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LUSZTIG'S a-FUNCTION IN TYPE B_n IN THE ASYMPTOTIC CASE

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To George Lusztig on his 60th birthday

Abstract. In this paper, we study Lusztig's **a**-function for a Coxeter group with unequal parameters. We determine that function explicitly in the "asymptotic case" in type B_n, where the left cells have been determined in terms of a generalized Robinson-Schensted correspondence by Bonnafé and the second author. As a consequence, we can show that all of Lusztig's conjectural properties (P1)–(P15) hold in this case, except possibly (P9), (P10) and (P15). Our methods rely on the "leading matrix coefficients" introduced by the first author. We also interprete the ideal structure defined by the two-sided cells in the associated Iwahori-Hecke algebra $\mathcal{H}_n$ in terms of the Dipper-James-Murphy basis of $\mathcal{H}_n$.

§1. Introduction

Let (W, S) be a Coxeter system where W is finite. We shall be interested in the Kazhdan-Lusztig cells and Lusztig's **a**-function on W, which play an important role in the representation theory of finite reductive groups. The notions of cells and **a**-functions are defined in terms of the Iwahori-Hecke algebra associated with W. Originally, Kazhdan-Lusztig [16] and Lusztig [18] only considered the case of a one-parameter Iwahori-Hecke algebra; subsequently, the theory has been extended to the case of unequal parameters by Lusztig [17], [20]. However, many results that are known to hold in the equal parameter case (thanks to a geometric interpretation of the Kazhdan-Lusztig basis) are only conjectural in the general case of unequal parameters. A precise set of conjectures has been formulated by Lusztig in [20, Chap. 14], (P1)–(P15). (We recall these conjectures in Section 2.)

The aim of this paper is to determine Lusztig's **a**-function explicitly in the case where $W = W_n$ is of type B_n and the parameters satisfy the

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"asymptotic" conditions in Bonnafé-Iancu [2]. As an application, we show that all of the conjectures in [20, Chap. 14] hold in this case, except possibly (P9), (P10) and (P15)[1]. We also determine the structure of the associated ring J. Our methods rely on the "leading matrix coefficients" introduced by the first named author [10]. It is our hope that similar methods may also be applied to other choices of parameters in type B_n where the left cell representations are expected to be irreducible.

In a different direction, we show that the ideal structure defined by the two-sided cells in the "asymptotic" case in type B_n corresponds precisely to the ideal structure given in terms of the Dipper-James-Murphy basis [6].

To state our main results more precisely, we have to introduce some notation. In [20], an Iwahori-Hecke algebra with possibly unequal parameters is defined with respect to an integer-valued weight function on W. Following a suggestion of Bonnafé [3], we can slightly modify Lusztig's definition so as to include the more general setting in [17] as well. Let Γ be an abelian group (written additively) and assume that there is a total order $\leqslant$ on Γ compatible with the group structure. (In the setting of [20], $\Gamma = \mathbb{Z}$ with the natural order.)

Let $A = \mathbb{Z}[\Gamma]$ be the free abelian group with basis $\{e^\gamma \mid \gamma \in \Gamma\}$. There is a well-defined ring structure on A such that $e^\gamma e^{\gamma'} = e^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. (Hence, if $\Gamma = \mathbb{Z}$, then A is nothing but the ring of Laurent polynomials in an indeterminate e.) We write $1 = e^0 \in A$. Given $a \in A$ we denote by a_γ the coefficient of e^γ, so that $a = \sum_{\gamma\in\Gamma} a_\gamma e^\gamma$. We denote by $A_{\geqslant 0}$ the set of $\mathbb{Z}$-linear combinations of elements e^γ where $\gamma \geqslant 0$. Similarly, we define $A_{>0}$, $A_{\leqslant 0}$ and $A_{<0}$. We say that a function

$$L\colon W \longrightarrow \Gamma$$

is a weight function if $L(ww') = L(w) + L(w')$ whenever we have $\ell(ww') = \ell(w) + \ell(w')$ where $\ell\colon W \to \mathbb{N}$ is the usual length function. (We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$.) We assume throughout that $L(s) > 0$ for all $s \in S$. Let $\mathcal{H} = \mathcal{H}(W, S, L)$ be the generic Iwahori-Hecke algebra over A with parameters $\{v_s \mid s \in S\}$ where $v_s := e^{L(s)}$ for $s \in S$. The algebra $\mathcal{H}$ is free over A with basis $\{T_w \mid w \in W\}$, and the multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

[1] In a subsequent paper [12], using completely different methods, the first author shows that (P9), (P10) and a weak version of (P15) also hold. Thus, eventually, (P1)–(P14) and a weak version of (P15) are known to hold in the "asymptotic case" in type B_n.

where $s \in S$ and $w \in W$. Having fixed a total order on Γ, we have a corresponding Kazhdan-Lusztig basis $\{C'_w \mid w \in W\}$ of $\mathcal{H}$; we have

$$C'_w = T_w + \sum_{\substack{y \in W \\ y < w}} P^*_{y,w}\, T_y \in \mathcal{H},$$

where $<$ denotes the Bruhat-Chevalley order on W and $P^*_{y,w} \in A_{<0}$ for all $y < w$ in W; see [17, §6]. (In the framework of [20], the polynomials $P^*_{y,w}$ are denoted $p_{y,w}$ and the basis elements C'_w are denoted c_w.) Given $x, y \in W$, we write

$$C'_x\, C'_y = \sum_{z \in W} h_{x,y,z}\, C'_z \quad \text{where } h_{x,y,z} \in A.$$

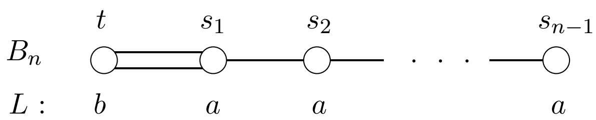
For a fixed $z \in W$, we set

$$\mathbf{a}(z) := \min\{\gamma \geqslant 0 \mid e^{\gamma}\, h_{x,y,z} \in A_{\geqslant 0} \text{ for all } x, y \in W\};$$

this is Lusztig's function $\mathbf{a}\colon W \to \Gamma$. (If $\Gamma = \mathbb{Z}$ with its natural order, then this reduces to the function defined by Lusztig [18].) In Section 2, we recall Lusztig's conjectures concerning the $\mathbf{a}$-function and its relation with the pre-order relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$. In the case where W is a Weyl group and L is constant on S, these conjectures are known to hold, thanks to a geometric interpretation which yields certain "positivity properties"; see Lusztig [18]. In the general case of unequal parameters, it is known that these positivity properties are no longer satisfied.

In this paper, we will be dealing with a Coxeter group of type B_n where the parameters are specified as follows.

Example 1.1. Let Γ be any totally ordered abelian group. Let W_n be a Coxeter group of type B_n ($n \geqslant 2$), with generators, relations and weight function $L\colon W_n \to \Gamma$ given by the following diagram:



where $a, b \in \Gamma$ are such that

$$\boxed{b > (n-1)a > 0.}$$

(Here, $(n-1)a$ means $a+\cdots+a$ in Γ, with $n-1$ summands.) We refer to this hypothesis as the "*asymptotic case*" in type B_n. Let $\mathcal{H}_n$ be the corresponding Iwahori-Hecke algebra over $A=\mathbb{Z}[\Gamma]$, where we set

$$V := v_t = e^b \quad \text{and} \quad v := v_{s_1} = \cdots = v_{s_{n-1}} = e^a.$$

We have the following special case worth mentioning: Let $\Gamma_0=\mathbb{Z}^2$. Let $\leqslant$ be the usual lexicographic order so that $(i,j)<(i',j')$ if $i<i'$ or if $i=i'$ and $j<j'$. Then $A_0=\mathbb{Z}[\Gamma_0]$ is nothing but the ring of Laurent polynomials in two independent indeterminates $V_0=e^{(1,0)}$ and $v_0=e^{(0,1)}$. This is the set-up originally considered by Bonnafé-Iancu [2]; we may refer to this case as the "*generic asymptotic case*" in type B_n.

In Bonnafé-Iancu [2] (for the "generic" case), the left cells of W_n are determined explicitly in terms of a generalized Robinson-Schensted correspondence. This correspondence associates to each element $w \in W_n$ a pair of standard bitableaux of the same shape and total size n. (By a bitableau, we mean an ordered pair of two tableaux; the shape of a bitableau is an ordered pair of partitions, that is, a bipartition.) Subsequently, Bonnafé [3] has shown that these results remain valid in the general "asymptotic case". (In Section 5, we recall in more detail the main results of [2], [3].)

Our first main result gives an explicit description of the **a**-function.

Theorem 1.2. *In the setting of Example* 1.1*, let $w \in W_n$ and assume that w corresponds to a pair of bitableaux of shape (λ_1,λ_2) by the generalized Robinson-Schensted correspondance defined in* [2]*, where λ_1 and λ_2 are partitions such that $|\lambda_1|+|\lambda_2|=n$. Then*

$$\mathbf{a}(w) = b\,|\lambda_2| + a\,(n(\lambda_1) + 2n(\lambda_2^*) - n(\lambda_2)).$$

Here, $n(\mu)=\sum_i (i-1)\mu^{(i)}$ for any partition $\mu=(\mu^{(1)} \geqslant \mu^{(2)} \geqslant \cdots \geqslant 0)$ and μ^* denotes the conjugate partition.

In the "generic asymptotic case", the above formula reads

$$\mathbf{a}(w) = (|\lambda_2|,\, n(\lambda_1) + 2n(\lambda_2^*) - n(\lambda_2)).$$

The proof will be given in Section 5, using the general methods developed in Section 4. The main ingredients in that proof are Bonnafé's results [3] on the two-sided cells in W_n, and the orthogonal representations and leading matrix coefficients introduced in [10]. These are generalizations of the leading coefficients of character values considered by Lusztig [19]. As an application, we obtain the following result. (See Section 5 for the proof.)

THEOREM 1.3. *In the setting of Example* 1.1, *all the conjectures* (P1)–(P15) *in* [20, Chap. 14] *hold except possibly* (P9), (P10) *and* (P15). *The set of "distinguished involutions" is given by* $\mathcal{D} = \{z \in W_n \mid z^2 = 1\}$.

Thanks to the validity of the properties in Theorem 1.3, we can construct the ring J as explained in [20, Chap. 18]. As an abelian group, J is free with a basis $\{t_w \mid w \in W_n\}$. The multiplication is given by

$$t_x \cdot t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}}\, t_z \qquad \text{for all } x, y \in W_n,$$

where $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ is the constant term of $e^{\mathbf{a}(z)}\, h_{x,y,z} \in A_{\geqslant 0}$.

THEOREM 1.4. *In the setting of Example* 1.1, *we have an isomorphism of rings* $J \cong \bigoplus_\lambda M_{d_\lambda}(\mathbb{Z})$, *where* $\lambda = (\lambda_1, \lambda_2)$ *runs over all bipartitions of* n *and* d_λ *is the number of standard bitableaux of shape* λ. *We have*

$$\gamma_{x,y,z} = \begin{cases} \pm 1 & \textit{if } x \sim_{\mathcal{L}} y^{-1},\ y \sim_{\mathcal{L}} z^{-1},\ z \sim_{\mathcal{L}} x^{-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

The proof in Proposition 4.9 actually yields an explicit isomorphism which shows that $\pm t_w$ ($w \in W_n$) corresponds to a matrix unit in $M_{d_\lambda}(\mathbb{Z})$ for some bipartition λ. Furthermore, the signs are interpreted in terms of leading matrix coefficients.

Finally, we show that the Kazhdan-Lusztig basis in the asymptotic case is compatible with the basis constructed by Dipper-James-Murphy [6]. That basis is denoted $\{x_{\mathfrak{st}}\}$ where $(\mathfrak{s}, \mathfrak{t})$ runs over all pairs of standard bitableaux of total size n and of the same shape; see [6, Theorem 4.14]. Note that the construction of the elements $x_{\mathfrak{st}}$ does not rely on the choice of any total order on Γ. Given a bipartition λ of n, let $N^\lambda \subseteq \mathcal{H}_n$ be the A-submodule spanned by all $x_{\mathfrak{st}}$ where the shape of $\mathfrak{s}$ and $\mathfrak{t}$ is a bipartition μ of n such that $\lambda \trianglelefteq \mu$. By [6, Cor. 4.13], N^λ is a two-sided ideal of $\mathcal{H}_n$. Now we can state (see the end of Section 5 for the proof):

THEOREM 1.5. *In the setting of Example* 1.1, *let* $\lambda = (\lambda_1, \lambda_2)$ *be a bipartition of* n. *Then* N^λ *is spanned by the basis elements* C'_w *where* $w \in W_n$ *corresponds, via the generalized Robinson-Schensted correspondence, to a bitableau of shape* $\nu = (\nu_1, \nu_2)$ *such that* $(\lambda_1, \lambda_2) \trianglelefteq (\nu_2, \nu_1^*)$.

This paper is organized as follows. In Section 2, we recall the basic definitions concerning the Kazhdan-Lusztig pre-orders on a finite Coxeter group W and state Lusztig's conjectures (P1)–(P15), following [20, Chap. 14].

In Section 3, we deal with the leading coefficients of the matrix representations of the Iwahori-Hecke algebra associated with W and show that, under suitable hypotheses, these leading coefficients can be used to detect left, right and two-sided cells. This is an elaboration, with some refinements, of the ideas in [10].

In Section 4, we present some criteria and tools for attacking Lusztig's conjectures. It is our hope that these methods will also be applicable to other situations where the left cell representations are expected to be irreducible.

In Section 5, we show that the hypotheses required for the criteria in Section 4 are all satisfied for the "asymptotic case" in type B_n. This heavily relies on the fact that Hoefsmit's [15] matrix representations in type B_n are "orthogonal representations" in the sense of [10]; hence the theory of leading coefficients and the results in Sections 3 and 4 can be applied in this case.

§2. Left cells and Lusztig's conjectures

We keep the basic set-up introduced in Section 1 where W is a finite Coxeter group and $\mathcal{H}$ is the corresponding Iwahori-Hecke algebra over A, with parameters $\{v_s \mid s \in S\}$ where $v_s = e^{L(s)}$ and $L(s) > 0$ for all $s \in S$.

Since we will be dealing with $\mathbf{a}$-invariants of elements in W and of irreducible representations, it will be technically more convenient to work with a slightly different version of the Kazhdan-Lusztig basis of $\mathcal{H}$. (The reasons can be seen, for example, in [20, Chap. 18].) For any $a \in A$, we define $\bar{a} := \sum_{\gamma\in\Gamma} a_\gamma e^{-\gamma}$. Then we have a unique ring involution $j\colon \mathcal{H} \to \mathcal{H}$ such that $j(a) = \bar{a}$ for $a \in A$ and $j(T_w) = \varepsilon_w T_w$ for $w \in W$, where we set $\varepsilon_w = (-1)^{\ell(w)}$. As in [17, §6], we set $C_w := \varepsilon_w j(C'_w)$. Then we have

$$C_w = T_w + \sum_{\substack{y\in W\\ y<w}} \varepsilon_y\, \varepsilon_w\, \overline{P}^{\,*}_{y,w}\, T_y \quad \text{for all } w \in W.$$

The multiplication rule now reads:

$$C_x\, C_y = \sum_{z\in W} \varepsilon_x\, \varepsilon_y\, \varepsilon_z\, h_{x,y,z}\, C_z \quad \text{for any } x, y \in W.$$

For example, if $x = s \in S$, we have (see [17, §6]):

$$C_s\, C_y = \begin{cases} C_{sy} - \displaystyle\sum_{\substack{z \in W \\ sz < z < y}} \varepsilon_y\, \varepsilon_z\, M^s_{z,y}\, C_z & \text{if } sy > y, \\ -(v_s + v_s^{-1})\, C_y & \text{if } sy < y, \end{cases}$$

where $M^s_{z,y} \in A$ is determined as in [17, §3]. Note that we have $\overline{M}^s_{z,y} = M^s_{z,y}$.

Throughout this paper, we will make use of another important feature of Iwahori-Hecke algebras, namely, the fact that these algebras carry a natural *symmetrizing trace*. Indeed, consider the linear map $\tau\colon \mathcal{H} \to A$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. Then we have

$$\tau(T_w T_{w'}) = \begin{cases} 1 & \text{if } w' = w^{-1}, \\ 0 & \text{if } w' \neq w^{-1}; \end{cases}$$

see [14, §8.1]. Thus, τ is a *symmetrizing trace* on $\mathcal{H}$. In the following discussion, we shall also need the basis of $\mathcal{H}$ which is dual to the basis $\{C_w\}$ with respect to the symmetrizing trace τ. For any $y \in W$ we set

$$D_y := T_y + \sum_{\substack{w \in W \\ y < w}} \overline{P}^*_{ww_0, yw_0}\, T_w \in \mathcal{H},$$

where $w_0 \in W$ is the unique element of maximal length in W. Then we have

$$\tau(C_w D_{y^{-1}}) = \begin{cases} 1 & \text{if } y = w, \\ 0 & \text{if } y \neq w. \end{cases}$$

(See [10, 2.4]; see also [20, Prop. 11.5] where the analogous statement is proved for the C'-basis.) This immediately yields the following result:

Corollary 2.1. *For any $z \in W$, we have*

$$\mathbf{a}(z) = \min\{\gamma \geqslant 0 \mid e^{\gamma}\, \tau(C_x C_y D_{z^{-1}}) \in A_{\geqslant 0} \textit{ for all } x, y \in W\}.$$

(Indeed, just note that $\tau(C_x C_y D_{z^{-1}}) = \varepsilon_x\, \varepsilon_y\, \varepsilon_z\, h_{x,y,z}$.)

We recall the definition of the left cells of W and the corresponding left cell representations of $\mathcal{H}$ (see [17] or [20]).

We write $z \leftarrow_{\mathcal{L}} y$ if there exists some $s \in S$ such that $h_{s,y,z} \neq 0$, that is, C'_z occurs in $C'_s\, C'_y$ (when expressed in the C'-basis) or, equivalently,

C_z occurs in $C_s\,C_y$ (when expressed in the C-basis). Let $\leqslant_{\mathcal{L}}$ be the pre-order relation on W generated by $\leftarrow_{\mathcal{L}}$, that is, we have $z \leqslant_{\mathcal{L}} y$ if there exist elements $z = z_0, z_1, \ldots, z_k = y$ such that $z_{i-1} \leftarrow_{\mathcal{L}} z_i$ for $1 \leqslant i \leqslant k$. The equivalence relation associated with $\leqslant_{\mathcal{L}}$ will be denoted by $\sim_{\mathcal{L}}$ and the corresponding equivalence classes are called the *left cells* of W.

Similarly, we can define a pre-order $\leqslant_{\mathcal{R}}$ by considering multiplication by C_s' on the right in the defining relation. The equivalence relation associated with $\leqslant_{\mathcal{R}}$ will be denoted by $\sim_{\mathcal{R}}$ and the corresponding equivalence classes are called the *right cells* of W. We have

$$x \leqslant_{\mathcal{R}} y \iff x^{-1} \leqslant_{\mathcal{L}} y^{-1}.$$

This follows by using the anti-automorphism $\flat\colon \mathcal{H} \to \mathcal{H}$ given by $T_w^\flat = T_{w^{-1}}$; we have $C_w'^\flat = C_{w^{-1}}'$ and $C_w^\flat = C_{w^{-1}}$ for all $w \in W$; see [20, 5.6]. Thus, any statement concerning the left pre-order relation $\leqslant_{\mathcal{L}}$ has an equivalent version for the right pre-order relation $\leqslant_{\mathcal{R}}$, via $\flat$.

Finally, we define a pre-order $\leqslant_{\mathcal{LR}}$ by the condition that $x \leqslant_{\mathcal{LR}} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leqslant_{\mathcal{L}} x_i$ or $x_{i-1} \leqslant_{\mathcal{R}} x_i$. The equivalence relation associated with $\leqslant_{\mathcal{LR}}$ will be denoted by $\sim_{\mathcal{LR}}$ and the corresponding equivalence classes are called the *two-sided cells* of W.

Each left cell $\mathfrak{C}$ gives rise to a representation of $\mathcal{H}$. This is constructed as follows (see [17, §7]). Let $[\mathfrak{C}]_A$ be an A-module with a free A-basis $\{c_w \mid w \in \mathfrak{C}\}$. Then the action of C_w ($w \in W$) on $[\mathfrak{C}]_A$ is given by the above multiplication formulas, i.e., we have

$$C_w.c_x = \sum_{y \in \mathfrak{C}} \varepsilon_w\,\varepsilon_x\,\varepsilon_y\,h_{w,x,y}\,c_y \quad \text{for all } x \in \mathfrak{C} \text{ and } w \in W.$$

Remark 2.2. It is also possible to define a left cell module using the C'-basis. Recall that $C_w = \varepsilon_w j(C_w')$ for all $w \in W$. Let $\mathfrak{C}$ be a left cell of W and let $[\mathfrak{C}]_A'$ be a free A-module with a basis $\{c_x' \mid x \in \mathfrak{C}\}$. Then we have an $\mathcal{H}$-module structure on $[\mathfrak{C}]_A'$ given by the formula

$$C_w'.c_x' = \sum_{y \in W} h_{w,x,y}\,c_y' \quad \text{for all } x \in \mathfrak{C} \text{ and } w \in W.$$

The passage between the two definitions can be performed using the A-algebra automorphism $\delta\colon \mathcal{H} \to \mathcal{H}$ given by $T_s \mapsto -T_s^{-1}$ for $s \in S$. Note that, by the definition of the Kazhdan-Lusztig basis, the elements C_w and C_w' are fixed under the composition $\delta\circ j$. Hence, we have $C'_w = \varepsilon_w j(C_w) = \varepsilon_w \delta(C_w)$ for all $w \in W$. This shows that we have an isomorphism of $\mathcal{H}$-modules

$$^{\delta}[\mathfrak{C}]_A \cong [\mathfrak{C}]'_A$$

where $^{\delta}[\mathfrak{C}]_A$ is the $\mathcal{H}$-module obtained from $[\mathfrak{C}]_A$ by composing the original action of $\mathcal{H}$ with δ. This remark will play a role in Example 3.11 below.

For the convenience of the reader, we restate here Lusztig's conjectures (P1)–(P15) in [20, Chap. 14] in the general framework involving a totally ordered abelian group Γ. For $z \in W$, we define an element $\mathbf{\Delta}(z) \in \Gamma$ and an integer $0 \neq n_z \in \mathbb{Z}$ by the condition

$$e^{\mathbf{\Delta}(z)} P^*_{1,z} \equiv n_z \mod A_{<0}; \quad \text{see [20, 14.1]}.$$

Note that $\mathbf{\Delta}(z) \geqslant 0$. Furthermore, given $x, y, z \in W$, we define $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ by

$$\gamma_{x,y,z^{-1}} = \text{constant term of } e^{\mathbf{a}(z)}\, h_{x,y,z} \in A_{\geqslant 0}.$$

CONJECTURE 2.3. (Lusztig [20, 14.2]) *Let $\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = \mathbf{\Delta}(z)\}$. Then the following properties hold.*

- **P1.** *For any $z \in W$ we have $\mathbf{a}(z) \leqslant \mathbf{\Delta}(z)$.*
- **P2.** *If $d \in \mathcal{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*
- **P3.** *If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.*
- **P4.** *If $z' \leqslant_{\mathcal{LR}} z$ then $\mathbf{a}(z') \geqslant \mathbf{a}(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$, then $\mathbf{a}(z) = \mathbf{a}(z')$.*
- **P5.** *If $d \in \mathcal{D}$, $y \in W$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*
- **P6.** *If $d \in \mathcal{D}$, then $d^2 = 1$.*
- **P7.** *For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.*
- **P8.** *Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.*
- **P9.** *If $z' \leqslant_{\mathcal{L}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{L}} z$.*
- **P10.** *If $z' \leqslant_{\mathcal{R}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{R}} z$.*
- **P11.** *If $z' \leqslant_{\mathcal{LR}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{LR}} z$.*
- **P12.** *Let $I \subset S$ and W_I be the parabolic subgroup generated by I. If $y \in W_I$, then $\mathbf{a}(y)$ computed in terms of W_I is equal to $\mathbf{a}(y)$ computed in terms of W.*

P13. *Any left cell* $\mathfrak{C}$ *of* W *contains a unique element* $d \in \mathcal{D}$. *We have* $\gamma_{x^{-1},x,d} \neq 0$ *for all* $x \in \mathfrak{C}$.

P14. *For any* $z \in W$, *we have* $z \sim_{\mathcal{LR}} z^{-1}$.

P15. *If* $x, x', y, w \in W$ *are such that* $\mathbf{a}(w) = \mathbf{a}(y)$, *then*

$$\sum_{y' \in W} h_{w,x',y'} \otimes_{\mathbb{Z}} h_{x,y',y} = \sum_{y' \in W} h_{x,w,y'} \otimes_{\mathbb{Z}} h_{y',x',y} \quad \text{in } A \otimes_{\mathbb{Z}} A.$$

(The above formulation of (P15) is taken from Bonnafé [3].)

Remark 2.4. For all $x, y, z \in W$, we have

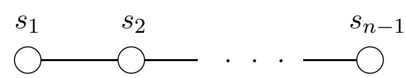
$$h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}} \quad \text{and} \quad P^*_{x,y} = P^*_{x^{-1},y^{-1}}.$$

Hence, we have $\mathbf{a}(z) = \mathbf{a}(z^{-1})$, $n_z = n_{z^{-1}}$, $\mathbf{\Delta}(z) = \mathbf{\Delta}(z^{-1})$, $\mathcal{D} = \mathcal{D}^{-1}$.

Proof. We have already remarked above that there is an anti-automorphism $\flat \colon \mathcal{H} \to \mathcal{H}$ such that $T_w^\flat = T_{w^{-1}}$ for all $w \in W$. By the argument in [20, 5.6], we have $C_w^\flat = C_{w^{-1}}$. This yields all the above statements. ∎

If W is the symmetric group $\mathfrak{S}_n$, the above conjectures are all known to hold; see [20, Chap. 15][2]. Hence the information about left, right and two-sided cells, as well as the $\mathbf{a}$-function, is rather complete in this case. We close this section by summarizing some known results on $\mathfrak{S}_n$. (This information will be needed in the proof of Theorem 1.2; see Section 5.)

EXAMPLE 2.5. Let $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ be the symmetric group, where $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. The diagram is given as follows.

A_{n-1} 

We consider the abelian group $\Gamma = \mathbb{Z}$ with its natural order and denote $v := e^1$. Then $A = \mathbb{Z}[v, v^{-1}]$ is the ring of Laurent polynomials in an indeterminate v. Let $L \colon W \to \mathbb{Z}$ be any weight function such that $L(s_i) > 0$ for all i. Since all generators are conjugate, L takes the same value on each s_i. Thus, we are in the case of "equal parameters", and we can assume $L(s_i) = 1$ for all i.

[2]In a recent preprint [13], the first named author has given elementary proofs of (P1)–(P15) for $W = \mathfrak{S}_n$.

The classical Robinson-Schensted correspondence associates with each element $\sigma \in \mathfrak{S}_n$ a pair of standard tableaux $(P(\sigma), Q(\sigma))$ of the same shape. The tableau $P(\sigma)$ is obtained by "row-insertion" of the numbers $\sigma.1, \ldots, \sigma.n$ (in this order) into an initially empty tableau; the tableau $Q(\sigma)$ "keeps the record" of the order by which the positions in $P(\sigma)$ have been filled; see Fulton [9, Chap. 4]. For any partition $\nu \vdash n$, we set

$$\mathfrak{R}_\nu := \{\sigma \in \mathfrak{S}_n \mid P(\sigma),\, Q(\sigma) \text{ have shape } \nu\}.$$

Thus, we have $\mathfrak{S}_n = \coprod_{\nu \vdash n} \mathfrak{R}_\nu$. Then the following hold.

(a) *For a fixed standard tableau T, the set $\{\sigma \in \mathfrak{S}_n \mid Q(\sigma) = T\}$ is a left cell of $\mathfrak{S}_n$ and $\{\sigma \in \mathfrak{S}_n \mid P(\sigma) = T\}$ is a right cell of $\mathfrak{S}_n$. Furthermore, all left cells and all right cells arise in this way.*

This was first proved by Kazhdan-Lusztig [16, §4]; for a more direct and elementary proof, see Ariki [1].

(b) *The sets $\mathfrak{R}_\nu$, $\nu \vdash n$ are precisely the two-sided cells of $\mathfrak{S}_n$.*

This is seen as follows. First note that the statements (P1)–(P15) in Conjecture 2.3 are known to hold for $W = \mathfrak{S}_n$ (see [20, Chap. 15] and the references there). Now (P4), (P9), (P10) imply that $x, y \in \mathfrak{S}_n$ lie in the same two-sided cell if and only if there exists a sequence of elements $x = x_0, x_1, \ldots, x_k = y$ in $\mathfrak{S}_n$ such that, for each i, we have $x_{i-1} \sim_{\mathcal{L}} x_i$ or $x_{i-1} \sim_{\mathcal{R}} x_i$. Now (b) follows from (a).

(c) *For any $\nu \vdash n$, we have $\sigma_{\nu^*} \in \mathfrak{R}_\nu$, where σ_{ν^*} is the longest element in the Young subgroup $\mathfrak{S}_{\nu^*} \subseteq \mathfrak{S}_n$ and ν^* denotes the conjugate partition.*

This is a purely combinatorial exercice: it is enough to apply the Robinson-Schensted correspondence to the element $\sigma_{\nu *}$ and to verify that the corresponding tableaux have shape ν.

(d) *If $\sigma \in \mathfrak{R}_\nu$, then $\mathbf{a}(\sigma) = n(\nu)$, where $n(\nu)$ is defined as in Theorem* 1.2.

This is seen as follows. Again, we use the fact that (P1)–(P15) in Conjecture 2.3 hold for $W = \mathfrak{S}_n$. Since, by (P4), the $\mathbf{a}$-function is constant on the two-sided cells, (c) shows that it is enough to compute $\mathbf{a}(\sigma_\nu)$ for any $\nu \vdash n$. But, since σ_ν is the longest element in a parabolic subgroup, we have $\mathbf{a}(\sigma_\nu) = \ell(\sigma_\nu)$ by (P12) and [20, 13.8]. It remains to note that $\ell(\sigma_\nu) = n(\nu^*)$.

(e) *If $\sigma \in \mathfrak{R}_\nu$ and $\sigma' \in \mathfrak{R}_{\nu'}$ are such that $\sigma \leqslant_{\mathcal{LR}} \sigma'$, then we have $\nu \trianglelefteq \nu'$, where $\trianglelefteq$ denotes the dominance order. This means that*

$$\sum_{i=1}^{k} \nu^{(i)} \leqslant \sum_{i=1}^{k} \nu'^{(i)} \quad \textit{for all } k > 0,$$

where $\nu^{(i)}$ and $\nu'^{(i)}$ are the parts of ν and ν', respectively.

This follows from a result of Lusztig-Xi [21, 3.2]; see Du-Parshall-Scott [8, 2.13.1] and the references there. (Note that, here again, (P1)–(P15) are used.)

§3. Leading matrix coefficients

We now recall the basic facts concerning the leading matrix coefficients introduced in [10]. We extend scalars from $\mathbb{Z}$ to $\mathbb{R}$ and consider the group algebra $\mathbb{R}[\Gamma]$. Since Γ is totally ordered, $\mathbb{R}[\Gamma]$ is an integral domain; let K be its field of fractions. We define $\mathfrak{I}_{>0} \subset \mathbb{R}[\Gamma]$ to be the set of all $f \in \mathbb{R}[\Gamma]$ such that

$$f = 1 + \mathbb{R}\text{-linear combination of elements of } e^{\gamma} \text{ where } \gamma > 0.$$

Note that $\mathfrak{I}_{>0}$ is multiplicatively closed. Furthermore, every element $x \in K$ can be written in the form

$$x = r_x\, e^{\gamma_x} f/g \quad \text{where } r_x \in \mathbb{R},\, \gamma_x \in \Gamma \text{ and } f, g \in \mathfrak{I}_{>0};$$

note that, if $x \neq 0$, then r_x and γ_x indeed are *uniquely determined* by x; if $x = 0$, we have $r_0 = 0$ and we set $\gamma_0 := +\infty$ by convention. We set

$$\mathcal{O} := \{x \in K \mid \gamma_x \geqslant 0\} \quad \text{and} \quad \mathfrak{p} := \{x \in K \mid \gamma_x > 0\}.$$

Then it is easily verified that $\mathcal{O}$ is a valuation ring in K, that is, $\mathcal{O}$ is a subring of K such that, for any $0 \neq x \in K$, we have $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Furthermore, $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{p}$. The group of units in $\mathcal{O}$ is given by

$$\mathcal{O}^{\times} = \{x \in \mathcal{O} \mid r_x \neq 0,\, \gamma_x = 0\}.$$

Note that we have

$$\begin{aligned} \mathcal{O} \cap \mathbb{R}[\Gamma] &= \mathbb{R}[\Gamma]_{\geqslant 0} := \langle e^{\gamma} \mid \gamma \geqslant 0 \rangle_{\mathbb{R}}, \\ \mathfrak{p} \cap \mathbb{R}[\Gamma] &= \mathbb{R}[\Gamma]_{>0} := \langle e^{\gamma} \mid \gamma > 0 \rangle_{\mathbb{R}}. \end{aligned}$$

We have a well-defined $\mathbb{R}$-linear ring homomorphism $\mathcal{O} \to \mathbb{R}$ with kernel $\mathfrak{p}$. The image of $x \in \mathcal{O}$ in $\mathbb{R}$ is called the *constant term* of x. Thus, the constant term of x is 0 if $x \in \mathfrak{p}$; the constant term equals r_x if $x \in \mathcal{O}^\times$.

Extending scalars from A to K, we obtain a finite dimensional K-algebra $\mathcal{H}_K$, with basis $\{T_w \mid w \in W\}$ and multiplication as specified in Section 1. We have:

Remark 3.1. The algebra $\mathcal{H}_K$ is split semisimple and abstractly isomorphic to the group algebra of W over K.

Proof. Since the situation here is somewhat more general than usual, let us indicate the main ingredients. To show that $\mathcal{H}_K$ is semisimple, we use the $\mathbb{R}$-linear ring homomorphism $\theta\colon \mathbb{R}[\Gamma] \to \mathbb{R}$ such that $\theta(e^\gamma) = 1$ for all $\gamma \in \Gamma$. By extension of scalars, we obtain $\mathbb{R} \otimes_{\mathbb{R}[\Gamma]} \mathcal{H} \cong \mathbb{R}[W]$, the group algebra of W over $\mathbb{R}$. Since the latter algebra is known to be semisimple, a standard argument (using Tits' Deformation Theorem) shows that $\mathcal{H}_K$ must be semisimple, too. (See, for example, [14, 7.4.6 and 8.1.7].) But then it is also known that $\mathcal{H}_K$ is split and abstractly isomorphic to $K[W]$; see [14, 9.3.5 and 9.3.9] and the references there. $\square$

Let $\mathrm{Irr}(\mathcal{H}_K)$ be the set of irreducible characters of $\mathcal{H}_K$. We write this set in the form

$$\mathrm{Irr}(\mathcal{H}_K) = \{\chi_\lambda \mid \lambda \in \Lambda\},$$

where Λ is some finite indexing set. The algebra $\mathcal{H}_K$ is *symmetric* with respect to the trace function $\tau\colon \mathcal{H}_K \to K$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$ (see Section 2). The fact that $\mathcal{H}_K$ is split semisimple yields that

$$\tau = \sum_{\lambda \in \Lambda} \frac{1}{c_\lambda}\, \chi_\lambda \quad \text{where } 0 \neq c_\lambda \in \mathbb{R}[\Gamma].$$

The elements c_λ are called the *Schur elements*. By [14, 8.1.8], we have $c_\lambda = P_{W,L}/D_\lambda$ where $P_{W,L} = \sum_{w \in W} e^{2L(w)}$ is the Poincaré polynomial of W, L and D_λ is the "generic degree" associated with χ_λ. We can write

$$c_\lambda = r_\lambda\, e^{-2\alpha_\lambda}\, f_\lambda \quad \text{where } r_\lambda \in \mathbb{R}_{>0},\ f_\lambda \in \mathfrak{I}_{>0} \text{ and } \alpha_\lambda \geqslant 0.$$

The element $\alpha_\lambda \in \Gamma$ is called the *generalized a-invariant* of χ_λ; see [10, §3]. (Note that the notation in [*loc. cit.*] has to be adapted to the present setting where we write the elements of $\mathbb{R}[\Gamma]$ exponentially.)

By [10, Prop. 4.3], every χ_λ is afforded by a so-called orthogonal representation. This means that there exists a matrix representation $\mathfrak{X}_\lambda \colon \mathcal{H}_K \to M_{d_\lambda}(K)$ with character χ_λ and an invertible diagonal matrix $P \in M_{d_\lambda}(K)$ such that the following conditions hold:

(O1) We have $\mathfrak{X}_\lambda(T_{w^{-1}}) = P^{-1} \cdot \mathfrak{X}_\lambda(T_w)^{\mathrm{tr}} \cdot P$ for all $w \in W$, and

(O2) the diagonal entries of P lie in $\mathfrak{I}_{>0}$.

This has the following consequence. Let $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$. For any $h \in \mathcal{H}_K$, we denote by $\mathfrak{X}_\lambda^{ij}(h)$ the (i,j)-entry of the matrix $\mathfrak{X}_\lambda(h)$. Then, by [10, Theorem 4.4 and Remark 4.5], we have

$$e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(T_w) \in \mathcal{O}, \quad e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(C_w) \in \mathcal{O}, \quad e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(D_w) \in \mathcal{O}$$

for any $w \in W$ and

$$e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(T_w) \equiv e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(C_w) \equiv e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(D_w) \mod \mathfrak{p}.$$

Hence, the above three elements of $\mathcal{O}$ have the same constant term which we write as $\varepsilon_w\, c_{w,\lambda}^{ij}$. The constants $c_{w,\lambda}^{ij} \in \mathbb{R}$ are called the *leading matrix coefficients* of $\mathfrak{X}_\lambda$. By [10, Theorem 4.4], these coefficients have the following property:

$$\begin{aligned} c_{w,\lambda}^{ij} &= c_{w^{-1},\lambda}^{ji} && \text{for all } w \in W,\\ c_{w,\lambda}^{ij} &\neq 0 && \text{for some } w \in W. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \alpha_\lambda &= \min\{\gamma \geqslant 0 \mid e^\gamma\, \mathfrak{X}_\lambda^{ij}(T_w) \in \mathcal{O} \text{ for all } w \in W \text{ and } 1 \leqslant i,j \leqslant d_\lambda\}\\ &= \min\{\gamma \geqslant 0 \mid e^\gamma\, \mathfrak{X}_\lambda^{ij}(C_w) \in \mathcal{O} \text{ for all } w \in W \text{ and } 1 \leqslant i,j \leqslant d_\lambda\}\\ &= \min\{\gamma \geqslant 0 \mid e^\gamma\, \mathfrak{X}_\lambda^{ij}(D_w) \in \mathcal{O} \text{ for all } w \in W \text{ and } 1 \leqslant i,j \leqslant d_\lambda\}. \end{aligned}$$

The leading matrix coefficients satisfy the following *Schur relations.* Let $\lambda, \mu \in \Lambda$, $1 \leqslant i, j \leqslant d_\lambda$ and $1 \leqslant k, l \leqslant d_\mu$; then

$$\sum_{w \in W} c_{w,\lambda}^{ij} c_{w,\mu}^{kl} = \begin{cases} \delta_{ik}\,\delta_{jl}\, r_\lambda & \text{if } \lambda = \mu,\\ \quad 0 & \text{if } \lambda \neq \mu; \end{cases}$$

see [10, Theorem 4.4]. Since $|W| = \sum_{\lambda \in \Lambda} d_\lambda^2$, we can invert the above relations and obtain another set of relations (analogous to the "second" orthogonality relations for the characters of a finite group): For any $y, w \in W$ we have

$$\sum_{\lambda \in \Lambda} \sum_{i,j=1}^{d_\lambda} \frac{1}{r_\lambda} c_{y,\lambda}^{ij} c_{w,\lambda}^{ij} = \begin{cases} 1 & \text{if } y = w, \\ 0 & \text{if } y \neq w. \end{cases}$$

The above relations immediately imply that $W = \bigcup_{\lambda \in \Lambda} \mathfrak{T}_\lambda$, where we set

$$\mathfrak{T}_\lambda := \{w \in W \mid c_{w,\lambda}^{ij} \neq 0 \text{ for some } 1 \leqslant i, j \leqslant d_\lambda\}.$$

The leading matrix coefficients are related to the left cells of W by the following result. Recall that, given a left cell $\mathfrak{C}$, we have a corresponding left cell module $[\mathfrak{C}]_A$. Extending scalars from A to K, we obtain an $\mathcal{H}_K$-module $[\mathfrak{C}]_K$. We denote by $\chi_{\mathfrak{C}}$ the character of $[\mathfrak{C}]_K$. Then we have:

PROPOSITION 3.2. (See [10, Prop. 4.7]) *Let $\lambda \in \Lambda$ and $\mathfrak{C}$ be a left cell in W. Denote by $[\chi_{\mathfrak{C}} : \chi_\lambda]$ the multiplicity of χ_λ in $\chi_{\mathfrak{C}}$. Then we have*

$$\sum_{k=1}^{d_\lambda} \sum_{y \in \mathfrak{C}} (c_{y,\lambda}^{ik})^2 = [\chi_{\mathfrak{C}} : \chi_\lambda]\, r_\lambda, \qquad \textit{for any } 1 \leqslant i \leqslant d_\lambda.$$

In particular, if $w \in \mathfrak{T}_\lambda$, then χ_λ occurs with non-zero multiplicity in $\chi_{\mathfrak{C}}$, where $\mathfrak{C}$ is the left cell containing w.

Proof. The formula is proved in [*loc. cit.*]. Now fix $\lambda \in \Lambda$ and let $w \in \mathfrak{T}_\lambda$; then $c_{w,\lambda}^{ij} \neq 0$ for some $1 \leqslant i, j \leqslant d_\lambda$. Let $\mathfrak{C}$ be the left cell containing w. Now all terms on the left hand side of the formula are non-negative, and the term corresponding to $y = w$ and $k = j$ is strictly positive. Hence the left hand side is non-zero and so $[\chi_{\mathfrak{C}} : \chi_\lambda] \neq 0$. ∎

The Schur relations lead to particularly strong results when some additional hypotheses are satisfied. These are isolated in the following definition.

DEFINITION 3.3. We say that $\mathcal{H}$ is *integral* if

$$c_{w,\lambda}^{ij} \in \mathbb{Z} \quad \text{for all } w \in W,\ \lambda \in \Lambda \text{ and } 1 \leqslant i, j \leqslant d_\lambda.$$

Furthermore, recall that $c_\lambda = r_\lambda e^{-2\alpha_\lambda} f_\lambda$, where r_λ is a positive real number and $f_\lambda \in \mathfrak{I}_{>0}$. We say that $\mathcal{H}$ is *normalized* if

$$r_\lambda = 1 \quad \text{for all } \lambda \in \Lambda.$$

Remark 3.4. Suppose that $\mathcal{H}$ is normalized and that Lusztig's conjectures (P1)–(P15) in [20, Chap. 14] are satisfied. Then we necessarily have that $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_K)$ for all left cells $\mathfrak{C}$ in W; see [11, Cor. 4.8]. Thus, the conditions in Definition 3.3 should be considered as rather severe restrictions on the structure of $\mathcal{H}$.

Since the Schur elements c_λ are known in all cases (see the appendices of Carter [4] and Geck-Pfeiffer [14], for example) the condition that $\mathcal{H}$ be *normalized* is rather straightforward to verify. The condition that $\mathcal{H}$ be *integral* is more subtle. Let us describe here a convenient setting in which this condition can be dealt with. Let $\lambda \in \Lambda$ and assume that we have a matrix representation $\mathfrak{Y}_\lambda\colon \mathcal{H}_K \to M_{d_\lambda}(K)$ affording χ_λ such that $\mathfrak{Y}_\lambda$ is *integral*, in the sense that

$$\mathfrak{Y}_\lambda(T_w) \in M_d(A) \quad \text{for all } w \in W.$$

Furthermore, assume that there is a *symmetric* matrix $\Omega = (\omega_{ij}) \in M_{d_\lambda}(A)$ satisfying the following two conditions:

(F1) We have $\mathfrak{Y}_\lambda(T_{w^{-1}}) = \Omega^{-1} \cdot \mathfrak{Y}_\lambda(T_w)^{\mathrm{tr}} \cdot \Omega$ for all $w \in W$, and

(F2) all principal minors of Ω lie in $1 + A_{>0}$.

Note that $1 + A_{>0} = A \cap \mathfrak{I}_{>0}$.

LEMMA 3.5. *In the above setting, there is an orthogonal representation $\mathfrak{X}_\lambda\colon \mathcal{H}_K \to M_{d_\lambda}(K)$ affording χ_λ such that the corresponding leading coefficients satisfy $c_{w,\lambda}^{ij} \in \mathbb{Z}$ for all $w \in W$ and $1 \leqslant i, j \leqslant d_\lambda$. We have $\mathfrak{X}_\lambda(T_w) \in M_{d_\lambda}(K_0)$, where K_0 is the field of fractions of A.*

Proof. Let $R := \{f/g \mid f \in A, g \in 1 + A_{>0}\} \subseteq K$, a subring of K.

Now we consider the system of equations $\Pi^{\mathrm{tr}} \cdot Z \cdot \Pi = \Omega$, where $\Pi = (\pi_{ij})$ is an *upper triangular* matrix with 1 on the diagonal (and unknown coefficients above the diagonal) and $Z = (z_{ij})$ is a *diagonal* matrix (with unknown coefficients on the diagonal). It is well-known and easy to see that the above system has a unique solution (Π, Z), where $z_{ii} \in R^\times$ for all i and $\pi_{ij} \in R$ for all $i < j$. (Note that all principal minors of Ω are invertible in R.) Since Ω and Z have the same principal minors, there exists some $f \in 1 + A_{>0}$ such that

$$f z_{ii} \in 1 + A_{>0} \subseteq \mathfrak{I}_{>0} \quad \text{for all } i.$$

We set $P := fZ$ and define $\mathfrak{X}_\lambda$ by

$$\mathfrak{X}_\lambda(T_w) := \Pi \cdot \mathfrak{Y}_\lambda(T_w) \cdot \Pi^{-1} \quad \text{for all } w \in W.$$

Then, clearly, $\mathfrak{X}_\lambda$ affords χ_λ, and a straightforward computation yields

$$\mathfrak{X}_\lambda(T_{w^{-1}}) = Z^{-1} \cdot \mathfrak{X}_\lambda(T_w)^{\mathrm{tr}} \cdot Z = P^{-1} \cdot \mathfrak{X}_\lambda(T_w)^{\mathrm{tr}} \cdot P.$$

So $\mathfrak{X}_\lambda$ and P satisfy the conditions (O1) and (O2). Now note that Π (being triangular with 1 on the diagonal) is invertible over R; let us denote

$$\Pi^{-1} = (\tilde{\pi}_{ij}) \quad \text{where } \tilde{\pi}_{ii} = 1 \text{ and } \tilde{\pi}_{ij} \in R \text{ for all } i, j.$$

Hence, for any $1 \leqslant i, j \leqslant d_\lambda$, we obtain

$$e^{\alpha_\lambda} \mathfrak{X}_\lambda^{ij}(T_w) = \sum_{k=1}^{d_\lambda} \sum_{l=1}^{d_\lambda} \pi_{ik}\, \tilde{\pi}_{lj} \Big(e^{\alpha_\lambda} \mathfrak{Y}_\lambda^{kl}(T_w) \Big) \in R.$$

On the other hand, $\mathfrak{X}_\lambda$ is an orthogonal representation; hence the above element lies in $\mathcal{O} \cap R$. Consequently, the corresponding leading matrix coefficient $c_{w,\lambda}^{ij}$ will lie in $\mathbb{Z}$, as required. ∎

EXAMPLE 3.6. Let $W = W_n$ and $L \colon W_n \to \Gamma$ be as in Example 1.1 (the "asymptotic case" in type B_n). Let $\mathcal{H}_n$ be the corresponding Iwahori-Hecke algebra and write $\mathcal{H}_{n,K} = K \otimes_A \mathcal{H}_n$. Then we have a natural parametrization

$$\mathrm{Irr}(\mathcal{H}_{n,K}) = \{\chi_\lambda \mid \lambda \in \Lambda_n\}$$

where Λ_n is the set of all bipartitions $\lambda = (\lambda_1, \lambda_2)$ such that $|\lambda_1| + |\lambda_2| = n$; see [14, Chap. 5]. For any $(\lambda_1, \lambda_2) \in \Lambda_n$, we have

$$r_{(\lambda_1,\lambda_2)} = 1 \quad \text{and} \quad \alpha_{(\lambda_1,\lambda_2)} = b\,|\lambda_2| + a\,(n(\lambda_1) + 2n(\lambda_2) - n(\lambda_2^*)),$$

where we set $n(\nu) = \sum_i (i-1)\nu^{(i)}$ for any partition $\nu = (\nu^{(1)} \geqslant \cdots \geqslant \nu^{(r)} \geqslant 0)$ and where ν^* denotes the conjugate partition. Thus, $\mathcal{H}_n$ is normalized. (For the proof, see [10, Remark 5.1] and the references there. Actually, in [*loc. cit.*], we only consider the "generic asymptotic case". But it is readily checked, using the formula for c_λ in [4, p. 447], that the above formulas also hold in the "asymptotic case". For a weight function with values in $\mathbb{Z}$, explicit formulas for $r_{(\lambda_1,\lambda_2)}$ and $\alpha_{(\lambda_1,\lambda_2)}$ are given by Lusztig [20, Prop. 22.14]; then it is a purely combinatorial exercise to show that, in the case $b > (n-1)a$, Lusztig's formulas can be rewritten as above.)

Furthermore, by the discussion in [10, §5], $\mathcal{H}_n$ is integral. Since that discussion is somewhat sketchy (and only deals with the "generic asymptotic case"), let us add a more rigorous argument, based on Dipper-James-Murphy [6], [7] and Lemma 3.5. That argument also shows that we can work with the field of fractions of A instead of K.

Let $\lambda \in \Lambda_n$ and $\tilde{S}^\lambda$ be the corresponding Specht module over K, as constructed in [6, 4.19]. The modules $\tilde{S}^\lambda$ are absolutely irreducible and pairwise non-isomorphic [6, 4.22]. Let $\mathbb{T}_\lambda$ be the set of standard bitableaux of shape λ. Then $\tilde{S}^\lambda$ has a standard basis $\{e_t \mid t \in \mathbb{T}_\lambda\}$ such that the corresponding matrix representation is *integral* in the above sense. Furthermore, by [6, §5], there is a non-degenerate bilinear form $\langle\ ,\ \rangle_\lambda$ on $\tilde{S}^\lambda$, satisfying the condition $\langle T_s x, x'\rangle_\lambda = \langle x, T_s x'\rangle_\lambda$ for all $x, x' \in \tilde{S}^\lambda$ and all $s \in S$. The Gram matrix of $\langle\ ,\ \rangle_\lambda$ with respect to the standard basis of $\tilde{S}^\lambda$ has coefficients in A and a non-zero determinant. We will have to slightly modify the standard basis of $\tilde{S}^\lambda$ in order to make sure that (F1) and (F2) hold for the corresponding matrix representation.

Now, by [6, Theorem 8.11], there is an orthogonal basis $\{f_t \mid t \in \mathbb{T}_\lambda\}$ with respect to the above bilinear form; moreover, the matrix transforming the standard basis into the orthogonal basis is triangular with 1 on the diagonal[3]. Using the recursion formula in [7, Proposition 3.8], it is straightforward to show that, for each basis element f_t, there exist integers $s_t, a_{ti}, b_{tj}, c_{tk}, d_{tl} \in \mathbb{Z}$ such that

$$a_{ti} \geqslant 0, \quad b_{tj} \geqslant 0, \quad b + c_{tk}a > 0, \quad b + d_{tl}a > 0,$$

and

$$\langle f_t, f_t\rangle_\lambda = e^{2s_t a} \cdot \frac{\prod_i (1 + e^{2a} + \cdots + e^{2a_{ti}a})}{\prod_j (1 + e^{2a} + \cdots + e^{2b_{tj}a})} \cdot \frac{\prod_k \bigl(1 + e^{2(b+c_{tk}a)}\bigr)}{\prod_l \bigl(1 + e^{2(b+d_{tl}a)}\bigr)}.$$

The fact that $b + c_{tk}a$ and $b + d_{tl}a$ are strictly positive essentially relies on the condition that $b > (n-1)a$. Hence, setting

$$\tilde{f}_t := e^{-s_t a} \cdot \Bigl(\prod_j (1 + e^{2a} + \cdots + e^{2b_{tj}a})\Bigr) \cdot \Bigl(\prod_l (1 + e^{2(b+d_{tl}a)})\Bigr) \cdot f_t,$$

we obtain $\langle \tilde{f}_t, \tilde{f}_t\rangle_\lambda \in 1 + A_{>0}$ for all t. Now let

$$\tilde{e}_t = e^{-s_t a} e_t \quad \text{for all } t \in \mathbb{T}_\lambda.$$

[3] It is remarked in [6] that the orthogonal representation with respect to the basis $\{f_t\}$ actually coincides with the one defined by Hoefsmit [15].

Then $\{\tilde{e}_t \mid t \in \mathbb{T}_\lambda\}$ is a new basis of $\tilde{S}^\lambda$. Let $\mathfrak{Y}_\lambda$ be the corresponding matrix representation and Ω be the corresponding Gram matrix of $\langle\ ,\ \rangle_\lambda$. Then $\mathfrak{Y}_\lambda$ is still integral, Ω has coefficients in A and (F1) holds. Now, the matrix transforming $\tilde{e}_t$ to the basis $\tilde{f}_t$ is triangular with quotients of elements from $1+A_{>0}$ on the diagonal. Hence, up to quotients of elements from $1+A_{>0}$, the principal minors of Ω are products of terms $\langle \tilde{f}_t, \tilde{f}_t\rangle_\lambda \in 1+A_{>0}$, for various t. We conclude that each principal minor of Ω is a quotient of elements from $1+A_{>0}$. Since, on the other hand, the coefficients of Ω lie in A, so do all principal minors of Ω. Hence (F2) holds. So Lemma 3.5 shows that $\mathcal{H}_n$ is integral.

Remark 3.7. In the above setting, consider the special case where $\Gamma = \mathbb{Z}$ and A is the ring of Laurent polynomials in $v = e^1$. The parabolic subgroup $\mathfrak{S}_n := \langle s_1, \ldots, s_{n-1}\rangle \subseteq W_n$ can be identified with the symmetric group on $\{1, \ldots, n\}$ where s_i corresponds to the basic transposition $(i, i+1)$ for $1 \leqslant i \leqslant n-1$. Let $\mathcal{H}(\mathfrak{S}_n) \subseteq \mathcal{H}_n$ be the corresponding parabolic subalgebra. We have the following diagram:

$$\begin{array}{cccccc} & s_1 & & s_2 & & s_{n-1} \\ \mathfrak{S}_n & \bigcirc & \!\!-\!\!\!-\!\!\!-\!\! & \bigcirc\!\!-\!\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\!- & & \bigcirc \\ L: & a & & a & & a \end{array}$$

Now, by Hoefsmit [15, §2.3], we obtain a complete set of irreducible representations of $\mathcal{H}(\mathfrak{S}_n)_K$ by restricting the irreducible representations of $\mathcal{H}_{n,K}$ with label of the form $\lambda = (\lambda_1, \varnothing)$, where λ_1 is a partition of n. Hence, the fact that $\mathcal{H}_n$ is integral and normalized immediately implies that $\mathcal{H}(\mathfrak{S}_n)$ is integral and normalized, too. Alternatively, one could also work directly with the Dipper-James construction of Hoefsmit's matrices for type A_{n-1} in [5, Theorem 4.9].

The following arguments are inspired from the proof of [10, Theorem 4.10].

LEMMA 3.8. *Assume that $\mathcal{H}$ is integral and normalized.*

(a) *We have $c_{w,\lambda}^{ij} \in \{0, \pm 1\}$ for all $w \in W$, $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$.*

(b) *For any $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$, there exists a unique $w \in W$ such that $c_{w,\lambda}^{ij} \neq 0$; we denote that element by $w = w_\lambda(i,j)$. The correspondence $(\lambda, i, j) \mapsto w_\lambda(i,j)$ defines a bijective map*

$$\{(\lambda, i, j)\} \mid \lambda \in \Lambda, 1 \leqslant i,j \leqslant d_\lambda\} \longrightarrow W.$$

In particular, the sets $\mathfrak{T}_\lambda$ defined above form a partition of W:

$$W = \coprod_{\lambda\in\Lambda} \mathfrak{T}_\lambda \qquad \textit{and} \qquad |\mathfrak{T}_\lambda| = d_\lambda^2 \quad \textit{for all } \lambda \in \Lambda.$$

Proof. First we construct the desired map in (b). Fix $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$. Then consider the Schur relation where $\lambda = \mu$, $i = l$ and $j = k$:

$$\sum_{w\in W} (c_{w,\lambda}^{ij})^2 = r_\lambda = 1.$$

Since the leading matrix coefficients are integers, we conclude that there exists a unique $w = w_\lambda(i,j)$ such that $c_{w,\lambda}^{ij} = \pm 1$ and $c_{y,\lambda}^{ij} = 0$ for all $y \in W \setminus \{w\}$. Thus, we have a map $(\lambda, i, j) \mapsto w = w_\lambda(i,j)$; furthermore, note that once we have shown that this map is bijective, then (a) follows.

Next we show that the above map is surjective. Let $w \in W$. Then the "second Schur relations" show that there exist some $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$ such that $c_{w,\lambda}^{ij} \neq 0$. The previous argument implies that $w = w_\lambda(i,j)$. Thus, the above map is surjective. Since $\dim \mathcal{H}_K = |W| = \sum_{\lambda\in\Lambda} d_\lambda^2$, that map is between finits sets of the same cardinality. Hence, the map is bijective. ∎

Remark 3.9. In the setting of Lemma 3.8, let $\lambda \in \Lambda$ and consider the set $\mathfrak{T}_\lambda$. First of all, we have a unique labelling of the elements in $\mathfrak{T}_\lambda$:

$$\mathfrak{T}_\lambda = \{w_\lambda(i,j) \mid 1 \leqslant i, j \leqslant d_\lambda\}.$$

It follows from [10, Theorem 4.4(b)] that

(a) $$w_\lambda(i,j)^{-1} = w_\lambda(j,i).$$

In particular, $w_\lambda(i,j)$ is an involution if and only of $i = j$. Furthermore, let $i, j \in \{1, \dots, d_\lambda\}$ and define

(b) $${}^j\mathfrak{T}_\lambda := \{w_\lambda(k,j) \mid 1 \leqslant k \leqslant d_\lambda\} \quad \text{and} \quad \mathfrak{T}_\lambda^i := \{w_\lambda(i,l) \mid 1 \leqslant l \leqslant d_\lambda\}.$$

It is shown in [10, Theorem 4.10] that ${}^j\mathfrak{T}_\lambda$ is contained in a left cell of W and $\mathfrak{T}_\lambda^j$ is contained in a right cell of W. In particular, the whole set $\mathfrak{T}_\lambda$ is contained in a two-sided cell of W.

LEMMA 3.10. *Assume that $\mathcal{H}$ is integral and normalized. Furthermore, assume that $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_K)$ for all left cells $\mathfrak{C}$ of W. Then the sets*

$$\{{}^j\mathfrak{T}_\lambda \mid \lambda \in \Lambda,\ 1 \leqslant j \leqslant d_\lambda\}$$

are precisely the left cells of W. The character of the left cell representation afforded by ${}^j\mathfrak{T}_\lambda$ is given by χ_λ. Furthermore, each left cell contains a unique involution.

Proof. Let $\lambda \in \Lambda$ and $1 \leqslant j \leqslant d_\lambda$. By Remark 3.9, there is a left cell $\mathfrak{C}$ of W such that ${}^j\mathfrak{T}_\lambda \subseteq \mathfrak{C}$. Now Proposition 3.2 shows that χ_λ occurs with non-zero multiplicity in $\chi_\mathfrak{C}$. Since $\chi_\mathfrak{C} \in \mathrm{Irr}(\mathcal{H}_K)$, we conclude that $\chi_\mathfrak{C} = \chi_\lambda$. In particular, this means that the underlying $\mathcal{H}_K$-modules have the same dimension and so $|\mathfrak{C}| = \chi_\lambda(1) = d_\lambda$. Consequently, we have $\mathfrak{C} = {}^j\mathfrak{T}_\lambda$. Thus, we have shown that each set ${}^j\mathfrak{T}_\lambda$ is a left cell of W. Since these sets form a partition of W, we conclude that they are precisely the left cells of W. The statement concerning involutions now follows from Remark 3.9(a). ∎

EXAMPLE 3.11. Let $W = W_n$ and $L\colon W_n \to \Gamma$ be as in Example 1.1 (the "asymptotic case" in type B_n). We have already noted in Example 3.6 that $\mathcal{H}_n$ is integral and normalized. Thus, by Lemma 3.8, we have a partition

$$W_n = \coprod_{(\lambda_1,\lambda_2)\in\Lambda_n} \mathfrak{T}_{(\lambda_1,\lambda_2)}.$$

Let us identify the sets $\mathfrak{T}_{(\lambda_1,\lambda_2)}$. For this purpose, we need the results of Bonnafé-Iancu [2] concerning the left cells of W_n and their characters. These results remain valid in the "asymptotic case" by Bonnafé [3, §5]. More precisely, in [2, Theorem 7.7], the left cells in the "generic asymptotic case" (that is, with respect to the weight function $L_0\colon W_n \to \Gamma_0$) are described in terms of a generalized Robinson-Schensted correspondence. In [3, Cor. 5.2], it is shown that two elements of W_n lie in the same left cell with respect to $L\colon W_n \to \Gamma$ if and only if this is the case with respect to L_0. Hence the combinatorial description of the left cells remains valid in the "asymptotic case". The fact that the characters afforded by the left cells are all irreducible now follows by exactly the same argument as in [2, Prop. 7.9].

Now let $\mathfrak{C}$ be a left cell. Then $\mathfrak{C}$ is precisely the set of elements in a generalized Robinson-Schensted cell (RS-cell for short). Such a cell is labelled by a pair of bitableaux of the same shape and size n. Let $(\lambda_1, \lambda_2) \in \Lambda_n$ be the bipartition specifying the shape of the bitableaux. Then we have

$$\chi_\mathfrak{C} = \chi_{(\lambda_1,\lambda_2^*)} \in \mathrm{Irr}(\mathcal{H}_{n,K}).$$

(Note that, in [2, Prop. 7.11], the labelling is given by (λ_2, λ_1^*). The reason for the different labelling is that, in [2], the left cell representations are defined using the C'-basis while here we use the C-basis. As explained in Remark 2.2, the character of the left cell representation defined with respect to the C'-basis is given by $\chi_{\mathfrak{C}} \circ \delta$. On the level of characters of W_n, this corresponds to tensoring with the sign character; see [14, 9.4.1] for a precise statement and more details. The effect of tensoring with the sign character is described in [14, 5.5.6].) Now we claim that

$$\mathfrak{T}_{(\lambda_1,\lambda_2^*)} = \{w \in W \mid w \text{ belongs to an RS-cell of shape } (\lambda_1, \lambda_2)\}.$$

Indeed, let $w \in \mathfrak{T}_{(\lambda_1,\lambda_2^*)}$. Then, by Proposition 3.2, the character $\chi_{(\lambda_1,\lambda_2^*)}$ occurs with non-zero multiplicity in $\chi_{\mathfrak{C}}$, where $\mathfrak{C}$ is the left cell containing w. So the above formula for $\chi_{\mathfrak{C}}$ shows that w belongs to an RS-cell of shape (λ_1, λ_2). The reverse implication now follows formally from the fact that both the sets $\mathfrak{T}_{(\lambda_1,\lambda_2)}$ and the RS-cells form a partition of W.

§4. On Lusztig's conjectures

We keep the setting of the previous section. Now our aim is to develop some tools and criteria for attacking the properties in Conjecture 2.3. Throughout this section, we assume that

$$\mathcal{H} \textit{ is integral and normalized (see Definition 3.3).}$$

We begin with an alternative characterization of the $\mathbf{a}$-function. First note that

$$\mathbf{a}(z) = \min\{\gamma \geqslant 0 \mid e^{\gamma} h_{x,y,z} \in \mathcal{O} \text{ for all } x, y \in W\}$$

for any $z \in W$. This simply follows from the equality $\mathcal{O} \cap A = A_{\geqslant 0}$.

Proposition 4.1. *For any $z \in W$, we have*

$$\mathbf{a}(z) = \min\{\gamma \geqslant 0 \mid e^{\gamma}\, \mathfrak{X}_\lambda^{ij}(D_{z^{-1}}) \in \mathcal{O} \textit{ for all } \lambda \in \Lambda,\ 1 \leqslant i, j \leqslant d_\lambda\}.$$

Furthermore, if $z \in \mathfrak{T}_{\lambda_0}$ (see Lemma 3.8), we have $\alpha_{\lambda_0} \leqslant \mathbf{a}(z)$.

Proof. Let $\gamma_0 \geqslant 0$ be minimal such that $e^{\gamma_0}\, \mathfrak{X}_\lambda^{ij}(D_{z^{-1}}) \in \mathcal{O}$ for all $\lambda \in \Lambda$, $1 \leqslant i, j \leqslant d_\lambda$. First we show that $\mathbf{a}(z) \leqslant \gamma_0$. For this purpose, it is enough to show that $e^{\gamma_0} h_{x,y,z} \in \mathcal{O}$ for all $x, y \in W$. Let $x, y \in W$. To evaluate $h_{x,y,z}$, we use the formula

$$\begin{aligned}\varepsilon_x\,\varepsilon_y\,\varepsilon_z\,h_{x,y,z} &= \tau(C_xC_yD_{z^{-1}}) = \sum_{\lambda\in\Lambda}\frac{1}{c_\lambda}\,\chi_\lambda(C_xC_yD_{z^{-1}})\\ &= \sum_{\lambda\in\Lambda}\frac{1}{c_\lambda}\,\mathrm{trace}\Big(\mathfrak{X}_\lambda(C_x)\mathfrak{X}_\lambda(C_y)\mathfrak{X}_\lambda(D_{z^{-1}})\Big)\\ &= \sum_{\lambda\in\Lambda}\sum_{i,j,k=1}^{d_\lambda}\frac{1}{c_\lambda}\,\mathfrak{X}_\lambda^{ij}(C_x)\,\mathfrak{X}_\lambda^{jk}(C_y)\,\mathfrak{X}_\lambda^{ki}(D_{z^{-1}}).\end{aligned}$$

Now, for any $\lambda\in\Lambda$, we have $r_\lambda=1$ and so $c_\lambda=e^{-2\alpha_\lambda}f_\lambda$ where $f_\lambda\in\mathfrak{I}_{>0}$. Thus, we obtain

$$e^{\gamma_0}h_{x,y,z}=\pm\sum_{\lambda\in\Lambda}\sum_{i,j,k=1}^{d_\lambda}f_\lambda^{-1}\Big(e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(C_x)\Big)\Big(e^{\alpha_\lambda}\mathfrak{X}_\lambda^{jk}(C_y)\Big)\Big(e^{\gamma_0}\mathfrak{X}_\lambda^{ki}(D_{z^{-1}})\Big).$$

As we have already noted, for all $w\in W$, we have

$$e^{\alpha_\lambda}\,\mathfrak{X}_\lambda^{ij}(C_w)\equiv\varepsilon_w\,c_{w,\lambda}^{ij}\mod\mathfrak{p}.$$

Hence, since $f_\lambda^{-1}\in\mathcal{O}$, all terms on the right hand side of the above identity lie in $\mathcal{O}$. So we obtain $e^{\gamma_0}h_{x,y,z}\in\mathcal{O}$ as desired. Next we show that $\gamma_0\leqslant\mathbf{a}(z)$. By the definition of γ_0, there exists some $\lambda_0\in\Lambda$ such that

$$e^{\gamma_0}\mathfrak{X}_{\lambda_0}^{k_0i_0}(D_{z^{-1}})\not\equiv 0\mod\mathfrak{p}\quad\text{for some }1\leqslant i_0,k_0\leqslant d_{\lambda_0}.$$

We also know by Lemma 3.8 that there exist $x_0,y_0\in W$ such that $c_{x_0,\lambda_0}^{i_0k_0}=\pm1$ and $c_{y_0,\lambda_0}^{k_0k_0}=\pm1$. As before, we obtain an identity

$$e^{\gamma_0}h_{x_0,y_0,z}=\pm\sum_{\lambda\in\Lambda}\sum_{i,j,k=1}^{d_\lambda}f_\lambda^{-1}\Big(e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(C_{x_0})\Big)\Big(e^{\alpha_\lambda}\mathfrak{X}_\lambda^{jk}(C_{y_0})\Big)\Big(e^{\gamma_0}\mathfrak{X}_\lambda^{ki}(D_{z^{-1}})\Big).$$

All terms on the right hand side lie in $\mathcal{O}$. So we obtain

$$e^{\gamma_0}h_{x_0,y_0,z}\equiv\pm\sum_{\lambda\in\Lambda}\sum_{i,j,k=1}^{d_\lambda}c_{x_0,\lambda}^{ij}\,c_{y_0,\lambda}^{jk}\Big(e^{\gamma_0}\mathfrak{X}_\lambda^{ki}(D_{z^{-1}})\Big)\mod\mathfrak{p};$$

note that $f_\lambda^{-1}\equiv 1\bmod\mathfrak{p}$ since $f_\lambda\in\mathfrak{I}_{>0}$. Now, by Lemma 3.8, we have

$$\begin{aligned}c_{x_0,\lambda}^{ij}\neq 0 &\implies (\lambda,i,j)=(\lambda_0,i_0,k_0),\\ c_{y_0,\lambda}^{jk}\neq 0 &\implies (\lambda,j,k)=(\lambda_0,k_0,k_0).\end{aligned}$$

Hence we obtain

$$\begin{aligned} e^{\gamma_0} h_{x_0,y_0,z} &\equiv \pm c^{i_0k_0}_{x_0,\lambda_0}\, c^{k_0k_0}_{y_0,\lambda_0} \Big(e^{\gamma_0} \mathfrak{X}^{k_0i_0}_{\lambda_0}(D_{z^{-1}}) \Big) \\ &\equiv \pm e^{\gamma_0} \mathfrak{X}^{k_0i_0}_{\lambda_0}(D_{z^{-1}}) \not\equiv 0 \mod \mathfrak{p}. \end{aligned}$$

Consequently, we must have $\gamma_0 \leqslant \mathbf{a}(z)$ and so $\gamma_0 = \mathbf{a}(z)$.

Finally, if $z \in \mathfrak{T}_{\lambda_0}$, then $c^{ij}_{z,\lambda_0} = \pm 1$ for some i, j. Since

$$\varepsilon_z\, e^{\alpha_{\lambda_0}} \mathfrak{X}^{ji}_{\lambda_0}(D_{z^{-1}}) \equiv c^{ji}_{z^{-1},\lambda_0} \equiv c^{ij}_{z,\lambda_0} \equiv \pm 1 \mod \mathfrak{p},$$

we conclude that $\mathbf{a}(z) \geqslant \alpha_{\lambda_0}$. ∎

Definition 4.2. Recall that $\mathcal{H}$ is assumed to be normalized and integral. Let $z \in W$. Then we set

$$\alpha_z := \alpha_\lambda,$$

where λ is the unique element of Λ such that $z \in \mathfrak{T}_\lambda$; see Lemma 3.8. By Proposition 4.1, we have $\alpha_z \leqslant \mathbf{a}(z)$ for all $z \in W$.

The following result shows that the identity $\mathbf{a}(z) = \alpha_z$ holds for $z \in W$ once we know that (P4) from the list of Lusztig's conjectures holds. Note, however, that it seems to be very hard to prove (P4) directly. Therefore, in Lemma 4.4 below, we shall establish a somewhat different criterion for proving that the identity $\mathbf{a}(z) = \alpha_z$ holds for all $z \in W$ (and it is this latter criterion which will be used in Section 5 in dealing with type B_n).

Lemma 4.3. *Assume that* (P4) *holds. Then* $\mathbf{a}(z) = \alpha_z$ *for all* $z \in W$.

Proof. Let $z \in W$. We already know by Proposition 4.1 that $\alpha_z \leqslant \mathbf{a}(z)$. To prove the reverse inequality, let $x, y \in W$ be such that $e^{\mathbf{a}(z)}\, h_{x,y,z} \in A_{\geqslant 0}$ has a non-zero constant term. As in the proof of Proposition 4.1, we have

$$e^{\mathbf{a}(z)} h_{x,y,z} \equiv \pm \sum_{\lambda \in \Lambda} \sum_{i,j,k=1}^{d_\lambda} c^{ij}_{x,\lambda}\, c^{jk}_{y,\lambda} \Big(e^{\mathbf{a}(z)}\, \mathfrak{X}^{ki}_\lambda(D_{z^{-1}}) \Big) \mod \mathfrak{p}.$$

We are assuming that the left hand side is $\not\equiv 0 \bmod \mathfrak{p}$. So there exists some $\lambda_0 \in \Lambda$ and $1 \leqslant i, j, k \leqslant d_{\lambda_0}$ such that $c^{ij}_{x,\lambda_0} \neq 0$, $c^{jk}_{y,\lambda_0} \neq 0$ and

$$(\dagger) \qquad e^{\mathbf{a}(z)}\, \mathfrak{X}^{ki}_{\lambda_0}(D_{z^{-1}}) \not\equiv 0 \mod \mathfrak{p}.$$

The condition (†) implies that $\alpha_{\lambda_0} \geqslant \mathbf{a}(z)$. Furthermore, we have $x, y \in \mathfrak{T}_{\lambda_0}$ and so $\mathbf{a}(x) \geqslant \alpha_{\lambda_0}$, $\mathbf{a}(y) \geqslant \alpha_{\lambda_0}$ by Proposition 4.1. Hence we obtain

$$\mathbf{a}(x) \geqslant \alpha_{\lambda_0} \geqslant \mathbf{a}(z).$$

But, since $h_{x,y,z} \neq 0$, we have $z \leqslant_{\mathcal{R}} x$ and so $\mathbf{a}(x) \leqslant \mathbf{a}(z)$, thanks to the assumption that (P4) holds. Thus, we conclude that $\mathbf{a}(z) = \alpha_{\lambda_0}$. But then (†) also yields that $c_{z,\lambda_0}^{ik} = c_{z^{-1},\lambda_0}^{ki} \neq 0$ and so $z \in \mathfrak{T}_{\lambda_0}$. Thus, we have $\mathbf{a}(z) = \alpha_{\lambda_0} = \alpha_z$, as claimed. ∎

LEMMA 4.4. *Assume that the following implication holds for any $x, y \in W$:*

$$(\clubsuit) \qquad x \leqslant_{\mathcal{LR}} y \implies \alpha_y \leqslant \alpha_x.$$

Then we have $\mathbf{a}(z) = \alpha_z$ for all $z \in W$ (and, consequently, (P4) *holds).*

Proof. Let $z \in W$. By Proposition 4.1, we already know that $\alpha_z \leqslant \mathbf{a}(z)$; furthermore, in order to prove equality, it will be enough to show that

$$e^{\alpha_z}\mathfrak{X}_\lambda^{ij}(D_{z^{-1}}) \in \mathcal{O} \quad \text{for all } \lambda \in \Lambda \text{ and } 1 \leqslant i, j \leqslant d_\lambda.$$

To prove this, let $\lambda \in \Lambda$ be such that $\mathfrak{X}_\lambda^{ij}(D_{z^{-1}}) \neq 0$ for some i, j. Let $\mathfrak{C}$ be a left cell such that $[\chi_{\mathfrak{C}} : \chi_\lambda] \neq 0$, that is, χ_λ occurs with non-zero multiplicity in the character afforded by $\mathfrak{C}$. Then $\mathfrak{X}_\lambda$ will occur (up to equivalence) in the decomposition of $\mathfrak{X}_{\mathfrak{C}}$ as a direct sum of irreducible representations. Hence, since $\mathfrak{X}_\lambda(D_{z^{-1}}) \neq 0$, we will also have $\mathfrak{X}_{\mathfrak{C}}(D_{z^{-1}}) \neq 0$. Recalling the definition of $\mathfrak{X}_{\mathfrak{C}}$, we deduce that there exists some $x \in \mathfrak{C}$ such that $D_{z^{-1}}C_x \neq 0$. Since τ is non-degenerate, we have

$$\tau(D_{z^{-1}}C_xC_w) \neq 0 \quad \text{for some } w \in W.$$

This yields $\pm h_{x,w,z} = \tau(C_xC_wD_{z^{-1}}) = \tau(D_{z^{-1}}C_xC_w) \neq 0$ and so $z \leqslant_{\mathcal{R}} x$. Hence ($\clubsuit$) implies that $\alpha_x \leqslant \alpha_z$. Now we claim that $\alpha_x = \alpha_\lambda$. This can be seen as follows. Let $i \in \{1, \ldots, d_\lambda\}$. Then the right hand side of the formula in Proposition 3.2 is non-zero and so there exists some $y \in \mathfrak{C}$ and some k such that $c_{y,\lambda}^{ik} \neq 0$. Consequently, we have $y \in \mathfrak{T}_\lambda$ and so $\alpha_y = \alpha_\lambda$. On the other hand, ($\clubsuit$) implies that the function $w \mapsto \alpha_w$ is constant on two-sided cells. Hence we can deduce that $\alpha_x = \alpha_y = \alpha_\lambda$, as claimed.

Now the inequality $\alpha_x \leqslant \alpha_z$ yields $\alpha_\lambda \leqslant \alpha_z$ and so

$$e^{\alpha_z}\mathfrak{X}_\lambda^{ij}(D_{z^{-1}}) = e^{\alpha_\lambda}\mathfrak{X}_\lambda^{ij}(D_{z^{-1}})\, e^{\alpha_z - \alpha_\lambda} \in \mathcal{O},$$

as desired. ∎

We will now show that the validity of the identity $\mathbf{a}(z) = \alpha_z$ for all $z \in W$ together with (P4) formally imply most of the remaining conjectures from the list (P1)–(P15).

We introduce the following notation. Let $x, y \in W$. By Lemma 3.8, we can write $x = w_\lambda(i_0, j_0)$ and $y = w_\mu(k_0, l_0)$ where $\lambda, \mu \in \Lambda$, $1 \leqslant i_0, j_0 \leqslant d_\lambda$ and $1 \leqslant k_0, l_0 \leqslant d_\mu$ are uniquely determined. Then we set

$$z = x \star y := w_\lambda(l_0, i_0) \quad \text{if } \lambda = \mu \text{ and } j_0 = k_0.$$

We note the following equivalences for $x, y, z \in \mathfrak{T}_\lambda$:

$$(\heartsuit) \qquad z = x \star y \iff x = y \star z \iff y = z \star x.$$

If $z = x \star y$ as above, we also set

$$n_{x,y,z} = c_{x,\lambda}^{i_0 j_0} c_{y,\lambda}^{j_0 l_0} c_{z,\lambda}^{l_0 i_0} = \pm 1.$$

Note that $(\heartsuit)$ implies $n_{x,y,z} = n_{y,z,x} = n_{z,x,y}$. With this notation, we now have the following result.

LEMMA 4.5. *Assume that* (P4) *holds. Let $x, y, z \in W$. If there is no $\lambda \in \Lambda$ such that $x, y, z \in \mathfrak{T}_\lambda$, then $\gamma_{x,y,z} = 0$. If $x, y, z \in \mathfrak{T}_\lambda$, then*

$$\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y} = \begin{cases} n_{x,y,z} & \textit{if } z = x \star y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Remark 2.4, we have $\mathbf{a}(z) = \mathbf{a}(z^{-1})$. This yields

$$\gamma_{x,y,z} \equiv e^{\mathbf{a}(z)} h_{x,y,z^{-1}} \mod \mathfrak{p}.$$

Writing $h_{x,y,z^{-1}} = \varepsilon_x \, \varepsilon_y \, \varepsilon_z \, \tau(C_x C_y D_z)$ and $\tau = \sum_\mu c_\mu^{-1} \chi_\mu$, we obtain

$$\gamma_{x,y,z} \equiv \sum_{\mu \in \Lambda} \sum_{i,j,k=1}^{d_\mu} c_{x,\mu}^{ij} \, c_{y,\mu}^{jk} \Big(\varepsilon_z \, e^{\mathbf{a}(z)} \, \mathfrak{X}_\mu^{ki}(D_z) \Big) \mod \mathfrak{p};$$

see the argument in the proof of Proposition 4.1. Now, if there is no λ such that $x, y \in \mathfrak{T}_\lambda$, then the above sum certainly is zero.

Hence it remains to consider the case where $x, y \in \mathfrak{T}_\lambda$ for some $\lambda \in \Lambda$ (which is uniquely determined). We write $x = w_\lambda(i_0, j_0)$ and $y = w_\lambda(k_0, l_0)$, where $1 \leqslant i_0, j_0, k_0, l_0 \leqslant d_\lambda$ are uniquely determined. Then the above sum reduces to

$$\gamma_{x,y,z} \equiv \delta_{j_0k_0}\, c_{x,\lambda}^{i_0j_0}\, c_{y,\lambda}^{k_0l_0} \Big(\varepsilon_z\, e^{\mathbf{a}(z)}\, \mathfrak{X}_\lambda^{l_0i_0}(D_z)\Big) \mod \mathfrak{p}.$$

First assume that $z = x \star y$, that is, we have $j_0 = k_0$ and $z = w_\lambda(l_0, i_0)$. In particular, $z \in \mathfrak{T}_\lambda$ and so $\mathbf{a}(z) = \alpha_\lambda$ by Lemma 4.3. Hence, we obtain

$$\gamma_{x,y,z} = c_{x,\lambda}^{i_0j_0}\, c_{y,\lambda}^{k_0l_0}\, c_{z,\lambda}^{l_0i_0} = n_{x,y,z} = \pm 1,$$

as desired.

Conversely, assume that $\gamma_{x,y,z} \neq 0$. First of all, this means that $j_0 = k_0$. Furthermore, the constant term of $e^{\mathbf{a}(z)}\, \mathfrak{X}_\lambda^{l_0i_0}(D_z)$ is non-zero and so $\alpha_\lambda \geqslant \mathbf{a}(z)$. On the other hand, we have $h_{x,y,z^{-1}} \neq 0$ and so $z^{-1} \leqslant_{\mathcal{R}} x$. Hence (P4) yields $\mathbf{a}(z) = \mathbf{a}(z^{-1}) \geqslant \mathbf{a}(x) = \alpha_\lambda$ (since $x \in \mathfrak{T}_\lambda$) and so $\mathbf{a}(z) = \alpha_\lambda$. Consequently, the constant term of $\varepsilon_z\, e^{\mathbf{a}(z)}\, \mathfrak{X}_\lambda^{l_0i_0}(D_z)$ is $c_{z,\lambda}^{l_0i_0}$ and so

$$0 \neq \gamma_{x,y,z} = c_{x,\lambda}^{i_0j_0}\, c_{y,\lambda}^{j_0l_0}\, c_{z,\lambda}^{l_0i_0}.$$

Hence, we must have $j_0 = k_0$ and $z = x \star y$.

The identity $\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}$ easily follows from the symmetry in ($\heartsuit$). $\square$

LEMMA 4.6. *Assume that* (P4) *holds. Then* (P1), (P2), (P3), (P5), (P6), (P7), (P8) *and* (P14) *hold.*

Proof. **(P1)** Let $z \in W$. We must show $\mathbf{a}(z) \leqslant \mathbf{\Delta}(z)$. Now, by the definition of the symmetrizing trace τ, we have

$$\tau(C_z) = \varepsilon_z\, \overline{P}_{1,z}^*.$$

On the other hand, we have $\tau = \sum_\lambda c_\lambda^{-1} \chi_\lambda$. Using the expression $c_\lambda = e^{-2\alpha_\lambda} f_\lambda$ where $f_\lambda^{-1} \in \mathcal{O}$, this yields the identity

$$\varepsilon_z\, \overline{P}_{1,z}^* = \sum_{\lambda \in \Lambda} \sum_{i=1}^{d_\lambda} f_\lambda^{-1} (e^{\alpha_\lambda}\, \mathfrak{X}_\lambda^{ii}(C_z))\, e^{\alpha_\lambda}.$$

Assume that the term in the sum corresponding to $\lambda \in \Lambda$ and $1 \leqslant i \leqslant d_\lambda$ is non-zero. Let $\mathfrak{C}$ be a left cell such that $[\chi_{\mathfrak{C}} : \chi_\lambda] \neq 0$, that is, χ_λ occurs with non-zero multiplicity in the character afforded by $\mathfrak{C}$. Then $\mathfrak{X}_\lambda$ will occur (up to equivalence) in the decomposition of $\mathfrak{X}_{\mathfrak{C}}$ as a direct sum of irreducible representations. Hence, since $\mathfrak{X}_\lambda(C_z) \neq 0$, we will also have $\mathfrak{X}_{\mathfrak{C}}(C_z) \neq 0$. Recalling the definition of $\mathfrak{X}_{\mathfrak{C}}$, we deduce that there exist some $x, y \in \mathfrak{C}$ such that $h_{z,x,y} \neq 0$. But then we have $y \leqslant_{\mathcal{R}} z$ and so $\mathbf{a}(z) \leqslant \mathbf{a}(y)$, by (P4). Now, as in the proof of Lemma 4.4, we conclude that $\mathbf{a}(y) = \mathbf{a}(y') = \alpha_\lambda$, where $y' \in \mathfrak{C}$ is chosen such that $c_{y',\lambda}^{ik} \neq 0$ for some k. Hence, we have $\mathbf{a}(z) \leqslant \alpha_\lambda$ for all non-zero terms in the above sum. Thus, we obtain

$$\varepsilon_z\, e^{-\mathbf{a}(z)}\, \overline{P}_{1,z}^{\,*} = \sum_{\substack{\lambda \in \Lambda \\ \mathbf{a}(z) \leqslant \alpha_\lambda}} \sum_{i=1}^{d_\lambda} f_\lambda^{-1}(e^{\alpha_\lambda}\, \mathfrak{X}_\lambda^{ii}(C_z))\, e^{\alpha_\lambda - \mathbf{a}(z)} \in \mathcal{O}.$$

Since the left hand side lies in A, we conclude that $e^{-\mathbf{a}(z)}\, \overline{P}_{1,z}^{\,*} \in A_{\geqslant 0}$ and so $\mathbf{a}(z) \leqslant \mathbf{\Delta}(z)$, as desired.

(P2), **(P5)** Let $x, y \in W$ and $d \in \mathcal{D}$ be such that $\gamma_{x,y,d} \neq 0$. We must show that $x = y^{-1}$ and $\gamma_{y^{-1},y,d} = n_d = \pm 1$. Now, using the expression of C_w in terms of the T-basis, we obtain

$$\begin{aligned} \tau(C_x C_y) &= \sum_{z \in W} \varepsilon_x\, \varepsilon_y\, \varepsilon_z\, h_{x,y,z}\, \tau(C_z) = \sum_{z \in W} \varepsilon_x\, \varepsilon_y\, h_{x,y,z}\, \overline{P}_{1,z}^{\,*} \\ &= \sum_{z \in W} \varepsilon_x\, \varepsilon_y\, e^{\mathbf{a}(z)}\, h_{x,y,z} (e^{-\mathbf{a}(z)}\, \overline{P}_{1,z}^{\,*}). \end{aligned}$$

By the definition of $\mathbf{a}(z)$ and (P1) (see the proof above), we have $e^{\mathbf{a}(z)}\, h_{x,y,z} \in A_{\geqslant 0}$ and $e^{-\mathbf{a}(z)}\, \overline{P}_{1,z}^{\,*} \in A_{\geqslant 0}$. So the terms of the above sum lie in $A_{\geqslant 0}$ and we have

$$\tau(C_x C_y) \equiv \sum_{z \in \mathcal{D}} \varepsilon_x\, \varepsilon_y\, \gamma_{x,y,z^{-1}}\, n_z \mod \mathfrak{p}.$$

By Remark 2.4, we have $n_z = n_{z^{-1}}$ and $\mathcal{D}^{-1} = \mathcal{D}$. So the above congruence can also be written in the form

$$\tau(C_x C_y) \equiv \sum_{z \in \mathcal{D}} \varepsilon_x\, \varepsilon_y\, \gamma_{x,y,z}\, n_z \mod \mathfrak{p}.$$

Now, Lemma 4.5 shows that the only non-zero term in the above sum is $\gamma_{x,y,d}$ and that $d = x \star y$. So we have

$$\tau(C_x C_y) \equiv \varepsilon_x\, \varepsilon_y\, \gamma_{x,y,d}\, n_d \not\equiv 0 \mod \mathfrak{p}.$$

On the other hand, we also have

$$\tau(C_x C_y) \equiv \delta_{x^{-1}y} \mod A_{>0},$$

where $\delta_{x^{-1}y}$ denotes the Kronecker symbol. (This easily follows from the defining formulas; see [20, 14.5(a)].) Hence we obtain the congruence

$$\delta_{x^{-1}y} \equiv \tau(C_x C_y) \equiv \varepsilon_x\, \varepsilon_y\, \gamma_{x,y,d}\, n_d \not\equiv 0 \mod \mathfrak{p}.$$

So we must have $x^{-1} = y$. But then we also get $\gamma_{y^{-1},y,d} n_d = 1$, as required.

(P3) Let $y \in W$. We want to show that there exists a unique $z \in \mathcal{D}$ such that $\gamma_{y^{-1},y,z} \neq 0$. As in the proof of (P2), we have

$$\sum_{z \in \mathcal{D}} \gamma_{y^{-1},y,z} n_z \equiv \tau(C_{y^{-1}} C_y) \equiv 1 \mod \mathfrak{p}.$$

Consequently, there exists some $z \in \mathcal{D}$ such that $\gamma_{y^{-1},y,z} \neq 0$. By Lemma 4.5, $z = y^{-1} \star y$ is uniquely determined with this property.

(P6) This is a formal consequence of (P2) and (P3); see [20, 14.6].

(P7), **(P8)** This is clear by Lemma 4.5 and Remark 3.9.

(P14) This is clear by Remark 3.9. ∎

COROLLARY 4.7. *We keep the hypotheses of Lemma* 4.6. *Let $z \in \mathcal{D}$ and $\lambda \in \Lambda$ be such that $z \in \mathfrak{T}_\lambda$. Then the constant $n_z = \pm 1$ is determined by the formula*

$$\varepsilon_z\, e^{\alpha_\lambda}\, \chi_\lambda(T_z) \equiv n_z \mod \mathfrak{p}.$$

Thus, n_z is precisely the leading coefficient of a character value as defined by Lusztig [19].

Proof. Since $z \in \mathcal{D}$, we have $z^2 = 1$ and so $z = w_\lambda(i_0, i_0)$ for a unique $i_0 \in \{1, \ldots, d_\lambda\}$. Hence we obtain

$$\varepsilon_z\, e^{\alpha_\lambda}\, \chi_\lambda(T_z) \equiv \varepsilon_z \sum_{i=1}^{d_\lambda} e^{\alpha_\lambda} \mathfrak{X}_\lambda^{ii}(T_z) \equiv \sum_{i=1}^{d_\lambda} c_{z,\lambda}^{ii} \equiv c_{z,\lambda}^{i_0 i_0} \mod \mathfrak{p}.$$

On the other hand, by (P5) and Lemma 4.5, we have $n_z = \gamma_{z,z,z} = c_{z,\lambda}^{i_0 i_0}$, as required. ∎

Lemma 4.8. *We keep the hypotheses of Lemma* 4.6 *and assume, in addition, that* $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_K)$ *for every left cell* $\mathfrak{C}$ *of* W. *Then* (P13) *holds and we have*

$$\mathcal{D} = \{z \in W \mid z^2 = 1\}.$$

Proof. Let $\mathfrak{C}$ be a left cell of W. Let $x \in \mathfrak{C}$. By Lemma 4.6, (P3) holds and so there exists a unique $d \in \mathcal{D}$ such that $\gamma_{x^{-1},x,d} \neq 0$. By (P8), we have $d \sim_{\mathcal{L}} x$ and so $d \in \mathfrak{C}$. Thus, each left cell contains an element of $\mathcal{D}$. Note that the above argument also shows that (P13) holds, once we know that each left cell contains a unique element of $\mathcal{D}$.

Now, by Lemma 3.10, the total number of left cells equals $\sum_{\lambda\in\Lambda} d_\lambda$. Thus, we have

$$|\mathcal{D}| \geqslant \sum_{\lambda\in\Lambda} d_\lambda.$$

On the other hand, by a well-known result due to Frobenius-Schur, the number on the right hand side is the number of all $z \in W$ such that $z^2 = 1$. (We also use the fact that every irreducible character of W can be realized over $\mathbb{R}$; see [14, 6.3.8].) Since $d^2 = 1$ for all $d \in \mathcal{D}$ by (P6), we conclude that

$$\mathcal{D} = \{z \in W \mid z^2 = 1\}$$

and that $\mathfrak{C}$ contains a unique element from $\mathcal{D}$. ∎

Finally, let J be the free abelian group with basis $\{t_w \mid w \in W\}$. We define a bilinear pairing on J by

$$t_x \cdot t_y = \sum_{z\in W} \gamma_{x,y,z^{-1}}\, t_z \quad \text{for all } x, y \in W,$$

where the constants $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ were introduced in Section 2.

Proposition 4.9. *Assume that* (P4) *holds. Then* J *is an associative ring with identity* $1_J = \sum_{z\in\mathcal{D}} n_z t_z$. *For any* $\lambda \in \Lambda$, *we set*

$$J_\lambda := \langle t_w \mid w \in \mathfrak{T}_\lambda \rangle \subseteq J.$$

Then J_λ *is a two-sided ideal which is isomorphic to the matrix ring* $M_{d_\lambda}(\mathbb{Z})$, *and we have* $J = \bigoplus_{\lambda\in\Lambda} J_\lambda$.

We have $t_z^2 = n_z t_z$ *for any* $z \in \mathcal{D}$.

In Example 5.7, we will see that negative coefficients actually do occur in 1_J.

Proof. By Lemma 4.6, we know that (P1)–(P8) hold. Hence J can be constructed as explained in [20, Chap. 18]. Now Lemma 4.5 immediately shows that J_λ is a two-sided ideal and so we have $J = \bigoplus_\lambda J_\lambda$. To establish the stated isomorphism $J_\lambda \cong M_{d_\lambda}(\mathbb{Z})$, we explicitly construct a set of "matrix units" in J_λ. This is done as follows. Let us fix $\lambda \in \Lambda$. For any $1 \leqslant i, j \leqslant d_\lambda$, we set

$$E_\lambda^{ij} := c_{w,\lambda}^{ij} t_w \quad \text{where } w = w_\lambda(i,j) \text{ and } c_{w,\lambda}^{ij} = \pm 1.$$

Now let $1 \leqslant i, j, k, l \leqslant d_\lambda$ and write $x = w_\lambda(i,j)$, $y = w_\lambda(k,l)$. Then we have

$$E_\lambda^{ij} \cdot E_\lambda^{kl} = c_{x,\lambda}^{ij}\, c_{y,\lambda}^{kl}\, t_x \cdot t_y = c_{x,\lambda}^{ij}\, c_{y,\lambda}^{kl} \sum_{z \in W} \gamma_{x,y,z}\, t_{z^{-1}}.$$

Now Lemma 4.5 shows that the result will be zero unless $j = k$. So let us finally assume that $j = k$. Then we obtain

$$\begin{aligned} E_\lambda^{ij} \cdot E_\lambda^{jl} &= c_{x,\lambda}^{ij} c_{y,\lambda}^{jl} \gamma_{x,y,z_0}\, t_{z_0^{-1}}, \\ &= c_{x,\lambda}^{ij} c_{y,\lambda}^{jl} c_{z_0,\lambda}^{li} \gamma_{x,y,z_0} E_\lambda^{il}, \end{aligned}$$

where $z_0 := x \star y = w_\lambda(l,i)$. By Lemma 4.5, the coefficient of E_λ^{il} in the above expression equals 1. Thus, we have shown that

$$E_\lambda^{ij} \cdot E_\lambda^{kl} = \delta_{jk} E_\lambda^{il} \quad \text{for } 1 \leqslant i, j, k, l \leqslant d_\lambda.$$

Hence, the elements E_λ^{ij} multiply in exactly the same way as the matrix units in $M_{d_\lambda}(\mathbb{Z})$, which yields the desired isomorphism.

The formula for t_z^2, where $z \in \mathcal{D}$, is obtained as follows. We have $t_z^2 = \gamma_{z,z,x} t_{x^{-1}}$ where $x = z \star z$ and $\gamma_{z,z,x} \neq 0$. By (P2), (P6), (P7), we have $\gamma_{x,z,z} \neq 0$ and so $x = z^{-1} = z$. Then (P5) yields $\gamma_{z,z,z} = n_z = \pm 1$. ∎

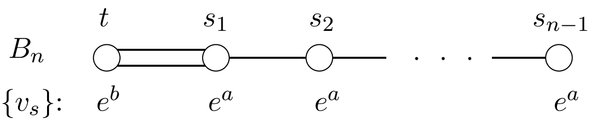
Remark 4.10. We keep the setting of Proposition 4.9. Let us also assume that $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_K)$ for all left cells $\mathfrak{C}$ in W. Let $x, y, z \in W$ be such that $x, y, z \in \mathfrak{T}_\lambda$ for some λ. Then note that, by Remark 3.9 and Lemma 3.10, the condition $z = x \star y$ is equivalent to the conditions $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$. Thus, the multiplication rule in J can now be formulated as follows:

$$t_x \cdot t_y = \begin{cases} \pm t_{z^{-1}} & \text{if } x \sim_{\mathcal{L}} y^{-1},\ y \sim_{\mathcal{L}} z^{-1},\ z \sim_{\mathcal{L}} x^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing the results in this section, we see that for a normalized and integral Iwahori-Hecke algebra, property (P4) (or the variant in Lemma 4.4) implies all the remaining properties in the list of Lusztig's conjectures except (P9)–(P12) and (P15).

§5. The a-function in the "asymptotic case" in type B_n

Throughout this section, we place ourselves in the setting of Example 1.1, where W_n is a Coxeter group of type B_n and the weight function $L\colon W_n \to \Gamma$ is given by the following diagram:



$b > (n-1)a > 0$

Let $\mathcal{H}_n$ be the corresponding Iwahori-Hecke algebra over A where $V = e^b$ is the parameter associated with the generator t and $v = e^a$ is the parameter associated with the generators $s_1, \ldots, s_{n-1}$ of W_n. Our aim is to see that we can apply the methods in Section 4. In Corollary 5.5, we will be able to show that the key condition (♣) in Lemma 4.4 holds in the present setting.

We shall need some notation from [2]. Given $w \in W_n$, we denote by $\ell_t(w)$ the number of occurrences of the generator t in a reduced expression for w, and call this the "t-length" of w.

The parabolic subgroup $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ is naturally isomorphic to the symmetric group on $\{1, \ldots, n\}$, where s_i corresponds to the basic transposition $(i, i+1)$. For $1 \leqslant l \leqslant n-1$, we set $\Sigma_{l,n-l} := \{s_1, \ldots, s_{n-1}\} \setminus \{s_l\}$. For $l = 0$ or $l = n$, we also set $\Sigma_{0,n} = \Sigma_{n,0} = \{s_1, \ldots, s_{n-1}\}$. Let $Y_{l,n-l}$ be the set of distinguished left coset representatives of the Young subgroup $\mathfrak{S}_{l,n-l} := \langle \Sigma_{l,n-l} \rangle$ in $\mathfrak{S}_n$. We have the parabolic subalgebra $\mathcal{H}_{l,n-l} = \langle T_\sigma \mid \sigma \in \mathfrak{S}_{l,n-l} \rangle_A \subseteq \mathcal{H}_n$.

We denote by $\leqslant_{\mathcal{L},l}$ the Kazhdan-Lusztig (left) pre-order relation on $\mathfrak{S}_{l,n-l}$ and by $\sim_{\mathcal{L},l}$ the corresponding equivalence relation. The symbols $\leqslant_{\mathcal{R},l}$, $\leqslant_{\mathcal{LR},l}$, $\sim_{\mathcal{R},l}$ and $\sim_{\mathcal{LR},l}$ have a similar meaning.

Furthermore, as in [2, §4], we set $a_0 = 1$ and

$$a_l := t(s_1 t)(s_2 s_1 t) \cdots (s_{l-1} s_{l-2} \cdots s_1 t) \quad \text{for } l > 0.$$

Then, by [2, Prop. 4.4], the set $Y_{l,n-l} a_l$ precisely is the set of distinguished left coset representatives of $\mathfrak{S}_n$ in W_n whose t-length equals l. Furthermore, every element $w \in W_n$ has a unique decomposition

$$w = a_w a_l \sigma_w b_w^{-1} \quad \text{where } l = \ell_t(w),\ \sigma_w \in \mathfrak{S}_{l,n-l} \text{ and } a_w, b_w \in Y_{l,n-l};$$

see [2, 4.6]. With this notation, we have the following result.

THEOREM 5.1. (Bonnafé-Iancu [2] and Bonnafé [3, §5]) *In the above setting, let $x, y \in W_n$. Then we have $x \sim_{\mathcal{L}} y$ if and only if $l := \ell_t(x) = \ell_t(y)$, $b_x = b_y$ and $\sigma_x \sim_{\mathcal{L},l} \sigma_y$.*

(In [2, Theorem 7.7], the above statement is proved in the "generic asymptotic case". As already discussed in Example 3.11, this remains valid in the "asymptotic case" by [3, Cor. 5.2].)

We shall also need the following result on the elementary steps in the relation $\leqslant_{\mathcal{L}}$.

PROPOSITION 5.2. (Bonnafé-Iancu [2] and Bonnafé [3, §5]) *In the above setting, let $x, y \in W_n$ be such that $x \leftarrow_{\mathcal{L}} y$. Then we have $\ell_t(x) = \ell_t(y)$ or $x = ty > y$. In particular, we always have $\ell_t(y) \leqslant \ell_t(x)$. (A similar result holds for $\leftarrow_{\mathcal{R}}$.)*

(The precise references are Theorems 6.3, 6.6 and Corollary 6.7 in [2] for the "generic asymptotic case". In [3, Cor. 5.2], it is shown that, if $x \leftarrow_{\mathcal{L}} y$ with respect to $L\colon W_n \to \Gamma$, then we also have $x \leftarrow_{\mathcal{L}} y$ with respect to $L_0\colon W_n \to \Gamma_0$ as in Example 1.1. Hence the assertions hold in the "asymptotic case" too.)

As discussed in Example 3.11, let Λ_n be the set of bipartitions of n. Then the partition

$$W_n = \coprod_{(\lambda_1,\lambda_2)\in\Lambda_n} \mathfrak{T}_{(\lambda_1,\lambda_2)}$$

is explicitly given by

$$\mathfrak{T}_{(\lambda_1,\lambda_2^*)} = \{w \in W_n \mid w \text{ belongs to an RS-cell of shape } (\lambda_1,\lambda_2)\}.$$

Furthermore, for $w \in \mathfrak{T}_{(\lambda_1,\lambda_2^*)}$, we have

$$\alpha_w = \alpha_{(\lambda_1,\lambda_2^*)} = b\,|\lambda_2| + a\,(n(\lambda_1) + 2n(\lambda_2^*) - n(\lambda_2)).$$

Finally, we need the following result concerning the relation $\leqslant_{\mathcal{LR}}$.

THEOREM 5.3. (Bonnafé [3]) *In the above setting, let $x, y \in W_n$ be such that $l := \ell_t(x) = \ell_t(y)$. Then we have $x \leqslant_{\mathcal{LR}} y$ if and only if $\sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$. Furthermore, the sets $\mathfrak{T}_{(\lambda_1,\lambda_2)}$, $(\lambda_1, \lambda_2) \in \Lambda_n$, are precisely the two-sided cells of W_n.*

Using known results on the two-sided cells in the symmetric group and the Robinson-Schensted correspondence, we can translate the above statement into a combinatorial description of the relation $\leqslant_{\mathcal{LR}}$ for W_n. To state this, we need to introduce some notation. Recall the definition of the dominance order on partitions in Example 2.5. Following Dipper-James-Murphy [6, §3], we can extend this partial order to bipartitions, as follows.

Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be bipartitions of n, with parts

$$\lambda_1 = (\lambda_1^{(1)} \geqslant \lambda_1^{(2)} \geqslant \cdots \geqslant 0), \quad \lambda_2 = (\lambda_2^{(1)} \geqslant \lambda_2^{(2)} \geqslant \cdots \geqslant 0),$$
$$\mu_1 = (\mu_1^{(1)} \geqslant \mu_1^{(2)} \geqslant \cdots \geqslant 0), \quad \mu_2 = (\mu_2^{(1)} \geqslant \mu_2^{(2)} \geqslant \cdots \geqslant 0).$$

Then we write $\lambda \trianglelefteq \mu$ if

$$\sum_{i=1}^{j} \lambda_1^{(j)} \leqslant \sum_{i=1}^{j} \mu_1^{(j)} \quad \text{for all } j$$

and

$$|\lambda_1| + \sum_{i=1}^{j} \lambda_2^{(j)} \leqslant |\mu_1| + \sum_{i=1}^{j} \mu_2^{(j)} \quad \text{for all } j.$$

Note that, if $|\lambda_1| = |\mu_1|$, then we have

$$\lambda \trianglelefteq \mu \iff \lambda_1 \trianglelefteq \mu_1 \quad \text{and} \quad \lambda_2 \trianglelefteq \mu_2,$$

where, on the right, the symbol $\trianglelefteq$ just denotes the usual dominance order on partitions, as in Example 2.5. The following result is a refinement of [3, Remark 3.7] (which only deals with elements of the same t-length).

PROPOSITION 5.4. *Let $x, y \in W_n$ be such that $x \leqslant_{\mathcal{LR}} y$. Assume that x belongs to an RS-cell of shape (λ_1, λ_2) and y belongs to an RS-cell of shape (μ_1, μ_2). Then we have*

$$(\lambda_1, \lambda_2^*) \trianglelefteq (\mu_1, \mu_2^*),$$

with equality only if $x \sim_{\mathcal{LR}} y$.

Proof. Let $w \in W_n$ and write $w = a_w a_l \sigma_w b_w^{-1}$ where $l = \ell_t(w)$. Now the parabolic subgroup $\mathfrak{S}_{l,n-l}$ is a direct product of $\mathfrak{S}_l = \langle s_1, \ldots, s_{l-1}\rangle$ and $\mathfrak{S}_{[l+1,n]} = \langle s_{l+1}, \ldots, s_{n-1}\rangle$. Thus, we have

$$\sigma_w = \sigma'_w \sigma''_w \quad \text{where } \sigma'_w \in \mathfrak{S}_l \text{ and } \sigma''_w \in \mathfrak{S}_{[l+1,n]}.$$

Since $\mathfrak{S}_l$ is a Coxeter group of type A_{l-1}, the classical Robinson-Schensted correspondence associates to σ'_w a pair of tableaux whose shape is a partition of l, say ν_2. Similarly, since $\mathfrak{S}_{[l+1,n]}$ is of type A_{n-l}, the classical Robinson-Schensted correspondence associates to σ''_w a pair of tableaux whose shape is a partition of $n-l$, say ν_1. Then, by the discussion in [2, 4.7], we have

(♠) w belongs to an RS-cell of shape (ν_1, ν_2^*).

Now consider the given two elements $x, y \in W_n$ such that $x \leqslant_{\mathcal{LR}} y$. By Proposition 5.2, this implies $\ell_t(y) \leqslant \ell_t(x)$. In particular, x and y cannot lie in the same two-sided cell unless x and y have the same t-length. We now distinguish two cases.

Case 1. Assume that $l := \ell_t(x) = \ell_t(y)$. By Theorem 5.3, we know that $x \leqslant_{\mathcal{LR}} y$ implies that $\sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$. Furthermore, it is well-known and easy to check that the Kazhdan-Lusztig pre-order relations are compatible with direct products; in particular, we have

$$\sigma'_x \leqslant'_{\mathcal{LR}} \sigma'_y \quad \text{and} \quad \sigma''_x \leqslant''_{\mathcal{LR}} \sigma''_y,$$

where a single dash denotes the pre-order relation on $\mathfrak{S}_l$ and a double-dash denotes the pre-order on $\mathfrak{S}_{[l+1,n]}$. Now (♠) shows that

- σ'_x is associated with the partition λ_2^*,
- σ'_y is associated with the partition μ_2^*,
- σ''_x is associated with the partition λ_1,
- σ''_y is associated with the partition μ_1.

Thus, we are reduced to statements concerning two-sided cells in the symmetric group. Now Example 2.5(e) shows that we have the implications

$$\sigma'_x \leqslant'_{\mathcal{LR}} \sigma'_y \implies \lambda_2^* \trianglelefteq \mu_2^* \quad \text{and} \quad \sigma''_x \leqslant'_{\mathcal{LR}} \sigma''_y \implies \lambda_1 \trianglelefteq \mu_1.$$

This yields $(\lambda_1, \lambda_2^*) \trianglelefteq (\mu_1, \mu_2^*)$ as required. Furthermore, if $(\lambda_1, \lambda_2^*) = (\mu_1, \mu_2^*)$, then $x \sim_{\mathcal{LR}} y$ by Theorem 5.3.

Case 2. Assume that $\ell_t(y) < \ell_t(x)$. By Case 1 and the definition of $\leqslant_{\mathcal{LR}}$, it is enough to consider an elementary step, where $\ell_t(y) < \ell_t(x)$ and $x \leftarrow_{\mathcal{L}} y$ or $x \leftarrow_{\mathcal{R}} y$. Since $w \sim_{\mathcal{LR}} w^{-1}$ for all $w \in W_n$, we can even assume that $x \leftarrow_{\mathcal{R}} y$, that is, $h_{y,s,x} \neq 0$ for some $s \in \{t, s_1, s_2, \ldots, s_{n-1}\}$. Since we are assuming $\ell_t(y) < \ell_t(x)$, we must have $s = t$ and $x = yt > y$ by Proposition 5.2. Thus, it remains to consider the effect of multiplying with t on the generalized Robinson-Schensted correspondence. We claim that, if $x = yt > y$, then

- λ_1 is obtained from μ_1 by decreasing one part by 1, and
- λ_2^* is obtained from μ_2^* by increasing one part by 1.

This is seen as follows. Recall the basic ingredients of the generalized Robinson-Schensted correspondence. We write $y \in W_n$ as a signed permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \varepsilon_1 \cdot p_1 & \varepsilon_2 \cdot p_2 & \cdots & \varepsilon_n \cdot p_n \end{pmatrix} \qquad (\varepsilon_1 = 1)$$

where the sequence $p_1, \ldots, p_n$ is a permutation of $1, \ldots, n$ and where $\varepsilon_i = \pm 1$ for all i. The fact that $\varepsilon_1 = 1$ follows from our assumption that $yt > y$. Let $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ be the sequence of indices where the sign is "+" and let $1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant n$ be the sequence of indices where the sign is "−". Applying the usual "insertion algorithm" to these two sequences of numbers, we obtain a standard bitableau $(A^+(y), A^-(y))$ of size n and shape (μ_1, μ_2), where $\ell_t(y) = |\mu_2| = l$.

Now let us multiply y on the right by t. Then the corresponding signed permutation is given by

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ -\varepsilon_1 \cdot p_1 & \varepsilon_2 \cdot p_2 & \cdots & \varepsilon_n \cdot p_n \end{pmatrix}.$$

Thus, the only effect of multiplying by t is to change the sign in the first position of the above array. Hence, in order to obtain the tableaux $A^+(yt)$ and $A^-(yt)$, we must apply the insertion algorithm to the sequences $p_{i_2}, \ldots, p_{i_k}$ and $p_{i_1}, p_{j_1}, p_{j_2}, \ldots, p_{j_l}$, respectively. Thus, we are reduced to a purely combinatorial statement. Using [9, §5, Prop. 1], one shows that the partition giving the shape of $A^+(yt)$ is obtained from the partition giving the shape of $A^+(y)$ by decreasing one part by 1. The same argument shows that the partition giving the shape of $A^-(yt)$ is obtained from the partition giving the shape of $A^-(y)$ by increasing one part by 1. (A much more general statement can be found in [22, Theorem 4.2].)

Now the definition of $\trianglelefteq$ immediately shows that, if (λ_1, λ_2) is obtained from (μ_1, μ_2) by the above procedure, then $(\lambda_1, \lambda_2^*) \trianglelefteq (\mu_1, \mu_2^*)$ as required. $\square$

COROLLARY 5.5. *Let $x, y \in W_n$ be such that $x \leqslant_{\mathcal{LR}} y$. Then we have $\alpha_y \leqslant \alpha_x$, with equality only if $x \sim_{\mathcal{LR}} y$.*

Proof. Assume that x belongs to an RS-cell of shape (λ_1, λ_2) and y belongs to an RS-cell of shape (μ_1, μ_2). Hence, by Example 3.11, we have

$$\alpha_x = \alpha_{(\lambda_1, \lambda_2^*)} \quad \text{and} \quad \alpha_y = \alpha_{(\mu_1, \mu_2^*)}.$$

The description of the generalized Robinson-Schensted correspondence in [2] shows that $\ell_t(x) = |\lambda_2| = |\lambda_2^*|$ and $\ell_t(y) = |\mu_2| = |\mu_2^*|$. Now, by Proposition 5.2, the condition $x \leqslant_{\mathcal{LR}} y$ implies that $\ell_t(y) \leqslant \ell_t(x)$.

Case 1. Assume that $l := \ell_t(x) = \ell_t(y)$. Then we have

$$\lambda_1 \trianglelefteq \mu_1 \quad \text{and} \quad \lambda_2^* \trianglelefteq \mu_2^*$$

by Proposition 5.4. Now note the following property of the dominance order. For any partitions ν and ν' of n, we have

$$\nu^* \trianglelefteq \nu'^* \iff \nu' \trianglelefteq \nu \implies n(\nu) \leqslant n(\nu'),$$

with equality only for $\nu = \nu'$; see, for example, [14, Exercise 5.6]. Using the above property, we conclude that

$$n(\mu_1) + 2n(\mu_2^*) - n(\mu_2) \leqslant n(\lambda_1) + 2n(\lambda_2^*) - n(\lambda_2),$$

with equality only if $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$. Hence, the formula for α_w shows that $\alpha_y \leqslant \alpha_x$, as required. Furthermore, if $\alpha_x = \alpha_y$, then we necessarily have $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$, and so $x \sim_{\mathcal{LR}} y$ by Theorem 5.3.

Case 2. Assume that $\ell_t(y) < \ell_t(x)$. As in the proof of Proposition 5.4, we can reduce to the case where $x = yt > y$.

In this case we have $\alpha_x - \alpha_y = b + a(2r' - m - r)$, where m, r, r' are integers determined by the conditions:

$$\lambda_1^{(m)} = \mu_1^{(m)} - 1, \quad \lambda_2^{(r)} = \mu_2^{(r)} + 1, \quad \lambda_2^{*(r')} = \mu_2^{*(r')} + 1.$$

Now note that we have $2r' - m - r + n - 1 \geqslant 0$, hence $\alpha_x > \alpha_y$ as desired. $\square$

Proofs of Theorem 1.2, *Theorem* 1.3 *and Theorem* 1.4. Let us recall the principal ingredients. By Example 3.6, we know that $\mathcal{H}_n$ is integral and normalized. Furthermore, by the discussion in Example 3.11, the characters afforded by all left cells are irreducible. Finally, the assumption (♣) in Lemma 4.4 holds by Corollary 5.5. Thus, we can conclude that

$$\mathbf{a}(w) = \alpha_{(\lambda_1,\lambda_2)} \quad (w \in \mathfrak{T}_{(\lambda_1,\lambda_2)}).$$

The identification of $\mathfrak{T}_{(\lambda_1,\lambda_2)}$ in Example 3.11 and the formula for $\alpha_{(\lambda_1,\lambda_2)}$ in Example 3.6 now yield the explicit description of the $\mathbf{a}$-function of W_n, proving Theorem 1.2.

Now, once the $\mathbf{a}$-function is determined, condition (♣) in Lemma 4.4 yields the implication "$x \leqslant_{\mathcal{LR}} y \Rightarrow \mathbf{a}(y) \leqslant \mathbf{a}(x)$" for any $x, y \in W_n$, that is, (P4) holds. But then Corollary 5.5 also yields the fact that, if $x \leqslant_{\mathcal{LR}} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_{\mathcal{LR}} y$, that is, (P11) holds. Now Lemma 4.6 and Lemma 4.8 show that all the other properties mentioned in Theorem 1.3 hold. As far as (P12) is concerned, note that every parabolic subgroup of W_n is a direct product of a group of type B_k and possibly some factors of type A_{n_i}. Since (P1)–(P15) are known to hold for groups of type A_{n_i}, we conclude that (P3), (P4), (P8) hold for every parabolic subgroup of W_n. This formally implies that (P12) holds; see [20, 14.12].

Finally, as explained in [20, 18.3], the ring J can be constructed once it is known that (P1)–(P8) are known to hold. The structure of J is now determined by Proposition 4.9 (and Remark 4.10).

Remark 5.6. Bonnafé has remarked at the end of [3, §4] that, once the equality $\mathbf{a}(z) = \alpha_z$ $(z \in W_n)$ is established, the methods in his paper [3] yield properties (P1), (P4), (P6), (P11), (P12) and the first assertion of (P13). However, it does not seem to be possible to gain control over the constants $\gamma_{x,y,z}$ in his approach, and this is where the leading matrix coefficients of orthogonal representations naturally come into play.

Example 5.7. Let us consider the case $n = 2$, where $W_2 = \langle t, s_1 \rangle$ is the dihedral group of order 8. We set $s_0 = t$. The polynomials $P^*_{y,w}$ and the left cells have already been determined by an explicit computation in [17, §6]. The left cells are

$$\{1\}, \quad \{s_1\}, \quad \{s_0, s_1s_0\}, \quad \{s_0s_1, s_1s_0s_1\}, \quad \{s_0s_1s_0\}, \quad \{w_0\}$$

where $w_0 = s_1s_0s_1s_0$ is the unique element of maximal length. Using the polynomials $P^*_{y,w}$, we compute:

$$\begin{aligned} \mathbf{\Delta}(1) &= 0, & n_1 &= 1,\\ \mathbf{\Delta}(s_1) &= a, & n_{s_1} &= 1,\\ \mathbf{\Delta}(s_0) &= b, & n_{s_0} &= 1,\\ \mathbf{\Delta}(s_1s_0s_1) &= b, & n_{s_1s_0s_2} &= 1,\\ \mathbf{\Delta}(s_0s_1s_0) &= 2b-a, & n_{s_0s_1s_0} &= -1,\\ \mathbf{\Delta}(w_0) &= 2b+2a, & n_{w_0} &= 1. \end{aligned}$$

In particular, we see that the coefficients n_z can be negative. Proposition 4.9 shows that

$$t_{s_0s_1s_0}^2 = -t_{s_0s_1s_0}; \quad \text{see also Lusztig [20, 18.7].}$$

There is a unique irreducible character χ_λ of degree 2; it is labelled by $\lambda = ((1),(1))$. A corresponding orthogonal representation and the leading matrix coefficients are explicitly described in [10, Exp. 5.5]. We have $\alpha_\lambda = b$ and the representation $\mathfrak{X}_\lambda$ is given by

$$\mathfrak{X}_\lambda\colon T_{s_0} \longmapsto \begin{bmatrix} V & 0 \\ 0 & -V^{-1}\end{bmatrix}, \quad T_{s_1} \longmapsto \frac{1}{V^2+1}\begin{bmatrix} v-v^{-1} & 1+V^2v^{-2} \\ 1+V^2v^2 & V^2(v-v^{-1})\end{bmatrix}.$$

Thus, we see that $c^{2,2}_{s_0,\lambda} = c^{1,1}_{s_1s_0s_1,\lambda} = 1$ and $c^{2,1}_{s_0s_1,\lambda} = c^{1,2}_{s_1s_0,\lambda} = -1$. We conclude that

$$s_1s_0s_1 = w_\lambda(1,1), \quad s_1s_0 = w_\lambda(1,2), \quad s_0s_1 = w_\lambda(2,1), \quad s_0 = w_\lambda(2,2).$$

Hence, Proposition 4.9 yields a ring isomorphism $J_\lambda \cong M_2(Z)$ where

$$\begin{aligned} t_{s_1s_0s_1} &\longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, & t_{s_1s_0} &\longmapsto \begin{bmatrix} 0 & -1 \\ 0 & 0\end{bmatrix},\\ t_{s_0s_1} &\longmapsto \begin{bmatrix} 0 & 0 \\ -1 & 0\end{bmatrix}, & t_{s_0} &\longmapsto \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}. \end{aligned}$$

Note that one also obtains an isomorphism $J_\lambda \cong M_2(\mathbb{Z})$ by sending the above four elements t_x directly to the corresponding matrix units (omitting the sign in the matrices), as in [20, 18.7]. The signs arise from the construction in the proof of Proposition 4.9 and the choice of the orthogonal representation. Note that the latter is not unique: for example, one can conjugate $\mathfrak{X}_\lambda$ by a diagonal matrix with ± 1 on the diagonal.

Proof of Theorem 1.5. Let us recall some ingredients of the construction of the Dipper-James-Murphy basis. Let $\lambda = (\lambda_1, \lambda_2)$ be a bipartition of n; let $(\mathfrak{s}, \mathfrak{t})$ be a pair of standard bitableaux of shape λ. By [6, §4], we have $x_{\mathfrak{st}} = T_d x_\lambda T_{d'}$ where d, d' are certain elements in $\mathfrak{S}_n$; the element x_λ is defined in [6, 4.1]. Since N^λ is a two-sided ideal, we conclude that

$$N^\lambda = \sum_{\mu \in \Lambda_n;\, \lambda \trianglelefteq \mu} \mathcal{H}_n x_\mu \mathcal{H}_n.$$

We must show that $N^\lambda = M^\lambda$, where we set

$$M^\lambda := \left\langle C'_w \;\middle|\; \begin{array}{l} w \text{ corresponds to an RS-cell of shape} \\ \nu = (\nu_1, \nu_2) \text{ where } (\lambda_1, \lambda_2) \trianglelefteq (\nu_2, \nu_1^*) \end{array} \right\rangle_A \subseteq \mathcal{H}_n.$$

One easily checks that $(\nu_1, \nu_2^*) \trianglelefteq (\lambda_2^*, \lambda_1^*) \Leftrightarrow (\lambda_1, \lambda_2) \trianglelefteq (\nu_2, \nu_1^*)$ (using the analogous statement for the dominance order on partitions; see [14, Exercise 5.6]). Hence, by Proposition 5.4, M^λ is a two-sided ideal of $\mathcal{H}_n$.

We now show that $N^\lambda \subseteq M^\lambda$. Let $\mu = (\mu_1, \mu_2)$ be a bipartition of n such that $(\lambda_1, \lambda_2) \trianglelefteq (\mu_1, \mu_2)$. Let $l := |\mu_1|$. The element x_μ is defined as the product of three factors u_l^+, x_{μ_1}, x_{μ_2}. The formula in Bonnafé [3, Prop. 2.5] shows that, up to multiplying by a monomial in V and v, the factor u_l^+ equals $T_{\sigma_l} C'_{a_l}$ where σ_l is the longest element in $\mathfrak{S}_l = \langle s_1, \ldots, s_{l-1} \rangle$. Furthermore, by Lusztig [20, Cor. 12.2], we have $x_{\mu_1} x_{\mu_2} = v^{l(w_\mu)} C'_{w_\mu}$ where w_μ is the longest element in the Young subgroup of $\mathfrak{S}_{l,n-l}$ given by $\mu = (\mu_1, \mu_2)$. Finally, by [3, Prop. 2.3], we have $C'_{a_l} C'_{w_\mu} = C'_{a_l w_\mu}$. Hence, we obtain

$$x_\mu = T_{\sigma_l} C'_{a_l w_\mu} \quad \text{(up to multiplying by a monomial in } V \text{ and } v\text{)}.$$

By relation (♠) in the proof of Proposition 5.4, $a_l w_\mu$ belongs to an RS-cell of shape (μ_2^*, μ_1). Hence, since $(\lambda_1, \lambda_2) \trianglelefteq (\mu_1, \mu_2)$ and since M^λ is an ideal, we obtain that $x_\mu \in M^\lambda$. As this holds for all μ such that $\lambda \trianglelefteq \mu$, we conclude that $N^\lambda \subseteq M^\lambda$.

In order to show equality, we note that M^λ is free over A of rank $\sum_{\lambda \trianglelefteq \nu} d_\nu^2$, where d_ν denotes the number of standard bitableaux of shape ν. By [6, 4.15], N^λ is free over A of the same rank. Consequently, we have $K_0 \otimes_A N^\lambda = K_0 \otimes_A M^\lambda$, where K_0 is the field of fractions of A. So there exists some $0 \neq f \in A$ such that $f M^\lambda \subseteq N^\lambda \subseteq M^\lambda$. Now, since the generators of N^λ can be extended to an A-basis of $\mathcal{H}_n$ (see [6, §4]), the quotient $\mathcal{H}_n / N^\lambda$ is a free A-module. Hence, $M^\lambda \subseteq N^\lambda$ and the conclusion follows.

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