\mathbf{b}_i}-q^2)=0$.

For $\mathbf{b}_2 = \mathbf{s}_1\mathbf{s}_4$, we use proposition [DMR, 5.2.8] with $\mathbf{x}=\mathbf{y}=\mathbf{s}_1$ and $I=\{s_1,s_4\}$. The subgroup $\mathbf{L}_I$ is of type $A_1\times A_1$; The element $z=s_1^{-1}w$ normalises $\mathbf{L}_I$, exchanging the two A_1 components. The variety $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1,\dot zF)$ is mapped by $D_{\mathbf{s}_1}$, which acts as $\dot zF$, to $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_4,\dot zF)$. Then $D_{\mathbf{s}_4}$ maps

$\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_4, \dot{z}F)$ to $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot{z}F)$, and $D_{\mathbf{s}_4} \circ D_{\mathbf{s}_1}$ induces F^2 on $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot{z}F)$; by *e.g.*, [DMR, 3.3.16] this variety has two non-zero cohomology groups with compact support: H^1_c on which F^2 acts as 1 and H^2_c on which F^2 acts as q^2. Thus $D_{\mathbf{b}_2}$ satisfies the quadratic relation $(D_{\mathbf{b}_2} - 1)(D_{\mathbf{b}_2} - q^2) = 0$ on the cohomology of $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot{z}F)$, and thus by [DMR, 2.3.13] and the Künneth formula, it also satisfies the same relation on the cohomology of $\mathbf{X}(w)$.

To show that $D_{\mathbf{b}_3}$ satisfies the same quadratic relation we use [DMR, 3.1.8] applied to the triality automorphism of $\mathbf{G}$. The triality maps $\mathbf{w}$ to $\mathbf{s}_1\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3$ and $D_{\mathbf{b}_2}$ to $D_{\mathbf{s}_2\mathbf{s}_4}$ acting on $\mathbf{X}(\mathbf{s}_1\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3)$ which by the same argument as above, with $\mathbf{x} = \mathbf{y} = \mathbf{s}_4$ and $I = \{s_2, s_4\}$, satisfies the expected quadratic relation. The conjugation by $\mathbf{s}_2\mathbf{s}_3 \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{w}, \mathbf{s}_1\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3)$ composed with the triality is equal to $\boldsymbol{\psi}$; and the conjugation by $\mathbf{s}_2\mathbf{s}_3$ maps back $D_{\mathbf{s}_2\mathbf{s}_4}$ to $D_{\mathbf{b}_3}$ which thus satisfies the quadratic relation.

One proceeds similarly for $D_{\mathbf{b}_1}$, using the square of the triality automorphism which maps $D_{\mathbf{s}_1\mathbf{s}_4}$ to $D_{\mathbf{s}_2\mathbf{s}_1}$ acting on $\mathbf{X}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3)$. One has that $\boldsymbol{\psi}^2$ is the square of triality followed by conjugation by $\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3$, and conjugating by $\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3 \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{w}, \mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3)$ maps $D_{\mathbf{s}_2\mathbf{s}_1}$ to $D_{\mathbf{b}_1}$. ∎

If we denote by T_1, T_2, T_3 the images in $\mathcal{H}_q(w)$ of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, we thus get a representation ρ of $\mathcal{H}_q(w)$ on $\bigoplus H^i_c(\mathbf{X}(w))$ which maps T_i to $D_{\mathbf{b}_i}$. We note that triality commutes with the automorphism of $\mathcal{H}_q(w)$ given by $T_1 \mapsto T_2 \mapsto T_3 \mapsto T_1$.

The next proposition and Theorem 12.4 show that Conjecture 2.5 holds.

Proposition 12.3. *The representation induced by* ρ *on* $\sum(-1)^i H^i_c(\mathbf{X}(w))$ *is special.*

Proof. Let $\Phi = T_1T_2T_3$. It follows from the computation of the generic degrees and the character table of $\mathcal{H}_q(w)$ in [BMa, 5.A] that a representation is special if (and only if) the elements $\{(T_i)_{i=1,2,3}, (T_iT_j)_{i=1,2,3,\, j=1+(i \bmod 3)}, (\Phi^i)_{i=1,2,3}\}$ have trace zero on it. The quadratic relations show that $\mathcal{B} = \{(T_i)_{i=1,2,3}, (T_i\Phi)_{i=1,2,3}, (\Phi^i)_{i=1,2,3}\}$ generates the same subspace of $\mathcal{H}_q(w)$ as these elements. We shall show that the trace of any element in $\mathcal{B}$ vanishes on $\sum(-1)^i H^i_c(\mathbf{X}(w))$. It is enough to check this on just one element in each orbit of $\mathcal{B}$ under triality, *e.g.*, for $T_2, T_1\Phi, \Phi^i$. We have $\rho(\Phi) = D_{\mathbf{w}} = F$. Thus $D_{\mathbf{w}}$ as well as its square and cube verify the trace formula, and by [DMR, 5.2.3] they all have trace zero on $\sum(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$. Thus it only remains to compute the traces of $D_{\mathbf{b}_2}$ and $D_{\mathbf{b}_1\mathbf{w}}$.

To compute the trace of $D_{\mathbf{b}_2} = D_{\mathbf{s}_1\mathbf{s}_4}$, we apply [DMR, 5.2.10] with $g = 1$, $I = \{s_1, s_4\}$ and $\mathbf{x} = \mathbf{s}_1\mathbf{s}_4$, which gives that the trace of $D_{\mathbf{s}_1\mathbf{s}_4}$ on

$\sum(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$ is the value at 1 of the class function

$$R^{\mathbf{G}^F}_{\mathbf{L}_I^{\dot z F}}\left(l \mapsto \sum_i (-1)^i \operatorname{Trace}(l D_{\mathbf{s}_1\mathbf{s}_4} \mid H^i_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot z F), \overline{\mathbb{Q}}_\ell))\right),$$

with z as in 12.2. By a general result on Lusztig induction (see *e.g.*, [DM, 12.17]), this value is a multiple of

$$\sum_i (-1)^i \operatorname{Trace}(D_{\mathbf{s}_1\mathbf{s}_4} \mid H^i_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot z F), \overline{\mathbb{Q}}_\ell)).$$

But this last trace is zero since $D_{\mathbf{s}_1\mathbf{s}_4}$ acts as F^2 on $\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_1, \dot z F)$, thus has no fixed points.

Using the square of the triality automorphism as in the proof of 12.2, we see that the trace of $D_{\mathbf{b}_1\mathbf{w}}$ on $\sum(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$ is equal to the trace of $D_{\mathbf{s}_2\mathbf{s}_1}F$ on $\sum(-1)^i H^i_c(\mathbf{X}(\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3\mathbf{s}_4))$. By [DMR, 2.3.13] and [DMR, 5.2.9] applied with $I = \{s_1, s_2\}$ this trace is equal to

$$\begin{aligned}|\mathbf{L}_I^{\dot z F}|^{-1} \sum_{l \in \mathbf{L}_I^{\dot z F}} \sum_i (-1)^i \operatorname{Trace}(l D_{\mathbf{s}_2\mathbf{s}_1} F \mid H^i_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{s}_2, \dot z F), \overline{\mathbb{Q}}_\ell))\\ \times \sum_i (-1)^i \operatorname{Trace}(lF \mid H^i_c(\tilde{\mathbf{X}}_{(I)}(\dot z), \overline{\mathbb{Q}}_\ell))\end{aligned}$$

where $z = s_3s_2s_1s_3s_4$ and where $\tilde{\mathbf{X}}_{(I)}(\dot z) = \{g\mathbf{U}_I \mid g^{-1}F(g) \in \mathbf{U}_I \dot z F(\mathbf{U}_I)\}$ (the action of F decomposes as a product since $\mathbf{U}_I$ is F-stable). By the Lefschetz trace formula one has $\sum_i (-1)^i \operatorname{Trace}(lF \mid H^i_c(\tilde{\mathbf{X}}_{(I)}(\dot z), \overline{\mathbb{Q}}_\ell)) = |\tilde{\mathbf{X}}_{(I)}(\dot z)^{lF}| = |\{g\mathbf{U}_I \mid g^{-1}{}^F g \in (\mathbf{U}_I \dot z \mathbf{U}_I) \cap (l\mathbf{U}_I)\}|$. But this last intersection is empty since $z \notin W_I$ so that $\mathbf{P}_I$ does not intersect $\mathbf{U}_I \dot z \mathbf{U}_I$. Thus the trace vanishes. ∎

We shall now prove Conjecture 2.6 by giving a full description of $H^*_c(\mathbf{X}(\mathbf{w}))$ as a $\mathbf{G}^F \times \langle F \rangle$-module. To give the result, we first introduce a notation for the irreducible characters of $W(D_4)$: they are parametrized by the pairs of partitions of total sum 4, except that a pair of equal partitions corresponds to 2 characters. We shall thus denote the characters by

$$1^2+,\ 1^2-,\ 1.1^3,\ 1^4,\ 1^2.2,\ 1.21,\ 21^2,\ 2+,\ 2-,\ 2^2,\ 1.3,\ 31,\ 4$$

where a missing "." means that one of the partitions is empty. If λ is a parameter as above, we shall denote by γ_λ the corresponding unipotent character of the principal series, except that we denote respectively by St and Id the characters γ_{1^4} and γ_4. There is one more unipotent character of D_4, a cuspidal one, that we shall denote by θ. We will use the same convention as in Theorem 9.2 to describe the cohomology by a two variable polynomial. With these notation, we have

THEOREM 12.4. *The cohomology of* $\mathbf{X}(\mathbf{w})$ *is given by*

$$h^6 \operatorname{St} + t^2h^7(\gamma_{1^2+} + \gamma_{1^2-} + \gamma_{21^2}) + 2t^3h^7\theta + 2t^3h^8\gamma_{1.21} \\ + t^4h^9(\gamma_{2+} + \gamma_{2-} + \gamma_{31}) + t^6h^{12} \operatorname{Id}.$$

Proof. We will use the parabolic subgroup of type A_3 generated by $I = \{s_1, s_3, s_4\}$. Let $w' = s_3s_1s_3s_4s_3$. We will need the value of the sets defined in Lemma 8.11: they are $E_{W_I}(w') = \{1, s_1, s_3, s_4\}$ and $E_W(w) = E_{W_I}(w') \cup \{s_2s_3\}$. The value for w' is obtained by a direct computation in the Hecke algebra of W_I and the value for $w = s_2w'$ comes from Proposition 8.12 whose assumptions are easily checked.

We first compute the cohomology of $\mathbf{X}(\mathbf{w'}^{\mathbf{w}_0^I}) = \mathbf{X}(\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3)$ in the Levi subgroup $\mathbf{L}_I$ corresponding to I. We use [DMR, 3.2.10]; with the notation of the proof of Theorem 9.2 (ρ_2 corresponds to the partition $2, 2$, γ_2 to $1, 1, 2$, and γ_3 to $1, 3$) we know by [Lu2] the cohomology of the Coxeter variety $\mathbf{X}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1)$:

$$\sum_i h^i \cdot H^i_c(\mathbf{X}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1)) = h^3 \operatorname{St} + h^4t\gamma_2 + h^5t^2\gamma_3 + h^6t^3 \operatorname{Id}$$

and since $\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1$ is a 3-rd root of $\boldsymbol{\pi}_I$ we know its cohomology by Theorem 9.2

$$\sum_i h^i \cdot H^i_c(\mathbf{X}(\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1)) = h^4 \operatorname{St} + t^2h^5\rho_2 + h^8t^4 \operatorname{Id}.$$

The exact sequence given by [DMR, 3.2.10] then completely determines $H^i_c(\mathbf{X}(\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1\mathbf{s}_3))$, except for the Id-isotypic and St-isotypic parts; we get these by [DMR, 3.3.14] and [DMR, 3.3.15], and we obtain

$$\sum_i h^i \cdot H^i_c(\mathbf{X}(w')) = h^5 \operatorname{St} + t^2h^6(\gamma_2 + \rho_2) + t^3h^7(\gamma_3 + \rho_2) + t^5h^{10} \operatorname{Id}.$$

We now determine the principal series part of the cohomology of $\mathbf{X}(\mathbf{w})$ by the same method as in the proof of Theorem 9.2. The value of $E_W(w)$ shows that

$$\mathbf{X}(w) = \mathbf{X}^{w_0}(w) \coprod \mathbf{X}^{w_0s_1}(w) \coprod \mathbf{X}^{w_0s_3}(w) \coprod \mathbf{X}^{w_0s_4}(w) \coprod \mathbf{X}^{w_0s_2s_3}(w).$$

We may apply Proposition 8.17 with $s = s_2$. If we set

$$\mathbf{Y} = \Big(\mathbf{X}^{w_0}(w) \coprod \mathbf{X}^{w_0 s_1}(w) \coprod \mathbf{X}^{w_0 s_3}(w) \coprod \mathbf{X}^{w_0 s_4}(w)\Big)/\mathbf{U}_{\mathbf{P}_I}^F,$$

we have thus an equality of $\mathbf{L}_I^F$-modules:

$$\begin{aligned}\sum_i h^i \cdot H_c^i(\mathbf{Y}) &= (th^2 + h) \times \Big(\sum_i h^i \cdot H_c^i(\mathbf{X}(w_0^I w' w_0^I))\Big) \\ &= h^6 \operatorname{St} + h^7(t \operatorname{St} + t^2(\gamma_2 + \rho_2)) + h^8 t^3(\gamma_2 + \gamma_3 + 2\rho_2) \\ &\qquad + h^9 t^4(\gamma_3 + \rho_2) + t^5 h^{11} \operatorname{Id} + t^6 h^{12} \operatorname{Id}.\end{aligned}$$

To study the remaining piece $\mathbf{X}^{w_0 s_2 s_3}(w)$ we use Proposition 8.25 with $v = w_0 s_2 s_3$ and taking for w' the element $w'' = s_2 s_3 s_1 s_4 s_3$. We show that w and w'' satisfy the assumptions of Proposition 8.25.

We first check the assumptions of 8.21 (iii) for w and w''. We check on $E_W(w)$ that v is the only element of $W_I v$ such that $\mathbf{X}^v(w) \neq \emptyset$. On the other hand, one gets by Proposition 8.12 whose assumptions are easily checked that $E_W(w'') = E_{W_I}(s_3 s_1 s_4 s_3) \cup \{s_2 s_3, s_2 s_3 s_1 s_4\}$, where $E_{W_I}(s_3 s_1 s_4 s_3) = \{1, s_1, s_3, s_4, s_1 s_4, s_3 s_1 s_4\}$, which results from a direct computation in the Hecke algebra of W_I. Thus v is indeed the only element of $W_I v$ such that $\mathbf{X}^v(w'') \neq \emptyset$.

We check now the assumption 8.21 (iv) for w and w''. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots of D_4, we have $v^{-1}(\{\alpha_1, \alpha_3, \alpha_4\}) = \{-\alpha_1 - \alpha_3, -\alpha_2, -\alpha_3 - \alpha_4\}$, thus ${}^{v^{-1}}\mathbf{U}_I \subset \mathbf{U}^-$. The projection onto ${}^{v^{-1}}\mathbf{U}_I$ of ${}^{v^{-1}}\mathbf{U} \cap (\mathbf{B}w\mathbf{B} \coprod \mathbf{B}w''\mathbf{B})$ is the same as that of ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^- \cap (\mathbf{B}w\mathbf{B} \coprod \mathbf{B}w''\mathbf{B})$; indeed each double coset is invariant by right multiplication by ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}$, thus its intersection with ${}^{v^{-1}}\mathbf{U}$ is the product of its respective intersections with ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^-$ and ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}$. As ${}^{v^{-1}}\mathbf{U}_I \subset \mathbf{U}^-$, the part in ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}$ has a trivial projection.

Let us compute ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^- \cap (\mathbf{B}w\mathbf{B} \coprod \mathbf{B}w''\mathbf{B})$. By Proposition 8.13 we have $\mathbf{B}w\mathbf{B} \cap \mathbf{U}^- = \mathbf{U}^*_{-\alpha_2}.(\mathbf{B}_I w' \mathbf{B}_I \cap \mathbf{U}_I^-)$ and $\mathbf{B}w''\mathbf{B} \cap \mathbf{U}^- = \mathbf{U}^*_{-\alpha_2}.(\mathbf{B}_I s_3 s_1 s_4 s_3 \mathbf{B}_I \cap \mathbf{U}_I^-)$, whence, from the decomposition ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^- = ({}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^-_{\mathbf{P}_I}).({}^{v^{-1}}\mathbf{U} \cap \mathbf{U}_I^-)$ we get

$$\begin{aligned}&{}^{v^{-1}}\mathbf{U} \cap \mathbf{U}^- \cap (\mathbf{B}w\mathbf{B} \coprod \mathbf{B}w''\mathbf{B}) \\ &\quad = \mathbf{U}^*_{-\alpha_2}.\Big({}^{v^{-1}}\mathbf{U} \cap \mathbf{U}_I^- \cap (\mathbf{B}_I w' \mathbf{B}_I \coprod \mathbf{B}_I s_3 s_1 s_4 s_3 \mathbf{B}_I)\Big).\end{aligned}$$

We have ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}_I^- = \prod_{\alpha \in \Phi_I^- - \{-\alpha_3\}} \mathbf{U}_\alpha$. In order to make explicit computations, we may replace $\mathbf{L}_I$ by the group GL_4 as the Borel subgroup varieties, as well as the groups $\mathbf{U}_I$ and $\mathbf{U}_\alpha$ depend only on the isogeny type together with the Frobenius action on the root system. The variety ${}^{v^{-1}}\mathbf{U}\cap\mathbf{U}_I^-$ is thus isomorphic to the variety of matrices $\begin{pmatrix}1&0&0&0\\ *&1&0&0\\ *&0&1&0\\ *&*&*&1\end{pmatrix}$.

We now determine the variety of matrices in GL_4 representing elements of ${}^{v^{-1}}\mathbf{U}\cap\mathbf{U}_I^-\cap(\mathbf{B}_I w'\mathbf{B}_I\coprod\mathbf{B}_I s_3s_1s_4s_3\mathbf{B}_I)$. For this we use the following:

Lemma 12.5. *Let* $\mathbf{B}$ *be the Borel subgroup of* GL_n *of upper triangular matrices; let* W *be the Weyl group relative to the torus of diagonal matrices; let* (w_{ij}) *be the matrix for* $w\in W$*; then* $(a_{ij})\in GL_n$ *is in* $\mathbf{B}w\mathbf{B}$ *if and only if the ranks of the submatrices* $(a_{ij})_{i=k,\dots,n,\,j=1,\dots,l}$ *and* $(w_{ij})_{i=k,\dots,n,\,j=1,\dots,l}$ *coincide for all* k *and* l.

Proof. The condition on ranks is invariant by left multiplication (resp. right multiplication) by $\mathbf{B}$ since this replaces each line (resp. column) by a non-zero multiple of itself plus a linear combination of the following lines (resp. columns). This condition defines thus a union of double $\mathbf{B}$-cosets. It remains to see that elements of W are determined by the rank conditions: but indeed, in line k, the position l of the non-zero coefficient is the smallest integer such that the rank of the matrices $(w_{ij})_{i=k,\dots,n,\,j=1,\dots,l}$ and $(w_{ij})_{i=k+1,\dots,n,\,j=1,\dots,l}$ differ. ∎

We thus obtain, characterizing w' and $s_3s_1s_4s_3$ by rank conditions, that ${}^{v^{-1}}\mathbf{U}\cap\mathbf{U}_I^-\cap(\mathbf{B}_I w'\mathbf{B}_I\coprod\mathbf{B}_I s_3s_1s_4s_3\mathbf{B}_I)$ is the variety of matrices $\begin{pmatrix}1&0&0&0\\ d&1&0&0\\ \alpha&0&1&0\\ 0&\beta&f&1\end{pmatrix}$, with α, β, d and f in $\overline{\mathbb{F}}_q$, such that $\begin{vmatrix}\alpha&0\\0&\beta\end{vmatrix}\neq 0$; the open subset corresponding to $\mathbf{B}_I w'\mathbf{B}_I$ is given by the condition $\begin{vmatrix}d&1&0\\ \alpha&0&1\\ 0&\beta&f\end{vmatrix}\neq 0$. These matrices may be written as $\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ \alpha&0&1&0\\ 0&\beta&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\ d&1&0&0\\ 0&0&1&0\\ -d\beta&0&f&1\end{pmatrix}\in{}^{v^{-1}}\mathbf{U}_I{}^{v^{-1}}\mathbf{U}_{\mathbf{P}_I}$. The projection on ${}^{v^{-1}}\mathbf{U}_I$ of ${}^{v^{-1}}\mathbf{U}\cap\mathbf{U}^-\cap\mathbf{B}w\mathbf{B}$ as well as that of ${}^{v^{-1}}\mathbf{U}\cap\mathbf{U}^-\cap\mathbf{B}w''\mathbf{B}$ is thus $\mathbf{U}^*_{-\alpha_2}\mathbf{U}^*_{-\alpha_1-\alpha_3}\mathbf{U}^*_{-\alpha_3-\alpha_4}$. The assumption 8.21 (iv) thus holds for w and w''.

To check the assumptions of Proposition 8.25, we must compute the fibers of the maps $\overline{\pi}'$ and $\overline{\pi}\coprod\overline{\pi}'$ of Proposition 8.25.

The above computations show that for $y\in{}^{v^{-1}}\mathbf{U}_{\mathbf{P}_I}\cap\mathbf{U}^-$ and $x\in{}^{v^{-1}}\mathbf{U}_I$, we have $y.x^{-1}{}^Fx\in\mathbf{B}w\mathbf{B}\coprod\mathbf{B}w''\mathbf{B}$ if and only if $x^{-1}.{}^Fx\in\prod_{\alpha\in{}^{v^{-1}}\mathbf{U}_I}\mathbf{U}^*_\alpha$ and ${}^{Fx^{-1}x}y$ is in $\mathbf{U}_{-\alpha_1}\mathbf{U}_{-\alpha_4}\mathbf{U}_{-\alpha_1-\alpha_4}$ and are such that the latter element corresponds to the matrix $\begin{pmatrix}1&0&0&0\\ d&1&0&0\\ 0&0&1&0\\ -d\beta&0&f&1\end{pmatrix}$ where the projection of $x^{-1F}x$ on

$\mathbf{U}_I^-$ is given by the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}$. The closed subset corresponding to $\mathbf{B}w''\mathbf{B}$ is given by $d\beta + f\alpha = 0$. We see thus that the fibers of the map $\overline{\pi} \coprod \overline{\pi}'$ of Proposition 8.25 are 2-dimensional affine spaces corresponding to the d and f coordinates of the matrix for y and that the fibers of the map $\overline{\pi}'$ are 1-dimensional affine subspaces corresponding to the equation $d\beta + f\alpha = 0$.

The assumptions of Proposition 8.25 thus hold, and we get the cohomology of $\mathbf{X}^v(w)/\mathbf{U}_{\mathbf{P}_I}^F$ as an $\mathbf{L}^F \times \langle F \rangle$-module by multiplying by $th^3 + t^2h^4$ the two-variable polynomial encoding the cohomology of the Coxeter variety for $\mathbf{L}_I$. We get

$$\begin{aligned}\sum_i h^i \cdot H_c^i(\mathbf{X}^v(w))^{\mathbf{U}_{\mathbf{P}_I}^F} = h^6 t \operatorname{St} &+ h^7 t^2(\gamma_2 + \operatorname{St}) + h^8 t^3 (\gamma_2 + \gamma_3) \\ &+ h^9 t^4 (\gamma_3 + \operatorname{Id}) + h^{10} t^5 \operatorname{Id}.\end{aligned}$$

The long exact sequence of $\mathbf{L}_I^F \times \langle F \rangle$-modules

$$\cdots \longrightarrow H_c^i(\mathbf{Y}) \longrightarrow H_c^i(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow H_c^i(\mathbf{X}^v(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow H_c^{i+1}(\mathbf{Y}) \longrightarrow \cdots$$

gives

$$\begin{gathered}
0 \longrightarrow \operatorname{St} \longrightarrow H_c^6(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow t \operatorname{St} \longrightarrow t^2(\gamma_2 + \rho_2) + t \operatorname{St} \\
\longrightarrow H_c^7(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow t^2(\gamma_2 + \operatorname{St}) \longrightarrow 0 \\
H_c^8(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} = t^3(2\gamma_2 + 2\rho_2 + 2\gamma_3) \\
H_c^9(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} = t^4(\rho_2 + 2\gamma_3 + \operatorname{Id}) \\
0 \longrightarrow H_c^{10}(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow t^5 \operatorname{Id} \longrightarrow t^5 \operatorname{Id} \longrightarrow H_c^{11}(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} \longrightarrow 0 \\
H_c^{12}(\mathbf{X}(w))^{\mathbf{U}_{\mathbf{P}_I}^F} = t^6 \operatorname{Id}
\end{gathered}$$

To obtain the non-cuspidal part of $H_c^*(\mathbf{X}(\mathbf{w}))$, we first use [DMR, 3.3.14] and [DMR, 3.3.15] which give the Id and St isotypic parts. We then consider for each b the t^b-isotypic part of the above exact sequences arguing as the proof of Theorem 9.2 and using the following table which describes

$\langle \operatorname{Ind}_{A_3}^{D_4} \chi, \psi\rangle_{D_4}$:

	1^2+	1^2-	1.1^3	St	$1^2.2$	1.21	21^2	$2+$	$2-$	2^2	1.3	31	Id
Id	0	0	0	0	0	0	0	0	1	0	1	0	1
γ_2	1	0	1	0	1	1	1	0	0	0	0	0	0
ρ_2	0	1	0	0	0	1	0	0	1	1	0	0	0
γ_3	0	0	0	0	1	1	0	1	0	0	1	1	0
St	0	1	1	1	0	0	0	0	0	0	0	0	0

We get:

$$\sum_i h^i \cdot H^i_c(\mathbf{X}(w))_{\text{non-cuspidal}} = h^6 \operatorname{St} + t^2h^7(\gamma_{1^2+} + \gamma_{1^2-} + \gamma_{21^2}) + 2t^3h^8\gamma_{1.21} + t^4h^9(\gamma_{2+} + \gamma_{2-} + \gamma_{31}) + t^6h^{12}\operatorname{Id}.$$

To study the θ-part of the cohomology of $\mathbf{X}(w)$, we will use the variety $\overline{\mathbf{X}}(w)$. One may check that all Kazhdan-Lusztig polynomials $P_{y,w}$ for $y \le w$ are 1, thus $\overline{\mathbf{X}}(w)$ is rationally smooth (*cf.* [DMR, 3.2.5]), which by [DMR, 3.3.8 (ii)] allows to compute $\sum_i h^i \cdot (H^i_c(\overline{\mathbf{X}}(w)))_\theta = t^3h^6\theta$.

By [DMR, 3.1.3], if y lies in a proper standard parabolic subgroup, $H^i_c(\mathbf{X}(y))$ cannot have a cuspidal part. The only $y < w$ for which this does not hold are the elements of

$$\mathcal{C} = \{s_2s_3s_1s_4s_3, s_2s_1s_3s_4s_3, s_2s_1s_3s_1s_4, s_2s_3s_1s_4, s_2s_1s_4s_3, s_2s_1s_3s_4\}.$$

Thus $\overline{\mathbf{X}}(w) = \mathbf{X}(w)\coprod \mathbf{X}\coprod \mathbf{Y}$ where $\mathbf{X} = \coprod_{v\in\mathcal{C}}\mathbf{X}(v)$ and where $\mathbf{Y}$ is a union of Deligne-Lusztig varieties, closed in $\overline{\mathbf{X}}(w)$ and such that $H^i_c(\mathbf{Y})_\theta = 0$ for any i. The long exact sequence corresponding to $\overline{\mathbf{X}}(w) = \mathbf{Y}\coprod(\overline{\mathbf{X}}(w) - \mathbf{Y})$ shows thus that for any i we have $H^i_c(\overline{\mathbf{X}}(w))_\theta = H^i_c(\mathbf{X}(w)\cup\mathbf{X})_\theta$.

The varieties $\mathbf{X}(s_2s_3s_1s_4)$, $\mathbf{X}(s_2s_1s_4s_3)$ and $\mathbf{X}(s_2s_1s_3s_4)$, corresponding to Coxeter elements, satisfy (*cf. e.g.*, [Lu2]) $\sum_i h^i \cdot (H^i_c)_\theta = t^2h^4$, and they are connected components of their union, thus

$$\sum_i h^i \cdot H^i_c(\mathbf{X}(s_2s_3s_1s_4)\cup\mathbf{X}(s_2s_1s_4s_3)\cup\mathbf{X}(s_2s_1s_3s_4))_\theta = 3t^2h^4.$$

The elements $s_2s_3s_1s_4s_3$, $s_2s_1s_3s_4s_3$ and $s_2s_1s_3s_1s_4$ are conjugate by cyclic permutation respectively to $s_2s_3s_1s_4s_2$, $s_4s_2s_1s_3s_4$ and $s_1s_2s_3s_4s_1$ which, by [DMR, 3.1.6] and [DMR, 3.2.10], allows to compute the cuspidal part of their cohomology For each of them we get $\sum_i h^i \cdot (H^i_c)_\theta = t^2h^5 + t^3h^6$, thus for their union (of which they are connected components) $\sum_i h^i \cdot (H^i_c)_\theta = 3t^2h^5 + 3t^3h^6$. The union

$$\mathbf{X}(s_2s_3s_1s_4s_3) \coprod \mathbf{X}(s_2s_1s_3s_4s_3) \coprod \mathbf{X}(s_2s_1s_3s_1s_4)$$

is open in $\mathbf{X}$; the corresponding long exact sequence gives

$$0 \longrightarrow H^4_c(\mathbf{X})_\theta \longrightarrow 3t^2\theta \longrightarrow 3t^2\theta \longrightarrow H^5_c(\mathbf{X})_\theta \longrightarrow 0,$$

and $H^6_c(\mathbf{X})_\theta = 3t^3\theta$. There exists thus an integer $\eta \leq 3$ such that $\sum_i h^i \cdot H^i_c(\mathbf{X})_\theta = \eta t^2(h^4 + h^5)\theta + 3t^3h^6\theta$. The long exact sequence corresponding to the union of $\mathbf{X}$ and $\mathbf{X}(w)$ shows then that only the characters t^2 and t^3 of $\langle F\rangle$ may occur in the cohomology of $\mathbf{X}(w)$, and gives for the corresponding isotypic parts of the cohomology: $H^5_c(\mathbf{X}(w))_{\theta,t^2} = H^6_c(\mathbf{X}(w))_{\theta,t^2} = \eta t^2\theta$, and

$$0 \longrightarrow H^6_c(\mathbf{X}(w))_{\theta,t^3} \longrightarrow t^3\theta \longrightarrow 3t^3\theta \longrightarrow H^7_c(\mathbf{X}(w))_{\theta,t^3} \longrightarrow 0.$$

By [DMR, 3.3.22] we have $H^5_c(\mathbf{X}(w)) = 0$, thus also $H^6_c(\mathbf{X}(w))_{\theta,t^2} = 0$.

To lift the ambiguity on (θ, t^3), we now use [DMR, 3.3.21]; we take for w' a Coxeter element, which is not conjugate to w; thus any eigenvalue of F on $H^6_c(\mathbf{X}(w))_\theta$ must have a module less than q^3. This shows that $H^6_c(\mathbf{X}(w))_\theta = 0$, thus $H^7_c(\mathbf{X}(w))_{\theta,t^3} = 2t^3\theta$. □

From the values of the generic degrees of $\mathcal{H}_q(w)$ (see [BMa, 5.A]), we see that if we denote by $\rho(x_1, x_2, x_3)$ where $x_i \in \{1, q^2\}$ the 1-dimensional representation of $\mathcal{H}_q(w)$ given by $T_i \mapsto x_i$, and ρ^+ (resp. ρ^-) the irreducible 2-dimensional representation where $T_1T_2T_3$ acts as the scalar q^3 (resp. $-q^3$), we have the following equalities, if we denote by m_ρ the multiplicity of ρ in $\sum_i(-1)^i H^i_c(\mathbf{X}(\mathbf{w}))$: $m_{\rho(1,1,1)} = \dim \mathrm{St}$, $m_{\rho(q^2,q^2,q^2)} = \dim \mathrm{Id}$, $m_{\rho(q^2,q^2,1)} = m_{\rho(1,q^2,q^2)} = m_{\rho(q^2,1,q^2)} = -\dim\gamma_{1^2+} = -\dim\gamma_{1^2-} = -\dim\gamma_{21^2}$, $m_{\rho(q^2,1,1)} = m_{\rho(1,q^2,1)} = m_{\rho(1,1,q^2)} = -\dim\gamma_{2+} = -\dim\gamma_{2-} = -\dim\gamma_{31}$, $m_{\rho^+} = \dim\gamma_{1.21}$ and $m_{\rho^-} = -\dim\theta$. Thus we can determine $\bigoplus_i H^i_c(\mathbf{X}(\mathbf{w}))$ as a $\mathbf{G}^F \times \mathcal{H}_q(w)$-module up to an ambiguity on the correspondence between characters of $\mathbf{G}^F$ and of $\mathcal{H}_q(w)$ which appear in the same degree.

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